Online Appendix to "Sovereign Default Risk and the US Equity Market"

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A Numeraire

This Appendix shows that the price index of the world consumption basket, which is denoted by P_t and used as the numeraire, can be written as

$$P_t = P_{d,t}^{\alpha} P_{f,t}^{1-\alpha}, \tag{1}$$

where α denotes the size of Domestic production as a share of world production, while $P_{d,t}$ and $P_{f,t}$ denote the prices of the Domestic and Foreign good, X_d and X_f, respectively.

The derivation is as follows. Consider a representative aggregate agent who consumes production in both countries. This agent maximizes the utility of total consumption, given by a constant-elasticity-of-substitution (CES) function Ω_t, such that

$$\Omega_t \left(X_d, X_f\right) = \left[\alpha^{\frac{1}{\vartheta}} X_{d,t}^{\frac{\vartheta-1}{\vartheta}} + (1-\alpha)^{\frac{1}{\vartheta}} X_{f,t}^{\frac{\vartheta-1}{\vartheta}}\right]^{\frac{\vartheta}{\vartheta-1}}, \tag{2}$$

given total expenditure N_t, equal to

$$N_t \equiv P_{d,t} X_{d,t} + P_{f,t} X_{f,t}, \tag{3}$$

where $\vartheta > 0$ is the elasticity of substitution between the Domestic and Foreign goods, $X_{d,t}$ and $X_{f,t}$.

Maximizing Equation 2 subject to the constraint (Equation 3) yields

$$\left(\frac{P_{d,t}}{P_{f,t}}\right)^{\vartheta} = \frac{(1-\alpha) X_{d,t}}{\alpha X_{f,t}}. \tag{4}$$

Combining Equations 3 and 4 generates the following demand functions for the Domestic and the Foreign goods :

$$X_{d,t} = \frac{N_t \alpha P_{d,t}^{-\vartheta}}{\alpha P_{d,t}^{1-\vartheta} + (1-\alpha) P_{f,t}^{1-\vartheta}}, \quad X_{f,t} = \frac{N_t (1-\alpha) P_{f,t}^{-\vartheta}}{\alpha P_{d,t}^{1-\vartheta} + (1-\alpha) P_{f,t}^{1-\vartheta}}. \tag{5}$$

The highest value of $\Omega_t \left(X_d, X_f\right)$ given N_t is found by substituting those demands (Equation 5) into the CES function (Equation 2), which yields

$$\left[\alpha^{\frac{1}{\vartheta}}\left[\frac{N_t\alpha P_{d't}^{-\vartheta}}{\alpha P_{d,t}^{1-\vartheta}+(1-\alpha)P_{f't}^{1-\vartheta}}\right]^{\frac{\vartheta-1}{\vartheta}}+(1-\alpha)^{\frac{1}{\vartheta}}\left[\frac{N_t(1-\alpha)P_{f,t}^{-\vartheta}}{\alpha P_{d,t}^{1-\vartheta}+(1-\alpha)P_{f,t}^{1-\vartheta}}\right]^{\frac{\vartheta-1}{\vartheta}}\right]^{\frac{\vartheta}{\vartheta-1}}. \tag{6}$$

Following Obstfeld and Rogoff (1996, p.227), the consumption-based price index P_t is defined as the minimum expenditure $N_t = P_{d,t}X_{d,t} + P_{f,t}X_{f,t}$ such that $\Omega_t(X_d, X_f) = 1$, given $P_{d,t}$ and $P_{f,t}$. The price index P_t thus satisfies

$$\left[\alpha^{\frac{1}{\vartheta}}\left[\frac{P_t\alpha P_{d,t}^{-\vartheta}}{\alpha P_{d,t}^{1-\vartheta}+(1-\alpha)P_{f,t}^{1-\vartheta}}\right]^{\frac{\vartheta-1}{\vartheta}}+(1-\alpha)^{\frac{1}{\vartheta}}\left[\frac{P_t(1-\alpha)P_{f,t}^{-\vartheta}}{\alpha P_{d,t}^{1-\vartheta}+(1-\alpha)P_{f,t}^{1-\vartheta}}\right]^{\frac{\vartheta-1}{\vartheta}}\right]^{\frac{\vartheta}{\vartheta-1}}=1, \tag{7}$$

which leads to the solution

$$P_t = \left[\alpha P_{d,t}^{1-\vartheta} + (1-\alpha)P_{f,t}^{1-\vartheta}\right]^{\frac{1}{1-\vartheta}}. \tag{8}$$

I now determine the price function P_t for the Cobb-Douglas case ($\vartheta = 1$). For convenience, take the logarithm of the above solution, which implies that

$$\ln(P_t) = \frac{\ln\left[\alpha P_{d,t}^{1-\vartheta} + (1-\alpha)P_{f,t}^{1-\vartheta}\right]}{1-\vartheta}, \tag{9}$$

and then apply L'Hospital's rule on Equation 9, which yields

$$\lim_{\vartheta\to 1}\ln(P_t) = \alpha\ln(P_{d,t}) + (1-\alpha)\ln(P_{f,t}) \tag{10}$$

and eventually

$$P_t = P_{d,t}^{\alpha}P_{f,t}^{1-\alpha}. \tag{11}$$

B Exchange Rate

In a competitive equilibrium, the price of one unit of the Domestic good to be delivered at time t in state w is $\xi_{d,t} = P_{d,t}\xi_t$ and the price of one unit of the Foreign good to be delivered at time t in state w is $\xi_{f,t} = P_{f,t}\xi_t$, where ξ_t is the equilibrium state-price density (in units of the world numeraire). Following Pavlova and Rigobon (2007), these prices are given by

$$\xi_{d,t} = P_{d,t}\xi_t = e^{-\beta t}\frac{\partial u\left(y_{d,t}\right)}{\partial y_{d,t}} = \frac{e^{-\beta t}}{y_{d,t}} = \frac{e^{-\beta t}}{X_{d,t}} \tag{12}$$

$$\xi_{f,t} = P_{f,t}\xi_t = e^{-\beta t}\frac{\partial u\left(y_{f,t}\right)}{\partial y_{f,t}} = \frac{e^{-\beta t}}{y_{f,t}} = \frac{e^{-\beta t}}{X_{f,t}}, \tag{13}$$

where β is the rate of time preference, y_j denotes consumption of country j's agent, and the subscript $j = \{d, f\}$ indicates the Domestic and the Foreign country, respectively. Finally, the last equalities obtain from the good market clearing conditions (i.e., consumption $y_{j,t}$ equals production $X_{j,t}$ in autarky).

The exchange rate, denoted by S_t, is defined as the ratio of the country's marginal utilities of the Domestic and Foreign goods, $\xi_{d,t}$ and $\xi_{f,t}$.[1] Given the preferences of agents, state prices are unique, as is the ratio of the two. The exchange rate is thus given by

$$S_t \equiv \frac{\xi_{d,t}}{\xi_{f,t}} = \frac{u_d\left(y_{d,t}\right)}{u_f\left(y_{f,t}\right)} = \frac{P_{d,t}}{P_{f,t}} = \frac{X_{f,t}}{X_{d,t}}, \tag{14}$$

which is also equal to the price of the Domestic good $P_{d,t}$ per unit of the price of the Foreign good $P_{f,t}$.

C Output Value

This Appendix determines the value of output in both countries and derives its dynamics.

C.1 Domestic and Foreign output value

From the definition of the exchange rate, $S_t = P_{d,t}/P_{f,t}$, and the normalized world basket numeraire $P_{d,t}^{\alpha}P_{f,t}^{1-\alpha} = 1$ (see Appendix A), prices of the Domestic and Foreign goods can be written $P_{d,t} = S_t^{1-\alpha}$ and $P_{f,t} = S_t^{-\alpha}$, respectively. Using the solution of the exchange rate

[1] For reference, see Dumas (1992), Backus, Foresi, and Telmer (2001), Brandt, Cochrane, and Santa-Clara (2006), and Bakshi, Carr, and Wu (2008), among others.

(Equation 14), the value of Domestic and Foreign output is given by

$$
\begin{aligned}
P_{d,t}X_{d,t} &= S_t^{1-\alpha}X_{d,t} = \left(\frac{X_{f,t}}{X_{d,t}}\right)^{1-\alpha} X_{d,t} = X_t && (15)\\
P_{f,t}X_{f,t} &= S_t^{-\alpha}X_{f,t} = \left(\frac{X_{f,t}}{X_{d,t}}\right)^{-\alpha} X_{f,t} = X_t, && (16)
\end{aligned}
$$

where $X_t \equiv X_{d,t}^{\alpha}X_{f,t}^{1-\alpha}$ is an aggregate measure of Domestic and Foreign output. Given that $P_{d,t}X_{d,t} = P_{f,t}X_{f,t}$, the value of output perfectly comoves internationally, even though the output level correlates imperfectly across countries.

C.2 Aggregate output value

The value of Domestic and Foreign output is solely determined by the aggregate output value X_t. To derive the dynamics of X_t, apply Itô's formula on $X_{d,t}^{\alpha}X_{f,t}^{1-\alpha}$, which yields

$$
\begin{aligned}
dX_t &= \alpha X_t\left(\theta_d dt + \sigma_d dW_t^d\right) + (1-\alpha)X_t\left(\theta_{f,s}dt + \sigma_f dW_t^f\right)\\
&\quad -\frac{\alpha(1-\alpha)}{2}X_t\left(\sigma_d^2 + \sigma_f^2 - 2\rho\sigma_d\sigma_f\right)dt && (17)\\
&= X_t\left\{\begin{array}{c}\left[\alpha\theta_d + (1-\alpha)\theta_{f,s} - \frac{\alpha(1-\alpha)}{2}\left(\sigma_d^2 + \sigma_f^2 - 2\rho\sigma_d\sigma_f\right)\right]dt\\ +\alpha\sigma_d dW_t^d + (1-\alpha)\sigma_f dW_t^f\end{array}\right\}. && (18)
\end{aligned}
$$

The dynamics of X_t are thus characterized by the process

$$
\frac{dX_t}{X_t} = \theta_{X,s}dt + \sigma_{X,d}dW_t^d + \sigma_{X,f}dW_t^f, \quad s = \{L, H\}, \tag{19}
$$

with

$$
\theta_{X,s} = \alpha\theta_d + (1-\alpha)\theta_{f,s} - \frac{\alpha(1-\alpha)}{2}\left(\sigma_d^2 + \sigma_f^2 - 2\rho\sigma_d\sigma_f\right), \tag{20}
$$

$$
\sigma_{X,f} = (1-\alpha)\sigma_f, \quad \text{and} \quad \sigma_{X,d} = \alpha\sigma_d. \tag{21}
$$

D State-Price Density

This Appendix derives the state-price density (or the state-price deflator), ξ_t, that prevails in a competitive equilibrium of the economy.

