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SUPPLEMENTARY MATERIAL: STRONG CONVERGENCE OF PEAKS OVER A THRESHOLD

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Abstract

This supplementary material document contains proofs of all technical lemmas presented in the main paper.

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1. Recall on important notation

For $y > 0$, we denote $T(y) = U(e^y)$ and, for $t < x^*$, we define the functions

$$p_t(y) = \begin{cases} \frac{T(y+T^{-1}(t))-t}{s(t)} - \frac{e^{\gamma y}-1}{\gamma}, & \gamma > 0 \\ \frac{T(y+T^{-1}(t))-t}{s(t)} - y, & \gamma = 0 \\ \frac{T(y+T^{-1}(t))-x^*-\gamma^{-1}s(t)}{s(t)} - \frac{e^{\gamma y}-1}{\gamma}, & \gamma < 0 \end{cases},$$

with $s(t) = (1 - F(t))/f(t)$, and

$$q_t(y) = \begin{cases} \frac{1}{\gamma} \ln\left[1 + \gamma e^{-\gamma y} p_t(y)\right], & \gamma \neq 0 \\ p_t(y), & \gamma = 0 \end{cases}.$$

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Finally, for $x \in \mathbb{R}$, $\gamma \in \mathbb{R}$, $\rho \leq 0$, we set

$$I_{\gamma,\rho}(x) = \begin{cases} \int_0^x e^{\gamma s} \int_0^s e^{\rho z} \mathrm{d}z \mathrm{d}s, & \gamma \geq 0 \\ -\int_x^\infty e^{\gamma s} \int_0^s e^{\rho z} \mathrm{d}z \mathrm{d}s, & \gamma < 0 \end{cases}.$$

2. Proof of Lemma 4.1

By Lemma 5 in [3], for all $\varepsilon > 0$ there exists x_0 such that, for all $t \in (x_0, x^*)$ and $y > 0$,

$$e^{-\gamma x}|p_t(y)| \leq (1+\varepsilon)|A(e^{T^{-1}(t)})| I_{\gamma,\rho}(y) e^{-(\gamma-\varepsilon)y}.$$

Moreover, for a positive constant ϑ_1

$$I_{\gamma,\rho}(y) e^{-(\gamma-\varepsilon)y} \leq \vartheta_1 e^{2\varepsilon y}.$$

Combining these two inequalities, we deduce that

$$e^{-\gamma y}|p_t(y)| \leq (1+\varepsilon)|A(e^{T^{-1}(t)})| \vartheta_1 e^{2\varepsilon y}. \tag{1}$$

As a consequence, for any $\alpha > 0$ there exists a constant ϑ_2 such that

$$\sup_{y \in (0, -\alpha \ln |A(e^{T^{-1}(t)})|)} e^{-\gamma y}|p_t(y)| \leq \vartheta_2 |A(e^{T^{-1}(t)})|^{1-2\varepsilon\alpha}. \tag{2}$$

Therefore, choosing ε sufficiently small, $e^{-\gamma y}|p_t(y)|$ converges to zero uniformly over the interval $(0, -\alpha \ln |A(e^{T^{-1}(t)})|)$ as $t \to x^*$.

It now follows that, if $y \in (0, -\alpha \ln |A(e^{T^{-1}(t)})|)$ and $t > x_1$ for a sufficiently large value $x_1 < x^*$, when $\gamma \neq 0$ a first-order Taylor expansion of the logarithm at 1 yields

$$\begin{aligned} |q_t(y)| &= \left| \frac{1}{\gamma} \frac{\gamma e^{-\gamma y} p_t(y)}{1 + \vartheta(t,y)\gamma e^{-\gamma y} p_t(y)} \right| \\ &\leq \vartheta_4 |A(e^{T^{-1}(t)})| e^{2\varepsilon y}, \end{aligned}$$

where $\vartheta(t,y) \in (0,1)$ and ϑ_4 is a positive constant, while when $\gamma = 0$ it holds that

$$\begin{aligned} |q_t(y)| &= e^{\gamma y} e^{-\gamma y} |p_t(y)| \\ &\leq \vartheta_5 |A(e^{T^{-1}(t)})| e^{2\varepsilon y}, \end{aligned}$$

where ϑ_5 is a positive constant. The result in the statement is a direct consequence of the last two inequalities.

