Online Appendix for "Solvency Constraint, Underdiversification, and Idiosyncratic Risks" by Hong Liu

In this appendix, we provide the proofs for Theorems 1-3.

PROOF OF THEOREM 1: Define the adjusted initial wealth as $W_p^A \equiv W_p - \underline{C}$. When $W_p^A = 0$, obviously, the investor can only invest in the risk-free asset. Now suppose his adjusted wealth increases to $W_p^A = \eta$, where $\eta > 0$ is small. Since he cannot borrow or shortsell, the most he can invest in stocks is η. Given this restriction, he will first invest in the stock that provides the highest marginal utility. To identify the stock that provides the highest marginal utility, suppose he invests $\underline{C}$ in the risk-free asset and η in Stock j with the end-of-period wealth

$$\widetilde{W}_1 = \underline{C} + \eta(\tilde{z}_j + 1).$$

As η approaches 0, the investor's marginal utility from investing in Stock j converges to

$$\text{(A-1)} \quad \lim_{\eta \downarrow 0} \frac{\partial E[u(\widetilde{W}_1)]}{\partial \eta} = \lim_{\eta \downarrow 0} E[h(\eta, \tilde{z}_j)] = E[h(0, \tilde{z}_j)] = E[u'(\underline{C})(\tilde{z}_j + 1)] = u'(\underline{C})(\mu_j + 1),$$

where

$$h(\eta, \tilde{z}_j) \equiv u'(\underline{C} + \eta(\tilde{z}_j + 1))(\tilde{z}_j + 1),$$

the first equality in equation (A-1) follows from Leibnitz's rule, and the second equality follows from the Monotone Convergence Theorem (e.g., Williams 1991, p. 59) because $h(\eta, \tilde{z}_j)$ is strictly decreasing in η since $\partial h(\eta, \tilde{z}_j)/\partial \eta = u''(\underline{C} + \eta(\tilde{z}_j + 1))(\tilde{z}_j + 1)^2 < 0$ by the strict concavity of $u(\cdot)$. Therefore when η is small enough, since $\infty > u'(\underline{C}) > 0$, the marginal utility from investing in the stock with the highest expected return is strictly the highest. Thus, if wealth W_p is slightly above the committed level $\underline{C}$, the investor invests $\underline{C}$ in the risk-free asset and the rest in the stock with the highest expected return, say, Stock 1. Higher moments, such as variance, skewness, and kurtosis, do not affect this choice.

We now prove by induction. Suppose that, given the initial adjusted wealth W_p^A, the investor optimally invests $\underline{C}$ in the risk-free asset, $\theta_i > 0$ in Stock i for $1 \leq i \leq m < n$, and 0 in the rest of the stocks. Since the investor is investing in only $m < n$ of n stocks, the constraint $\sum_{i=1}^{m} \theta_i \leq W_p - \underline{C} = W_p^A$ must be still binding. This is because the marginal utility of investing a bit in the rest of the stocks is strictly greater than that of investing more in the risk-free asset, since the expected stock returns are strictly greater than the risk-free rate. The end-of-period wealth is $\widetilde{W_1} = \underline{C} + \sum_{i=1}^{m} \theta_i(\tilde{z}_i + 1)$. The Lagrangian is

$$V(W_p^A) = \max_{\theta \geq 0} \left\{ E\left[u\left(\underline{C} + \sum_{i=1}^{m} \theta_i(\tilde{z}_i + 1) \right) \right] + \nu \left(W_p^A - \sum_{i=1}^{m} \theta_i \right) \right\},$$

where ν is the Lagrangian multiplier. From the first-order conditions, we have that the marginal utility of investing in Stock i for $1 \leq i \leq m$ is

$$\text{(A-2)} \qquad \frac{\partial E\left[u\left(\widetilde{W_1} \right) \right]}{\partial \theta_i} = E[u'(\widetilde{W_1})(\tilde{z}_i + 1)] = \nu.$$