In the familiar single-good asset-pricing framework, the consumption good's price is normalized to one and the agents' marginal utilities are proportional to the state-price density. Here, one needs to use the generalization of the standard argument to multiple-good economies. Given that the world numeraire basket is of the form $P_t = P_{d,t}^{\alpha} P_{f,t}^{1-\alpha}$, the state-price density ξ_t can be written as

$$\xi_t \quad = \left(\xi_t P_{d,t}\right)^{\alpha} \quad \left(\xi_t P_{f,t}\right)^{1-\alpha} = e^{-\beta t} \left(\frac{1}{X_{d,t}}\right)^{\alpha} \left(\frac{1}{X_{f,t}}\right)^{1-\alpha} = \frac{e^{-\beta t}}{X_t}, \tag{22}$$

where the first equality of Equation 22 follows from the price normalization $P_{d,t}^{\alpha} P_{f,t}^{1-\alpha} = 1$, the second obtains by substituting Equations 12 and 13 in Equation 22, and the last equality is obtained from $X_t \equiv X_{d,t}^{\alpha} X_{f,t}^{1-\alpha}$ (see Appendix C).

Applying Itô's formula to $f(t, X_t) = \xi_t$ yields

$$\begin{aligned} df(t,X) \quad &= \quad -\beta f_t dt - \frac{f_t}{X_t}\left(\theta_{X,s} X_t dt + \sigma_{X,d} X_t dW_t^d + \sigma_{X,f} X_t dW_t^f\right) \\ &\quad + \frac{f_t}{X_t^2}\left[\left(\sigma_{X,d} X_t\right)^2 dt + \left(\sigma_{X,f} X_t\right)^2 dt + 2\rho \sigma_{X,d} \sigma_{X,f} dt\right]. \end{aligned} \tag{23}$$

The state-price density thus follows the process defined by

$$\frac{d\xi_t}{\xi_t} = -r_s dt - \sigma_{X,d} dW_t^d - \sigma_{X,f} dW_t^f, \quad s = \{L, H\}, \tag{24}$$

where r_s is the risk-free rate prevailing in equilibrium in state s, which equals

$$r_s = \beta + \theta_{X,s} - \left(\sigma_{X,d}^2 + \sigma_{X,f}^2 + 2\rho \sigma_{X,d} \sigma_{X,f}\right). \tag{25}$$

E Firm Revenue

This Appendix derives the dynamics of firm revenue under the physical and the risk-neutral probability measures.

E.1 Physical revenue

Firm i in country j produces a quantity $X_{ij,t}$ that is sold at a price $P_{j,t}$. This firm's revenue, denoted by $R_{i,t} \equiv P_{j,t}X_{ij,t}$, satisfy the dynamics

$$\frac{dR_{i,t}}{R_{i,t}} = \theta_{X,s}dt + \sigma_{X,d}dW_t^d + \sigma_{X,f}dW_t^f, \quad s = \{L, H\}, \tag{26}$$

where the country's subscript (j) can be ignored when denoting firm revenue because of the perfect co-movement of revenue across countries (see Appendix C). The aggregation of firm revenue within a country equals the country's output value, such that

$$\int R_{i,t}dG(I_j) \equiv X_t, \tag{27}$$

where $G(I_j)$ denotes the distribution of firms in country j.

E.2 Risk-neutral measure

Let us now derive the dynamics of firm revenue under the risk-neutral measure $\mathbb{Q}$. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be the probability space on which the Brownian motions are defined. The corresponding information filtration is $F = \{\mathscr{F}_t : t \geq 0\}$. For convenience, start by rewriting the state-price density as follows

$$\begin{aligned} \frac{d\xi_t}{\xi_t} &= -r_s dt - \sigma_{X,d}dW_t^d - \sigma_{X,f}dW_t^f & (28) \\ &= -r_s dt - (\sigma_{X,d} + \rho\sigma_{X,f})\, dW_t^d - \sigma_{X,f}\sqrt{1-\rho^2}dW_t, & (29) \end{aligned}$$

using the notation $dW_t^f = \rho dW_t^d + \sqrt{1-\rho^2}dW_t$, where W_t denotes a standard Brownian motion independent of W_t^d, such that $dW_t dW_t^d = 0$.

I define the risk-neutral measure $\mathbb{Q}$ associated with the pricing kernel under the world basket numeraire by specifying the density process φ_t, which satisfies

$$\varphi_t = \mathbb{E}_t \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \right] \tag{30}$$

and evolves as follows

$$\frac{d\varphi_t}{\varphi_t} = - \left(\sigma_{X,d} + \rho \sigma_{X,f} \right) dW_t^d - \sigma_{X,f} \sqrt{1-\rho^2} dW_t. \tag{31}$$

Applying the Girsanov theorem, one obtains new Brownian motions under $\mathbb{Q}$, $\tilde{W}_t^d$ and $\tilde{W}_t$, which solve

$$\begin{aligned} dW_t^d &= d\tilde{W}_t^d - \left(\sigma_{X,d} + \rho \sigma_{X,f} \right) dt & (32) \\ dW_t &= d\tilde{W}_t - \sigma_{X,f} \sqrt{1-\rho^2} dt. & (33) \end{aligned}$$

Finally, by substitution, firm revenue $R_{i,t}$ under the risk-neutral probability measure $\mathbb{Q}$ follow the dynamics

$$\begin{aligned} \frac{dR_{i,t}}{R_{i,t}} &= \theta_{X,s} dt + \left(\sigma_{X,d} + \rho \sigma_{X,f} \right) \left[d\tilde{W}_t^d - \left(\sigma_{X,d} + \rho \sigma_{X,f} \right) dt \right] \\ &\quad + \sigma_{X,d} \sqrt{1-\rho^2} \left[d\tilde{W}_t - \sigma_{X,f} \sqrt{1-\rho^2} dt \right] & (34) \\ &= \tilde{\theta}_{X,s} dt + \left(\sigma_{X,d} + \rho \sigma_{X,f} \right) d\tilde{W}_t^d + \sigma_{X,f} \sqrt{1-\rho^2} d\tilde{W}_t, \quad s = \{L, H\}, & (35) \end{aligned}$$

with

$$\tilde{\theta}_{X,s} = \theta_{X,s} - \left(\sigma_{X,d}^2 + \sigma_{X,f}^2 + 2\rho \sigma_{X,d} \sigma_{X,f} \right). \tag{36}$$

F Stationary Leverage Property and Change of Variables

This Appendix first describes the firm capital structure and the operating costs that generate stationary financial and operational leverage ratios. It then presents a change of variables that helps solve the model conveniently.

F.1 Capital structure and operating costs

Empirical observation indicates that leverage ratios are stationary (see Hovakimian, Opler, and Titman, 2001). Thus, firms tend to issue additional debt in response to a growth in firm value. This empirical evidence is consistent with Collin-Dufresne and Goldstein (2001) and Goldstein, Ju, and Leland (2001), who demonstrate that models with stationary leverage ratios help better explain financial leverage and credit spreads. Consequently, stationary leverage (i.e., which does not vanish over time) is an important property to capture when valuing firm assets.

To account for this feature, the model assumes that firms maintain a long-term stationary leverage ratio. That is, their debt coupon relative to firm revenue, $C^F_{j,t}/R_{i,t}$, must remain stationary. A capital structure characterized by a debt coupon that displays the same long-term growth rate as that of firm revenue satisfies this leverage stationarity. Hence, I consider a debt coupon $C^F_{j,t}$ in country j determined by

$$C^F_{j,t} \equiv g_t C^F_j, \tag{37}$$

where, for ease of notation, $C^F_j = C^F_{j,0}$, and g_t denotes the expected growth of firm revenue up to time t, which is given by

$$g_t \equiv \mathbb{E}_0 \left[\frac{R_{i,t}}{R_{i,0}} \mid_{s=H} \right] = e^{\theta_{X,H} t}. \tag{38}$$

The expected growth of firm revenue g_t depends only on the state $s = H$. The reason is that g_t captures the growth of firm revenue over the long run, which is almost surely characterized by the state $s = H$ when $t \to \infty$ and $\lambda_{LH} > 0$ (i.e., the state $s = L$ is temporary).

The firm operating costs are determined similarly to generate a stationary operational leverage ratio. Following the reasoning above, the firm i in country j has operating costs $I_{ij,t}$ that are given by

$$I_{ij,t} \equiv g_t I_{ij}, \tag{39}$$

where, for ease of notation, $I_{ij} = I_{ij,0}$.

When the default policy is endogenous, the default boundary becomes linear in the debt coupon and the level of operating costs.[2] Hence, a firm's default boundary increases exponentially when its debt coupons and operating costs grow over time. Solving a first hitting time problem is not trivial in this case.

However, a suitable change of variables and an application of Itô's Lemma allow deriving the model in the more standard case of a first-passage time problem with constant boundary. This approach, which is described in the next section, follows Ju, Parrino, Poteshman, and Weisbach (2005).

F.2 Variable scaling

Let us define the following set of variables

$$C_j^F \equiv C_{j,t}^F/g_t, \quad I_{ij} \equiv I_{ij,t}/g_t, \quad Z_{i,t} \equiv R_{i,t}/g_t, \tag{40}$$

which are obtained by scaling the debt coupon $C_{j,t}^F$, the operating costs $I_{ij,t}$, and the firm revenue $R_{i,t}$, respectively, by the growth of firm revenue g_t, which is a deterministic function of time. The scaled debt coupon C_j^F and operating costs I_{ij} of firm i in country j are constant and equal to the levels of $C_{j,t}^F$ and $I_{ij,t}$ observed at time $t = 0$.

The scaled revenue $Z_{i,t}$ of firm i satisfy, from Itô's Lemma, the dynamics

$$\frac{dZ_{i,t}}{Z_{i,t}} = \theta_{Z,s}dt + \sigma_{X,d}dW_t^d + \sigma_{X,f}dW_t^f, \quad s = \{L, H\}, \tag{41}$$

with $\theta_{Z,s} = \theta_{X,s} - \theta_{X,H}$, while the corresponding risk-neutral growth rate is

$$\tilde{\theta}_{Z,s} = \theta_{Z,s} - \left(\sigma_{X,d}^2 + \sigma_{X,f}^2 + 2\rho\sigma_{X,d}\sigma_{X,f}\right). \tag{42}$$

Finally, let us define a scaled measure of the risk-free rate, denoted by $\bar{r}_s$, and equal to

$$\bar{r}_s \equiv r_s - \theta_{X,H}, \tag{43}$$

[2] Leland (1994) shows that the optimal default threshold linearly depends on the debt coupon, in absence of operational leverage. Appendix H shows that the optimal default threshold linearly depends on both the debt coupon and the level of operating costs when operational leverage is introduced in the model (see Equation 95).

which will be convenient to price assets in the economy.