3. Proof of Lemma 4.2

If $\gamma \neq 0$

$$1 + q_t'(y) = \frac{\exp\left\{\int_{e^{T^{-1}(t)}}^{e^{y+T^{-1}(t)}} \frac{A(u)}{u} \mathrm{d}u\right\}}{1 + \gamma e^{-\gamma y} p_t(y)},$$

while if $\gamma = 0$

$$1 + q_t'(y) = \exp\left\{\int_{e^{T^{-1}(t)}}^{e^{y+T^{-1}(t)}} \frac{A(u)}{u} \mathrm{d}u\right\}.$$

Therefore, if $y \in (0, -\alpha \ln |A(e^{T^{-1}(t)})|)$ and $t > x_2$ for a sufficiently large value $x_2 < x^*$, using the bounds in formulas (1)–(2) and choosing a suitably small ε we deduce

$$\begin{aligned}
1 + q_t'(y) &\leq \frac{\exp\left\{\int_{e^{T^{-1}(t)}}^{e^{y+T^{-1}(t)}} \frac{A(u)}{u} \mathrm{d}u\right\}}{1 - \mathbb{1}(\gamma \neq 0)|\gamma| e^{-\gamma y} |p_t(y)|} \\
&\leq \exp\left\{y |A(e^{T^{-1}(t)})|\right\} \frac{1}{1 - \omega_1 |A(e^{T^{-1}(t)})| e^{2\varepsilon y}} \\
&\leq \exp\left\{\omega_2 |A(e^{T^{-1}(t)})| e^{2\varepsilon y}\right\},
\end{aligned}$$

for positive constants ω_i, $i = 1, 2$. Similarly,

$$\begin{aligned}
1 + q_t'(y) &\geq \frac{\exp\left\{\int_{e^{T^{-1}(t)}}^{e^{y+T^{-1}(t)}} \frac{A(u)}{u} \mathrm{d}u\right\}}{1 + \mathbb{1}(\gamma \neq 0)|\gamma| e^{-\gamma y} |p_t(y)|} \\
&\geq \exp\left\{-y |A(e^{T^{-1}(t)})|\right\} \frac{1}{1 + \omega_3 |A(e^{T^{-1}(t)})| e^{2\varepsilon y}} \\
&\geq \exp\left\{-\omega_4 |A(e^{T^{-1}(t)})| e^{2\varepsilon y}\right\},
\end{aligned}$$

for positive constants ω_i, $i = 3, 4$. The result now follows.

4. Proof of Lemma 4.3

Let $v_0 > 0$ satisfy $U(v_0) \neq 0$ and $U'(v_0) \neq 0$. Then, for $v > v_0$ it holds that

$$\begin{aligned}
\eta(U(v)) &= \frac{1 + \gamma U(v)}{v U'(v)} - 1 \\
&= \frac{1 + \gamma U(v_0)}{v U'(v)} + \gamma \int_{v_0}^{v} \frac{U'(r)}{v U'(v)} \mathrm{d}r - 1.
\end{aligned}$$

Moreover, by definition of A, we have the identity

$$\gamma\int_{v_0}^{v}\frac{U'(r)}{vU'(v)}\mathrm{d}r-1=\gamma\int_{v_0/v}^{1}\frac{U'(zv)}{U'(v)}\mathrm{d}z-1$$
$$=\int_{v_0/v}^{1}\gamma z^{\gamma-1}\left[\exp\left\{-\int_z^1\frac{A(vu)}{u}\mathrm{d}u\right\}-1\right]\mathrm{d}z-\left(\frac{v_0}{v}\right)^{\gamma}.$$

