Supporting Information for "Induced Aseismic Slip and the Onset of Seismicity in Displaced Faults"

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April 27, 2022

Introduction This Supporting Information (SI) consists of four sections: Section S1 describes how the nucleus of strain concept, as applied to a double-couple representation of a dislocation, can be used to derive the stress and displacement fields resulting from that dislocation. Section S2 treats some mathematical aspects of Cauchy-type singular integral equations (Cauchy equations for short) of the type as encountered in equation (24) in the paper. In particular it gives proofs for the inverse in equation (29) and for conditions (30) and (31), making use of earlier results from Bilby & Eshelby (1968) and Estrada & Kanwal (2000). Section S3 is concerned with the specific use of Cauchy equations in the paper. In particular, it treats the nature of the singularities and discontinuities in functions given by equations (5) to (10) in the paper, and the consequences for use of these functions with Cauchy equations. Section S4 provides an alternative route to solve Cauchy-type integral equations, using a "forward formulation" rather than the analytical "inverse formulation" by Muskhelishvili (1953) as used in the paper.

S1 - Double Couple Representation

Consider a dislocation represented in 2D as a concentrated unit double couple per unit length, $\{f\lambda, -f\lambda\} = \{1, -1\}$, acting at point $(0, \xi)$ in a right-hand $x - y$ coordinate system under plane-strain conditions; see Figure S1-1. The product $f\lambda$ of a force per unit length f and a displacement λ serves to derive Green's functions for the corresponding displacements and stresses through a limiting process as described below.

We start from another set of Green's functions $g_{ij}(x, y, \zeta, \xi)$ that give the displacement at point (x, y) in direction $i \in \{x, y\}$ resulting from a unit force per unit length acting at point

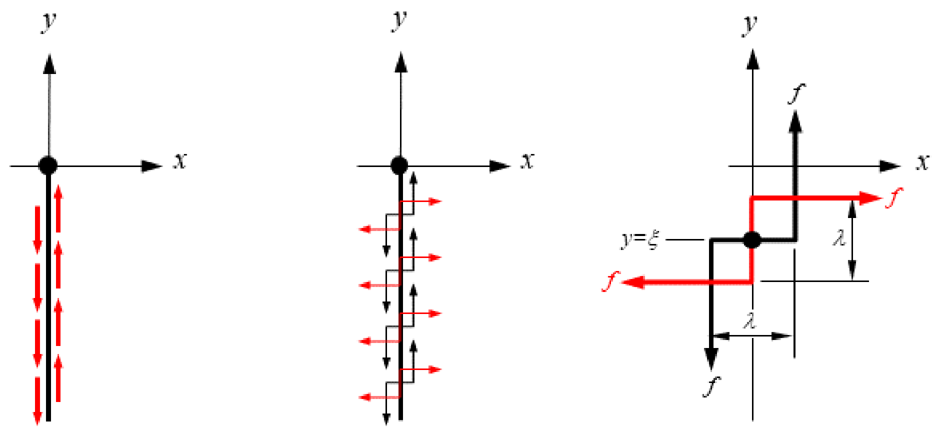

Figure S1-1: Left: 2D view of an edge dislocation at the origin with displacements along the half line $\{x = 0, y < 0\}$ indicated in red. Middle: representation as an infinite array of double couples each consisting of a positive (black) and a negative (red) couple. Right: Detail of two couples of magnitude $\pm f\lambda = \pm m$ with λ approaching zero while the product $f\lambda$ remains equal to m.

(ζ, ξ) in direction $j \in \{x, y\}$:

$$g_{xx}(x, y, \zeta, \xi) = \frac{1}{8\pi(1-\nu)G}\left[\frac{(x-\zeta)^2}{R^2} - (3-4\nu)\ln R\right], \tag{S1-1}$$

$$g_{yy}(x, y, \zeta, \xi) = \frac{1}{8\pi(1-\nu)G}\left[\frac{(y-\xi)^2}{R^2} - (3-4\nu)\ln R\right], \tag{S1-2}$$

$$g_{xy}(x, y, \zeta, \xi) = \frac{1}{8\pi(1-\nu)G}\frac{(x-\zeta)(y-\xi)}{R^2}, \tag{S1-3}$$

where

$$R = \sqrt{(x-\zeta)^2 + (y-\xi)^2}. \tag{S1-4}$$

Equations (S1-1) to (S1-3) represent the plane-strain version of the "Kelvin solution" which gives the displacements resulting from a point force in an infinite solid (**?**Love 1927, Barber 2010). Green's functions for the displacements resulting from a unit double couple per unit length positioned at the y axis (i.e. for $\zeta = 0$) can now be obtained as

$$\begin{aligned}
g_x(x,y,\xi) &= \lim_{\lambda\downarrow 0}\Big|_{f\lambda=1}\Big[g_{xx}(x,y,0,\xi+\frac{\lambda}{2}) - g_{xx}(x,y,0,\xi-\frac{\lambda}{2}) \\
&\qquad + g_{xy}(x,y,\frac{\lambda}{2},\xi) - g_{xy}(x,y,-\frac{\lambda}{2},\xi)\Big] \\
&= \frac{1}{2\pi(1-\nu)G}\left[\frac{(1-2\nu)(y-\xi)}{2R^2} + \frac{x^2(y-\xi)}{R^4}\right], \qquad \text{(S1-5)} \\
g_y(x,y,\xi) &= \lim_{\lambda\downarrow 0}\Big|_{f\lambda=1}\Big[g_{yx}(x,y,0,\xi+\frac{\lambda}{2}) - g_{yx}(x,y,0,\xi-\frac{\lambda}{2}) \\
&\qquad + g_{yy}(x,y,\frac{\lambda}{2},\xi) - g_{yy}(x,y,-\frac{\lambda}{2},\xi)\Big] \\
&= \frac{1}{2\pi(1-\nu)G}\left[\frac{(1-2\nu)x}{2R^2} + \frac{x(y-\xi)^2}{R^4}\right], \qquad \text{(S1-6)}
\end{aligned}$$

where we used Taylor expansions with respect to λ before taking the limits. The same expressions were derived by Steketee (1958), although with x and y swapped because he considered an array of dislocations along the x axis. Green's functions for the stresses can be obtained from equations (S1-5) and (S1-6) with the aid of kinematic equations and Hooke's law for plane strain as

$$g_{xx}(x,y,\xi) = -\frac{1}{\pi(1-\nu)}\frac{x(y-\xi)\left[3x^2-(y-\xi)^2\right]}{R^6}, \qquad \text{(S1-7)}$$

$$g_{yy}(x,y,\xi) = -\frac{1}{\pi(1-\nu)}\frac{x(y-\xi)\left[3(y-\xi)^2-x^2\right]}{R^6}, \qquad \text{(S1-8)}$$

$$g_{xy}(x,y,\xi) = \frac{1}{2\pi(1-\nu)}\frac{\left[x^4-6x^2(y-\xi)^2+(y-\xi)^4\right]}{R^6}. \qquad \text{(S1-9)}$$

Note that the subscripts in these Green's functions refer to the corresponding stresses $\sigma_{ij}, i \in \{x,y\}, j \in \{x,y\}$ and thus have a different meaning than in the Green's functions g_{ij} defined in equations (S1-1) to (S1-3).