Firm assets can then be evaluated using the scaled firm revenue $Z_{i,t}$, a capital structure with constant debt coupon C_j^F and operating costs I_{ij}, and constant optimal default boundaries.

G Firm Asset Valuation

This Appendix derives the value of a firm's equity and debt under the risk of Foreign government default. In the absence of arbitrage, the levered firm value equals the sum of debt and equity values $V_{ij,t}(Z_i) = E_{ij,t}(Z_i) + D_{ij,t}(Z_i)$, which depend on the scaled firm revenue $Z_{i,t}$ (see Appendix F).

Consider for now that firm i in country j defaults at time $T_{ij}^D = \inf\{t \geq 0 \mid Z_{i,t} \leq Z_{ij}^D\}$, when $Z_{i,t}$ falls to the constant default threshold Z_{ij}^D for the first time, while the Foreign government defaults when $Z_{i,t}$ falls to the default threshold $Z_i^G < Z_{ij}^D$, which occurs at time $T^G = \inf\{t \geq 0 \mid Z_{i,t} \leq Z_i^G\}$.[3]

G.1 Firm defaults before the Foreign government defaults

This section provides the asset valuation of firm i in country j when this firm defaults before the Foreign government defaults, i.e. $T_{ij}^D < T^G$.

[3] The firm optimal default policy is determined subsequently (see Appendix H). The details of the Foreign government's default policy will be examined in Appendix J.

G.1.1 Equity value

The value of equity for firm i in country j is given by

$$\begin{aligned}
E_{ij,t}(Z_i) &= \mathbb{E}_t\left[\int_t^{T_{ij}^D} \frac{\xi_u}{\xi_t}(1-\tau_j)\left(R_{i,u} - C_{j,u}^F - I_{ij,u}\right) du\right] && (44)\\
&= \mathbb{E}_t\left[\int_t^{T_{ij}^D} \frac{\xi_u}{\xi_t}(1-\tau_j)\left(\frac{R_{i,u}}{e^{\theta_{X,H}u}} - \frac{C_j^F}{e^{\theta_{X,H}u}} - \frac{I_{ij}}{e^{\theta_{X,H}u}}\right) e^{\theta_{X,H}u} du\right] && (45)\\
&= \mathbb{E}_t\left[g_t \int_t^{T_{ij}^D} \frac{\xi_u}{\xi_t}(1-\tau_j)\left(Z_{i,u} - C_j^F - I_{ij}\right) e^{\theta_{X,H}(u-t)} du\right] && (46)\\
&= \mathbb{E}_t^{\mathbb{Q}}\left[g_t \int_t^{T_{ij}^D} (1-\tau_j)\left(Z_{i,u} - C_j^F - I_{ij}\right) e^{-\bar{r}_H(u-t)} du\right], && (47)
\end{aligned}$$

where τ_j is the tax rate in country j and Equation 45 introduces the change of variables proposed in Appendix F.

Solving for the equity value yields

$$\begin{aligned}
E_{ij,t}(Z_i) &= \mathbb{E}_t^{\mathbb{Q}}\left[g_t \int_t^{\infty} (1-\tau_j)\left(Z_{i,u} - C_j^F - I_{ij}\right) e^{-\bar{r}_H(u-t)} du\right]\\
&\quad - \mathbb{E}_t^{\mathbb{Q}}\left[g_t \int_{T_{ij}^D}^{\infty} (1-\tau_j)\left(Z_{i,u} - C_j^F - I_{ij}\right) e^{-\bar{r}_H(u-t)} du\right] && (48)\\
&= (1-\tau_j) g_t \left(\frac{Z_{i,t}}{\bar{r}_H - \tilde{\theta}_{Z,H}} - \frac{C_j^F + I_{ij}}{\bar{r}_H}\right)\\
&\quad - (1-\tau_j) g_t \left(\frac{Z_{ij}^D}{\bar{r}_H - \tilde{\theta}_{Z,H}} - \frac{C_j^F + I_{ij}}{\bar{r}_H}\right)\left(\frac{Z_{i,t}}{Z_{ij}^D}\right)^{\omega_H}, && (49)
\end{aligned}$$

where the present value of $Z_{i,t}$ is given by the standard Gordon formula

$$\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\infty} Z_{i,u} e^{-\bar{r}_H(u-t)} du\right] = \frac{Z_{i,t}}{\bar{r}_H - \tilde{\theta}_{Z,H}}, \qquad (50)$$

while the present value of the perpetual stream of payments $C_j^F + I_{ij}$ is given by

$$\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\infty} \left(C_j^F + I_{ij}\right) e^{-\bar{r}_H(u-t)} du\right] = \frac{C_j^F + I_{ij}}{\bar{r}_H} \qquad (51)$$

and the second part of the Equation 49 is obtained as follows: First, it is known from Karatzas and Shreve (1991, p.197) that

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\bar{r}_H\left(T_{ij}^D-t\right)}\right]=\left(\frac{Z_{i,t}}{Z_{ij}^D}\right)^{\omega_H}, \tag{52}$$

where ω_s is the negative root of the quadratic equation $\frac{1}{2}\sigma_Z^2\omega_s(\omega_s-1)+\tilde{\theta}_{Z,s}\omega-\bar{r}_s=0$ in state s, defined by

$$\omega_s=\frac{1}{2}-\frac{\tilde{\theta}_{Z,s}}{\sigma_Z^2}-\sqrt{\left(\frac{1}{2}-\frac{\tilde{\theta}_{Z,s}}{\sigma_Z^2}\right)^2+\frac{2\bar{r}_s}{\sigma_Z^2}}<0,\quad s=\{L,H\}, \tag{53}$$

with $\sigma_Z=\sqrt{\sigma_{X,d}^2+\sigma_{X,f}^2+2\rho\sigma_{X,d}\sigma_{X,f}}$. Then, from the strong Markov property for Brownian motion,

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}\left[\int_{T_{ij}^D}^{\infty}Z_{i,u}e^{-\bar{r}_H(u-t)}du\right] &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\bar{r}_H\left(T_{ij}^D-t\right)}Z_{ij}^D\int_t^{\infty}e^{\sigma_Z Z_{i,u}-\left(\bar{r}_H-\tilde{\theta}_{Z,H}+\frac{\sigma_Z^2}{2}\right)(u-t)}du\right]\\ &= \left(\frac{Z_{ij}^D}{\bar{r}_H-\tilde{\theta}_{Z,H}}\right)\left(\frac{Z_{i,t}}{Z_{ij}^D}\right)^{\omega_H}. \end{aligned}\tag{54}$$

G.1.2 Corporate debt

Bondholders receive the coupon $C_{j,t}^F$ as long as the firm does not default. In the case of default, bondholders are entitled to perpetual value of the after-tax cash flows of the unlevered firm $X_{i,t}-I_{ij,t}$, net of a fraction $\eta\in(0,1)$ that consists of default costs.

Using the same change of variables and the Brownian motion properties discussed in Appendix G.1.1, the value of corporate debt is given by

$$\begin{aligned}D_{ij,t}(Z_i) &= \mathbb{E}_t\left[g_t\int_t^{T_{ij}^D}\frac{\xi_u}{\xi_t}C_j^F e^{\theta_{X,H}(u-t)}du\right]\\ &\quad+\mathbb{E}_t\left[g_t\int_{T_{ij}^D}^{\infty}\frac{\xi_u}{\xi_t}(1-\eta)(1-\tau_j)(Z_{i,u}-I_{ij})e^{\theta_{X,H}(u-t)}du\right] \qquad (55)\\ &= \frac{C_j^F g_t}{\bar{r}_H}\left[1-\left(\frac{Z_{i,t}}{Z_{ij}^D}\right)^{\omega_H}\right]+V_{ij,T_{ij}^D}(Z_i)g_t\left(\frac{Z_{i,t}}{Z_{ij}^D}\right)^{\omega_H}, \qquad (56)\end{aligned}$$

with $V_{ij,T_{ij}^D}(Z_i)$ being the unlevered firm asset value at the time of default, which equals

$$\begin{aligned}
V_{ij,T_{ij}^D}(Z_i) &= +\mathbb{E}_t\left[\int_{T_{ij}^D}^{T^G}\frac{\xi_u}{\xi_t}(1-\eta)(1-\tau_j)(Z_{i,u}-I_{ij})e^{\theta_{X,H}(u-t)}du\right] \\
&\quad +\mathbb{E}_t\left[\int_{T^G}^{\infty}\frac{\xi_u}{\xi_t}(1-\eta)(1-\tau_j)(Z_{i,u}-I_{ij})e^{\theta_{X,H}(u-t)}du\right] && (57) \\
&= (1-\tau_j)(1-\eta)\left(\frac{Z_{ij}^D}{\bar{r}_H-\tilde{\theta}_{Z,H}}-\frac{I_{ij}}{\bar{r}_H}\right)^+ \\
&\quad -(1-\tau_j)(1-\eta)\left(\frac{Z_i^G}{\bar{r}_H-\tilde{\theta}_{Z,H}}-\frac{I_{ij}}{\bar{r}_H}\right)^+\left(\frac{Z_{ij}^D}{Z_i^G}\right)^{\omega_H} \\
&\quad +V_{ij,T^G}(Z_i)\left(\frac{Z_{ij}^D}{Z_i^G}\right)^{\omega_H}, && (58)
\end{aligned}$$

where the upperscript $^+$ indicates that debtholders are only entitled to the unlevered firm value if it is positive, Z_i^G is the level of the firm i's scaled revenue at time of sovereign default T^G, while $V_{ij,T^G}(Z_i)$ denotes unlevered firm asset value at time of the sovereign default. This value is derived as follows.

Once the Foreign government has defaulted, the state of the economy temporarily switches from $s=H$ to $s=L$, which decreases the cash flow growth rate and the risk-free rate. There is an exogenous intensity λ_{LH} such that the probability that the regime returns to the pre-crisis state over the next infinitesimal time instant dt is $\lambda_{LH}dt$.[4] The discounting value of the after-tax cash flows of the unlevered firm after sovereign default must then account for the stochastic change in regime from $s=L$ to $s=H$. To do so, I follow the derivation of Bhamra, Kuehn, and Strebulaev (hereafter BKS) (2010a,b).[5] The unlevered value of the firm at time of the sovereign default, when the state is $s=L$, satisfies

$$\begin{aligned}
V_{ij,T^G}(Z_i) &= \mathbb{E}_t\left[\int_{T^G}^{\infty}\frac{\xi_u}{\xi_{T^G}}(1-\tau_j)(1-\eta)(Z_{i,u}-I_{ij})e^{\theta_{X,H}\left(u-T^G\right)}du\right] \\
&= (1-\tau_j)(1-\eta)\left(\frac{Z_i^G}{r_{A,L}}-\frac{I_{ij}}{r_{P,L}}\right)^+,
\end{aligned}$$

[4]With logarithmic utility (and more generally under CRRA preferences), the risk-neutral and the actual switching probabilities are identical.

