Nuclear Brinkmanship, Limited War, and Military Power

Online Appendix

The first step in characterizing the equilibria of the asymmetric-information game is describing the equilibria of the brinkmanship subgame. Let $\Gamma_B(p)$ denote the brinkmanship continuation game given p. A pure strategy in this subgame for D is a pair $\{\rho(p), q_D(r|p)\}$ where $\rho(p) \in [\underline{r}(p), \overline{r}(p)]$ is D's bid and $q_D(r|p) \in \{0, 1\}$ indicates whether D quits ($q_D(r|p) = 1$) or stands firm ($q_D(r|p) = 0$) after r. (When we consider mixed strategies, $q_D(r|p) \in [0, 1]$ will be the probability that D quits.) Similarly, a pure strategy for D' is the analogous pair $\{\rho'(p), q'_D(r|p)\}$. A strategy for C is a function $q_C(r|p)$ for all $r \in [\underline{r}, \overline{r}]$ where $q_C(r|p)$ is the probability C quits after bid r given p. We ease the notation by suppressing the argument "p" when it is not needed for clarity. A belief system for the challenger is a function $t(r)$ which is the conditional probability of facing D' given a bid of r. Finally, a PBE of the brinkmanship continuation game is an assessment $\Delta = \{\rho, q_D, \rho', q'_D, q_C, t\}$ which is sequentially rational and in which t is derived from C's prior beliefs by Bayes' rule when possible.

Three lemmas help characterize the PBEs of the brinkmanship game. Lemma 1A demonstrates that neither D nor D' ever bids an $r \in (\underline{r}, R_C(p))$. C is sure to stand firm after such a bid and, consequently, D and D' would have done better by bidding $\underline{r}$. Lemma 2A shows that at most one $r \in (R_C, \overline{r}]$ is played with positive probability in a PBE. Lemma 3A shows no $r \in (R_C, \overline{r}]$ is played with positive probability in any PBE satisfying D1. Taken together, these lemmas imply that a PBE satisfying D1 can put positive probability on at most $\underline{r}$ and $R_C(p)$.

Lemma 1A: *Let $\Delta = \{\rho, q_D, \rho', q'_D, q_C, t\}$ be a PBE of $\Gamma_B(p)$. Then $\rho \notin (\underline{r}, R_C(p))$ and $\rho' \notin (\underline{r}, R_C(p))$.*

Proof: Arguing by contradiction, suppose D bids an $r \in (\underline{r}, R_C)$. Since $r < R_C$, C strictly prefers to stand firm after r regardless of C's beliefs about the defender's type. It follows that D's payoff to bidding r is $\max\{-k_D - \underline{r}n_D, (1-p)(1-r)v_D - k_D - rn_D\}$. Given that $p > 0$, D would have done strictly better by bidding $\underline{r}$ and then standing firm to obtain

$(1-p)(1-\underline{r})v_D - k_D - \underline{r}n_D$. A similar argument holds for D'.■

To ease the proof of Lemma 2A, observe that irresolute defender's preference over any two distinct bids $r \geq R_C$ and $\widehat{r} \geq R_C$ depends solely on the probability that C backs down after r and $\widehat{r}$. Indeed, D strictly prefers r to $\widehat{r}$ if and only if $q_C(r) > q_C(\widehat{r})$. To see why, note that D is bluffing, i.e., sure to quit, whenever at least R_C since $R_C > R_D$ by assumption (ii). More specifically, $R_C(\widetilde{p}) - R_D(\widetilde{p}) > 0$ by assumption (ii), and $R_C(p) - R_D(p) > R_C(\widetilde{p}) - R_D(\widetilde{p})$ for $p > \widetilde{p}$ since R_C is increasing in p and R_D is decreasing. Given that D is sure to quit following any $r \geq R_C$, D strictly prefers r to $\widehat{r}$ if and only if $q_C(r)[(1-\underline{r})v_D - k_D - \underline{r}n_D] + [1-q_C(r)][-k_D - \underline{r}n_C] > q_C(\widehat{r})[(1-\underline{r})v_D - k_D - \underline{r}n_D] + [1-q_C(\widehat{r})][-k_D - \underline{r}n_C]$ or $q_C(r) > q_C(\widehat{r})$.