By the Envelop theorem, we have

$$\nu = \frac{\partial V(W_p^A)}{\partial W_p^A} > 0,$$

since utility is strictly increasing in wealth. Therefore, at the initial level W_p^A, the marginal utilities of investing in these m stocks are equal, strictly positive, and strictly greater than the marginal utilities of investing more than $\underline{C}$ in the risk-free asset and the rest of the stocks (because it is optimal for the investor not to hold the rest of the stocks). Since

$$\frac{\partial E\left[u'\left(\underline{C} + \sum_{i=1}^{m} \theta_i(\tilde{z}_i + 1) \right) (\tilde{z}_i + 1) \right]}{\partial \theta_i} = E[u''(\widetilde{W_1})(\tilde{z}_i + 1)^2] < 0,$$

the marginal utility of investing in Stock i decreases as the investment in the stock increases.

Thus, as the adjusted initial wealth increases from W_p^A, the investor will increase the investment in all these m stocks, i.e.,[18]

$$\frac{\partial \theta_i}{\partial W_p^A} > 0, \quad i = 1, 2, ..., m. \tag{A-3}$$

Similar to equation (A-1), the marginal utility of investing a small amount in a new Stock j ($j > m$) at W_p^A is

$$\begin{aligned} E\left[u'\left(\underline{C} + \sum_{i=1}^{m} \theta_i(\tilde{z}_i + 1)\right)(\tilde{z}_j + 1)\right] &= E\left[u'\left(\underline{C} + \sum_{i=1}^{m} \theta_i(\tilde{z}_i + 1)\right)\right](\mu_j + 1) \\ &+ \mathrm{Cov}\left(u'\left(\underline{C} + \sum_{i=1}^{m} \theta_i(\tilde{z}_i + 1)\right), \tilde{z}_j\right). \end{aligned} \tag{A-4}$$

Therefore, when it becomes optimal to add another stock to the portfolio, which stock to add only depends on its expected return and its covariance with the stocks in the current portfolio. Other moments of Stock j, such as variance and skewness, are irrelevant for this choice. If stocks are uncorrelated, then this choice only depends on the expected return of Stock j.

We next derive the critical adjusted wealth level $\hat{W}_{m+1}^A$ above which it is optimal to add another stock, say, Stock $m + 1$. By equations (A-2) and (A-4) with $j = m + 1$, we must have that at this critical level, the marginal utilities of investing in these $m + 1$ stocks are

[18] At W_p^A, the marginal utilities across all the m stocks are the same. When the wealth increases above W_p^A, if the investor increases investment only in some of the m stocks, then the marginal utilities from these stocks would be lowered and the marginal utilities from the rest of the m stocks would be strictly higher, which is a contradiction to optimality.

exactly the same, i.e., for $i = 1, 2, ..., m$, we have

$$(\text{A-5}) \qquad E\left[u'\left(\underline{C} + \sum_{i=1}^{m} \theta_i(\tilde{z}_i + 1)\right)(\tilde{z}_i + 1)\right] = E\left[u'\left(\underline{C} + \sum_{i=1}^{m} \theta_i(\tilde{z}_i + 1)\right)(\tilde{z}_{m+1} + 1)\right],$$

which can be simplified to

$$(\text{A-6}) \qquad E\left[u'\left(\underline{C} + \sum_{i=1}^{m} \theta_i(\tilde{z}_i + 1)\right)(\tilde{z}_i - \tilde{z}_{m+1})\right] = 0, \quad i = 1, 2, ..., m.$$

Therefore, we have $\hat{W}_{m+1}^{A} = \sum_{i=1}^{m} \theta_i$ where θ_i's are the solution to equation (A-6). By equation (A-3), we have that as long as the constraint is binding, the critical wealth level $\hat{W}_i^A$ at which it is optimal to add Stock i strictly increases with i. Combining these results shows that both the number and the dollar amount of stocks optimally held increase as the adjusted wealth W_p^A increases. This completes the proof of Parts 1-3.

