Appendices

Internet Appendix for

Portfolio Diversification and International Corporate Bonds

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A Mean Adjusted Risk Reduction

The significant risk reduction gains ψ for the minimum variance portfolio with international corporate bonds is accompanied by lower expected returns. To understand that risk reduction potential when taking into account changes in expected return, I evaluate the risk reduction for a mean equivalent portfolio on the efficient frontier. Although the mean adjustment raises the problem of estimation risk around the expected return, this section evaluates the effect of a mean-adjusted measure of risk reduction, δ. The mean-adjusted risk reduction is a more conservative measure of diversification gain than the pure risk reduction ψ. The mean-adjusted risk reduction is defined to be the volatility decrease of moving from the benchmark minimum variance portfolio to a diversified portfolio on the efficient frontier with an equivalent mean as the benchmark.

It is important to note that the definition of δ limits the investor to only mean-adjusted portfolios on the efficient frontier. The limitation to only efficient frontier portfolios implies that US investors cannot borrow at the US risk-free rate and invest additional funds into the tangency portfolio to achieve portfolios on the capital allocation line. The constraint that investors can only stay on the efficient frontier, combined with no short-sale constraints imposed on the risky assets that generate the efficient frontier, implies an extreme case when investors have no borrowing capacity of any kind. Therefore, δ represents a very conservative estimate of risk reduction in the face of heavily constrained portfolio allocation.

Panel A of Table A1 reports the in-sample mean adjusted risk reduction gains when portfolio weights are unconstrained [30]. The posterior distribution for δ shows an average risk reduction of 45.7% with the lowest 1% risk reduction gain at 7.8%. As compared with the large ψ gains of 80% and above in Table 3, the mean adjustment clearly reduces the diversification benefits of holding international corporate bonds. However, beyond the drop in the posterior mean of the risk reduction distribution, the mean adjustment also increases the

[30] The expected return and volatility stated in the table are for the minimum variance portfolio, and are repeated here from previous tables for convenience.

standard deviation of the posterior distribution and dramatically reduces the lowest 1% of gains of the posterior distribution.

With the multi-asset benchmarks, the effect of the mean adjustment on the standard deviation of the posterior distribution is less dramatic. The mean-adjusted risk reduction gain δ has a higher posterior mean of 56.8% as compared with a posterior mean of 45.7% under the US corporate bond only benchmark. The lowest 1% of the posterior distribution of δ is also higher at 17.7% as compared with 7.8%. The increase in the δ gain is due partly to the lower expected return for the minimum variance portfolio of the US multi-asset benchmark. Since the mean-adjusted portfolio is constructed to match the expected return of the benchmark, less mean adjustment is required when the benchmark return is lower. Consequently, the δ risk reduction gain will be closer to the pure risk reduction gain, ψ.

For out-of-sample δ gains, the mean-adjusted efficient portfolio is evaluated for the testing period of either the crisis of 2008–2010 or the non-crisis period of 2003–2007. For the crisis period, I find significant mean-adjusted risk reduction of 53.7%, 65.4%, and 73.8% for the US corporate bond only benchmark, the US multi-asset benchmark, and the US multi-asset with international equities benchmark, respectively. While these results are surprisingly strong, it is important to keep in mind that the out-of-sample expected return of the mean-adjusted efficient portfolio may not be as high as for the benchmark portfolios. In other words, the out-of-sample δ gains represent both the accuracy of the mean adjustment and the risk reduction[31].

Similarly, for the US multi-asset and international equities benchmark, the mean-adjusted diversified portfolio would have reduced volatility by 69.0% in the out-of-sample period. The only notable difference between the crisis and the non-crisis out-of-sample results is the mean-adjusted risk reduction δ for the US corporate bond only benchmark. The measured 5.7% δ gain for 2003–2007 is much smaller than the 53.7% gain for the crisis period. Finally, the non-crisis out-of-sample risk reduction of the initial equal-weight portfolio shows that even though expected returns are not markedly different from risk reduction gains, they are still above 39% and statistically significant.