Recall that in 2D the seismic moment per unit length is defined as $m = \delta G$. The stresses resulting from an array of double couples of magnitude $\{\delta G, -\delta G\}$ distributed over the half line $y \leqslant 0$ (see Figure S1-1) now follow as

$$\sigma_{ij}(x,y) = \delta G \int_{-\infty}^{0} g_{ij}(x,y,\xi)\, d\xi, \qquad \text{(S1-10)}$$

which leads to

$$\bar{\sigma}_{xx}(x,y) = \frac{\delta G}{2\pi(1-\nu)}\frac{x(y^2-x^2)}{R^4}, \qquad \text{(S1-11)}$$

$$\bar{\sigma}_{yy}(x,y) = \frac{-\delta G}{2\pi(1-\nu)}\frac{x(x^2+3y^2)}{R^4}, \qquad \text{(S1-12)}$$

$$\bar{\sigma}_{xy}(x,y) = \frac{\delta G}{2\pi(1-\nu)}\frac{y(y^2-x^2)}{R^4}, \qquad \text{(S1-13)}$$

with R now reduced to

$$R = \sqrt{x^2 + y^2}. \tag{S1-14}$$

In the same way, the vertical displacement can be obtained as

$$\bar{u}_y(x, y) = \delta G \int_{-\infty}^{0} g_y(x, y, \xi)\, d\xi, \tag{S1-15}$$

resulting in

$$\bar{u}_y(x, y) = \frac{\delta}{4\pi(1-\nu)} \left\{ 2(1-\nu) \left[\mathrm{sgn}(x)\frac{\pi}{2} - \arctan\left(\frac{y}{x}\right) \right] + \frac{xy}{x^2 + y^2} \right\}. \tag{S1-16}$$

Equations (S1-11) to (S1-13) are well known results from dislocation theory; see, e.g., Burgers (1939), Nabarro (1952), Barber (2010) or Cai & Nix (2016), where they are typically obtained with the aid of an Airy stress function for an array of edge (glide) dislocations along the x axis resulting in swapped x and y values compared to our results. It can be shown that the stress field defined by these equations is in internal equilibrium, i.e. the total stresses and moments on any closed curve around the dislocation (i.e. around the origin) vanish.

Expressions for the displacements $\bar{u}_y$ are usually also identical (apart from the swapping of x and y) to our equation (S1-16), although slight differences occur because the displacement field resulting from a stress function is only defined up to an arbitrary translation and rotation. Similar slightly different results occur in the literature for displacements $\bar{u}_x$ (not shown here) which, moreover, contain a logarithmic singularity at infinity.

Equations (S1-11) to (S1-13) and (S1-16) are valid for a vertical fault. For $x = 0$ they reduce to

$$\bar{u}_y(0, 0 < y) = 0, \tag{S1-17}$$

$$\bar{u}_y(0^{\mp}, y < 0) = \mp\frac{\delta}{2}, \tag{S1-18}$$

$$\bar{\sigma}_{xx}(0, y) = 0, \tag{S1-19}$$

$$\bar{\sigma}_{yy}(0, y) = 0, \tag{S1-20}$$

$$\bar{\sigma}_{xy}(0, y) = \frac{\delta G}{2\pi(1-\nu)y}, \tag{S1-21}$$

where $x = 0^-$ and $x = 0^+$ indicate locations just to the left and just to the right of the fault. For an inclined fault passing through the origin, and with an edge dislocation just there, the expressions for $\bar{\sigma}_\perp$ and $\bar{\sigma}_{||}$ now follow as in equations (21) and (22) in the paper.

S2 - Cauchy-Type Singular Integral Equations - Mathematical Aspects

S2.1 - Overview

In this Section we will derive an expression for the function $g(s')$, that has a non-zero value in-between and at endpoints a and b, at which two points it is bounded, has a zero value otherwise, and satisfies

$$\mathrm{PV}\int_a^b \frac{\hat{g}(t')dt'}{t'-s'} = \hat{f}(s'), \quad a < t' < b,\ -\infty < s' < \infty. \tag{S2-1}$$

(Note that in this Section, which treats generic aspects of Cauchy equations, we use a notation that differs from the one used in the paper. For the paper-specific aspects of Cauchy equations, as treated in Section S3, we will return to the paper-specific notation.) The solution to equation (S2-1) is given by

$$\hat{g}(s') = \frac{\sqrt{s'-a}\sqrt{b-s'}}{\pi^2}\ \mathrm{PV}\int_a^b \frac{f(t'),\ dt'}{\sqrt{t'-a}\sqrt{b-t'}(s'-t')}, \tag{S2-2}$$

and is bounded provided f satisfies a few requirements. In order to have a bounded solution at the endpoints we need

$$I_1 = \int_a^b \frac{f(t')\ dt'}{\sqrt{t'-a}\sqrt{b-t'}} = 0, \tag{S2-3}$$

and

$$I_2 = \int_a^b \frac{t' f(t')\ dt'}{\sqrt{t'-a}\sqrt{b-t'}} = 0. \tag{S2-4}$$

Furthermore we need to ensure convergence of the integral (S2-2) around the endpoints and in the interior. A *sufficient* condition is Lipschitz-continuity of $f(t')$ for all $a \leqslant t' \leqslant b$ in order to guarantee convergence around the endpoints. We note that slightly weaker conditions on $f(t')$ are also be sufficient; around the endpoints Hölder continuity with $\alpha > \frac{1}{2}$ is sufficient. In the interior we may even allow discontinuities, as long as the PV of the integral in equation (S2-2) can still be computed.

The solution (S2-2) and the conditions (S2-3)-(S2-4) can be found in the literature, see, e.g., Muskhelishvili (1953), Bilby & Eshelby (1968), Weertman (1996), Estrada & Kanwal (2000) and Segall (2010), but various elements, as relevant for our paper, are not readily available. Here we aim to present the material in a complete and easily accessible fashion.

S2.2 - A Few Useful Integrals

In this Subsection we list a few integrals, with clues for their derivation, for future reference:

$$\int_0^\infty \frac{x^{\alpha-1}}{x+1}\ dx = \frac{\pi}{\sin \pi\alpha},\ 0 < \alpha < 1; \tag{S2-5}$$

use contour integration; for full details see, e.g., *Playlist "Complex Analysis - Contour integrals"* (n.d.).

$$\mathrm{PV}\int_0^\infty \frac{x^{\alpha-1}}{1-x}\ dx = \frac{\pi\cos\pi\alpha}{\sin\pi\alpha},\ 0 < \alpha < 1; \tag{S2-6}$$

use contour integration and equation (S2-5); for full details see *Playlist "Complex Analysis - Contour integrals"* (n.d.).

$$\int_s^1 \frac{1}{\sqrt{u-s}\sqrt{u-t}}\ du = \ln\left(\frac{\sqrt{1-t}+\sqrt{1-s}}{\sqrt{1-t}-\sqrt{1-s}}\right),\ s > t, \tag{S2-7}$$

and (interchanging the roles of s and t)

$$\int_t^1 \frac{1}{\sqrt{u-s}\sqrt{u-t}}\,du = \ln\left(\frac{\sqrt{1-s}+\sqrt{1-t}}{\sqrt{1-s}-\sqrt{1-t}}\right),\ t > s; \tag{S2-8}$$

see (Estrada & Kanwal 2000).