Now we show Part 4. If the investor has a CRRA utility, i.e.,

$$u(W) = \frac{W^{1-\gamma}}{1-\gamma},$$

then equation (A-6) is equivalent to

$$(\text{A-7}) \qquad E\left[u'\left(1 + \sum_{i=1}^{m} w_i \tilde{z}_i\right)(\tilde{z}_i - \tilde{z}_{m+1})\right] = 0, \quad i = 1, 2, ..., m,$$

where $w_i \equiv \theta_i / W_p$ represent the fraction of the initial wealth W_p invested in stock i. And the constraint (10) become

$$\sum_{i=1}^{n} w_i \leq 1 - \frac{\underline{C}}{W_p} \ \text{ and } \ w_i \geq 0, \quad i = 1, 2, ..., n.$$

Due to the no-borrowing constraint, if the w_i's that solve equation (A-7) are such that

$\sum_{i=1}^{m} w_i > 1$ for some $m < n$, then the investor never holds more than m stocks, no matter how wealthy he is. To show that this can happen, suppose we have

$$E\left[u'\left(1 + w_1 \tilde{z}_1\right)\left(\tilde{z}_1 - \tilde{z}_j\right)\right] > 0 \tag{A-8}$$

for all $w_1 \in [0, 1]$ and $j = 2, 3, ..., n$, which can hold if $\mu_1 - \mu_j$ is large enough, Stock 1's volatility is small enough, and Stock 1 is independent of other stocks. Then it is never optimal for the investor to hold more than one stock. This is due to the fact that the marginal utility of investing in Stock 1 is strictly greater than that of investing in any other stocks for all feasible w_1 (i.e., for $w_1 \in [0, 1]$). A similar proof applies to mean-variance preferences.

For Part 5, if $u'(\underline{C})$ is infinite, then equation (A-1) implies that all stocks have the same (infinite) marginal utility at $\underline{C}$ irrespective of their expected returns and risks. It is therefore optimal to hold all the stocks even when the wealth of the less wealthy investor is just slightly above the committed level $\underline{C}$. As the wealth of the less wealthy increases, he will increase the investment in every stock such that the marginal utility from the investment in each stock stays the same. Therefore, if $u'(\underline{C})$ is infinite, an investor always invests in all the stocks as long as his wealth $W_p > \underline{C}$, as predicted in the standard portfolio selection theory. □

Proof of Theorem 2: The case where $W_p = 0$ is the same as the case without the less wealthy, which is shown in the text. We now consider the case where $W_p = \eta > 0$. Suppose first η is small. Given the solvency constraint, the less wealthy cannot borrow or shortsell and thus the maximum amount that they can invest in any stock is η. Therefore, as long as investing in different stocks provides different marginal utilities, the investor will choose sequentially the stocks that provide the next highest marginal utility until his budget η is exhausted. We first examine the portfolio allocation problem of the less wealthy investors without taking into account the equilibrium price impact of their trades. The utility from investing a dollar amount $\eta > 0$ in Stock j is

$$U_j(\eta) = E\left[-e^{-A\frac{\eta}{p_j}\tilde{P}_j}\right] = -\left(1 + A\frac{\eta}{p_j}\beta_j\right)^{-\alpha_j}.$$

Accordingly, the marginal utility from investing η in Stock j is

$$U_j'(\eta) = \frac{\frac{A\alpha_j\beta_j}{p_j}}{\left(1 + A\frac{\eta}{p_j}\beta_j\right)^{\alpha_j+1}}, \tag{A-9}$$

which, as $\eta \downarrow 0$, converges to

$$\lim_{\eta\downarrow 0} U_j'(\eta) = \frac{A\alpha_j\beta_j}{p_j} = A\frac{\kappa_j}{p_j} = A(\mu_j + 1), \tag{A-10}$$

where the last equality follows from equation (24). Therefore, irrespective of other moments (e.g., variance and skewness), the stock with the highest expected return yields the greatest marginal utility when the amount of investment η is small. By inequality (26) and the continuity of $U_j'(\eta)$, a less wealthy investor invests the entire amount η in Stock 1, when η is small enough.