B Appendix: Bayesian Inference and Portfolio Gains

For statistical inference on diversification gains, I follow Wang (1998) in using a combination of Bayesian inference and Monte Carlo simulation to generate the posterior distributions for φ and ψ, and I evaluate the diversification gains against the null hypothesis that diversification gains are zero. This inference method provides a unified way to evaluate

[31] The mean-adjusted portfolio will in general not be as stable as the minimum variance portfolio, since the small sample mean estimate is unlikely to be accurate and perform well in out-of-sample period.

diversification gains, even when the sampling distribution of the gain is unknown, which is the case for φ and ψ under no short-sale constraints[32]. The combination of Bayesian inference and Monte Carlo methods provides the posterior distributions for φ and ψ, on which the null hypothesis can be evaluated.

I assume that the representative US investor has diffuse prior, or a lack of a priori beliefs, on the means and covariance structure of risky asset returns. Furthermore, I assume the following prior probability density:

$$p(\mu, \Sigma) = p(\mu) * p(\Sigma), \quad p(\mu) \propto \text{constant}, \quad p(\Sigma) \propto |\Sigma|^{-(N+1)/2} \tag{B-1}$$

where $R = (r_1, \ldots, r_N)'$ denotes a sample of N risky asset returns with dimension N x T, and a mean vector μ and covariance matrix Σ. Let $\hat{\mu}$ and $\hat{\Sigma}$ be the Nx1 vector of sample means and the N x N matrix of sample covariances, respectively. Then the posterior density is a function of the marginal densities:

$$p(\mu, \Sigma | R) = p(\mu | \Sigma, \hat{\mu}, T) * p(\Sigma | \hat{\Sigma}, T) \tag{B-2}$$

Under these assumptions, marginal posterior density $p(\Sigma|\hat{\Sigma}, T)$ is known to be an inverted Wishart distribution with the parameter matrix $T\hat{\Sigma}$ and T-1 degrees of freedom. Then the conditional distribution $p(\mu|\Sigma, \hat{\mu}, T)$ is multivariate normal with mean $\hat{\mu}$ and variance $1/T\Sigma$. With the marginal distributions, the posterior distribution is particularly easy to implement using Monte Carlo methods. Specifically, I generate a draw of Σ from an inverted Wishart distribution with the parameter matrix $T\Sigma$ and T-1 degrees of freedom. Then conditional on the realization of Σ, I draw a random μ from a multivariate distribution with mean $\hat{\mu}$ and variance $1/T\Sigma$. For each pair (μ, Σ), the diversification gain measures φ and ψ can be computed using the optimization outlined above. Repeating this procedure for 250,000 independent draws, the resulting draws of φ and ψ are obtained from their respective posterior distributions[33]. For inference against the null hypothesis that there is no diversification gain, I report the lowest 1% and 5% diversification gain, as well as the mean and standard deviation of the posterior distribution.

[32] Basak et al. have derived the asymptotic distribution for variance reduction in the case of short-sale constraints

[33] The diffuse prior was placed on the mean and covariance of returns, which does not necessarily translate to a diffuse prior on φ or ψ. However, as discussed in Li et al Li et al. (2003), in simulations they find the implied priors for the diversification gain measures are also diffuse.

C Appendix: Longer Sample of Equities and International Corporate Bond Returns

Following Stambaugh (1997), this section outlines the maximum likelihood estimation of corporate bond return moments that combine longer sample returns of equities and sovereign bonds with the truncated sample of corporate bond returns. Let $R_{1,t}$ is a vector of asset returns for the first N_1 assets that has a longer sample period with $t = 1...T$, and $R_{2,t}$ contain N_2 assets with a shorter sample length where $t = s...T$, where $1 \geq s \leq T$, and the following moments:

$$\begin{aligned} E[R_{1,t}] &= \mu_1 \\ \text{cov}[R_{1,t}, R'_{1,t}] &= \Sigma_{11} \\ E[R_{2,t}] &= \mu_2 \\ \text{cov}[R_{2,t}, R'_{2,t}] &= \Sigma_{22} \end{aligned}$$