It is also useful to determine when D' prefers higher bids to lower bids. Suppose $r > \widehat{r} > R_C$. Assumption (i) ensures that $R'_D(p) \geq R'_D(\overline{p}) > \overline{r}(\overline{p}) \geq \overline{r}(p)$ and hence that D' is sure to stand firm after r or $\widehat{r}$. Because D' always stands firm, a higher bid brings a higher cost if C stands firm, i.e., $(1-p)v'_D - k_D - r[(1-p)v'_D + n_D]$ is decreasing in r. As a result, D' will only be willing to run risk $r > \widehat{r}$ if C is more more likely to quit after r than after $\widehat{r}$. To be more precise, D' strictly prefers r to $\widehat{r}$ if and only if $q_C(r)[(1-\underline{r})v'_D - k_D - \underline{r}n_D] + [1-q_C(r)][(1-p)v'_D - k_D - r[(1-p)v'_D + n_D]] > q_C(\widehat{r})[(1-\underline{r})v'_D - k_D - \underline{r}n_D] + [1-q_C(\widehat{r})][(1-p)v'_D - k_D - \widehat{r}[(1-p)v'_D + n_D]]$. This is equivalent to $q_C(r) > \widetilde{q}'_C(r,\widehat{r}) \equiv q_C(\widehat{r}) + [1-q_C(\widehat{r})(r-\widehat{r})[(1-p)v'_D + n_D]/[(1-\underline{r})v'_D - [(1-p)v'_D + n_D](R'_D - r)]$. D' is indifferent when $q_C(r) = \widetilde{q}'_C(r,\widehat{r})$.

LEMMA 2A: *Let Δ be a PBE in which $r \in (R_C, \overline{r}]$ is played with positive probability. Then no other $\widehat{r} \in (R_C, \overline{r}]$ is played with positive probability.*

Proof: The lemma holds vacuously if $p \leq \widetilde{p}$ as $(R_C, \overline{r}] = \varnothing$. Assume $p > \widetilde{p}$, and suppose that r and $\widehat{r}$ are played with positive probability in Δ with $r > \widehat{r} > R_C$. Then D' must put positive probability on both offers. Suppose not. If D alone put positive probability on r, then $t(r) = 0$ and C is sure to stand firm ($q_C(r) = 0$) since D is certain to back down as $r > R_D$. But if $q_C(r) = 0$, D can profitably deviate from r to bidding $\underline{r}$ and then standing firm. Hence D' must put positive probability on r. Repeating the argument for

$\widehat{r}$ establishes that D' must also put positive weight on $\widehat{r}$.

In order to put positive weight on r and $\widehat{r}$, D' must be indifferent between them. This implies $1 \geq q_C(r) = \widetilde{q}'_C(r, \widehat{r}) > q_C(\widehat{r}) \geq 0$. But this means that D strictly prefers r to $\widehat{r}$ as C is more likely to quit. This leaves $t(\widehat{r}) = 1$ and yields the contradiction $q_C(\widehat{r}) = 1$.■

LEMMA 3A: *Let Δ be a PBE satisfying D1. Then no $r \in (R_C, \overline{r}]$ is played with positive probability Δ.*

Proof: The lemma again holds vacuously if $p \leq \widetilde{p}$. Arguing by contradiction when $p > \widetilde{p}$, assume $r \in (R_C, \overline{r}]$ is played with positive probability. Lemmas 1A and 2A imply that the only other bids that might be played with positive probability are $\underline{r}$ and R_C.

Both D and D' must put positive probability r. Observe first that $q_C(r) > 0$. Otherwise both types prefer to deviate to $\underline{r}$ and r would not be played with positive probability. If only D plays r, $t(r) = 0$ and this leads to the contradiction $q_C(r) = 0$. If D' alone plays r, then $t(r) = 1$ and $q_C(r) = 1$. Moreover, D' must at least weakly prefer r to R_C, so $q_C(r) \geq \widetilde{q}'_C(r, R_C) > q_C(R_C)$. However, $q_C(r) > q_C(R_C)$ implies that D strictly prefers r to R_C. D must therefore at least weakly prefer $\underline{r}$ to r. This yields the contradiction $q_C(r) \leq 1 - p$.