We now derive the new equilibrium price for Stock 1, taking into account the equilibrium price impact of the less wealthy investor's purchase of Stock 1. Let $\hat{p}_1$ be the new equilibrium price of Stock 1. By equation (21), the market clearing condition for Stock 1 becomes

$$\lambda \times \frac{\eta}{\hat{p}_1} + 1 \times \left(\frac{\kappa_1}{\hat{p}_1} - 1\right)\frac{\kappa_1}{A\varphi_1^2} = \bar{\omega}_1,$$

which implies that the new equilibrium price for Stock 1 is

$$\hat{p}_1 = \frac{\kappa_1^2 + A\lambda\eta\varphi_1^2}{\kappa_1 + A\bar{\omega}_1\varphi_1^2} = p_1\left(1 + A\lambda\eta\frac{\varphi_1^2}{\kappa_1^2}\right), \tag{A-11}$$

and the new equilibrium expected return becomes

$$\hat{\mu}_1 = \frac{\kappa_1}{\hat{p}_1} - 1 = \mu_1 - \frac{A\lambda\eta\frac{\varphi_1^2}{\kappa_1^2}}{1 + A\lambda\eta\frac{\varphi_1^2}{\kappa_1^2}}(\mu_1 + 1).$$

Therefore, by inequality (26), there exists a small enough $\eta > 0$ such that inequality (26) still holds with μ_1 replaced by $\hat{\mu}_1$.[19] This shows that indeed when their wealth is low enough, less wealthy investors buy only Stock 1 in equilibrium and thus underdiversify.

We next show that as their wealth increases, less wealthy investors first increase the investment in Stock 1, then add the stock with the second highest expected return (i.e., Stock 2), then increase the investment in both Stock 1 and Stock 2, then add the stock with the third highest expected return (i.e., Stock 3), and so on until they are rich enough to hold the same portfolio as the wealthy and thus become fully diversified.

We show this by induction. Suppose at a higher wealth W_p a less wealthy investor invests only in the first $i-1$ stocks for $i \geq 2$. Let $\delta_j \geq 0$ denote the dollar amount invested in Stock j ($j = 1, 2, ..., n+1$) with $\delta_j > 0$ only for $j \leq i-1$ (recall that "Stock" $n+1$ is the risk free asset). Let $\bar{p}_j$ be the new equilibrium price for Stock j for $1 \leq j \leq n+1$. For $m \geq i$, since $\delta_m = 0$ and the wealthy's demand for a stock is independent of other stocks (as shown in equation (21)), the equilibrium price for Stock m remains the same as in the case with $\lambda = 0$, i.e., $\bar{p}_m = p_m$. Let

$$\begin{aligned} V(\delta_1, \delta_2, ..., \delta_n, \delta_{n+1}) &= E\left[-\exp\left(-A\sum_{j=1}^{n}\frac{\delta_j}{\bar{p}_j}\tilde{P}_j - A\delta_{n+1}\right)\right] \\ &= -\prod_{j=1}^{n}\left(1 + A\frac{\delta_j}{\bar{p}_j}\beta_j\right)^{-\alpha_j} e^{-A\delta_{n+1}} \end{aligned} \qquad \text{(A-12)}$$

[19]Since the less wealthy will only buy Stock 1 and by equation (21), the wealthy's demand for a stock is independent of any other stocks, the equilibrium prices and expected returns of other stocks remain the same.

be the value function of the less wealthy. Equation (A-12) implies that the marginal utility from investing δ_j in Stock j is

$$\text{(A-13)} \qquad \frac{\partial V(\delta_1, \delta_2, ..., \delta_n, \delta_{n+1})}{\partial \delta_j} = A|V(\delta_1, \delta_2, ..., \delta_n, \delta_{n+1})| \frac{\kappa_j}{\bar{p}_j + A\delta_j \varphi_j^2/\kappa_j},$$

which shows that for $m \geq i$, the marginal utility at $\delta_m = 0$ is

$$\text{(A-14)} \qquad \frac{\partial V(\delta_1, \delta_2, ..., \delta_n, \delta_{n+1})}{\partial \delta_m} = A|V(\delta_1, \delta_2, ..., \delta_n, \delta_{n+1})| \frac{\kappa_m}{p_m} = A|V(\delta_1, \delta_2, ..., \delta_n, \delta_{n+1})|(\mu_m + 1).$$

Similar to equation (A-10), equation (A-14) shows that the marginal utility from investing δ_m in Stock m at $\delta_m = 0$ only depends on its expected return μ_m, but not on any of its higher moments. Equations (A-14) and inequality (26) then imply that the next stock the less wealthy investor is going to add when his wealth increases enough beyond W_p will be Stock i, the stock with the next highest expected return and thus the highest marginal utility among the remaining stocks.

