Appendix: The Theoretical Model of Bimetallic Stability with Bullion Market Integration

This Appendix develops a general equilibrium model for a world bimetallic economy. The model herein, which is an adaptation of the models that were developed by I. Fisher (1894) and M. Flandreau (2004), determines where the permissible bimetallic ratios lie in the context of an open world economy comprising two bimetallic centers, England and the Dutch Republic. This economy is composed of three tradable goods: gold and silver, which are used for both monetary and nonmonetary purposes, and one representative consumer good that is not used for any monetary purpose. These three goods are available in quantities that are exogenously determined. International arbitrage ensures uniformity in the market price of gold, silver, and the consumer good between the two centers.

According to Walras' Law, any particular market must be in equilibrium if all of the other markets in the economy are in equilibrium. Therefore, we can omit one market (for example, the representative good market) because the general equilibrium in this world economy is entirely described by three markets: the money market, the gold commodity market, and the silver commodity market. Gold and silver are perfect substitutes for monetary purposes but imperfect substitutes for nonmonetary purposes. The clear distinction between monetary and nonmonetary purposes is that the utility of the monetary market depends on purchasing power, whereas the utility of the nonmonetary market depends on physical quantities.

The equilibrium conditions for the world economy under the bimetallic standard are described by the following equations.

First, let us describe the nonmonetary demand function for gold and silver. The demand for the gold commodity in center *i* (G_c^i) is a function of the gold market price (p_G), and the demand for the silver commodity in center *i* (S_c^i) is a function of the silver market price (p_S)

$$G_c^A = \mu_G^A \frac{P}{p_G} Y^A \tag{A1}$$

$$G_c^L = \mu_G^L \frac{P}{p_G} Y^L \tag{A2}$$

$$S_c^A = \mu_S^A \frac{P}{p_S} Y^A \tag{A3}$$

$$S_c^L = \mu_S^L \frac{P}{p_S} Y^L \tag{A4}$$

where μ_G^i and μ_S^i are positive constants for center *i* (*i* = Amsterdam, London), *P* is the general price level (the price of the representative consumer good), p_G is the market price of gold as a commodity, p_S is the market price of silver as a commodity, and Y^i is the real income in center *i* (the quantity of the representative consumer good).

We deem that the real incomes of the two centers remain proportional to one another to preserve tractability

$$Y^L = \beta Y^A \tag{A5}$$

Merging equations A1, A2, and A5 provides the world demand for the gold commodity (equation A6), and merging equations A3, A4, and A5 provides the world demand for the silver commodity (equation A7)

$$G_c^W = (\mu_G^A + \beta\mu_G^L)\frac{P}{p_G}Y^A = \mu_G^W \frac{P}{p_G}Y^A \tag{A6}$$

$$S_c^W = (\mu_S^A + \beta\mu_S^L)\frac{P}{p_S}Y^A = \mu_S^W \frac{P}{p_S}Y^A \tag{A7}$$

Second, let us describe the monetary demand function. The nominal amount of money that is demanded in center *i* (*i* = Amsterdam, London) is the quantity of gold that is used for monetary purposes (G_m^i) multiplied by the gold price (p_G) plus the quantity of silver that is used for monetary purposes (S_m^i) multiplied by the silver price (p_S). Recall that gold currency and silver currency are perfect substitutes for payments; thus, the money demand is expressed in purchasing power units. The money demand is defined in accordance with the Cambridge equation (in which *k* is a positive constant)

$$p_G G_m^A + p_S S_m^A = k^A \cdot P \cdot Y^A \tag{A8}$$

$$p_G G_m^L + p_S S_m^L = k^L \cdot P \cdot Y^L \tag{A9}$$

Merging equations A8, A9, and A5 provides the world demand for money (equation A10)

$$p_G(G_m^A + G_m^L) + p_S(S_m^A + S_m^L) = (k^A + \beta k^L)P \cdot Y^A = k^W \cdot P \cdot Y^A \tag{A10}$$

The model is closed by equating the world bullion supply and demand (*G* and *S* represent the total outstanding stocks of gold and silver)

$$G_c^W + G_m^A + G_m^L = G \tag{A11}$$

$$S_c^W + S_m^A + S_m^L = S \tag{A12}$$

The model explained in equations A1 to A12 can be reduced to a system that describes the world economy's gold and silver monetary holdings as a function of the world's stocks of these two metals. The prices p_G and p_S are the equilibrium prices, and the parameters result from the combination of the different propensities to hold bullion (as money or as a commodity) in the two bimetallic centers.