Because both D and D' play r with positive probability, their respective equilibrium payoffs are $q_C(r)[(1-\underline{r})v_D - k_D - \underline{r}n_{-D}] + [1-q_C(r)][-k_D - \underline{r}n_C] = -k_D - \underline{r}n_C + q_C(r)(1-\underline{r})v_D$ and $q_C(r)[(1-\underline{r})v'_D - k_D - \underline{r}n_D] + [1-q_C(r)][(1-p)v'_D - k_D - r[(1-p)v'_D + n_D]]$. Now consider any downward deviation $z \in (R_C, r)$. We show that D1 requires C to believe that it is facing D' for sure (i.e., $t(z) = 1$). C's best response given this belief is to quit with $q_C(z) = 1$. But this would make z a profitable deviation for both D and D', and this contradiction would establish the lemma.

To see that D1 eliminates D at z, observe first that C is indifferent between standing firm and quitting after z if it believes it is facing D' with probability $(pv_C + n_C)(z - R_C)/[(1-\underline{r})v_C + (pv_C + n_C)(z - R_C)]$. Hence, any $q_C(z) \in [0, 1]$ can be rationalized as a best response to some beliefs about the deviator's type.

Moreover, D' strictly prefers deviating to z if $q_C(z) > \widetilde{q}'_C(z, r)$ where $\widetilde{q}'_C(z, r) < q_C(r)$ when $z < r$. D weakly prefers bluffing at z to bluffing at r when $q_C(z) \geq q_C(r)$. The set of C's weakly profitable deviations for D is a strict subset of the set of deviations that are strictly profitable for D'. Hence, D1 eliminates D. ■

The previous lemmas make it easy to specify a PBE satisfying D1. Lemma 1A implies that both D and D' bid $\underline{r}$ and all states subsequently stand firm whenever $p < \widetilde{p}$ as this implies $\overline{r} < R_C$. Proposition 1A describes a separating PBE satisfying D1 when $p \in (\widetilde{p}, \overline{p}]$. D and D' respectively bid $\underline{r}$ and R_C, and both types then stand firm. C stands firm after $\underline{r}$ and does so with probability p after R_C. More precisely, define the assessment Δ_0 in which D plays according to $\rho = \underline{r}$, $q_D(r) = 0$ for $r \leq R_D$ and $q_D(r) = 1$ for $r > R_D$; D' plays according to $\rho' = R_C$, $q'_D(r) = 0$ for $r \leq R'_D$ and $q'_D(r) = 1$ for $r > R'_D$; and C follows $q_C(r) = 0$ for $r \neq R_C$ and $q_C(R_C) = 1 - p$. C's beliefs are $t(r) = 0$ if $r = \underline{r}$, $t(r) = \tau$ for $r \in (\underline{r}, R_C)$, $t(R_C) = 1$, and $t(r) = 0$ for $r \in (R_C, \overline{r}]$.

Proposition 1A: *If* $p \in (\widetilde{p}, \overline{p}]$, Δ_0 *is a PBE satisfying D1.*

Proof: Verifying that Δ_0 is a PBE is straightforward. Given $q_C(r) = 0$ for $r \neq R_C$, D and D' will bid either $\underline{r}$ or R_C since the payoff to $\underline{r}$ is strictly better than the payoff to bidding any $r \notin \{\underline{r}, R_C\}$. At $q_C(R_C) = 1 - p$, D is indifferent between $\underline{r}$ and R_C, so $\underline{r}$ is a best reply. D' strictly prefers R_C to $\underline{r}$. C in turn strictly prefers standing firm after $\underline{r}$ and is indifferent after R_C given that it believes it is facing D' for sure ($t(R_C) = 1$). Accordingly, $q_C(\underline{r}) = 0$ and $q_C(R_C) = 1 - p$ are best replies. C's beliefs are also clearly consistent with Bayes' rule.

To demonstrate that Δ_0 satisfies D1 consider any deviation $r > R_C$. As shown in the proof of Lemma 3A, any $q_C(r) \in [0, 1]$ can be rationalized as a best response to some beliefs about the deviator's type. Moreover, D' stands firm after bidding r since $R'_D(p) \geq R'_D(\overline{p}) > \overline{r}(p) \geq r$. This implies the responses $q_C(r)$ for which r is weakly profitable are defined by $q_C(r)[(1-\underline{r})v'_D - k_D - \underline{r}n_D] + [1 - q_C(r)][(1-p)v'_D - k_D - r[(1-p)v'_D + \underline{r}n_D] \geq (1-p)[(1-\underline{r})v'_D - k_D - \underline{r}n_D] + p[(1-p)v'_D - k_D - R_C[(1-p)v'_D + \underline{r}n_D]]$. This simplifies to $q_C(r) \geq \widetilde{q}_0(r) \equiv 1 - p + p[(1-p)v'_D + \underline{r}n_D](r - R_C)$.