Therefore, denoting by $\mathcal{R}_2(v)$ the first term on the right-hand side and setting

$$\mathcal{R}_1(v)=\frac{1+\gamma U(v_0)}{vU'(v)}-\left(\frac{v_0}{v}\right)^{\gamma},$$

we have $\eta(U(v))=\mathcal{R}_1(v)+\mathcal{R}_2(v)$. On one hand, the function $\mathcal{R}_1(v)$ is regularly varying of order $-\gamma$. On the other hand, for any $\beta\in(0,1)$, the function $\mathcal{R}_2(v)$ can be decomposed as follows

$$\mathcal{R}_2(v)=\int_{v_0/v}^{v^{-(1-\beta)}}+\int_{v^{-(1-\beta)}}^{1}\gamma z^{\gamma-1}\left[\exp\left\{-\int_z^1\frac{A(vu)}{u}\mathrm{d}u\right\}-1\right]\mathrm{d}z$$
$$=:\mathcal{R}_{2,1}(v)+\mathcal{R}_{2,2}(v).$$

Assuming that A is ultimately positive and selecting v_0 suitably large, we have

$$|\mathcal{R}_{2,1}(v)|\le\int_{v_0/v}^{v^{-(1-\beta)}}\gamma z^{\gamma-1}\left[1-\exp\left\{-\frac{A(vz)}{z}\right\}\right]\mathrm{d}z$$
$$=O(v^{-\gamma(1-\beta)})$$

and

$$|\mathcal{R}_{2,2}(v)|\le\int_{v^{-(1-\beta)}}^{1}\gamma z^{\gamma-1}\left[1-z^{A(v^{\beta})}\right]\mathrm{d}z$$
$$=O(v^{-\gamma(1-\beta)}\vee A(v^{\beta})).$$

Consequently, there exists a regularly varying function $\mathcal{R}$ of index $\varrho=\gamma(\beta-1)\vee\rho\beta$ complying with the property in the statement as $v\to\infty$.

Similarly, if A is ultimately negative, choosing β such that $\beta<2\gamma$ and v_0 suitably large, we have

$$|\mathcal{R}_{2,1}(v)|\le\int_{v_0/v}^{v^{-(1-\beta)}}\gamma z^{\gamma-1}\left[u^{A(v_0)}-1\right]\mathrm{d}z$$
$$=O(v^{-(\gamma-\beta/2)(1-\beta)})$$

and

$$|\mathcal{R}_{2,2}(v)|\le\int_{v^{-(1-\beta)}}^{1}\gamma z^{\gamma-1}\left[z^{A(v^{\beta})}-1\right]\mathrm{d}z$$
$$=O(v^{-(\gamma-\beta/2)(1-\beta)}\vee|A(v^{\beta})|)$$

as $v \to \infty$. Hence, there exists a regularly varying function $\mathcal{R}$ of index $\varrho = (\beta - 1)(\gamma - \beta/2) \vee \rho\beta$ complying with the property in the statement. The proof is now complete.

5. Proof of Lemma 4.4

Let $\mathcal{R}^*(t) := \mathcal{R}(1/(1-F(t)))$, where $\mathcal{R}$ is as in Lemma 4.3. Then $\mathcal{R}^*(t)$ is regularly varying of index ϱ/γ (see, e.g., [4], Proposition 0.8(iv)). In turn, by Karamata's theorem (e.g., [4], Proposition 0.6(a)) we have that for a large t^*

$$\int_{t^*}^{\infty} \frac{|\eta(t)|}{1+\gamma t} \mathrm{d}t < \infty$$

and thus, by Proposition 2.1.4 in [2], we conclude that

$$\tau := \lim_{t \to \infty} \frac{1 - F(t)}{1 - H_\gamma(t)} \in (0, \infty). \tag{3}$$

As a consequence, for any $\delta \in (0, -\varrho)$, as $t \to \infty$

$$\begin{aligned} \mathcal{R}^*(t) &\sim \mathcal{R}\left(\frac{1}{\tau(1 - H_\gamma(t))}\right) \\ &= O(\{1 - H_\gamma(t)\}^\delta). \end{aligned}$$

The conclusion now follows by Proposition 2.1.5 in [2].