For investing δ_j in Stock j $(j = 1, 2, ..., n+1)$ at W_p to be optimal, the marginal utility from each of the first $i - 1$ stocks must be the same and must also be greater than the marginal utility from investing any positive amount in the rest of the stocks. By equations (A-13) and (A-14), these optimality conditions then imply that

$$\text{(A-15)} \qquad \frac{\kappa_j}{\bar{p}_j + A\delta_j \varphi_j^2/\kappa_j} = k + 1, j = 1, 2, ..., i - 1,$$

for some $k \in [\mu_i, \mu_{i-1})$ that is to be determined later.[20]

Given the investment of δ_j in Stock j, a similar argument to that for equation (A-11)

[20] $k < \mu_{i-1}$ is because equation (A-15) shows that at $\delta_{i-1} = 0$, $k = \mu_{i-1}$, and the left-hand side of (A-15) is decreasing in δ_j as implied by (A-15) and (A-16).

implies that the new equilibrium price of Stock j is

$$\text{(A-16)} \qquad \bar{p}_j = \frac{\kappa_j^2 + A\lambda\delta_j\varphi_j^2}{\kappa_j + A\bar{\omega}_j\varphi_j^2}, \quad j = 1, 2, ..., i-1.$$

Solving equations (A-16) and (A-15) for $\bar{p}_j$ and δ_j and simplifying, we have that the new equilibrium prices are

$$\text{(A-17)} \qquad \bar{p}_j = p_j \frac{\lambda/(k+1)+1}{\lambda/(\mu_j+1)+1}, \quad j = 1, 2, ..., i-1,$$

which yields respectively the new expected returns and volatilities:

$$\text{(A-18)} \qquad \bar{\mu}_j = \nu_k k + (1-\nu_k)\mu_j, \quad j = 1, 2, ..., i-1,$$

and

$$\text{(A-19)} \qquad \bar{\sigma}_j = \frac{\varphi_j}{\kappa_j}(\bar{\mu}_j + 1), \quad j = 1, 2, ..., i-1,$$

where

$$\nu_k = \frac{\lambda}{\lambda + k + 1} \in [0, 1].$$

Thus, the new equilibrium expected returns are weighted averages of the original expected returns (μ_j's) and k. Since the weight ν_k is the same across all the first $i-1$ stocks, equation (A-18), inequality (26), and $k \in [\mu_i, \mu_{i-1})$ imply that $\bar{\mu}_1 > \bar{\mu}_2 > ... > \bar{\mu}_{i-1} > k \geq \mu_i > ... > \mu_n > \mu_{n+1}$. So the less wealthy will indeed invest only in the first $i-1$ stocks at W_p in equilibrium.

Plugging equation (A-17) back into equation (A-15) and simplifying, we have that the equilibrium dollar amount invested in Stock j is

$$\text{(A-20)} \qquad \delta_j = \frac{\alpha_j(\mu_j - k)}{A(k+1)(\lambda + \mu_j + 1)} = \frac{\bar{\mu}_j + 1}{k+1}\frac{\bar{\mu}_j - k}{A\bar{\sigma}_j^2}, \quad j = 1, 2, ..., i-1.$$

Without borrowing or shortselling, we must have the following budget constraint

$$\text{(A-21)} \qquad W_p = \sum_{j=1}^{i-1} \delta_j = \sum_{j=1}^{i-1} \frac{\alpha_j(\mu_j - k)}{A(k+1)(\lambda + \mu_j + 1)},$$

which yields that

$$\text{(A-22)} \qquad k = \frac{\sum_{j=1}^{i-1} \frac{\alpha_j \mu_j}{\lambda + \mu_j + 1} - A\,W_p}{\sum_{j=1}^{i-1} \frac{\alpha_j}{\lambda + \mu_j + 1} + A\,W_p}.$$