$$0 < t < u: \int_0^u \left(\frac{u-s}{s}\right)^\alpha \frac{ds}{(u-s)(s-t)} = \pi\cot\alpha\pi\frac{(u-t)^{\alpha-1}}{t^\alpha}, \tag{S2-9}$$

which simplifies for $\alpha = \frac{1}{2}$ to

$$0 < t < u: \int_0^u \frac{ds}{\sqrt{s(u-s)}(s-t)} = 0, \tag{S2-10}$$

$$t > u: \int_0^u \left(\frac{u-s}{s}\right)^\alpha \frac{ds}{(u-s)(s-t)} = \frac{-\pi}{\sin\alpha\pi}\frac{(t-u)^{\alpha-1}}{t^\alpha}, \tag{S2-11}$$

which further simplifies for $\alpha = \frac{1}{2}$ to

$$t > u: \int_0^u \frac{ds}{\sqrt{s(u-s)}(s-t)} = \frac{-\pi}{\sqrt{t(t-u)}}; \tag{S2-12}$$

see (Estrada & Kanwal 2000).

$$\int_0^u \frac{1}{\sqrt{s(u-s)}}\,ds = \pi; \tag{S2-13}$$

complete the square and use a suitable substitution.

$$\int_0^u \frac{1}{\sqrt{s(u-s)}}\,ds = \pi; \tag{S2-14}$$

use a suitable substitution and equation (S2-13).

S2.3 - Abel's Equations

In this Subsection we will show that the solution $g(t)$ of the Abel's problem

$$f(s) = \int_s^b \frac{g(t)}{\sqrt{t-s}}\,dt \tag{S2-15}$$

is given by

$$g(t) = -\frac{1}{\pi}\frac{d}{dt}\int_t^b \frac{f(s)}{\sqrt{s-t}}\,ds. \tag{S2-16}$$

Multiply both sides of equation (S2-15) with $\frac{ds}{\sqrt{s-u}}$ and integrate from u to b with respect to$s$:

$$\int_u^b \frac{ds}{\sqrt{s-u}}\int_s^b \frac{g(t)}{\sqrt{t-s}}\,dt = \int_u^b \frac{ds}{\sqrt{s-u}}f(s). \tag{S2-17}$$

Interchange the order of integration on the left-hand side (LHS):

$$\text{LHS} = \int_{t=u}^{b} dt \int_{s=u}^{t} ds \frac{g(t)}{\sqrt{t-s}\sqrt{s-u}} = \int_{t=u}^{b} dt\; g(t) \int_{s=u}^{t} ds \frac{1}{\sqrt{t-s}\sqrt{s-u}}. \tag{S2-18}$$

We can now evaluate the integral with respect to s using two consecutive substitutions. First we set

$$y = \frac{s-u}{t-u} \Rightarrow 1-y = \frac{t-s}{t-u}, \tag{S2-19}$$

which means that we have

$$t-s = (t-u)(1-y),\; s-u = (t=u)y,\; dy = \frac{ds}{t-u}, \tag{S2-20}$$

such that all the factors $t-u$ cancel out. This then means that we have

$$\int_{s=u}^{t} ds \frac{1}{\sqrt{t-s}\sqrt{s-u}} = \int_0^1 \frac{dy}{\sqrt{1-y}\sqrt{y}} = \int_0^1 \frac{2d\tau}{\sqrt{1-\tau^2}} = \pi, \tag{S2-21}$$

where we used the substitution $\tau = \sqrt{y}$ in the last step.

Substitution of equation (S2-21) in equation (S2-18) yields

$$\text{LHS} = \pi \int_{t=u}^{b} dt\; g(t) = \int_u^b \frac{ds}{\sqrt{s-u}} f(s), \tag{S2-22}$$

and differentiation with respect to u yields

$$-\pi g(u) = \frac{d}{du} \int_u^b \frac{ds}{\sqrt{s-u}} f(s). \tag{S2-23}$$

Finally we rename $u = t$ and solve for $g(t)$ to find

$$g(t) = -\frac{1}{\pi}\frac{d}{dt} \int_t^b \frac{ds}{\sqrt{s-t}} f(s). \tag{S2-24}$$

S2.4 - Cauchy Equation

In this Subsection we will show that the solution $g(s)$ of the problem

$$f(s) = \text{PV} \int_0^1 \frac{g(t)\; dt}{t-s},\; 0 < s < 1, \tag{S2-25}$$

is given by

$$g(s) = \frac{c}{\pi\sqrt{s(1-s)}} + \frac{1}{\pi^2\sqrt{s(1-s)}}\; \text{PV} \int_0^1 \frac{\sqrt{t(1-t)} f(t) dt}{s-t}, \tag{S2-26}$$

where

$$c = \int_0^1 g(t)\; dt. \tag{S2-27}$$

Note that the roles of the variables s and t are interchanged.

Step 1: Convert to Abel's Problem

Multiply both sides by s and use

$$\frac{sg(t)}{t-s} = \frac{s-t+t}{t-s} g(t) = -g(t) + \frac{tg(t)}{t-s}, \tag{S2-28}$$

to obtain

$$\int_0^1 \frac{tg(t)\;dt}{t-s} = sf(s) + c, \;\; c = \int_0^1 g(t)\;dt. \tag{S2-29}$$

Multiply both sides by

$$\frac{ds}{\sqrt{s(u-s)}}, \tag{S2-30}$$

and integrate from 0 to u. The LHS simplifies as follows:

$$\int_0^u \frac{ds}{\sqrt{s(u-s)}} \int_0^1 \frac{tg(t)\;dt}{t-s} = \int_0^1 \;dt\;tg(t) \int_0^u \frac{ds}{(t-s)\sqrt{s(u-s)}}, \tag{S2-31}$$

where

$$\int_0^u \frac{ds}{(t-s)\sqrt{s(u-s)}} = 0, \tag{S2-32}$$

for $0 < t < u$ according to equation (S2-10), and

$$\int_0^u \frac{ds}{(t-s)\sqrt{s(u-s)}} = \frac{\pi}{\sqrt{t(t-u)}}, \tag{S2-33}$$

for $t > u$ according to equation (S2-12) (the sign difference is due to the fact that we have $t-s$ here instead of $s-t$). Using this we find

$$\int_0^1 \;dt\;tg(t) \int_0^u \frac{ds}{(t-s)\sqrt{s(u-s)}} = \int_u^1 \;dt\;tg(t)\frac{\pi}{\sqrt{t(t-u)}}. \tag{S2-34}$$

At the RHS we obtain

$$\int_0^u \frac{sf(s)\;ds}{\sqrt{s(u-s}} + \int_0^u \frac{c\;ds}{\sqrt{s(u-s)}} = \int_0^u \frac{\sqrt{s}f(s)\;ds}{\sqrt{u-s}} + \pi c, \tag{S2-35}$$

where we used the integral from equation (S2-13). Equating LHS (S2-34) and RHS (S2-35) we find

$$\int_u^1 \frac{\sqrt{t}g(t)}{\sqrt{t-u}}\;dt = c + \frac{1}{\pi}\int_0^u \frac{\sqrt{s}f(s)\;ds}{\sqrt{u-s}}, \tag{S2-36}$$

which is Abel's equation.