[5]In BKS (2010a,b), the economy stochastically switches from a state to another. In the present paper, there is only one stochastic regime change that is absorbing.

where the discount rate of $Z_{i,t}$, when the current state is s, is given by

$$r_{A,s} = \bar{r}_s - \tilde{\theta}_{Z,s} + \frac{\left(\bar{r}_H - \tilde{\theta}_{Z,H}\right) - \left(\bar{r}_L - \tilde{\theta}_{Z,L}\right)}{\lambda_{LH} + \bar{r}_H - \tilde{\theta}_{Z,H}} \lambda_{LH} \mathbf{1}_{\{s=L\}}, \tag{59}$$

where $\mathbf{1}_{\{s=L\}}$ equals one if the current state is $s = L$ and zero otherwise. The discount rate $r_{A,L}$, relevant at time T^G, accounts for the possibility that the economy quits the low state and returns to the normal situation with an intensity $\lambda_{LH} > 0$.

Finally, the discount rate for a riskless perpetuity, when the current state is s, is given by

$$r_{P,s} = \bar{r}_s + \frac{\bar{r}_H - \bar{r}_L}{\lambda_{LH} + \bar{r}_H} \lambda_{LH} \mathbf{1}_{\{s=L\}}, \tag{60}$$

which indicates that the discount rate $r_{P,L}$ is greater than $\bar{r}_L$ because the risk-free rate is expected to increase in the future when economic growth returns to the normal state.

G.2 Firm defaults after the Foreign government defaults

This section examines the case in which a firm i in country j defaults after the Foreign government defaults, i.e. $T_{ij}^D > T^G$. Consider the same change in variables as in section G.1.

G.2.1 Equity

The value of equity for firm i in country j is given by

$$\begin{aligned}
E_{ij,t}(Z_i) &= \mathbb{E}_t \left[g_t \int_t^{T^G} \frac{\xi_u}{\xi_t} (1-\tau_j) \left(Z_{i,u} - C_j^F - I_{ij} \right) e^{\theta_{X,H}(u-t)} du \right] \\
&\quad + \mathbb{E}_t \left[\frac{\xi_{T^G}}{\xi_t} e^{\theta_{X,H}\left(T^G - t\right)} E_{ij,T^G}(Z_i) \right] \qquad (61) \\
&= (1-\tau_j) g_t \left(\frac{Z_{i,t}}{\bar{r}_H - \tilde{\theta}_{Z,H}} - \frac{C_j^F + I_{ij}}{\bar{r}_H} \right) \\
&\quad - (1-\tau) g_t \left(\frac{Z_i^G}{\bar{r}_H - \tilde{\theta}_{Z,H}} - \frac{C_j^F + I_{ij}}{\bar{r}_H} \right) \left(\frac{Z_{i,t}}{Z_i^G} \right)^{\omega_H} \\
&\quad + g_t E_{ij,L,T^G}(Z_i) \left(\frac{Z_{i,t}}{Z_i^G} \right)^{\omega_H}, \qquad (62)
\end{aligned}$$

where the equity value at time of the sovereign default, denoted by $E_{ij,L,T^G}(Z_i)$, accounts for the fact that $s = L$ at T^G but can change stochastically to $s = H$. As in BKS (2010a,b), the firm can default in either state.

Firm asset valuation in a regime-switching model is determined in terms of the prices of a set of Arrow-Debreu corporate default claims. The Arrow-Debreu corporate default claim, denoted by q_{D,ij,s,s_D}, is the value of a unit of consumption paid if a firm i in country j defaults in state $s_D = \{L, H\}$ when the current state is $s = \{L, H\}$. Since each Arrow-Debreu corporate default claim is effectively a perpetual digital put, their values can be derived by solving a system of ordinary differential equations (see Appendix G.3).

The value of equity for firm i located in country j at the time of sovereign default, when the state is $s = L$, is equal to

$$E_{ij,L,T^G}(Z_i) = (1-\tau_j)\mathbb{E}_t\left[\int_{T^G}^{T_{ij}^D} \frac{\xi_u}{\xi_{T^G}} \left(Z_{i,u} - C_j^F - I_{ij}\right) e^{\theta_{X,H}\left(u-T^G\right)} du\right] \tag{63}$$

$$= A_{ij,L}\left(Z_i^G\right) - (1-\tau_j)\frac{C_j^F}{r_{P,L}} + \sum_{s_D=L}^{H} q_{D,ij,L,s_D}\left[(1-\tau_j)\frac{C_j^F}{r_{P,s_D}} - A_{ij,s_D}\left(Z_{ij,s_D}^D\right)\right], \tag{64}$$

where $A_{ij,s}(Z_{i,t})$ denotes the firm liquidation value at time t and in state s, which is given by

$$A_{ij,s}(Z_{i,t}) = \mathbb{E}_t\left[\int_t^\infty \frac{\xi_u}{\xi_t}(1-\tau_j)(Z_{i,u} - I_{ij})\, e^{\theta_{X,H}(u-t)} du\right] \tag{65}$$

$$= (1-\tau_j)\left(\frac{Z_{i,t}}{r_{A,s}} - \frac{I_{ij}}{r_{P,s}}\right), \quad s = \{L, H\}. \tag{66}$$

As Equation 64 indicates, the firm default policy is characterized by two boundaries Z_{ij,s_D}^D, depending on whether the firm defaults before ($s_D = L$) or after ($s_D = H$) the economy returns to the normal state following the Foreign government's default.

G.2.2 Corporate debt

The value of corporate debt for firm i located in country j is determined by

$$
\begin{aligned}
D_{ij,t}(Z_i) &= \mathbb{E}_t\left[g_t\int_t^{T^G}\frac{\xi_u}{\xi_t}C_j^F e^{\theta_{X,H}(u-t)}du\right] \\
&\quad +\mathbb{E}_t\left[g_t\frac{\xi_{T^G}}{\xi_t}e^{\theta_{X,H}\left(T^G-t\right)}D_{ij,T^G}(Z_i)\right] && (67)\\
&= \frac{C_j^F g_t}{\bar{r}_H}\left[1-\left(\frac{Z_{i,t}}{Z_i^G}\right)^{\omega_H}\right]+D_{ij,T^G}(Z_i)g_t\left(\frac{Z_{i,t}}{Z_i^G}\right)^{\omega_H}, && (68)
\end{aligned}
$$

where the value of corporate debt at time of sovereign default is given by

$$
\begin{aligned}
D_{ij,T^G} &= \mathbb{E}_t\left[\int_{T^G}^{T_{ij}^D}\frac{\xi_u}{\xi_{T^G}}C_j^F e^{\theta_{X,H}\left(u-T^G\right)}du\right]+\mathbb{E}_t\left[\frac{\xi_{T_{ij}^D}}{\xi_{T^G}}(1-\eta)A_{ij,T_{ij}^D}\right] && (69)\\
&= \frac{C_j^F}{r_{P,L}}\left(1-\sum_{s_D=L}^{H}l_{D,s_D}q_{D,ij,L,s_D}\right), && (70)
\end{aligned}
$$

where

$$
l_{D,s_D}=\frac{\frac{C_j^F}{r_{P,s_D}}-(1-\eta)A_{ij,s_D}\left(Z_{ij,s_D}^D\right)}{\frac{C_j^F}{r_{P,L}}} \tag{71}
$$

is the loss ratio when the firm defaults in state $s_D=\{L,H\}$ (see BKS, 2010a,b).

G.3 Arrow-Debreu corporate default claims

To compute the Arrow-Debreu corporate default claims, I build on the derivation proposed by BKS (2010a,b). I impose two specific assumptions: i) the initial state at the time of sovereign default is $s=L$; ii) the change in regime ($s=L\rightarrow H$) is absorbing.

From no-arbitrage conditions, the Arrow-Debreu corporate default claims can be derived by solving a system of ordinary differential equations

$$
\begin{aligned}
&\frac{dq_{D,ij,s,s_D,t}}{dZ_{i,t}}\tilde{\theta}_{Z,s}Z_{i,t}+\frac{1}{2}\frac{d^2q_{D,ij,s,s_D,t}}{dZ_{i,t}^2}\sigma_Z^2Z_{i,t}^2+\lambda_{LH}\mathbf{1}_{\{s=L\}}\left(q_{D,ij,\bar{s},s_D,t}-q_{D,ij,s,s_D,t}\right)\\
&\qquad = \bar{r}_s q_{D,ij,s,s_D,t}, \quad s,s_D=\{L,H\},\bar{s}\neq s, \qquad (72)
\end{aligned}
$$

which are obtained, using Itô's Lemma, from the set of equations

$$\mathbb{E}_t^{\mathbb{Q}}\left[dq_{D,ij,s,s_D} - \bar{r}_s dq_{D,ij,s,s_D} dt\right] = 0, \quad s, s_D = \{L, H\}. \tag{73}$$

Given that the initial state at the time of sovereign default is $s = L$ and that the change in regime (from $s = L$ to $s = H$) is absorbing, one can directly set $q_{D,ij,H,L} = 0$.