As for D, assumption (ii) ensures that D is certain to quit if it bids r and C stands firm. That is, $r > R_C(p) > R_C(\widetilde{p})$ and $R_D(\widetilde{p}) > R_D(p)$ since R_C is increasing and R_D is decreasing and, by assumption (ii), $R_C(\widetilde{p}) > R_D(\widetilde{p})$. According, deviating to r is strictly profitable for D if $q_C(r)[(1-\underline{r})v_D - k_D - \underline{r}n_D] + [1-q_C(r)](-k_D - \underline{r}n_D) > (1-p)(1-\underline{r})v_D - k_D - \underline{r}n_D$ or $q_C(r) > 1-p$.

The set of C's responses to r for which r is weakly profitable for D' is a strict subset of the set of responses for which r is strictly profitable for D. That is, $\{q_C(r) : q_C(r) \geq \widetilde{q}_0(r)\} \subset \{q_C(r) : q_C(r) > 1-p\}$. D1 therefore eliminates D' and requires C to put probability one on D after $r > R_C$. Δ_0 does this as $t(r) = 0$ for $r > R_C$.

For $r \in (\underline{r}, R_C)$, C's unique best response is $q_C(r) = 0$ regardless of its beliefs. As a result, D1 has no bite, and any $t(r) \in [0,1]$ is consistent with D1. Hence, C's out-of-equilibrium beliefs in Δ_0 satisfy D1.■

D1 does not pin down a unique PBE. As Corollary 1A shows, other equilibria satisfying D1 exist. This multiplicity of equilibria arises from C's indifference between quitting and standing firm following a bid of $r = R_C$. To establish that equilibria other than Δ_0 exist and satisfy D1, observe first that C is indifferent only if C believes that it is facing D' with probability one after a bid of R_C. Any positive probability of facing D breaks C's indifference and leads C to stand firm. It follows that C must quit after a bid of R_C with a high enough probability that D' is willing to bid R_C rather than $\underline{r}$ but not so high that D prefers bidding R_C to $\underline{r}$. To define this range, let q_b be the smallest probability of quitting for which D' is willing to bid R_C. Then q_b is the smallest q satisfying $(1-\underline{r})[(1-p)v'_D - k_D] - \underline{r}[k_D + n_D] \leq q[-k_D - \underline{r}n_D] + (1-q)[(1-R_C)[(1-p)v'_D - k_D] - R_C[k_D + n_D]$.

Corollary 1A shows that a PBE satisfying D1 exists for all $q_C(R_C) \in [q_b, 1-p]$. Define the PBE Δ_q to be the same as Δ_0 except that $q_C(R_C) = q$ for any $q \in [q_b, 1-p)$. Then

COROLLARY 1A: Δ_q *for* $q \in [q_b, 1-p)$ *are PBEs satisfying D1.*[1]

Proof: By construction, D' at least weakly prefers bidding R_C and then standing firm to bidding $\underline{r}$ for $q \in [q_b, 1-p)$. Verifying that Δ_q is a PBE is straightforward. Repeating the argument in Proposition 1A also shows that D1 eliminates D' for all $r > R_C$. D1 thus requires $t(r) = 0$ at $r > R_C$ as is the case in Δ_q. ■

Although many PBEs satisfy D1, only Δ_0 satisfies an additional limit-point criterion.

[1] Other PBEs exist as well, e.g., at $q_C(R_C) = q_b$ where D' is indifferent between bidding R_C and $\underline{r}$.

The main idea underlying this criterion is that if the set of bids was discrete, it would be very unlikely that the defender could bid exactly R_C. The discrete-bid criterion imposes this condition.