Step 2: Use the Solution of Abel's Equation

Use the solution (S2-16) of Abels's equation (with $b = 1$)

$$\int_s^1 \frac{g(t)}{\sqrt{t-s}}\,dt = \hat{f}(s) \Rightarrow g(t) = \frac{-1}{\pi}\frac{d}{dt}\int_t^1 \frac{\hat{f}(s)}{\sqrt{s-t}}\,ds, \tag{S2-37}$$

applied to our problem (S2-36),

$$\int_s^1 \frac{\sqrt{t}g(t)}{\sqrt{t-s}}\,dt = \hat{f}(s) = c + \frac{1}{\pi}\int_0^s \frac{\sqrt{s'}f(s')\,ds'}{\sqrt{s-s'}}, \tag{S2-38}$$

to find the following expression for $\sqrt{t}g(t)$:

$$\sqrt{t}g(t) = -\frac{1}{\pi}\frac{d}{dt}\int_t^1 \left(\frac{c + \frac{1}{\pi}\int_0^s \frac{\sqrt{s'}f(s')\,ds'}{\sqrt{s-s'}}}{\sqrt{s-t}}\right). \tag{S2-39}$$

Rename according to

$$t \to s,\ s \to u,\ s' \to t, \tag{S2-40}$$

$$\sqrt{s}g(s) = -\frac{1}{\pi}\frac{d}{ds}\int_s^1 \frac{c\,du}{\sqrt{u-s}} - \frac{1}{\pi^2}\frac{d}{ds}\int_s^1 \left(\frac{du}{\sqrt{u-s}}\int_0^u \frac{\sqrt{t}f(t)}{\sqrt{u-t}}\right). \tag{S2-41}$$

Compute the first integral directly and invert the order of integration in the double integral:

$$\sqrt{s}g(s) = \frac{c}{\pi\sqrt{1-s}} - \frac{1}{\pi^2}\frac{d}{ds}\left(\int_0^s dt\int_s^1 du\frac{\sqrt{t}f(t)}{\sqrt{u-s}\sqrt{u-t}} + \int_s^1 dt\int_t^1 du\frac{\sqrt{t}f(t)}{\sqrt{u-s}\sqrt{u-t}}\right). \tag{S2-42}$$

For the first integral, we have $s > t$ which means that we have

$$\int_0^s dt\int_s^1 du\frac{\sqrt{t}f(t)}{\sqrt{u-s}\sqrt{u-t}} = \int_0^s dt\,\sqrt{t}f(t)\ln\left(\frac{\sqrt{1-t}+\sqrt{1-s}}{\sqrt{1-t}-\sqrt{1-s}}\right), \tag{S2-43}$$

using the integral from equation (S2-7). For the second integral, we have $t > s$ which means that we can write

$$\int_s^1 dt\int_t^1 du\frac{\sqrt{t}f(t)}{\sqrt{u-s}\sqrt{u-t}} = \int_0^s dt\,\sqrt{t}f(t)\ln\left(\frac{\sqrt{1-t}+\sqrt{1-s}}{\sqrt{1-s}-\sqrt{1-t}}\right), \tag{S2-44}$$

using the second integral from equation (S2-8). Combining both terms yields

$$\sqrt{s}g(s) = \frac{c}{\pi\sqrt{1-s}} - \frac{1}{\pi^2}\frac{d}{ds}\left(\int_0^1 dt\sqrt{t}f(t)\ln\left|\frac{\sqrt{1-t}+\sqrt{1-s}}{\sqrt{1-t}-\sqrt{1-s}}\right|\right). \tag{S2-45}$$

We use

$$\frac{d}{ds}\ln\left|\frac{\sqrt{1-t}+\sqrt{1-s}}{\sqrt{1-t}-\sqrt{1-s}}\right| = -\frac{\sqrt{1-t}}{(s-t)\sqrt{1-s}}, \tag{S2-46}$$

which can be verified by direct differentiation; (checking $s > t$ and $t > s$ separately) and differentiate under the integral sign to find

$$\sqrt{s}g(s) = \frac{c}{\pi\sqrt{1-s}} + \frac{1}{\pi^2}\int_0^1 \frac{\sqrt{t(1-t)}f(t)}{(s-t)\sqrt{1-s}}\,dt. \tag{S2-47}$$

We finally obtain

$$g(s) = \frac{c}{\pi\sqrt{s(1-s)}} + \frac{1}{\pi^2\sqrt{s(1-s)}} \int_0^1 \frac{\sqrt{t(1-t)}f(t)}{s-t}\, dt. \tag{S2-48}$$

Generalization is straightforward; setting

$$s = \frac{s'-a}{b-a},\ t = \frac{t'-a}{b-a},\ \hat{f}(s') = f(s),\ \hat{g}(s') = g(s), \tag{S2-49}$$

we observe that the solution of

$$\mathrm{PV}\int_a^b \frac{\hat{g}(t')dt'}{t'-s'} = \hat{f}(s') \quad a < t' < b, \tag{S2-50}$$

is given by

$$\hat{g}(s') = \frac{c}{\pi\sqrt{(s'-a)(b-s')}} + \frac{1}{\pi^2\sqrt{(s'-a)(b-s')}} \int_a^b \frac{\sqrt{(t'-a)(b-t')}\hat{f}(t')}{s'-t'}\, dt'. \tag{S2-51}$$

S2.5 - Conditions for a Bounded Solution at Both Endpoints

The solution (S2-51) is, in general, not bounded at $s' = a$ or $s' = b$. We will need to impose a number of constraints on f in order to have a solution that is bounded at both endpoints. In this section we will derive these requirements; the derivation is similar to the derivation presented in Bilby & Eshelby (1968) for a slightly different case.

We use the following identity:

$$\frac{\sqrt{t'-a}\sqrt{b-t'}}{\sqrt{s'-a}\sqrt{b-s'}} - \frac{\sqrt{s'-a}\sqrt{b-s'}}{\sqrt{t'-a}\sqrt{b-t'}} = \frac{(t'-a)(b-t')-(s'-a)(b-s')}{\sqrt{s'-a}\sqrt{b-s'}\sqrt{t'-a}\sqrt{b-t'}}, \tag{S2-52}$$

which can be rewritten as

$$\frac{\sqrt{t'-a}\sqrt{b-t'}}{\sqrt{s'-a}\sqrt{b-s'}} - \frac{\sqrt{s'-a}\sqrt{b-s'}}{\sqrt{t'-a}\sqrt{b-t'}} = \frac{(s'-t')(s'+t'-a-b)}{\sqrt{s'-a}\sqrt{b-s'}\sqrt{t'-a}\sqrt{b-t'}}. \tag{S2-53}$$

This means that we can rewrite the integral in equation (S2-51) as

$$\begin{aligned}\int_a^b \frac{\sqrt{t'-a}\sqrt{b-t'}}{\sqrt{s'-a}\sqrt{b-s'}} \frac{f(t')\, dt'}{s'-t'} &= \int_a^b \frac{\sqrt{s'-a}\sqrt{b-s'}}{\sqrt{t'-a}\sqrt{b-t'}} \frac{f(t')\, dt'}{s'-t'} \\ &+ \int_a^b \frac{(s'+t'-a-b)f(t')dt'}{\sqrt{s'-a}\sqrt{b-s'}\sqrt{t'-a}\sqrt{b-t'}},\end{aligned} \tag{S2-54}$$

which, in turn, means that $\hat{g}(s')$ can be rewritten as

$$\hat{g}(s') = \frac{1}{\sqrt{s'-a}\sqrt{b-s'}}\left(\frac{c}{\pi} + \frac{(s'-a-b)I_1}{\pi^2} + \frac{I_2}{\pi^2}\right) + \frac{\sqrt{s'-a}\sqrt{b-s'}}{\pi^2} I_3, \tag{S2-55}$$

where

$$I_1 =< f, p_0 >= \int_a^b \frac{f(t')\, dt'}{\sqrt{t'-a}\sqrt{b-t'}}, \tag{S2-56}$$

$$I_2 =< f, p_1 >= \int_a^b \frac{t' f(t')\, dt'}{\sqrt{t'-a}\sqrt{b-t'}}, \tag{S2-57}$$

and

$$I_3 = \int_a^b \frac{f(t')\, dt'}{\sqrt{t'-a}\sqrt{b-t'}(s'-t')}. \tag{S2-58}$$