The matrix form of Equation 72 is then given by

$$\left(\begin{bmatrix} \tilde{\theta}_{Z,L} & 0 \\ 0 & \tilde{\theta}_{Z,H} \end{bmatrix} Z_{i,t}\frac{d}{dZ_{i,t}} + \frac{1}{2}\sigma_Z^2 Z_{i,t}^2 I_{2\times 2}\frac{d^2}{dZ_{i,t}^2} + \begin{bmatrix} -\lambda_{LH} & \lambda_{LH} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \bar{r}_L & 0 \\ 0 & \bar{r}_H \end{bmatrix}\right) * \begin{bmatrix} q_{D,ij,L,L} & q_{D,ij,L,H} \\ 0 & q_{D,ij,H,H} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{74}$$

where $I_{2\times 2}$ is an identity matrix. The definitions of the payoffs of the Arrow-Debreu default claims give the boundary conditions

$$q_{D,ij,s,s_D} = \begin{cases} 1, & s = s_D, Z_{i,t} \leq Z_{ij,s}^D, \\ 0, & s \neq s_D, Z_{i,t} \leq Z_{ij,s}^D, \end{cases} \tag{75}$$

which are used to solve Equation 74 for the region $Z_{i,t} > Z_{ij,L}^D$. If we conjecture that the Arrow-Debreu corporate default claims have the form

$$q_{D,ij,s,s_D} = h_{ij,ss_D} Z_{i,t}^k, \quad s, s_D = \{L, H\}, \tag{76}$$

Equation 74 becomes

$$\left(\begin{bmatrix} \tilde{\theta}_{Z,L} & 0 \\ 0 & \tilde{\theta}_{Z,H} \end{bmatrix} k + \frac{1}{2}\sigma_Z^2 I_{2\times 2} k(k-1) + \begin{bmatrix} -\lambda_{LH} & \lambda_{LH} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \bar{r}_L & 0 \\ 0 & \bar{r}_H \end{bmatrix}\right) * \begin{bmatrix} h_{ij,LL} & h_{ij,LH} \\ 0 & h_{ij,HH} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{77}$$

There exists a solution in the above equation if

$$det\left(\begin{bmatrix} \tilde{\theta}_{Z,L} & 0 \\ 0 & \tilde{\theta}_{Z,H} \end{bmatrix} k + \frac{1}{2}\sigma_Z^2 I_{2\times 2} k(k-1) + \begin{bmatrix} -\lambda_{LH} & \lambda_{LH} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \bar{r}_L & 0 \\ 0 & \bar{r}_H \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{78}$$

which is given by solving the following quartic polynomial

$$\left[\frac{1}{2}\sigma_Z^2 k(k-1) + \tilde{\theta}_{Z,L} k - \bar{r}_L - \lambda_{LH}\right]\left[\frac{1}{2}\sigma_Z^2 k(k-1) + \tilde{\theta}_{Z,H} k - \bar{r}_H\right] = 0. \tag{79}$$

Because there are four distinct real roots in Equation 79, the general solution of the Arrow-Debreu corporate default claims should be

$$q_{D,ij,s,s_D} = \sum_m^4 h_{ij,ss_D m} Z_{i,t}^{k_m}, \tag{80}$$

where k_1, $k_2 < 0$ and k_3, $k_4 > 0$. Several restrictions help reduce the number of coefficients $h_{ij,ss_D m}$ to evaluate.

First, given that $\operatorname*{Lim}_{Z_i \to \infty} q_{D,ij,s,s_D} = 0$, only the negative roots become relevant to the solution. The Arrow-Debreu corporate default claims therefore have the following form

$$q_{D,ij,s,s_D} = \sum_m^2 h_{ij,ss_D m} Z_{i,t}^{k_m}, \tag{81}$$

where k_1 and k_2 denote, respectively, the negative roots of the quadratic equations

$$\left[\frac{1}{2}\sigma_Z^2 k(k-1) + \tilde{\theta}_{Z,L} k - \bar{r}_L - \lambda_{LH}\right] = 0 \tag{82}$$

and

$$\left[\frac{1}{2}\sigma_Z^2 k(k-1) + \tilde{\theta}_{Z,H} k - \bar{r}_H\right] = 0. \tag{83}$$

Second, the default probability in state $s = L$ only depends on the information related to that state, which suggests that $h_{ij,LL2} = 0$.

Third, Equation 77 indicates that

$$h_{ij,HHm} = h_{ij,LHm}\epsilon(k_m), \quad m = \{1, 2\} \tag{84}$$

with

$$\epsilon(k_m) = -\frac{\frac{1}{2}\sigma_Z^2 k(k-1) + \tilde{\theta}_{Z,L}k - \bar{r}_L - \lambda_{LH}}{\lambda_{LH}}, \tag{85}$$

which yields $h_{ij,HH1} = 0$ given that $\epsilon(k_1) = 0$, by definition of k (Equation 82).

Finally, under these conditions, the Arrow-Debreu corporate default claims are given by the following expressions:

$$q_{D,ij,L,L} = \begin{cases} h_{ij,LL1} Z_{i,t}^{k_1}, & Z_{i,t} > Z_{ij,L}^D, \\ 1, & Z_{i,t} \le Z_{ij,L}^D, \end{cases} \tag{86}$$

$$q_{D,ij,L,H} = \begin{cases} h_{ij,LH1} Z_{i,t}^{k_1} + h_{ij,LH2} Z_{i,t}^{k_2}, & Z_{i,t} > Z_{ij,L}^D, \\ 0, & Z_{i,t} \le Z_{ij,L}^D, \end{cases} \tag{87}$$

$$q_{D,ij,H,H} = \begin{cases} h_{ij,LH2}\epsilon(k_2) Z_{i,t}^{k_2}, & Z_{i,t} > Z_{ij,H}^D, \\ 1, & Z_{i,t} \le Z_{ij,H}^D, \end{cases} \tag{88}$$

$$q_{D,ij,H,L} = 0, \tag{89}$$

where the three constants $h_{ij,LL1}$, $h_{ij,LH1}$, and $h_{ij,LH2}$ are the solutions of the three simultaneous linear equations

$$q_{D,ij,L,L} \mid_{Z_i = Z_{ij,L}^D} = 1, \quad q_{D,ij,H,H} \mid_{Z_i = Z_{ij,H}^D} = 1, \quad q_{D,ij,L,H} \mid_{Z_i = Z_{ij,L}^D} = 0 \ . \tag{90}$$

Solving for the above coefficients is straightforward. The first two boundary conditions in

Equation 90 yield

$$h_{ij,LL1} = \left(Z_{ij,L}^{D}\right)^{-k_1}, \tag{91}$$

$$h_{ij,LH2} = \frac{1}{\epsilon(k_2)}\left(Z_{ij,H}^{D}\right)^{-k_2}, \tag{92}$$

while, after substituing Equation 92 in Equation 87, in combination with the third boundary condition of Equation 90, the last coefficient is given by

$$h_{ij,LH1} = -\frac{1}{{Z_{ij,L}^{D}}^{k_1}\epsilon(k_2)}\left(\frac{Z_{ij,L}^{D}}{Z_{ij,H}^{D}}\right)^{k_2}. \tag{93}$$

H Default policy

The determination of a firm's default policy requires knowledge on the default boundary that prevails in each possible case. Consider first the case in which the firm defaults before the Foreign government defaults.

H.1 Firm defaults before the Foreign government defaults

The firm i's default policy in country j is characterized by a constant boundary Z_{ij}^{D} that maximizes the value of equity $E_{ij,0}$ at time $t=0$, along the standard smooth-pasting condition (see Merton, 1973; Dumas, 1991). The first-order maximization yields

$$\frac{\partial E_{ij,0}(Z_i)}{\partial Z_i} = \frac{(1-\tau_j)}{\bar{r}_H-\tilde{\theta}_{Z,H}} - \frac{\omega_H}{Z_{ij}^{D}}(1-\tau_j)\left(\frac{Z_{ij}^{D}}{\bar{r}_H-\tilde{\theta}_{Z,H}} - \frac{C_j^F+I_{ij}}{\bar{r}_H}\right)\left(\frac{Z_i}{Z_{ij}^{D}}\right)^{\omega_H-1}. \tag{94}$$

Using the smooth-pasting condition $\frac{\partial[E_{ij,0}(Z_i)]}{\partial Z_i}\mid_{Z_i=Z_{ij}^{D}}=0$, the default boundary is given by

$$Z_{ij}^{D}\mid_{T^{D_{ij}}<T^G} = \frac{\omega_H\left(C_j^F+I_{ij}\right)\left(\bar{r}_H-\tilde{\theta}_{Z,H}\right)}{(\omega_H-1)\,\bar{r}_H}. \tag{95}$$

H.2 Firm defaults after the Foreign government defaults

If a firm chooses to default after the government defaults, it can occur in either regime. Hence, firm i located in country j has a set of two default boundaries, with Z_{ij,s_D}^D being the default boundary that prevails when the firm defaults in state $s_D = \{L, H\}$.

If we assume that firm i does not default before the Foreign government defaults, the cash flows entitled to shareholders before T^G become irrelevant for the timing of default. Hence, the optimal default policy is the one that maximizes equity value at time T^G, which is given by E_{ij,s,T^G} when the state is $s = \{L, H\}$. The default boundaries satisfy, for each state s, the following standard smooth-pasting condition:

$$\frac{\partial E_{ij,s,T^G}(Z_i)}{\partial Z_i} \mid_{Z_i = Z_{ij,s_D}^D} = 0, \quad s = s_D = \{L, H\}. \tag{96}$$

These state-contingent boundaries do not have analytical solutions and but can be obtained numerically.

H.3 Timing of default

The following rules determine a firm's final optimal default policy, which can be to default before, during, or after the Foreign government defaults. The optimal default policy is characterized by the default boundary Z_{ij}^D that yields the greater value of equity at time $t = 0$. It satisfies the following conditions:

$$Z_{ij}^D = \begin{cases} Z_{ij}^D \mid_{T_{ij}^D < T^G} & \text{if } Z_{ij}^D \mid_{T_{ij}^D < T^G} > Z_i^G \text{ and } E_{ij,0}(Z_i) \mid_{T_{ij}^D < T^G} \geq E_{ji,0}(Z_i) \mid_{T_{ij}^D > T^G}, \\ Z_{ij,s}^D \mid_{T_{ij}^D > T^G} & \text{if } Z_{ij,L}^D \mid_{T_{ij}^D > T^G} < Z_i^G \text{ and } E_{ij,0}(Z_i) \mid_{T_{ij}^D > T^G} > E_{ji,0}(Z_i) \mid_{T_{ij}^D < T^G}, \\ Z_i^G & \text{otherwise.} \end{cases} \tag{97}$$

I Equity Return Volatility

This Appendix derives the level of equity return volatility. Applying Itô's formula to the value of equity $E_{ij,t}$ yields

$$\begin{aligned} dE_{ij}(t, Z_i) &= E_{ij,Z,t} Z_{i,t} \left(\theta_{Z,s} dt + \sigma_{X,d} dW_t^d + \sigma_{X,f} dW_t^f\right) \\ &\quad + E_{ij,ZZ,t} Z_{i,t}^2 \left[\sigma_{X,d}^2 dt + \sigma_{X,f}^2 dt + 2\rho\sigma_{X,d}\sigma_{X,f} dt\right] \qquad (98) \\ &= \left[\theta_{Z,s} E_{ij,Z,t} Z_{i,t} + \left(\sigma_{X,d}^2 + \sigma_{X,f}^2 + 2\rho\sigma_{X,d}\sigma_{X,f}\right) E_{ij,ZZ,t} Z_{i,t}^2\right] dt \\ &\quad + E_{ij,Z,t} Z_{i,t} \left(\sigma_{X,d} dW_t^d + \sigma_{X,f} dW_t^f\right), \qquad (99) \end{aligned}$$

where $E_{ij,Z,t}$ and $E_{ij,ZZ,t}$ denote the first and second derivatives of equity value $E_{ij,t}$ with respect to $Z_{i,t}$, respectively. Hence, the dynamics of the equity return at time t are given by

$$\begin{aligned} \frac{dE_{ij,t}}{E_{ij,t}} &= \frac{1}{E_{ij,t}} \left[\theta_{Z,s} E_{ij,Z,t} Z_{i,t} + \left(\sigma_{X,d}^2 + \sigma_{X,f}^2 + 2\rho\sigma_{X,d}\sigma_{X,f}\right) E_{ij,ZZ,t} Z_{i,t}^2\right] dt \\ &\quad + \frac{E_{ij,Z,t} Z_{i,t}}{E_{ij,t}} \left(\sigma_{X,d} dW^d + \sigma_{X,f} dW^f\right). \qquad (100) \end{aligned}$$