6. Proof of Lemma 4.5

By definition,

$$\begin{aligned} \tilde{\eta}\,(y) &= \frac{f(x^* - 1/y)}{[1 - F(x^* - 1/y)]y^2} - \gamma\left[\frac{f(x^* - 1/y)}{y(1 - F(x^* - 1/y))} + \frac{1}{\gamma}\right] \\ &=: \tilde{\eta}_1\,(y) + \tilde{\eta}_2\,(y)\,. \end{aligned}$$

On one hand, we have that as $y \to \infty$

$$\tilde{\eta}_1\,(y) = O(1/y).$$

On the other hand, for $v > 1$ we have the identity

$$\tilde{\eta}_2\left(\frac{1}{x^* - U(v)}\right) = \int_1^\infty \gamma z^{\gamma - 1}\left[1 - \exp\left\{\int_1^z \frac{A(uv)}{u}\mathrm{d}u\right\}\right]\mathrm{d}z.$$

Hence, if A is ultimately positive,

$$\begin{aligned} \tilde{\eta}_2\left(\frac{1}{x^* - U(v)}\right) &\leq -\gamma \int_1^\infty z^{\gamma-1}(z^{A(v)} - 1)\mathrm{d}z \\ &= O(A(v)) \end{aligned}$$

while, if A is ultimately negative,

$$\left|\tilde{\eta}_2\left(\frac{1}{x^*-U(v)}\right)\right| \leq \gamma A(v)\int_1^\infty z^{\gamma-1}\ln z\mathrm{d}z$$
$$= O(|A(v)|).$$

As a result of the two above inequalities, as $v\to\infty$

$$\tilde{\eta}_2(t) = O\left(\left|A\left(\frac{1}{1-F(x^*-1/y)}\right)\right|\right),$$

Therefore, by regular variation of $1/(1-F(x^*-1/y))$ with index $-1/\gamma$, $\tilde{\eta}_2(y)$ is eventually dominated by a regularly varing function of index $-\rho/\gamma$. The final result now follows.

7. Proof of Lemma 4.6

The function $\tilde{f}(y) := f(x^*-1/y)y^{-2}$ is the density of the distribution function $\widetilde{F}(y) := F(x^*-1/y)$, which is in the domain of attraction of $G_{\tilde{\gamma}}$, with $\tilde{\gamma} = -\gamma$. Moreover,

$$\tilde{\eta}(y) = \frac{(1+\tilde{\gamma}y)\tilde{f}(y)}{1-\widetilde{F}(y)} - 1.$$

By Lemma 4.5 and regular variation of $1-H_{\tilde{\gamma}}$ with index $-1/\tilde{\gamma}$, we have

$$\tilde{\eta}(y) = O(\{1-H_{\tilde{\gamma}}(y)\}^{\tilde{\delta}})$$

for any $\tilde{\delta}>0$ such that $-\tilde{\delta}/\tilde{\gamma} > \tilde{\varrho}$. Therefore, by Proposition 2.1.5 in [2], as $y\to\infty$ it holds that

$$\tilde{f}(y) = h_{\tilde{\gamma}}(y)[1+O(\{1-H_{\tilde{\gamma}}(y)\}^{\tilde{\delta}})],$$

which is the result.

8. Proof of Lemma 4.7

We analyse the cases where $\gamma>0$ and $\gamma<0$ separately.