Equations (A-22), (A-20), and (A-13) show that as the wealth increases, k decreases and for all $j \leq i-1$, the dollar amount δ_j invested in Stock j increases, and the marginal utility of investing in Stock j decreases. When the wealth reaches a threshold level at which the marginal utility of investing more in each of the first $i-1$ stocks is equal to the marginal utility of investing a small amount in Stock i, the investor adds Stock i to his portfolio. By equations (A-13), (A-14), and (A-15), this threshold wealth level $\hat{W}_i$ above which the less wealthy investor holds Stock i must be such that $k = \mu_i$, which combined with equation (A-21) implies that

$$\text{(A-23)} \qquad \hat{W}_i = \sum_{j=1}^{i-1} \frac{\alpha_j(\mu_j - \mu_i)}{A(\mu_i + 1)(\lambda + \mu_j + 1)}.$$

By equation (A-22), we have $k \in [\mu_i, \mu_{i-1})$ if and only if $W_p \in (\hat{W}_{i-1}, \hat{W}_i]$. As i increases, μ_i decreases, so the threshold wealth level $\hat{W}_i$ increases. Because the above derivation applies to any $i = 2, 3, ..., n$, we have shown that for $2 \leq i \leq n$, the less wealthy holds only the stocks with the highest $i-1$ expected returns if and only if the initial wealth $W_p \in (\hat{W}_{i-1}, \hat{W}_i]$, equivalently if and only if $k \in [\mu_i, \mu_{i-1})$.

Therefore, as wealth increases, the less wealthy investors sequentially add stocks with the next highest expected returns. Eventually, the less wealthy will start to invest in the risk-free asset (which has the lowest expected return) when the less wealthy's wealth increases to a critical level $\hat{W}_{n+1}$. Beyond this critical level, because investing more in the risk-free asset does not increase risk, equations (A-13) and (A-14) imply that the marginal utility of investing more in the risk-free asset decreases less than that of investing more in any of the risky stocks. Therefore, the investor optimally invests any amount above $\hat{W}_{n+1}$ in the risk-free asset and no additional amount in any of the risky stocks (a standard result for CARA preferences). This implies that $\hat{W}_{n+1}$ is also the critical wealth level at which the less wealthy hold the same unconstrained optimal stock portfolio as the wealthy. Therefore, to get $\hat{W}_{n+1}$, we can simply set $k = \mu_{n+1} = 0$ and $i = n+1$ in equation (A-23). This shows that as W_p approaches $\hat{W}_{n+1}$, the less wealthy's portfolio converges to that of the wealthy.

Finally, as shown above, as long as $W_p \in (0, \hat{W}_{n+1})$, the less wealthy and the wealthy hold different portfolios and thus no one in the economy holds the market portfolio in equilibrium. The market portfolio's weight on Stock j is

$$w_j^M = \frac{\bar{\omega}_j \bar{p}_j}{\sum_{i=1}^{n} \bar{\omega}_i \bar{p}_i}, j = 1, 2, ..., n.$$

Direct computation using equations (A-17), (A-18), and (A-19) shows that CAPM does not hold, i.e.,

$$\bar{\mu}_j - r \neq \boldsymbol{\beta}_{jM}(\bar{\mu}_M - r),$$

where $r = 0$,

$$\bar{\mu}_M = \sum_{i=1}^{n} w_i^M \bar{\mu}_i,$$

$$\boldsymbol{\beta}_{jM} = \frac{w_j^M \bar{\sigma}_j^2}{\sum_{i=1}^n (w_i^M)^2 \bar{\sigma}_i^2}.$$

Thus idiosyncratic risks are priced. What holds is a modified CAPM equation with a nonzero alpha term. Specifically, we have for $j = 1, 2, ..., n$,

$$\tilde{r}_j - r = \boldsymbol{\alpha}_j + \boldsymbol{\beta}_j(\tilde{r}_M - r) + \tilde{\varepsilon}_j, \tag{A-24}$$

where $\tilde{r}_j$ is the Stock j's return, $\tilde{r}_M$ is the market portfolio return, $\tilde{\varepsilon}_j$ is the mean-zero error term, and

$$\boldsymbol{\alpha}_j \equiv (\bar{\mu}_j - r) - \boldsymbol{\beta}_{jM}(\bar{\mu}_M - r). \tag{A-25}$$

□

The following lemma will be used repeatedly in the proof of Theorem 3.