Next, we introduce the inner product

$$< f, g >= \int_a^b \frac{f(t')g(t')\, dt'}{\sqrt{t'-a}\sqrt{b-t'}} \tag{S2-59}$$

and the first two Chebyshev polynomials of the first kind (Mason & Handscomb 2003)

$$T_0 = 1,\ T_1 = t. \tag{S2-60}$$

In order to get rid of the singularities at the boundaries, we set

$$c = 0,\ < f, T_0 >= 0,\ < f, T_1 >= 0, \tag{S2-61}$$

which means that we are left with

$$\hat{g}(s') = \frac{\sqrt{s'-a}\sqrt{b-s'}}{\pi^2} \int_a^b \frac{f(t')\, dt'}{\sqrt{t'-a}\sqrt{b-t'}(s'-t')}. \tag{S2-62}$$

Imposing (S2-61) is necessary to obtain a bounded solution; in the following Subsection we will show that the integral in equation (S2-62) converges, provided $f(t')$ is Lipschitz continuous. This means that we have a bounded solution at both endpoints if both conditions are met.

Convergence I_3

Rewrite I_3 as a sum of two integrals:

$$I_3 = \int_a^b \frac{f(t')\, dt'}{\sqrt{t'-a}\sqrt{b-t'}(s'-t')} = \int_a^b \frac{f(t') - f(s') + f(s')\, dt'}{\sqrt{t'-a}\sqrt{b-t'}(s'-t')} = -I_4 + I_5,$$

where

$$I_4 = \int_a^b \frac{f(t') - f(s')}{t'-s'} \frac{1}{\sqrt{t'-a}\sqrt{b-t'}} dt', \tag{S2-63}$$

and

$$I_5 = f(s') \int_a^b \frac{dt'}{\sqrt{t'-a}\sqrt{b-t'}(s'-t')}. \tag{S2-64}$$

Use equation (S2-10) with $u = 1$ and $0 < t < 1$,

$$\int_0^1 \frac{ds}{\sqrt{s}\sqrt{1-s}(s-t)} = 0, \tag{S2-65}$$

to show that $I_5 = 0$. Substitute

$$s = \frac{t'-a}{b-a},\ t = \frac{s'-a}{b-a}, \tag{S2-66}$$

in equation (S2-65) and rewrite

$$0 = \int_0^1 \frac{ds}{\sqrt{s}\sqrt{1-s}(s-t)} = (b-a)\int_a^b \frac{dt'}{\sqrt{t'-a}\sqrt{b-t'}(t'-s')} = -\frac{b-a}{f(s')} I_5, \tag{S2-67}$$

(because $a < s' < b$).

The integral I_4 is more delicate though; if $f(t')$ is Lipschitz continuous then the integral converges, because

$$|f(t') - f(s')| \leqslant |t' - s'| \Rightarrow \left| \frac{f(t') - f(s')}{t' - s'} \right| \leqslant M, \tag{S2-68}$$

and we find

$$|I_4| \leqslant \int_a^b \left| \frac{f(t') - f(s')}{t' - s'} \right| \frac{1}{\sqrt{t'-a}\sqrt{b-t'}} dt' \leqslant M \int_a^b \frac{1}{\sqrt{t'-a}\sqrt{b-t'}} dt' = M\pi. \tag{S2-69}$$

S3 - Application to Describe Slip in a Displaced Fault

We set

$$t' = \xi,\ s' = y,\ \hat{g}(t') = A\delta'(\xi),\ f(s') = \sigma_C(y),\ a = y_-,\ b = y_+ \tag{S3-1}$$

in equation (S2-50), leading to

$$\text{PV} \int_{y_-}^{y_+} \frac{A\delta'(\xi) d\xi}{\xi - y} = \sigma_C(y), \quad y_- < \xi < y_+, \tag{S3-2}$$

which has the solution (S2-62)

$$\delta'(y) = \frac{\sqrt{(y-y_-)(y_+-y)}}{A\pi^2} \text{PV} \int_{y_-}^{y_+} \frac{\sigma_C(\xi)}{(\sqrt{(\xi-y_-)(y_+-\xi)})(y-\xi)} d\xi, \tag{S3-3}$$

provided that the following conditions hold:

$$\int_{y_-}^{y_+} \frac{\sigma_C(\xi)}{\sqrt{(\xi-y_-)(y_+-\xi)}} d\xi = 0, \tag{S3-4}$$

and

$$\int_{y_-}^{y_+} \frac{\xi\ \sigma_C(\xi)}{\sqrt{(\xi-y_-)(y_+-\xi)}} d\xi = 0. \tag{S3-5}$$

Equations (S3-3) to (S3-5) are identical to equations (C1) to (C3) in the paper.

Note that we *always* need to be careful with the numerical integration of equation (S3-3) around $\xi = y$, because the integral only converges in the PV sense. Furthermore two types of singularities may be present in $\sigma_C(\xi)$; see also Section 2.5 in the paper. First of all we may encounter logarithmic terms, i.e.,

$$\sigma_C \sim \ln((\xi - a)^2 + \epsilon^2), \tag{S3-6}$$

(regularized, ϵ small) or

$$\sigma_C \sim \ln(\xi - a)^2, \tag{S3-7}$$

(without regularization, $\epsilon = 0$). In the regularized case, $\sigma_C(\xi)$ is continuously differentiable, which means that we can use equations (S2-1)-(S2-4). The unregularized case is analyzed in Section where we will show that we can use (S2-1)-(S2-4) also in the unregularized case.

Secondly we may encounter jump discontinuities of $\sigma_C(\xi)$, i.e.,

$$\sigma_C \sim \arctan \frac{\xi - a}{\epsilon}, \tag{S3-8}$$

(regularized, ϵ small) and

$$\sigma_C(\xi) \sim H(\xi - a) = \begin{cases} 0 & \text{if} \quad \xi < a \\ 1 & \text{if} \quad \xi > a \end{cases} \tag{S3-9}$$

without regularization. In the regularized case, $\sigma_C(\xi)$ is continuously differentiable, which means that we can use equations (S2-1)-(S2-4). The unregularized case is analyzed in Section where we will show that we can *not* use (S2-1)-(S2-4) in the unregularized case.