When we combine assets, the vector of combined asset returns $R_t = [R'_{1,t}, R'_{2,t}]$ with N assets where $N = N_1 + N_2$, will have the following the expected return and covariance matrix:

(C-1) $$\hat{\mu} = \begin{bmatrix} R_{1,t} \\ R_{2,t} \end{bmatrix} = \mu_L$$

(C-2) $$\text{cov}\left\{ \begin{bmatrix} R_{1,t} \\ R_{2,t} \end{bmatrix}, R_{1,t} R_{2,t} \right\} = \begin{bmatrix} \Sigma_{11} & \Sigma'_{21} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \Sigma_L$$

To estimate the predictive distribution of R_{T+1}, we begin by regressing $R_{1,s}$ on $R_{2,s}$ and collecting the estimated regression results as:

(C-3) $$\hat{C} = \begin{bmatrix} \hat{\alpha} \\ \hat{B} \end{bmatrix} = (X' * X)^{-1}(X * R_{2,s})$$

where $X = [i_s \quad R_{1,s}]$ and the covariance of the error is $\hat{V} = 1/S * (R_{2,s} - X\hat{C})'(R_{2,s} - X\hat{C})$. As explained in detail in Stambaugh (1997), the longer sample asset returns provide additional information for the moments of corporate bond returns. Combining the above definitions with equations (8), (9), and (10) in the paper, it is easy to see how the long run asset estimates of $\hat{\Sigma}_{11}$ and $\hat{\mu}_1$ revises the estimates $\hat{\mu}_2$, $\hat{\Sigma}_{22}$, and $\hat{\Sigma}_{21}$, of the short sample asset.

Table A1: **Diversified Gains and Mean Adjusted Risk Reduction**

This table reports mean-adjusted risk reduction gain, δ, for three different US Benchmark portfolios: investment grade corporate bonds only (US IG), US equities and US corporate bonds (US (IG + Eq)), and the diversified portfolio of US equities, international equities, and US corporate bonds (US (IG + Eq) + Intl Eq). Diversified portfolios with benchmark assets and international corporate bonds are reported under columns Bench + Intl IG. Panels A and B report the minimum variance portfolio, with and without short sale constraints respectively. Panels C and D report the out of sample minimum variance portfolio for the crisis and pre-crisis periods respectively. All returns are expressed in monthly percent, in U.S. dollars, and span the sample period Jan. 2000–Dec. 2010 unless otherwise indicated.

	US IG	Bench+ Intl IG	US (Eq+ Gov + IG)	Bench+ Intl IG	US (Eq + Gov IG) + Intl Eq	Bench+ +Intl IG
In Sample						
Panel A. Min Variance (Unconstrained)						
Expected Return	0.643	0.374	0.571	0.363	0.615	0.347
Volatility	1.645	0.334	1.544	0.312	1.451	0.293
Posterior δ Mean		0.457		0.568		0.581
Posterior δ Stdev		0.190		0.167		0.132
Posterior δ 1%		0.078		0.177		0.274
Posterior δ 5%		0.156		0.276		0.356
Panel B. Min Variance (No Short Sale)						
Expected Return	0.643	0.370	0.571	0.361	0.592	0.358
Volatility	1.645	0.346	1.544	0.333	1.460	0.320
Posterior δ Mean		0.176		0.433		0.432
Posterior δ Stdev		0.269		0.212		0.191
Posterior δ 1%		0.000		0.028		0.069
Posterior δ 5%		0.000		0.098		0.138
Out of Sample						
Panel C. Min Variance (Crisis 2008–2010)						
Expected Return	0.776	0.289	0.697	0.300	0.522	0.274
Volatility	2.316	0.389	2.499	0.342	2.373	0.320
Out Sample δ		0.537		0.654		0.738
Bootstrap pVal		0.000		0.000		0.000
Panel D. Min Variance (2003–2007)						
Expected Return	0.433	0.377	0.487	0.381	0.565	0.338
Volatility	1.359	0.351	1.352	0.342	1.287	0.375
Out Sample δ		0.057		0.610		0.690
Bootstrap pVal		0.000		0.000		0.000