The system can be formally summarized by two equilibrium relations

$$\begin{cases} p_G \cdot (G_m^A + G_m^L) = p_G \cdot G \cdot \left[1 - \left(\dfrac{\mu_G^W}{k^W + \mu_G^W + \mu_S^W}\right)\right] - p_S S \cdot \dfrac{\mu_G^W}{k^W + \mu_G^W + \mu_S^W} & \text{(A13a)} \\ \\ p_S \cdot (S_m^A + S_m^L) = -p_G \cdot G \cdot \dfrac{\mu_S^W}{k^W + \mu_G^W + \mu_S^W} + p_S \cdot S \cdot \left(1 - \dfrac{\mu_S^W}{k^W + \mu_G^W + \mu_S^W}\right) & \text{(A13b)} \end{cases}$$

The bimetallic economies can be in equilibrium on an effective bimetallic standard, a *de facto* monometallic standard, or a combination of both types, whereby one center is on bimetallism and the other is on monometalism, depending on the legal ratios defined by the English and Dutch governments ($\dfrac{\overline{p}_G^L}{\overline{p}_S^L}, \dfrac{\overline{p}_G^A}{\overline{p}_S^A}$). Let us observe the different equilibrium ratios as a function of the relative gold and silver resources.

First, let us suppose that both Amsterdam and London are on a *de facto* gold standard. The legal ratio is excessively high in relation to the market ratio such that the use of silver as money becomes impossible ($S_m^L = S_m^A = 0$). This case is represented by the line "Gold" (Figures 1 and 2 in section "Modeling Bimetallic Stability"). Substituting $S_m^L = S_m^A = 0$, I resolve the model for the equilibrium ratio as a function of the relative gold and silver resources

$$\frac{p_G}{p_S} = \frac{S}{G} \cdot \frac{\mu_G^W + k^W}{\mu_S^W} \tag{A14}$$

Second, let us suppose that both Amsterdam and London are on a *de facto* silver standard. The legal ratio is excessively low in relation to the market ratio such that the use of gold as money becomes impossible ($G_m^L = G_m^A = 0$). This case is represented by the line "Silver" (Figures 1 and 2 in section "Modeling Bimetallic Stability").

APPENDIX TABLE 1
MONETARY REGIMES WHEN AMSTERDAM AND LONDON HAVE THE SAME BIMETALLIC RATIO

Situation 1 (segment 1, Figure 1)	Amsterdam and London *de facto* gold standard
$\frac{S}{G} < min \frac{S^*}{G^*}\left[\frac{\overline{p}_G}{\overline{p}_S}\right]$	$\frac{p_G}{p_S} < \frac{\overline{p}_G}{\overline{p}_S}$ because $\overline{p}_S < p_S$ ($\overline{p}_G > p_G$)
Situation 2 (segment 2, Figure 1)	Amsterdam and London bimetallic standard
$min \frac{S^*}{G^*}\left[\frac{\overline{p}_G}{\overline{p}_S}\right] < \frac{S}{G} < max \frac{S^*}{G^*}\left[\frac{\overline{p}_G}{\overline{p}_S}\right]$	$\frac{\overline{p}}{\overline{p}_S} = \frac{p_G}{p_S}$ because $\overline{p}_S = p_S$ & $\overline{p}_G = p_G$
Situation 3 (segment 3, Figure 1)	Amsterdam and London *de facto* silver standard
$\frac{S}{G} > max \frac{S^*}{G^*}\left[\frac{\overline{p}_G}{\overline{p}_S}\right]$	$\frac{p_G}{p_S} > \frac{\overline{p}_G}{\overline{p}_S}$ because $\overline{p}_G < p_G$ ($\overline{p}_S > p_S$)

Notes: See Figure 1 in section "Modeling Bimetallic Stability."
Sources: Self-elaboration.

Substituting $G_m^L = G_m^A = 0$, I resolve the model for the equilibrium ratio as a function of the relative gold and silver resources

$$\frac{p_G}{p_S} = \frac{S}{G} \cdot \frac{\mu_G^W}{\mu_S^W + k^W} \tag{A15}$$

The effective monetary regime when Amsterdam and London have the same bimetallic ratio will depend on the equilibrium ratio for the given level of resources (see Figure 1 in section "Modeling Bimetallic Stability" for the graphical representation). The three possible equilibria are summarized in Appendix Table 1.