Case 1: $\gamma>0$. In this case, $\tilde{l}_t = l_t$. By Lemma 4.4, there are positive constants κ, δ and ϵ such that, for all large t and all $x>0$

$$\frac{l_t(x)}{h_\gamma(x)} \leq \frac{h_\gamma(s(t)x+t)}{h_\gamma(x)}\frac{s(t)}{1-F(t)}\left[1+\kappa\left\{1-H_\gamma(s(t)x+t)\right\}^{\delta}\right]$$
$$\leq \left[\frac{1+\gamma x}{(1+\gamma t)/s(t)+\gamma x}\right]^{1+1/\gamma}\frac{1+\epsilon}{(s(t))^{1/\gamma}(1-F(t))}.$$

Moreover, by Lemma 4.3 it holds that as $t \to \infty$

$$\frac{1+\gamma t}{s(t)} = 1 + \eta(t) = 1 + o(1)$$

and, in turn, $(s(t))^{1/\gamma} \sim (1+\gamma t)^{1/\gamma}$. These two facts, combined with the tail equivalence relation in formula (3), imply that for all sufficiently large t and all $x > 0$

$$\begin{aligned}\frac{l_t(x)}{h_\gamma(x)} &\leq \left[\frac{1+\gamma x}{1-\epsilon+\gamma x}\right]^{1+1/\gamma} \frac{1+\epsilon}{(1-\epsilon)\tau} \\ &\leq \left[\frac{1}{1-\epsilon}\right]^{1+1/\gamma} \frac{1+\epsilon}{(1-\epsilon)\tau}.\end{aligned}$$

The result now follows.

Case 2: $\gamma < 0$. In this case, for any $x \in (0, -1/\gamma)$

$$\tilde{l}_t(x) = f\left(x^* - \frac{1}{y}\right) \frac{1}{y^2} \frac{y^2 \tilde{s}(t)}{1-F(t)}$$

where

$$y \equiv y(x,t) := \frac{1}{s(t)}\left[-\frac{1}{\gamma} - x\right]^{-1}$$

Note that y is bounded from below by $-\gamma/s(t)$, which converges to ∞ as $t \to x^*$. Thus, by Lemma 4.6 there are positive constants $\tilde{\delta}$, ϵ and $\tilde{\kappa}$ such that

$$\begin{aligned}\tilde{l}_t(x) &\leq (1-\gamma y)^{1/\gamma - 1}[1 + \tilde{\kappa}\{1 - H_{-\gamma}(y)\}^{\tilde{\delta}}] \frac{y^2 \tilde{s}(t)}{1-F(t)} \\ &\leq h_\gamma(x)\left[s(t)\left(-\frac{1}{\gamma} - x\right) - \gamma\right]^{\frac{1}{\gamma}-1} \frac{(1+\epsilon)(-\gamma^{-1}s(t))^{-1/\gamma}}{1-F(t)}.\end{aligned}$$

By hypothesis, it holds that $x < -1/\gamma$, thus

$$\left[s(t)\left(-\frac{1}{\gamma} - x\right) - \gamma\right]^{\frac{1}{\gamma}-1} \leq (-\gamma)^{\frac{1}{\gamma}-1}.$$

Finally, for all large t,

$$\frac{-\gamma^{-1}s(t)}{x^* - t} \leq (1+\epsilon)$$

Combining all the above inequalities we can now conclude that, for all large t and for any $x \in (0, (x^* - t)/\tilde{s}(t))$,

$$\frac{\tilde{l}_t(x)}{h_\gamma(x)} \leq (1+\epsilon)^{1-1/\gamma}(-\gamma)^{\frac{1}{\gamma}} \frac{(x^* - t)^{-\frac{1}{\gamma}}}{1-F(t)}.$$

Now, setting $t = U(v)$, we have that $v \to \infty$ if and only if $t \to x^*$ and, by Theorem 2.3.6 in [1], there is a constant $\varpi > 0$ such that for all large t

$$\frac{(x^* - t)^{-\frac{1}{\gamma}}}{1 - F(t)} \leq v[(1+\epsilon)\varpi v^\gamma]^{-\frac{1}{\gamma}} = [(1+\epsilon)\varpi]^{-\frac{1}{\gamma}}$$

The result now follows.

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