Lemma 1 *Given positive integer i, if $x_j > 0$, $b_j > 0$, $\hat{b}_j > 0$, and both x_j and $b_j/\hat{b}_j$ decrease with j for $j = 1, 2, ..., i$, then*

$$\frac{\sum_{j=1}^i b_j\, x_j}{\sum_{j=1}^i b_j} \geq \frac{\sum_{j=1}^i \hat{b}_j\, x_j}{\sum_{j=1}^i \hat{b}_j}. \tag{A-26}$$

Proof of Lemma 1: Let $\pi_j = \frac{b_j}{\sum_{l=1}^i b_l}$ and $\hat{\pi}_j = \frac{\hat{b}_j}{\sum_{l=1}^i \hat{b}_l}$. Then the left- (right-) hand side of inequality (A-26) can be viewed as the mean of a random variable $\tilde{x}$ with support $\{x_1, x_2, ..., x_i\}$ and a probability of π_j ($\hat{\pi}_j$, respectively) for x_j ($j = 1, 2, ..., i$). Next we show that the assumption that both x_j and $b_j/\hat{b}_j$ decrease with j implies that the probability distribution for the left-hand side of equation (A-26) first stochastic dominates that for the right-hand side and thus the mean on the left-hand side is greater than that on the right-hand side, and accordingly, inequality (A-26) holds. Since $x_j > 0$ decreases with j, for the first stochastic dominance we only need to show that for any $1 \leq m \leq i-1$,

$$\sum_{j=1}^{m} \pi_j \geq \sum_{j=1}^{m} \hat{\pi}_j, \tag{A-27}$$

which is equivalent to

$$\frac{\sum_{j=1}^{m} b_j}{\sum_{j=1}^{m} \hat{b}_j} \geq \frac{\sum_{j=m+1}^{i} b_j}{\sum_{j=m+1}^{i} \hat{b}_j}. \tag{A-28}$$

Since $b_j/\hat{b}_j > 0$ decreases with j, we have

(A-29)

$$\frac{\sum_{j=1}^{m} b_j}{\sum_{j=1}^{m} \hat{b}_j} = \frac{\sum_{j=1}^{m} \hat{b}_j \frac{b_j}{\hat{b}_j}}{\sum_{j=1}^{m} \hat{b}_j} \geq \frac{\sum_{j=1}^{m} \hat{b}_j \frac{b_m}{\hat{b}_m}}{\sum_{j=1}^{m} \hat{b}_j} = \frac{b_m}{\hat{b}_m} = \frac{\sum_{j=m+1}^{i} \hat{b}_j \frac{b_m}{\hat{b}_m}}{\sum_{j=m+1}^{i} \hat{b}_j} \geq \frac{\sum_{j=m+1}^{i} \hat{b}_j \frac{b_j}{\hat{b}_j}}{\sum_{j=m+1}^{i} \hat{b}_j} = \frac{\sum_{j=m+1}^{i} b_j}{\sum_{j=m+1}^{i} \hat{b}_j}.$$

Therefore inequality (A-28) indeed holds and thus inequality (A-26) also holds. □

Intuitively, Lemma 1 holds because b_j assigns higher weights to larger values of $\tilde{x}$ than $\hat{b}_j$. We are now ready to prove Theorem 3.

Proof of Theorem 3: Since as shown in equation (A-22), k decreases as W_p increases, we can equivalently show how the properties of the less wealthy's portfolio change with k. The mth central moment of the less wealthy's stock portfolio gross return is

$$\xi_m(k) = \sum_{j=1}^{i} w_j^m \frac{M_{mj}}{\bar{p}_j^m}, \tag{A-30}$$

where the portfolio weight

$$w_j = \frac{\bar{\delta}_j}{\sum_{j=1}^{i} \bar{\delta}_j}, \quad j = 1, 2, ..., i,$$

and M_{mj} is the mth central moment of the payoff of the jth stock. Using equations (A-17) and (A-20), we have