S3.1 - Logarithmic Singularities

In the unregularized case we have

$$\sigma_C \sim \ln(\xi - a)^2. \tag{S3-10}$$

We need to distinguish two situations: $y = a$ and $y \neq a$. In the first situation we need to show that the integral

$$\text{PV} \int_{y_-}^{y_+} \frac{\ln(\xi - a)^2}{(\sqrt{(\xi - y_-)(y_+ - \xi)})(a - \xi)} \, d\xi \tag{S3-11}$$

converges. We split the integral in three parts

$$\text{PV} \int_{y_-}^{y_+} \dots d\xi = \text{PV} \int_{y_-}^{a_-} \dots d\xi + \text{PV} \int_{a_-}^{a_+} \dots d\xi + \text{PV} \int_{a_+}^{y_+} \dots d\xi, \tag{S3-12}$$

where $a_- = a - \eta$ and $a_+ = a + \eta$; η does not need to be small, just small enough so that the endpoints $y_\pm$ stay far enough out of $[a_-, a_+]$.

The first and third integral converge because σ_C is continuously differentiable; for the second integral we have

$$\frac{\ln(\xi - a)^2}{(\sqrt{(\xi - y_-)(y_+ - \xi)})(a - \xi)} \sim \frac{\ln(\xi - a)^2}{(a - \xi)}, \tag{S3-13}$$

which converges in the PV sense, because

$$\begin{aligned} \text{PV} \int_{a_-}^{a_+} \frac{\ln(\xi - a)^2}{(a - \xi)} d\xi &= 2 \lim_{\epsilon \to 0} \left(\int_{a-\eta}^{a-\epsilon} \frac{\ln(a - \xi)}{a - \xi} d\xi + \int_{a+\epsilon}^{a+\eta} \frac{\ln(\xi - a)}{a - \xi} d\xi \right) \\ &= 2 \lim_{\epsilon \to 0} \left(- \int_{\eta}^{\epsilon} \frac{\ln u}{u} du - \int_{\epsilon}^{\eta} \frac{\ln u}{u} \right) \\ &= -2 \lim_{\epsilon \to 0} \left(\frac{1}{2} (\ln \epsilon)^2 - \frac{1}{2} (\ln \eta)^2 + \frac{1}{2} (\ln \eta)^2 - \frac{1}{2} (\ln \epsilon)^2 \right) = 0. \end{aligned} \tag{S3-14}$$

Note that the integrals here converge due to the fact that we take the PV. We do have that the second integral converges too, which means that our integral (S3-12) converges.

We will consider the second case next. In this case we have some $\eta > 0$ such that $y \notin (a_-, a_+)$, where $a_- = a - \eta$ and $a_+ = a + \eta$. We can split the integral (S3-3) in three parts:

$$\int_{y_-}^{y_+} \ldots d\xi = \int_{y_-}^{a_-} \ldots d\xi + \int_{a_-}^{a_+} \ldots d\xi \int_{a_+}^{y_+} \ldots d\xi. \tag{S3-15}$$

The first and third integral converge because σ_C is continuously differentiable over their integration intervals; the second integral converges as well, because we have

$$\frac{\sigma_C(\xi)}{\sqrt{(\xi - y_-)(y_+ - \xi)}(y - \xi)} \sim \ln(\xi - a)^2 \tag{S3-16}$$

which means that we are away from the zeros of the denominator (the prefactor remains bounded in $[a_-, a_+]$). We can integrate (even without PV)

$$\int_{a_-}^{a_+} \ln(\xi - a)^2 \, d\xi = \lim_{\epsilon \to 0} \left(2 \int_{a_-}^{a-\epsilon} \ln(a - \xi) \, d\xi \right) + \lim_{\epsilon \to 0} \left(2 \int_{a+\epsilon}^{a_+} \ln(\xi - a) \, d\xi \right), \tag{S3-17}$$

because

$$\int_{a_-}^{a-\epsilon} \ln(a - \xi) \, d\xi \underset{u = a - \xi}{=} - \int_{\eta}^{\epsilon} \ln u \, du = -u \ln u + u|_{\eta}^{\epsilon} \underset{\epsilon \to 0}{=} \delta \ln \eta - \eta \tag{S3-18}$$

Similarly we find

$$\int_{a+\epsilon}^{a_+} \ln(\xi - a) \, d\xi \underset{u = \xi - a}{=} \int_{\epsilon}^{\eta} \ln u \, du = u \ln u - u|_{\epsilon}^{\eta} \underset{\epsilon \to 0}{=} \eta \ln \eta - \eta. \tag{S3-19}$$

Note that it is not the PV that ensures convergence; convergence is due to the fact that both integrals converge themselves, even though the integrands both tend to infinity.

Now we do have that the second integral converges too, which means that our integral (S3-15) converges.

S3.2 - Jump Discontinuity

In the unregularized case we have behaviour like

$$\sigma_C(\xi) \sim H(\xi - a) = \begin{cases} 0 & \xi < a \\ 1 & \xi > a \end{cases} \tag{S3-20}$$

We only have a divergence if we take $y = a$ in this case.

For $y \neq a$ the integral

$$\text{PV} \int_{y_-}^{y_+} \frac{\sigma(\xi)}{(\sqrt{(\xi - y_-)(y_+ - \xi)})(y - \xi)} \, d\xi \tag{S3-21}$$

converges, because we have (similar to the previous section, by choosing a_- and a_+ close enough to a such that either $y \in [y_-, a_-)$ or $y \in (a_+, y_+]$)

$$\int_{y_-}^{y_+} \ldots d\xi = \int_{y_-}^{a_-} \ldots d\xi + \int_{a_-}^{a_+} \ldots d\xi + \int_{a_+}^{y_+} \ldots d\xi. \tag{S3-22}$$

This yields

$$\mathrm{PV}\int_{y_-}^{y_+}\frac{H(\xi-a)}{(\sqrt{(\xi-y_-)(y_+-\xi)})(y-\xi)}d\xi=\int_{y_-}^{a_+}\frac{1}{(\sqrt{(\xi-y_-)(y_+-\xi)})(y-\xi)}d\xi$$
$$+\ \mathrm{PV}\int_{a_+}^{y_+}\frac{1}{(\sqrt{(\xi-y_-)(y_+-\xi)})(y-\xi)}d\xi. \tag{S3-23}$$

The first integral converges because the denominator is nonzero in the integration interval, which means that we have a continuous function on a finite interval. For the second integral we have to be a bit more careful. If $y \in [y_-, a_-)$, then the denominator is only zero at the boundary point and the integral converges. If $y \in (a_+, y_+]$ then we take the PV as follows

$$\mathrm{PV}\int_{a_+}^{y_+}\frac{1}{(\sqrt{(\xi-y_-)(y_+-\xi)})(y-\xi)}d\xi$$
$$=\lim_{\epsilon\to 0}\left[\int_{y-\eta}^{y-\epsilon}\frac{1}{(\sqrt{(\xi-y_-)(y_+-\xi)})(y-\xi)}d\xi+\int_{y+\epsilon}^{y+\eta}\frac{1}{(\sqrt{(\xi-y_-)(y_+-\xi)})(y-\xi)}d\xi\right]$$
$$\int_{a_+}^{y-\eta}\frac{1}{(\sqrt{(\xi-y_-)(y_+-\xi)})(y-\xi)}d\xi+\int_{y+\eta}^{y_+}\frac{1}{(\sqrt{(\xi-y_-)(y_+-\xi)})(y-\xi)}d\xi. \tag{S3-24}$$

The third and fourth integral clearly converge because they only have singularities at the boundary. The singular contributions of the first and second integral cancel out, because we have

$$\frac{1}{(\sqrt{(\xi-y_-)(y_+-\xi)})(y-\xi)}\sim\frac{1}{y-\xi}, \tag{S3-25}$$

which means that we find

$$\int_{y-\eta}^{y-\epsilon}\frac{1}{(\sqrt{(\xi-y_-)(y_+-\xi)})(y-\xi)}d\xi+\int_{y+\epsilon}^{y+\eta}\frac{1}{(\sqrt{(\xi-y_-)(y_+-\xi)})(y-\xi)}d\xi$$
$$\sim \ln|y-\xi|\big|_{y-\eta}^{y-\epsilon}+\ln|y-\xi|\big|_{y+\epsilon}^{y+\eta}=0, \tag{S3-26}$$

such that that the first and second integral converge as well.