To define this criterion, consider a discrete-bid analogue of the brinkmanship game when $p > \widetilde{p}$. C must now select a bid from a finite set of offers $r_0, ..., r_n$ such that $\underline{r} = r_0 < \cdots < r_{m-1} < R_C < r_m < \cdots < r_n = \overline{r}$ with $r_j - r_{j-1} \leq \delta$ for all j and a $\delta > 0$. (The models of brinkmanship in Powell 1990 have this structure. Every step toward the brink raises the risk of disaster by a fixed amount δ.) The key features of this discrete set of offers is that the defender is no longer able to bid exactly R_C and there is a well defined next highest bid above R_C, namely, r_m. At r_m, C is no longer indifferent between quitting and standing firm if it is certain that it is facing D'. Rather, C strictly prefers to quit.

Call the discrete-bid brinkmanship game described above $\Gamma_B^\delta(p)$. Define the assessment Δ_δ to be: D plays r_m with probability μ_δ and $\underline{r}$ with probability $1 - \mu_\delta$ where $\mu_\delta \equiv \tau(pv_C + n_C)(r_m - R_C)/[(1-\tau)(1-\underline{r})v_C]$, $q_D(r_j) = 0$ for any $r_j \leq R_D$ and $q_D(r_j) = 1$ for $r_j > R_D$. D' plays according to $\rho' = r_m$, $q'_D(r_j) = 0$ for $r_j \leq R'_D$ and $q'_D(r_j) = 1$ for $r_j > R'_D$. C follows $q_C(r_j) = 0$ for $r \neq r_m$ and $q_C(r_m) = 1 - p$. C's beliefs are $t(\underline{r}) = 0$, $t(r) = \tau$ for $\underline{r} < r_j \leq r_{m-1}$, $t(r_m) = \tau/[\tau + (1-\tau)\mu_\delta]$, and $t(r_j) = 0$ for $r_j > r_m$.

The next lemma shows that all the PBEs of Γ_B^δ satisfying D1 are the same as Δ_δ except possibly at C's beliefs following an out of equilibrium offer less than R_C. These beliefs have no effect on subsequent play. C and both D and D' stand firm after this bid. Proposition 2A demonstrates that if we let the maximal distance between adjacent offers δ go to zero, then the limit of Δ_δ is identical to Δ_0 except possibly for C's beliefs at $t(r)$ for $r \in (\underline{r}, R_C)$. In this sense, D1 and the limit-criterion uniquely select Δ_0.

LEMMA 4A: *Assume $p > \widetilde{p}$ and let Δ be any PBE of Γ_B^δ satisfying D1. Then Δ is identical to Δ_δ except possibly for C's beliefs $t(r_j)$ for $\underline{r} < r_j \leq r_{m-1}$.*

Proof: Let Δ be a PBE satisfying D1. Repeating the argument in the proofs of Lemmas 1A and 2A shows that Δ can only put positive weight on $\underline{r}$ and on one $r_j \geq r_m$. If $r_j > r_m$, repeating the argument in the proof of Lemma 3A shows that C must believe that it is facing D' after the downward deviation to $r_{j-1} \geq r_m$. That is, $t(r_{j-1}) = 1$.

C's best response given this belief and $r_m > R_C$ is to quit, $q_C(r_{j-1}) = 1$. This, however, makes r_{j-1} a profitable deviation. Hence, putting positive weight on $r_j > r_m$ yields a contradiction. As a result, Δ can put positive probability on at most $\underline{r}$ and r_m.

In fact, Δ must put positive weight on r_m. Suppose not. Then r_m is an out-of-equilibrium bid, and $t(r_m) = 1$ by D1. To see that D1 eliminates D, note that D quits after r_m if C stands firm. D therefore weakly prefers to deviate to r_m from $\underline{r}$ if $q_C(r_m) \geq 1 - p$. Algebra shows that D' strictly prefers bidding r_m and standing firm when $q_C(r_m) \geq 1 - p$. D1 therefore eliminates D and leaves $t(r_m) = 1$.

A contradiction follows. If $t(r_m) = 1$, C's best reply is $q_C(r_m) = 1$. This, however, makes r_m a profitable deviation, and this contradiction ensures that Δ must put positive weight on r_m.

Δ must put positive probability on $\underline{r}$ as well. Arguing again by contradiction, suppose D and D' pool on r_m. Then $t(r_m) = \tau$. D's weak preference for r_m also implies $q_C(r_m) \geq 1 - p$. But as shown below, C strictly prefers to stand firm if D and D' pool on r_m and δ is sufficiently small. This yields the contradiction $q_C(r_m) = 0$.