$$\xi_m(k) = \frac{\sum_{j=1}^{i} C_{mj}(\mu_j - k)^m}{((\lambda/(k+1)+1)\sum_{j=1}^{i} a_j(\mu_j - k))^m}, \tag{A-31}$$

where $C_{mj} \equiv M_{mj}/\beta_j^m$ and $a_j \equiv \alpha_j/(\lambda + \mu_j + 1)$. Computing $\xi_m'(k)$ and rearranging yield that $\xi_m(k)$ is strictly increasing in k for all $\lambda > 0$ if and only if

$$\frac{\sum_{j=1}^{i} b_{mj}(\mu_j + 1)}{\sum_{j=1}^{i} b_{mj}} \geq \frac{\sum_{j=1}^{i} a_j(\mu_j + 1)}{\sum_{j=1}^{i} a_j}, \tag{A-32}$$

where

$$b_{mj} = C_{mj}(\mu_j - k)^{m-1}.$$

By Lemma 1, for inequality (A-32) to hold, we only need to show b_{mj}/a_j decreases with j. First, consider the expected return. Since $M_{1j} = \alpha_j \beta_j$, we have $b_{1j} = \alpha_j$. So

$$\frac{b_{1j}}{a_j} = \lambda + \mu_j + 1,$$

which indeed decreases with j by inequality (26) and thus the expected return decreases as k decreases. Next, consider the return volatility. Since $M_{2j} = \alpha_j \beta_j^2$, we have $b_{2j} = \alpha_j(\mu_j - k)$ and

$$\frac{b_{2j}}{a_j} = (\mu_j - k)(\lambda + \mu_j + 1),$$

which also decreases with j by inequality (26) and thus the volatility also decreases as k decreases.

The skewness of the stock portfolio is equal to

$$s(k) \equiv \frac{\xi_3(k)}{\xi_2(k)^{3/2}} = \frac{\sum_{j=1}^{i} C_{3j}(\mu_j - k)^3}{\left(\sum_{j=1}^{i} C_{2j}(\mu_j - k)^2\right)^{3/2}}.$$

It is easy to verify that $C_{3j} = 2\alpha_j$, $C_{2j} = \alpha_j$, and the skewness is increasing in k if and only if

$$\text{(A-33)} \qquad \frac{\sum_{j=1}^{i} b_j(\mu_j - k)}{\sum_{j=1}^{i} b_j} \geq \frac{\sum_{j=1}^{i} \hat{b}_j(\mu_j - k)}{\sum_{j=1}^{i} \hat{b}_j},$$

where

$$b_j = \alpha_j(\mu_j - k)^2, \quad \hat{b}_j = \alpha_j(\mu_j - k).$$

Since $b_j/\hat{b}_j = \mu_j - k$, which decreases with j, by Lemma 1, we have inequality (A-33) holds and therefore the skewness of the stock portfolio also decreases as k decreases.

For $W_p > \hat{W}_2$, the less wealthy investor holds at least two stocks, i.e., $i \geq 2$. The Sharpe ratio of the stock portfolio at $\lambda = 0$ is

$$SR(k) \equiv \frac{\xi_1(k) - 1}{\sqrt{\xi_2(k)}} = \frac{\sum_{j=1}^{i}(C_{1j} - a_j)(\mu_j - k)}{\sqrt{\sum_{j=1}^{i} C_{2j}(\mu_j - k)^2}}.$$

Computing $SR'(k)$ shows that the Sharpe ratio strictly *decreases* in k if and only if

$$\text{(A-34)} \qquad \frac{\sum_{j=1}^{i} d_j(\mu_j - k)}{\sum_{j=1}^{i} d_j} > \frac{\sum_{j=1}^{i} \hat{d}_j(\mu_j - k)}{\sum_{j=1}^{i} \hat{d}_j},$$

where

$$d_j = \alpha_j(\mu_j - k), \quad \hat{d}_j = \alpha_j \frac{\mu_j}{\mu_j + 1}.$$

Since $d_j/\hat{d}_j = \mu_j - k + 1 - k/\mu_j$, which strictly decreases with j, by Lemma 1, we have that inequality (A-34) holds and therefore by continuity the Sharpe ratio of the stock portfolio increases as k decreases when λ is small. $\square$