Finally, the equity return volatility of firm i in country j is given by

$$\sigma_{E_{ij,t}} = \frac{Z_{i,t} E_{ij,Z,t}}{E_{ij,t}} \sqrt{\sigma_{X,d}^2 + \sigma_{X,f}^2 + 2\rho\sigma_{X,d}\sigma_{X,f}}, \qquad (101)$$

where the first derivative $E_{ij,Z,t}$ equals

$$\begin{aligned} E_{ij,Z,t}(Z_i) &= \frac{1-\tau_j}{\bar{r}_H - \tilde{\theta}_{Z,H}} e^{\theta_{X,H} t} \qquad (102) \\ &\quad + e^{\theta_{X,H} t} \frac{\omega_H}{Z_i^G} \left(\frac{Z_{i,t}}{Z_i^G}\right)^{\omega_H - 1} \left[E_{ij,L,T^G}(Z_i^G) - (1-\tau_j) \left(\frac{Z_i^G}{\bar{r}_H - \tilde{\theta}_{Z,H}} - \frac{C_j^F + I_{ij}}{\bar{r}_H}\right)\right] \end{aligned}$$

if the firm defaults after the Foreign government defaults $\left(T_{ij}^D \geq T^G\right)$ and

$$\begin{aligned} E_{ij,Z,t}(Z_i) &= \frac{1-\tau_j}{\bar{r}_H - \tilde{\theta}_{Z,H}} e^{\theta_{X,H} t} \\ &\quad - e^{\theta_{X,H} t} \frac{\omega_H}{Z_{ij}^D} (1-\tau_j) \left(\frac{Z_{ij}^D}{\bar{r}_H - \tilde{\theta}_{Z,H}} - \frac{C_j^F + I_{ij}}{\bar{r}_H}\right) \left(\frac{Z_{i,t}}{Z_{ij}^D}\right)^{\omega_H - 1} \qquad (103) \end{aligned}$$

if it defaults before the Foreign government defaults $\left(T_{ij}^{D} \leq T^{G}\right)$.

J Government

This Appendix computes the present value of the fiscal revenue received by the government, derives the valuation of government debt, and determines the government's default policy.

J.1 Fiscal revenue

The country j's government receives corporate taxes at time t, denoted by $TC_{j,t}$ and equal to

$$\begin{aligned} TC_{j,t} &= \underbrace{\int \tau_j \left(P_{j,t}X_{ij,t} - I_{ij,t} - C_{j,t}^{F}\right)\left(1 - \mathbf{1}_{\mathrm{def},ij}\right) dG(I_j)}_{\text{Corporate taxes before firms default}} \\ &\quad \underbrace{+\int \tau_j \left(1-\eta\right)\left(P_{j,t}X_{ij,t} - \upsilon I_{ij,t} - C_{j,t}^{F}\right) \mathbf{1}_{\mathrm{def},ij} dG(I_j)}_{\text{Corporate taxes from reorganized firms}} \\ &\quad \underbrace{+\int \tau_j \eta \left(P_{j,t}X_{ij,t} - \upsilon I_{ij,t} - C_{j,t}^{F}\right) \mathbf{1}_{\mathrm{def},ij} dG(I_j)}_{\text{Corporate taxes from new firms}}, \end{aligned} \tag{104}$$

where the indicator $\mathbf{1}_{\mathrm{def},ij}$ equals one if firm i in country j has defaulted (i.e., $t \geq T_{ij}^{D}$) and zero otherwise, while $G(I_j)$ denotes the distribution of firms in country j. When a firm defaults, its size is reduced by the default costs and thus equals a fraction $(1-\eta)$ of the former firm (second term of Equation 104). A new firm immediately enters the market to compensate for the loss in output, which equals a fraction η of the former firm's production (third term of Equation 104). The debt coupon levels of the reorganized and the new firms are $(1-\eta)\, C_{j,t}^{F}$ and $\eta C_{j,t}^{F}$, respectively, while the corresponding operating costs are $(1-\eta)\, \upsilon I_{ij,t}$ and $\eta \upsilon I_{ij,t}$. The scaling parameter $\upsilon < 1$ ensures that default does not immediately reoccur after a firm is reorganized. Its value is chosen such that the reorganized and the new firms start with the same default probability as that of the former firm at $t = 0$.

Labor income taxes related to employment in country j at time t equal to

$$
\begin{aligned}
TL_{j,t} &= \underbrace{\int \tau_j I_{ij,t} \left(1 - \mathbf{1}_{\mathrm{def},ij}\right) dG(I_j)}_{\text{Labor taxes from firms before default}} + \underbrace{\int \tau_j \left(1-\eta\right) \upsilon I_{ij,t} \mathbf{1}_{\mathrm{def},ij} dG(I_j)}_{\text{Labor taxes from reorganized firms}} \\
&\quad + \underbrace{\int \tau_j \eta \upsilon I_{ij,t} \mathbf{1}_{\mathrm{def},ij} dG(I_j)}_{\text{Labor taxes from new firms}}. \qquad (105)
\end{aligned}
$$

Total fiscal revenue in country j, denoted by $FR_{j,t}$, combine Equations 104 and 105:

$$
\begin{aligned}
FR_{j,t} &= TC_{j,t} + TL_{j,t} && (106) \\
&= \tau_j \int \left[R_{i,t} - C^F_{j,t}\right] dG(I_j) && (107) \\
&= \tau_j \left(X_t - \bar{C}^F_{j,t}\right) && (108)
\end{aligned}
$$

where $R_{i,t}$ denotes the firm i's revenue, $X_t \equiv \int R_{i,t} dG(I_j)$ represents total revenue in the country, and $\bar{C}^F_{j,t} = \int C^F_{j,t} dG(I_j)$ aggregates all corporate debt coupons in country j.

The discounted fiscal revenue in country j, denoted by $TR_{j,t}$, are given by

$$
\begin{aligned}
TR_{j,t} &= \mathbb{E}^{\mathbb{Q}}_t \left[\int_t^\infty \tau_j \left(X_u - \bar{C}^F_{j,u}\right) e^{-\beta_g (u-t)} du\right] && (109) \\
&= \mathbb{E}^{\mathbb{Q}}_t \left[g_t \int_t^\infty \tau_j \left(Z_u - \bar{C}^F_j\right) e^{-\bar{\beta}_g (u-t)} du\right], && (110)
\end{aligned}
$$

where the last equation is obtained by scaling the variables X_t and $\bar{C}^F_{j,t}$ by the growth of firm revenue g_t, as proposed in Appendix F, such that

$$
Z_t \equiv \frac{X_t}{g_t} \text{ and } \bar{C}^F_j \equiv \frac{\bar{C}^F_{j,t}}{g_t}, \qquad (111)
$$

while β_g is the government's rate of preference for time. The corresponding discount rate that can be used to conveniently solve the scaled version of the model is given by

$$
\bar{\beta}_g \equiv \beta_g - \theta_{X,H}. \qquad (112)
$$

I now solve for the present value of the Foreign government's fiscal revenue, $TR_{f,t}$, assuming that the Foreign government defaults at time $T^G = \inf\{t \geq 0 \mid Z_{i,t} \leq Z_i^G\} = \inf\{t \geq 0 \mid Z_t \leq Z^G\}$. Accounting for a temporary change in state s at time T^G, the present value of the Foreign government's fiscal revenue is given by

$$
\begin{aligned}
TR_{f,t} &= \tau_f g_t \left\{ \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^{T^G} Z_u e^{-\bar{\beta}_g(u-t)} du \right] + \mathbb{E}_t^{\mathbb{Q}} \left[\int_{T^G}^{\infty} Z_u e^{-\bar{\beta}_g(u-t)} du \right] \right\} \\
&\quad -\tau_f g_t \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^{\infty} \bar{C}_f^F e^{-\bar{\beta}_g(u-t)} du \right] \qquad (113) \\
&= \tau_f g_t \left\{ \frac{Z_t}{r_{G,H}} + Z^G \left(\frac{1}{r_{G,L}} - \frac{1}{r_{G,H}} \right) \left(\frac{Z_t}{Z^G} \right)^{\omega_g} - \frac{\bar{C}_f^F}{\bar{\beta}_g} \right\}, \qquad (114)
\end{aligned}
$$

with

$$
\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}} \left[\int_t^{T^G} \left(Z_u - \bar{C}_f^F \right) e^{-\bar{\beta}_g(u-t)} du \right] &= \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^{\infty} Z_u e^{-\bar{\beta}_g(u-t)} du \mid s = H \right] \\
&\quad -\mathbb{E}_t^{\mathbb{Q}} \left[\int_{T^G}^{\infty} Z_u e^{-\bar{\beta}_g(u-t)} du \mid s = H \right] \qquad (115) \\
&= \frac{Z_t}{r_{G,H}} - \frac{Z^G}{r_{G,H}} \left(\frac{Z_t}{Z^G} \right)^{\omega_g}, \qquad (116)
\end{aligned}
$$

and

$$
\mathbb{E}_t^{\mathbb{Q}} \left[\int_{T^G}^{\infty} Z_u e^{-\bar{\beta}_g(u-t)} du \right] = \frac{Z^G}{r_{G,L}} \left(\frac{Z_t}{Z^G} \right)^{\omega_g} \qquad (117)
$$

from the strong Markov property for Brownian motion, with

$$
\omega_g = \frac{1}{2} - \frac{\tilde{\theta}_{Z,H}}{\sigma_Z^2} - \sqrt{\left(\frac{1}{2} - \frac{\tilde{\theta}_{Z,H}}{\sigma_Z^2} \right)^2 + \frac{2\bar{\beta}_g}{\sigma_Z^2}} < 0, \qquad (118)
$$

where the government's discount rate, which accounts for a stochastic change in regime from $s = L$ to $s = H$, is given by

$$
r_{G,s} = \bar{\beta}_g - \tilde{\theta}_{Z,s} + \frac{\left(\bar{\beta}_g - \tilde{\theta}_{Z,H} \right) - \left(\bar{\beta}_g - \tilde{\theta}_{Z,L} \right)}{\lambda_{LH} + \bar{\beta}_g - \tilde{\theta}_{Z,H}} \lambda_{LH} \mathbf{1}_{\{s=L\}}. \qquad (119)
$$

J.2 Tax transfer

From Appendix J.1, the level of fiscal revenue, $FR_{j,t}$, raised in country j at time t is given by

$$FR_{j,t} = \tau_j \int \left[P_{j,t}X_{ij,t} - C^F_{j,t}\right] dG(I_j), \tag{120}$$

while the fiscal expenses, $FE_{j,t}$, of the country j's government at time t are given by

$$FE_{j,t} = C_{j,t}\left(1 - \phi \mathbf{1}_{\{t \geq T^G \cap j=f\}}\right) + TT_{j,t} \tag{121}$$

where $TT_{j,t}$ denotes the excess tax revenue transferred to the country's representative resident.