To establish that C stands firm after r_m if D and D' pool on this bid and δ is sufficiently small, note that C stands firm if $(1-\tau)[v_C - \underline{r}n_C] + \tau[(1-r_m)pv_C - r_m n_C] > -\underline{r}n_C$. This is equivalent to $\tau < v_C/[v_C + (pv_C + n_C)(r_m - R_C(p))]$. As δ goes to zero, this constraint goes to $\tau < 1$ and is sure to hold.

That both $\underline{r}$ and r_m are played with positive probability implies that D' bids r_m and D mixes between $\underline{r}$ and r_m. Clearly the types cannot separate. If D plays $\underline{r}$ and D' plays r_m, then $q_C(r_m) = 1$ and D prefers to deviate. If D plays r_m and D' plays $\underline{r}$, then $q_C(r_m) = 0$ and D prefers to deviate to $\underline{r}$. Given that the types cannot separate and that D' strictly prefers r_m whenever D weakly prefers r_m, D' must play r_m, i.e., $\rho' = r_m$, and D must mix between $\underline{r}$ and r_m.

In order for D to mix, it must be indifferent. This implies that C must mix after r_m with $q_C(r_m) = 1 - p$. Because C is mixing, it must be indifferent between quitting and standing firm. Let μ_δ be the probability that D bids r_m. Then C's indifference gives $-k_C - \underline{r}n_C = (1-\tau\mu_\delta)/[(1-\tau)(1-\mu_\delta)+\tau][(1-\underline{r})v_C - k_C - \underline{r}n_C] + \tau/[(1-\tau)(1-\mu_\delta)+\tau][pv_C - k_C - r_m(pv_C + n_C)]$ where $(1-\tau\mu_\delta)/[(1-\tau)(1-\mu_\delta)+\tau]$ and $\tau/[(1-\tau)(1-\mu_\delta)+\tau]$ are the posteriors of facing D and D' given r_m. This yields $\mu_\delta = (pv_C + n_C)(r_m - R_C)/[(1-\tau)((1-\underline{r})v_C + (pv_C + n_C)(r_m - R_C))]$.

Finally consider C's out-of-equilibrium beliefs and actions $t(r_j)$ and $q_C(r_j)$ for $j > m$. Deviation r_j is strictly profitable for D if $q_C(r_j) > 1-p$. D' weakly prefers to deviate if $q_C(r_j) \geq \widetilde{q}'_C(r_j, r_m) > q_C(r_m) = 1-p$. Thus, D1 eliminates D' at r_j. This leaves $t(r_j) = 0$, $q_C(r_j) = 0$, and establishes the lemma. ■

It immediately follows that Δ_δ converges to Δ_0 except possibly for C's beliefs at $r \in (\underline{r}, R_C)$. Since $\delta \geq r_m - r_{m-1} > r_m - R_C$, we have $\mu_\delta \to 0$ and $t(r_m) \to 0$ as $\delta \to 0$. This leaves

Proposition 2A: *Assume $p > \widetilde{p}$. Then the assessment* $\lim_{\delta\to 0}\Delta_\delta$ *is identical to* Δ_0 *except possibly at the irrelevant beliefs $t(r)$ for $r \in (\underline{r}, R_C)$.*

Turning to a determination of C's choice of p at the outset of the game, Lemma 1A and Proposition 2A imply that the challenger's payoff to bring p to bear is:

$$U_C(p) = \begin{cases} pv_C - k_C - \underline{r}(p)n_C & \text{if} \quad 0 < p < \widetilde{p} \\ (1-\tau)[1-\underline{r}(p)]pv_C - k_C - \underline{r}(p)n_C & \text{if} \quad \widetilde{p} < p \leq \overline{p}. \end{cases}$$

As for the optimal p, define $U_C(0) = 0$ and $U_C(\widetilde{p}) = \lim_{p\uparrow\widetilde{p}} U_C(p) = [1-\underline{r}(\widetilde{p})]\widetilde{p}v_C - k_C - \underline{r}(\widetilde{p})n_C$. We justify this specification of $U_C(\widetilde{p})$ below. For now, observe that $U_C(p)$ defined in this way over $[0, \overline{p}]$ has a well defined global maximizer. This follows from the fact that U_C is weakly concave over $(0, \widetilde{p})$ and strictly concave over $(0, \overline{p}]$ with $U_C(0) = 0 > \lim_{p\to 0} U_C(p) = -k_C$ and $\lim_{p\uparrow\widetilde{p}} U_C(p) > \lim_{p\downarrow\widetilde{p}} U_C(p)$. Moreover, this maximizer is generically unique. That is, the set of feasible parameter values v_C and n_C for which there are multiple maximizers is a set of measure zero. Let p^{**} denote this maximizer.