For $y = a$ we have a problem though, because the integral

$$I=\mathrm{PV}\int_{y_-}^{y_+}\frac{H(\xi-a)}{(\sqrt{(\xi-y_-)(y_+-\xi)})(a-\xi)}\ d\xi \tag{S3-27}$$

does *not* converge, because we have

$$I=\lim_{\epsilon\to 0}\int_{a+\epsilon}^{y_+}\frac{1}{(\sqrt{(\xi-y_-)(y_+-\xi)})(a-\xi)}\ d\xi\sim-\lim_{\epsilon\to 0}\ln\epsilon\to\infty. \tag{S3-28}$$

The problem arises because the divergence in the integral is not canceled anymore. This means that $\delta'(y)$ blows up logarithmically; after integration however this blow-up disappears and $\delta(y)$ is continuous - also in the case of an unregularized jump discontinuity - as we will show in Section . Replacing the Heaviside function by a smooth (arbitrarily) steep function resolves the blow up in δ', but is not necessary to obtain a continuous δ.

S3.3 - Computing $\delta(y)$ Close to the Jump Discontinuity

If we have $y \gtrapprox a$ in problem (S3-20)-(S3-21) we can split the integral (S3-21) in two parts

$$\int_{y_-}^{y_+} \frac{\sigma(\xi)}{(\sqrt{(\xi-y_-)(y_+-\xi)})(y-\xi)}\,d\xi = \int_{y_-}^{a+\eta} \frac{1}{(\sqrt{(\xi-y_-)(y_+-\xi)})(y-\xi)}\,d\xi + \int_{a+\eta}^{y_+} \frac{1}{(\sqrt{(\xi-y_-)(y_+-\xi)})(y-\xi)}\,d\xi, \quad \text{(S3-29)}$$

where η is chosen small enough, e.g., $\eta = 2(y-a)$. The second integral on the RHS of equation (S3-29) is now convergent. For the first integral on the RHS we can use a Taylor-approximation of a part of the integrand $h(\xi)$ around y as follows

$$h(\xi) = \frac{1}{\sqrt{(\xi-y_-)(y_+-\xi)}} = h(y) + h'(y)(\xi-y) + \text{ h.o.t.} = \frac{1}{\sqrt{(y-y_-)(y_+-y)}} + h'(y)(\xi-y) + \text{ h.o.t.}, \quad \text{(S3-30)}$$

which means that we find for the first integral (S3-29)

$$\int_{y_-}^{a+\eta} \frac{1}{(\sqrt{(\xi-y_-)(y_+-\xi)})(y-\xi)}\,d\xi = \int_{y_-}^{a+\eta} \frac{1}{\sqrt{(y-y_-)(y_+-y)}}\frac{1}{y-\xi} - h'(y) + \text{ h.o.t. } d\xi = \frac{1}{\sqrt{(y-y_-)(y_+-y)}}\left(\ln(y-a) - \ln(a+\eta-y)\right) - h'(y)\eta \ + \text{h.o.t.}, \quad \text{(S3-31)}$$

which means that we find (keeping only terms to leading order in η)

$$\int_{y_-}^{a+\eta} \frac{1}{(\sqrt{(\xi-y_-)(y_+-\xi)})(y-\xi)}\,d\xi \approx \frac{\ln(y-a) - \ln(a+\eta-y)}{\sqrt{(y-y_-)(y_+-y)}}, \quad \text{(S3-32)}$$

which means that we can approximate the full integral (S3-29)

$$\int_{y_-}^{y_+} \frac{\sigma(\xi)}{(\sqrt{(\xi-y_-)(y_+-\xi)})(y-\xi)}\,d\xi \approx \frac{\ln(y-a) - \ln(a+\eta-y)}{\sqrt{(y-y_-)(y_+-y)}} + I_2(y), \quad \text{(S3-33)}$$

where $I_2(y)$ contains the contributions of the second integral, which is some continuous function of y. Using this we can now approximate $\delta'(y)$ from equation (S3-3) as

$$\delta'(y) = \frac{\ln(y-a) - \ln(a+\eta-y)}{A\pi^2} + \frac{\sqrt{(y-y_-)(y_+-y)}}{A\pi^2} I_2(y) \text{ if } y > a, \quad \text{(S3-34)}$$

which is singular as $y \to a$. The function $\delta(y)$, however, is continuous as $y \to a$. This is because the second term and the second part of the first term are continuous, while for the first part of the first term we find the contribution

$$\delta_1(y) = \frac{1}{A\pi^2}\int^y \ln(y'-a)\,dy' = \frac{1}{A\pi^2}\left((y-a)\ln(y-a) - (y-a)\right), \quad \text{(S3-35)}$$

which is continuous as $y \to a$, because

$$\lim_{y\to a} \lambda_1(y) = 0. \quad \text{(S3-36)}$$

S4 - Alternative Formulation Using a "Forward equation"

As an alternative to the analytical "inverse formulation" by Muskhelishvili (1953), as used in the paper, this Section provides an alternative route to solve Cauchy-type integral equations, using a "forward formulation".

S4.1 - Definitions

Coulomb stress can be written as

$$\Sigma_C(y,p,\delta(y,p)) = R(y,p) + W(y,p)\delta(y,p), \tag{S4-1}$$

where R and W are given as

$$R(y,p) = \Sigma_{\|}(y,p) - \mu_{st}\Sigma'_{\perp}(y,p) \tag{S4-2}$$

$$W(y,p) = \Sigma'_{\perp}(y,p)\left(\mu_{st} - \mu_{dyn}\right)/\delta_c. \tag{S4-3}$$

Explicit expressions for $\Sigma_{\|}(y,p)$ and $\Sigma'_{\perp}(y,p)$ are given in equations (1)-(12) of our paper; values of the parameters are given in Table 1 of our paper.

Using dislocation theory we can couple the Coulomb stress and the slip gradient as follows

$$-\Sigma_C(y,p) = A\int_{y_-(p)}^{y_+(p)} \frac{\nabla\delta(\xi,p)}{\xi - y}\, d\xi. \tag{S4-4}$$

Combining equations (S4-1)-(S4-4) we obtain

$$R(y,p) + W(y,p)\delta(y,p) = -A\int_{y_-(p)}^{y_+(p)} \frac{\nabla\delta(\xi,p)}{\xi - y}\, d\xi, \tag{S4-5}$$

which is equation (C10) of our paper.