In equilibrium, government revenue must equal expenses in each country such that government's budget is balanced at each point in time, i.e., $FR_{j,t} = FE_{j,t}$ (see Aguiar and Amador, 2011). Hence, there is a tax transfer $TT_{j,t}$ that satisfies this condition, which is given by

$$TT_{j,t} = \tau_j \int \left[P_{j,t}X_{ij,t} - C^F_{j,t}\right] dG(I_j) - C_{j,t}\left(1 - \phi \mathbf{1}_{\{t \geq T^G \cap j=f\}}\right). \tag{122}$$

J.3 Sovereign debt

This section derives the value of the Foreign country's government debt, which is characterized by a time-dependent debt coupon $C_{f,t} = C_f e^{\theta_{X,H} t}$.

The solution to the value of sovereign debt $D_t(Z)$ is subject to a number of conditions. First, when Z_t tends to infinity, the value of the sovereign debt tends to the value of a risk-free debt. Second, when default occurs at time T^G, the debt payment is reduced from $C_{f,t}$ to $(1-\phi)\,C_{f,t}$.

The value of sovereign debt associated with the above boundary conditions, under the risk-neutral measure $\mathbb{Q}$, is given by

$$D_t(Z) = \mathbb{E}_t\left[\int_t^{T^G} \frac{\xi_u}{\xi_t} C_{f,t} du\right] + \mathbb{E}_t\left[\int_{T^G}^{\infty} \frac{\xi_u}{\xi_t}(1-\phi) C_{f,t} du\right] \tag{123}$$

$$= C_f g_t \left\{ \frac{1}{\bar{r}_H}\left[1 - \left(\frac{Z_t}{Z^G}\right)^{\omega_H}\right] + \frac{(1-\phi)}{r_{P,L}}\left(\frac{Z_t}{Z^G}\right)^{\omega_H}\right\}, \tag{124}$$

where the debt value depends on the scaled debt coupon $C_f = C_{f,t}/e^{\theta_{X,H}t}$, following the change of variables discussed in Appendix F.

J.4 Sovereign wealth and default policy

Sovereign wealth in the Foreign country, denoted by $SW_t(Z)$, can be written as the present value of fiscal revenue $TR_{f,t}$ net of the country's debt $D_t(Z)$:

$$SW_t(Z) \;=\; TR_{f,t}(Z) - D_t(Z). \tag{125}$$

The default policy, characterized by the default boundary Z^G, maximizes sovereign wealth $SW_t(Z)$ at time $t = 0$ subject to the usual smooth-pasting condition (see Merton, 1973; Dumas, 1991).

The first-order maximization of sovereign wealth yields

$$\begin{aligned} \frac{\partial SW_t(Z)\mid_{t=0}}{\partial Z} \;=\; & \frac{\tau_f}{r_{G,H}} + \omega_g \tau_f \left(\frac{Z}{Z^G}\right)^{\omega_g - 1} \left(\frac{1}{r_{G,L}} - \frac{1}{r_{G,H}}\right) \\ & + \frac{\omega_H C_f}{Z^G} \left(\frac{Z}{Z^G}\right)^{\omega_H - 1} \left(\frac{1}{\bar{r}_H} - \frac{1-\phi}{r_{P,L}}\right). \end{aligned} \tag{126}$$

Using the smooth-pasting condition $\frac{\partial SW(Z)}{\partial Z}\mid_{Z=Z^G} = \frac{\tau_f}{r_{G,L}}$, we have

$$0 \;= \tau_f\,(1-\omega_g)\;\left(\frac{1}{r_{G,H}} - \frac{1}{r_{G,L}}\right) + \frac{\omega_H C_f}{Z^G}\left(\frac{1}{\bar{r}_H} - \frac{1-\phi}{r_{P,L}}\right). \tag{127}$$

Solving Equation 127, we obtain the optimal sovereign default policy, which satisfies

$$Z^{G*} \;=\; \frac{\omega_H C_f \left(\frac{1}{\bar{r}_H} - \frac{1-\phi}{r_{P,L}}\right)}{\tau_f\,(\omega_g - 1)\left(\frac{1}{r_{G,H}} - \frac{1}{r_{G,L}}\right)}. \tag{128}$$

J.5 Probability of sovereign default

Given the default policy discussed in Appendix J.4, the probability that the Foreign government defaults within a time period T is defined by

$$\mathbb{P}\left(\inf_{0\leq t\leq T} Z_t \leq Z^G \mid Z > Z^G\right) = \Phi\left(\frac{\ln(\frac{Z^G}{Z}) - \left(\theta_{Z,H} - \frac{\sigma_Z^2}{2}\right)T}{\sigma_Z\sqrt{T}}\right) + \left(\frac{Z^G}{Z}\right)^{\frac{2\theta_{Z,H}}{\sigma_Z^2}-1} \Phi\left(\frac{\ln(\frac{Z^G}{Z}) + \left(\theta_{Z,H} - \frac{\sigma_Z^2}{2}\right)T}{\sigma_Z\sqrt{T}}\right), \quad (129)$$

where $\Phi(\cdot)$ is the cumulative density of a standard normal distribution, and $\theta_{Z,H}$ is the physical growth rate of the process Z_t. The risk-neutral probability of defaulting obtains when the risk-neutral growth rate, $\tilde{\theta}_{Z,H}$, replaces the physical one, $\theta_{Z,H}$.

K Welfare Impact of Sovereign Default

This Appendix derives the welfare effect of sovereign default risk for the Foreign representative agent. It determines the compensation in consumption such that this agent is indifferent between an economy without sovereign default risk and a compensated path with a temporary lower growth rate due to a sovereign default.

K.1 Compensation factor and consumption path

Consider that the representative agent in the Foreign country has CRRA preferences and maximizes expected utility

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{y_t^{1-\gamma}}{1-\gamma}dt\right], \quad (130)$$

where γ is the coefficient of relative risk aversion. It is convenient to derive the general solution and then analyze the case of logarithmic preferences (i.e., $\gamma = 1$).

A risk-averse consumer would prefer a stable consumption growth rate to a path that depends on sovereign default risk. Following Lucas (1987), I quantify this utility difference by multiplying the path subject to sovereign default risk by a constant factor $1+\lambda$, choosing λ so that the agent is indifferent between the constant growth rate and the compensated path with a temporary lower growth rate during the sovereign crisis.

Define the consumption path starting at the time of sovereign default T^G. Under the hypothesis that the Foreign government never defaults, the level of consumption in this economy follows

$$\overline{y}_t = \overline{X}_{f,t} \quad \text{with} \quad \ln\left(\overline{X}_{f,t}\right) \sim N\left(\theta_{f,H} - \frac{1}{2}\sigma_f^2, \sigma_f^2\right), \tag{131}$$

while it is assumed that, if the sovereign defaults, there exists a compensated consumption stream given by

$$\widetilde{y}_t = (1+\lambda) X_{f,t} \quad \text{with} \quad \ln\left(X_{f,t}\right) \sim N\left(\theta_{f,s} - \frac{1}{2}\sigma_f^2, \sigma_f^2\right), \quad s = \{L, H\}, \tag{132}$$

where the growth rate starts at a reduced level $\theta_{f,L}$ and increases to $\theta_{f,H}$ with probability $\lambda_{LH} > 0$ per unit of time, with $\theta_{f,L} < \theta_{f,H}$.

K.2 Expected welfare costs

Proposition: The Lucas' welfare compensation required by the Foreign representative agent to bear the risk of sovereign default at time T^G is given by

$$\lambda \cong \frac{\triangle\theta}{\beta + \lambda_{LH}} \tag{133}$$

in the case of logarithmic preferences ($\gamma = 1$).

The welfare costs at time $t = 0$, per unit of consumption at sovereign default T^G, are given by

$$\begin{aligned} \mathbb{E}_0^{\mathbb{Q}}\left[\lambda e^{-\beta T^G}\right] &\cong \mathbb{E}_0^{\mathbb{Q}}\left[\left(\frac{\triangle\theta}{\beta + \lambda_{LH}}\right) e^{-\beta T^G}\right] & (134) \\ &= \left(\frac{\triangle\theta}{\beta + \lambda_{LH}}\right)\left(\frac{Z_0}{Z^G}\right)^{\omega'}, & (135) \end{aligned}$$

where ω' is the negative root of the quadratic equation $\frac{1}{2}\sigma_Z^2\omega'(\omega'-1)+\tilde{\theta}_{Z,H}\omega'-\beta=0$, defined by

$$\omega' = \frac{1}{2} - \frac{\tilde{\theta}_{Z,H}}{\sigma_Z^2} - \sqrt{\left(\frac{1}{2} - \frac{\tilde{\theta}_{Z,H}}{\sigma_Z^2}\right)^2 + \frac{2\beta}{\sigma_Z^2}} < 0. \tag{136}$$

Proof: When equalizing the lifetime utility in both cases (i.e., with and without sovereign default), starting at time T^G, we have

$$\mathbb{E}\left[\int_{T^G}^{\infty} e^{-\beta\left(t-T^G\right)}\frac{\overline{X}_{f,t}^{1-\gamma}}{1-\gamma}dt\right] = \mathbb{E}\left[\int_{T^G}^{\infty} e^{-\beta\left(t-T^G\right)}\frac{\left(\left(1+\lambda\right)X_{f,t}\right)^{1-\gamma}}{1-\gamma}dt\right], \tag{137}$$

which yields, after multiplying by $1-\gamma$, to

$$\mathbb{E}\left[\int_{T^G}^{\infty} e^{-\beta\left(t-T^G\right)}\overline{X}_{f,t}^{1-\gamma}dt\right] = \mathbb{E}\left[\int_{T^G}^{\infty} e^{-\beta\left(t-T^G\right)}\left(\left(1+\lambda\right)X_{f,t}\right)^{1-\gamma}dt\right]. \tag{138}$$

Let us compute the left side of Equation 138 first. From Fubini's theorem,[6]

$$\begin{aligned}
\mathbb{E}\left[\int_{T^G}^{\infty} e^{-\beta\left(t-T^G\right)}\overline{X}_{f,t}^{1-\gamma}dt\right] &= \int_{T^G}^{\infty} e^{-\beta\left(t-T^G\right)}\mathbb{E}\left[\overline{X}_{f,t}^{1-\gamma}\right]dt &(139)\\
&= \overline{X}_{f,T^G}^{1-\gamma}\int_{T^G}^{\infty} e^{-(\beta-a)\left(t-T^G\right)}dt &(140)\\
&= \frac{1}{\beta-a}\overline{X}_{f,T^G}^{1-\gamma}, &(141)
\end{aligned}$$

with

$$a \equiv (1-\gamma)\left(\theta_{f,H} - \frac{1}{2}\sigma_f^2\right) + \frac{1}{2}\left(1-\gamma\right)^2\sigma_f^2 < \beta, \tag{142}$$

using the property that $\mathbb{E}_0\left[u_t^m\right] = e^{\left(m\mu+\frac{1}{2}m^2\sigma^2\right)t}$ if $\ln\left(u_t\right) \sim N\left(\mu t, \sigma^2 t\right)$.