To justify the specification of $U_C(\widetilde{p})$, we show below that the brinkmanship game following $\widetilde{p}$ has multiple equilibria satisfying D1, and C's equilibrium payoffs vary across these equilibria. However, these payoffs are above below by $U_C(\widetilde{p})$, and a unique equilibrium path yields $U_C(\widetilde{p})$. Hence which equilibrium is played after $\widetilde{p}$ has no effect on the optimal choice of p if $p^{**} \neq \widetilde{p}$. If $p^{**} = \widetilde{p}$, the states must play an equilibrium with the unique path giving C a payoff of $U_C(\widetilde{p})$. Otherwise C's payoff to p would discontinuously jump down at $\widetilde{p}$ and C would not have a best reply to the defender's strategy.

To see that $\Delta(\widetilde{p})$ has multiple equilibria satisfying D1, let $\Delta_{\widetilde{p}}$ be the assessment in which both D and D' bid $\underline{r}$, and all three states subsequently stand firm after any $r < R_C$. C quits after $r = R_C$ with any $q_C(R_C) \leq q_b$. C's beliefs at $r \in (\underline{r}, R_C)$ can be anything, and $t(R_C) = 1$. These equilibria have the same equilibrium path and give a payoff of $U_C(\widetilde{p})$ to C. By contrast, Δ_0 is also an equilibrium satisfying D1 and yields a payoff of $\lim_{p \downarrow \widetilde{p}} U_C(p)$ to C.

To establish that C's equilibrium payoffs are bounded above by $U_C(\widetilde{p})$, let z be the equilibrium probability that D or D' bids $\underline{r}$. Then C's equilibrium payoff is $z[\widetilde{p}v_C - k_C - \underline{r}(\widetilde{p})(\widetilde{p}v_C + n_C)] - (1-z)[k_C + \underline{r}(\widetilde{p})n_C]$. This payoff is increasing in z and equal to $U_C(\widetilde{p})$ at $z = 1$. Finally, it is easy to see that the equilibrium path of $\Delta_{\widetilde{p}}$ is the unique path giving C the payoff $U_C(\widetilde{p})$. Since $z = 1$, D and D' must bid $\underline{r}$ which corresponds to the path in $\Delta_{\widetilde{p}}$.

In determining the comparative statics, assume an interior solution, $U'_C(p^{**}) = 0$. Using $U''_C < 0$ gives $\text{sgn}\{\partial p^{**}/\partial n_C\} = \text{sgn}\{\partial^2 U_C(p^{**})/\partial n_C \partial p\}$. Differentiation gives $\partial U_C/\partial p = -\underline{r}' n_C + (1-\tau)[(1-\underline{r})v_C - \underline{r}' p v_C]$. Trivially, $\partial^2 U_C/\partial n_C \partial p = -\underline{r}' < 0$ and $\partial p^{**}/\partial n_C < 0$. Further, $\partial^2 U_C/\partial v_C \partial p = (1-\tau)[1 - \underline{r} - \underline{r}'p]$. But $U'_C(p^{**}) = 0$ ensures $(1-\tau)[1-\underline{r}-\underline{r}'p] = \underline{r}' n_C/v_C > 0$. Thus $\partial p^{**}/\partial v_C > 0$. And, $\partial^2 U_C/\partial \tau \partial p = \underline{r}' p v_C - (1-\underline{r})v_C = -\underline{r}' n_C/(1-\tau) < 0$, so $\partial p^{**}/\partial \tau < 0$. As for $\partial R_C(p^{**})/\partial v_C$ and $\partial R_C(p^{**})/\partial n_C$, write $R_C(p^{**}) = 1 - (1 - \underline{r}(p^{**}))/[1 + p^{**} v_C/n_C]$. The results follow immediately.