Finally, suppose that each center has a different legal ratio, as occurred in London and Amsterdam during the mid-eighteenth century. Amsterdam's ratio was lower than London's ratio (14.65 versus 15.21, respectively).[1] The line "London Gold and Amsterdam Silver" indicates that London is on a *de facto* gold standard ($S_m^L = 0$) and that Amsterdam is on a *de facto* silver standard ($G_m^A = 0$) (Figure 2 in section "Modeling Bimetallic Stability"). The model is resolved for the equilibrium ratio as a function of the relative gold and silver resources by substituting $G_m^A = S_m^L = 0$

$$\frac{p_G}{p_S} = \frac{S}{G} \cdot \frac{\mu_G^W + \beta k^L}{\mu_S^W + k^A} \tag{A16}$$

[1] The case of different legal ratios when the legal ratio in Amsterdam is higher than in London would be symmetrical.

APPENDIX TABLE 2
MONETARY REGIMES WHEN LONDON'S RATIO IS HIGHER THAN AMSTERDAM'S RATIO

Situation 1 (segment 1, Figure 2) $\frac{S}{G} < min \frac{S^*}{G^*}\left[\frac{\overline{p}_G^A}{\overline{p}_S^A}\right]$	Amsterdam and London *de facto* gold standard $\frac{p_G}{p_S} < \frac{\overline{p}_G^L}{\overline{p}_S^L}$ & $\frac{p_G}{p_S} < \frac{\overline{p}_G^A}{\overline{p}_S^A}$ because $\overline{p}_S^L < p_S(\overline{p}_G^L > p_G)$ & $\overline{p}_S^A < p_S(\overline{p}_G^A > p_G)$
Situation 2 (segment 2, Figure 2) $min \frac{S^*}{G^*}\left[\frac{\overline{p}_G^A}{\overline{p}_S^A}\right] < \frac{S}{G} < min \frac{S^*}{G^*}\left[\frac{\overline{p}_G^L}{\overline{p}_S^A}\right]$	Amsterdam bimetallic and London *de facto* gold standard $\frac{\overline{p}_G^A}{\overline{p}_S^A} = \frac{p_G}{p_S} < \frac{\overline{p}_G^L}{\overline{p}_S^L}$ because $\overline{p}_S^L < p_S(\overline{p}_G^L > p_G)$
Situation 3 (segment 3, Figure 2) $min \frac{S^*}{G^*}\left[\frac{\overline{p}_G^L}{\overline{p}_S^A}\right] < \frac{S}{G} < max \frac{S^*}{G^*}\left[\frac{\overline{p}_G^L}{\overline{p}_S^A}\right]$	Amsterdam *de facto* silver st. and London *de facto* gold st. $\frac{\overline{p}_G^A}{\overline{p}_S^A} < \frac{p_G}{p_S} < \frac{\overline{p}_G^L}{\overline{p}_S^L}$ because $\overline{p}_S^L < p_S(\overline{p}_G^L > p_G)$ & $\overline{p}_G^A < p_G(\overline{p}_S^A > p_S)$
Situation 4 (segment 4, Figure 2) $max \frac{S^*}{G^*}\left[\frac{\overline{p}_G^L}{\overline{p}_S^A}\right] < \frac{S}{G} < max \frac{S^*}{G^*}\left[\frac{\overline{p}_G^L}{\overline{p}_S^L}\right]$	Amsterdam *de facto* silver standard and London bimetallic $\frac{\overline{p}_G^A}{\overline{p}_S^A} < \frac{p_G}{p_S} = \frac{\overline{p}_G^L}{\overline{p}_S^L}$ because $\overline{p}_G^A < p_G(\overline{p}_S^A > p_S)$
Situation 5 (segment 5, Figure 2) $\frac{S}{G} > max \frac{S^*}{G^*}\left[\frac{\overline{p}_G^L}{\overline{p}_S^L}\right]$	Amsterdam and London *de facto* silver standard $\frac{p_G}{p_S} > \frac{\overline{p}_G^L}{\overline{p}_S^L}$ & $\frac{p_G}{p_S} > \frac{\overline{p}_G^A}{\overline{p}_S^A}$ because $\overline{p}_G^L < p_G(\overline{p}_S^L > p_S)$& $\overline{p}_G^A < p_G(\overline{p}_S^A > p_S)$

Notes: See Figure 2 in section "Modeling Bimetallic Stability."
Sources: Self-elaboration.

The effective monetary regime when Amsterdam's ratio is lower than London's ratio will depend on the equilibrium ratio for the given level of resources (see Appendix Figure 2 in section "Modeling Bimetallic Stability" for the graphical representation). The five possible equilibria are summarized in Appendix Table 2.

REFERENCES

Fisher, I. "The Mechanics of Bimetallism." *The Economic Journal* 4, no. 15 (1894): 527–37.

Flandreau, M. *The Glitter of Gold: France, Bimetallism, and the Emergence of the International Gold Standard, 1848–1873*. Oxford: Oxford University Press, 2004.