Let us now compute the right side of Equation 138, which is the case with sovereign default. Denote the time of a change in state from $s=L$ to $s=H$ by T^λ, which is an exponentially distributed random variable with expected value $\frac{1}{\lambda_{LH}}$. Using the law if iterated

[6] The order of integration can be interchanged because the integrand is absolutely integrable.

expectations, we can first write

$$\begin{aligned}
\mathbb{E}\left[X_{f,t}^{1-\gamma}\right] =&\mathbb{E}\left[\mathbb{E}\left[X_{f,t}^{1-\gamma} \mid T^{\lambda}\right]\right] && (143)\\
=&\int_{T^G}^{t} X_{f,T^G}^{1-\gamma} e^{a\left(t-T^G\right)} e^{(1-\gamma)\left(\theta_{f,L}-\theta_{f,H}\right)\left(u-T^G\right)} F_{\lambda}(u)du \\
&+\int_{t}^{\infty} X_{f,0}^{1-\gamma} e^{a'\left(t-T^G\right)} F_{\lambda}(u)du && (144)\\
=&X_{f,T^G}^{1-\gamma} e^{a\left(t-T^G\right)} \int_{T^G}^{t} \lambda_{LH} e^{b\left(u-T^G\right)} du \\
&+ X_{f,T^G}^{1-\gamma} e^{a'\left(t-T^G\right)} \int_{t}^{\infty} \lambda_{LH} e^{-\lambda_{LH}\left(u-T^G\right)} du && (145)\\
=&X_{f,T^G}^{1-\gamma}\left[e^{a\left(t-T^G\right)} \frac{\lambda_{LH}\left(e^{b\left(t-T^G\right)}-1\right)}{b} + e^{(a'-\lambda_{LH})\left(t-T^G\right)}\right] && (146)\\
=&X_{f,T^G}^{1-\gamma}\left[\left(\frac{\lambda_{LH}}{b}+1\right) e^{(a+b)\left(t-T^G\right)} - \frac{\lambda_{LH}}{b} e^{a\left(t-T^G\right)}\right], && (147)
\end{aligned}$$

where $F_{\lambda}(u) = \lambda_{LH} e^{-\lambda_{LH} u}$ is the probability density function of an exponential distribution,

$$a' \equiv a + (1-\gamma)\left(\theta_{f,L}-\theta_{f,H}\right) \tag{148}$$

$$b \equiv (1-\gamma)\left(\theta_{f,L}-\theta_{f,H}\right) - \lambda_{LH} < 0 \tag{149}$$

and $b < 0$ given that $\theta_{f,L} < \theta_{f,H}$ and $\lambda_{LH} > 0$.

Therefore, the expected utility is given by

$$\begin{aligned}
\mathbb{E}\left[\int_{T^G}^{\infty} e^{-\beta t}\left(\left(1+\lambda\right) X_{f,t}\right)^{1-\gamma} dt\right] &= \left(1+\lambda\right)^{1-\gamma} \int_{T^G}^{\infty} \left\{e^{-\beta\left(t-T^G\right)} \mathbb{E}\left[X_{f,t}^{1-\gamma}\right]\right\} dt && (150)\\
&= \left(1+\lambda\right)^{1-\gamma} X_{f,T^G}^{1-\gamma} \frac{1}{b}\left[\frac{b+\lambda_{LH}}{\beta-a-b} - \frac{\lambda_{LH}}{\beta-a}\right], && (151)
\end{aligned}$$

where the necessary conditions $a+b-\beta < 0$ and $a-\beta < 0$ are satisfied given Equations 142 and 149.

When combining Equations 141 and 151, the value of λ satisfies the following equality:

$$\frac{1}{\beta-a} = \left(1+\lambda\right)^{1-\gamma} \frac{1}{b}\left(\frac{b+\lambda_{LH}}{\beta-a-b} - \frac{\lambda_{LH}}{\beta-a}\right), \tag{152}$$

which yields, after simplifications, to

$$1 = (1+\lambda)^{1-\gamma}\left(\frac{\beta - a + \lambda_{LH}}{\beta - a - b}\right) \tag{153}$$

$$= (1+\lambda)^{1-\gamma}\left(\frac{\beta - a + \lambda_{LH}}{\beta - a' + \lambda_{LH}}\right), \tag{154}$$

where $a + b = a' - \lambda_{LH}$ when combining Equations 148 and 149.

Hence, for $\gamma > 1$, the solution of λ satisfies

$$1+\lambda = \left[\frac{\beta - a + \lambda_{LH}}{\beta - a' + \lambda_{LH}}\right]^{\frac{1}{\gamma-1}}. \tag{155}$$

Consider the case with logarithmic preferences ($\gamma = 1$). For convenience, first take the logarithm of the above solution, which yields

$$\ln(1+\lambda) = \frac{\ln(\beta - a + \lambda_{LH}) - \ln(\beta - a' + \lambda_{LH})}{\gamma - 1}. \tag{156}$$

Using L'Hospital's rule,

$$\lim_{\gamma\to 1}\ln(1+\lambda) = \frac{\theta_{f,H} - \theta_{f,L}}{\beta + \lambda_{LH}}, \tag{157}$$

and the final solution is given by

$$\lambda \cong \frac{\theta_{f,H} - \theta_{f,L}}{\beta + \lambda_{LH}} = \frac{\triangle\theta}{\beta + \lambda_{LH}}. \tag{158}$$

L Markov-switching Probability of Recession

This Appendix describes the Markov regime-switching model used to compute the probability of an economic crisis in the US. Consider that two regimes characterize the mean and the volatility of the US economy's growth rate. The time-series of monthly industrial production in the US, denoted by $X_{d,t}$, from January 1950 to December 2013, identifies the different regimes:

$$\frac{dX_{d,t}}{X_{d,t}} = \theta_{d,s_t} dt + \sigma_{d,s_t} dW_t^d \tag{159}$$

$$s_t = 1, 2 \text{ with } \theta_{d,1} > \theta_{d,2} \;\text{ and } \sigma_{d,1} < \sigma_{d,2} \tag{160}$$

where the transition of regimes is stochastic. The dynamics behind the switching process is known and driven by the transition matrix:

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0.96 & 0.11 \\ 0.04 & 0.89 \end{bmatrix} \tag{161}$$

where p_{ij} denotes the probability of a switch from regime i to regime j. The volatility and the mean of the growth rate and the transition probability matrix are obtained by maximum likelihood.[7] The annualized estimates for the mean growth rate are $\theta_{Y,1} = 3.82\%$ and $\theta_{Y,2} = 0.94\%$, while they are $\sigma_{Y,1} = 1.85\%$ and $\sigma_{Y,2} = 5.77\%$ for the volatility. The two regimes obtained from this estimation essentially capture a recession regime and growth regime. The expected duration of a crisis (state 2) is 8.7 months over the period 1950-2013.

M Structural Estimation

The econometric methodology involves testing a set of over-identifying restrictions on a system of moment equations, using the generalized method of moments (GMM) developed by Hansen (1982).[8] The GMM estimation procedure chooses the parameter estimates that

[7] I use the code developed by Marcelo Perlin, which is generously posted on James D. Hamilton's home page (http://weber.ucsd.edu/~jhamilto/software.htm).

[8] Compared with the Maximum Likelihood estimation, the GMM technique is particularly attractive in this setting: first, the GMM approach does not require the distribution of equity return volatility and corporate leverage to be normal. The asymptotic justification for the GMM procedure requires only that the distribution of these series be stationary and ergodic, and that the relevant expectations exist; second, the GMM estimators and their standard errors are consistent, even if the assumed disturbances are conditionally heteroskedastic, which is the case for the series under consideration.

minimize the quadratic form $J(\Delta\theta) = m'(\Omega)W(\Omega)m(\Omega)$ with

$$m(\Omega) = \begin{cases} \sigma_{E_d,t} - \sigma_{r_{us}} \\ \frac{D_{d,t}}{V_{d,t}} - \frac{D_{us}}{V_{us}} \\ \sigma_{E_f,t} - \sigma_{r_{eu}} \\ \frac{D_{f,t}}{V_{f,t}} - \frac{D_{eu}}{V_{eu}} \\ \mathbb{P} - \text{default rate} \end{cases} \tag{162}$$

where Ω is the set of parameters to estimate, $W(\Omega)$ is a positive-definite symmetric weighting matrix, and $m(\Omega)$ is a vector of orthogonality conditions corresponding to the model's pricing errors (i.e., theoretical minus empirical moments).

The weighting matrix $W(\Omega)$ determines the relative importance of the various moment conditions to give more weight to the moment conditions with less uncertainty.[9] I estimate the covariance matrix using the Newey and West (1987) approach to account for heteroskedasticity and serial correlation (with 6 lags) with a correction for small samples. This covariance matrix is used to test the significance of the parameters, whereas the covariance matrix of the moments is used to test the significance of individual pricing errors.

As it is not possible to set every moment to zero, the key concern is the distance from zero. The minimized value of the quadratic form $J(\Omega)$, which is χ^2-distributed under the null hypothesis that the model is true, provides the goodness-of-fit test for the model.

[9]The optimal weighting matrix $W(\Omega)$ requires an estimate of the set of parameters Ω; at the same time, estimating the set of parameter Ω requires the weighting matrix. To solve this dependency, I account for a two-stage estimation method. I first set the initial weighting matrix to be equal to the identity matrix $W_0 = I$ and then calculate the parameter estimates. I then compute a new weighting matrix with the parameter estimates obtained at the first stage.

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