S4.2 - Scaling

We define the size of the slip patch Δy and its average position $\bar{\bar{y}}$ as follows

$$\Delta y = y_+ - y_-,\ \bar{\bar{y}} = \frac{y_+ + y_-}{2} \tag{S4-6}$$

and we introduce the scaled variables z and ζ

$$z(y;\Delta y,\bar{\bar{y}}) = \frac{2}{\Delta y}(y - \bar{\bar{y}}) \Rightarrow y = \frac{\Delta y}{2}z + \bar{\bar{y}} \tag{S4-7}$$

$$\zeta(\xi;\Delta y,\bar{\bar{y}}) = \frac{2}{\Delta y}(\xi - \bar{\bar{y}}) \Rightarrow \xi = \frac{\Delta y}{2}\zeta + \bar{\bar{y}} \tag{S4-8}$$

Note that with this scalings we map the interval $[y_-, y_+]$ to the interval $[-1, 1]$.

Furthermore we rescale δ and W

$$w(z;p,\Delta y,\bar{\bar{y}}) = W(y = y(z;\Delta y,\bar{\bar{y}}),p)\cdot\Delta y/2, \tag{S4-9}$$

$$d(z; p, \Delta y, \bar{\bar{y}}) = \frac{\delta(y = y(z; \Delta y, \bar{\bar{y}}), p)}{\Delta y/2} \tag{S4-10}$$

and we introduce

$$r(z; p, \Delta y, \bar{\bar{y}}) = R(y = y(z; \Delta y, \bar{\bar{y}}), p). \tag{S4-11}$$

This means that the LHS of equation (S4-5) is rewritten as

$$LHS = r(z; p, \Delta y, \bar{\bar{y}}) + w(z; p, \Delta y, \bar{\bar{y}}) d(z; p, \Delta y, \bar{\bar{y}}). \tag{S4-12}$$

Similarly we obtain for the RHS

$$RHS = -A \int_{-1}^{1} \frac{\frac{dd}{dz}(\zeta; p, \Delta y, \bar{\bar{y}})}{\zeta - z}\, d\zeta. \tag{S4-13}$$

We view $p, \Delta y$ and $\bar{\bar{y}}$ as parameters; we will solve the problem for a (number of) given value(s) of p; the parameters Δy and $\bar{\bar{y}}$ will be determined as part of the solution. We will suppress the dependence of the dependent variables on the parameters in the notation in order to avoid clutter and improve readability and we write the rescaled equation (S4-5) as

$$r(z) + w(z) d(z) = -A \int_{-1}^{1} \frac{\frac{dd}{dz}(z)}{\zeta - z}\, d\zeta. \tag{S4-14}$$

S4.3 - Functional Form of the Slip Gradient

From the inversion formula, we know that the slip gradient has the following form

$$\frac{dd}{dz} = \frac{A + Bz}{\sqrt{1 - z^2}} + f(z), \tag{S4-15}$$

where $f(z)$ is a continuous function, which means that the function

$$g(z) = \sqrt{1 - z^2} \frac{dd}{dz} \tag{S4-16}$$

is a continuous function, i.e.,

$$g(z) = a_0 + a_1 z + \ldots \tag{S4-17}$$

We can factorize $g(z)$ as follows

$$g(z) = r_0 + r_1 z + (1 - z^2)(b_0 + b_1 z + \ldots), \tag{S4-18}$$

where both the remainder $r_0 + r_1 z$ and the coefficients b_i can (in principle) be found by long division of $g(z)$ by $1 - z^2$. This means that we obtain the following functional form for the slip gradient

$$\frac{dd}{dz} = \frac{r_0 + r_1 z + (1 - z^2)((b_0 + b_1 z + \ldots)}{\sqrt{1 - z^2}}. \tag{S4-19}$$

In order to have a finite derivative at the endpoints, we have two conditions (similar to the conditions in the inverse problem):

$$r_0 = 0 \text{ and } r_1 = 0. \tag{S4-20}$$

We note that the numerator in equation (S4-19) is a power series in z, which we approximate using a (finite) sum of Chebyshev-polynomials

$$\frac{dd}{dz} = \frac{\sum_{n=0}^{N} c_n T_n(z)}{\sqrt{1-z^2}}. \tag{S4-21}$$

Note 1: I do not halve the first term here; we have to treat the first term with great care already anyway.
Note 2: Taking $N \to \infty$ allows us to approximate the numerator with arbitrary precision. In practice we just take N large enough.

S4.4 - Computation of the Slip

Integration of equation (S4-21) yields

$$d(z) - d(1) = \int_1^z \frac{dd}{d\xi}\, d\xi \Rightarrow d(z) = d(1) + \sum_{n=0}^{N} c_n \int_1^z \frac{T_n(\xi)}{\sqrt{1-\xi^2}}\, d\xi, \tag{S4-22}$$

which yields (after substitution of $\xi = \cos\chi$)

$$d(z) = d(1) + \sum_{n=0}^{N} c_n \int_0^{\arccos z} -\cos(n\chi) d\chi = \sum_{n=0}^{N} c_n \int_0^{\arccos z} -\cos(n\chi) d\chi, \tag{S4-23}$$

because $\delta(1) = 0$.

Note that we need to be careful here with the $n = 0$ term.

$$d(z) = -c_0 \arccos(z) - \sum_{n=1}^{N} c_n \frac{\sin n \arccos z}{n}. \tag{S4-24}$$

Note furthermore that we can simplify/rewrite ($z = \cos\chi$)

$$\sin n \arccos z = \sin\chi \frac{\sin n\chi}{\sin\chi} = \sqrt{1-z^2}\, U_{n-1}(z), \tag{S4-25}$$

which means that we obtain

$$d(z) = -c_0 \arccos(z) - \sum_{n=1}^{N} c_n \sqrt{1-z^2}\, U_{n-1}(z). \tag{S4-26}$$

S4.5 - Finite Slip Gradient at the Boundaries - Conditions

In order to have a finite slip gradient at the boundaries we need to satisfy condition (S4-20); this condition can be translated into a condition on the coefficients c_n as follows. Multiplication of equation (S4-19) by $\sqrt{1-z^2}$ yields

$$\sqrt{1-z^2}\frac{dd}{dz} = r_0 + r_1 z + (1-z^2)((b_0 + b_1 z + \ldots). \tag{S4-27}$$

Taking $z \to \pm 1$ yields

$$\lim_{z\to\pm 1}\left(\sqrt{1-z^2}\frac{dd}{dz}\right) = r_0 \pm r_1, \tag{S4-28}$$

which means that we find

$$r_0 = \frac{1}{2}\left(\lim_{z\to 1}\left(\sqrt{1-z^2}\frac{dd}{dz}\right) + \lim_{z\to -1}\left(\sqrt{1-z^2}\frac{dd}{dz}\right)\right) \tag{S4-29}$$

and

$$r_1 = \frac{1}{2}\left(\lim_{z\to 1}\left(\sqrt{1-z^2}\frac{dd}{dz}\right) - \lim_{z\to -1}\left(\sqrt{1-z^2}\frac{dd}{dz}\right)\right) \tag{S4-30}$$

Using equation (S4-21) and noting that

$$T_n(1) = \cos 0 = 1,\ T_n(-1) = \cos n\pi = (-1)^n, \tag{S4-31}$$

we have

$$\lim_{z\to 1}\left(\sqrt{1-z^2}\frac{dd}{dz}\right) = \sum_{n=0}^{N} c_n \text{ and } \lim_{z\to -1}\left(\sqrt{1-z^2}\frac{dd}{dz}\right) = \sum_{n=0}^{N}(-1)^n c_n \tag{S4-32}$$

This means that our conditions for a finite slip gradient read

$$r_0 = \frac{1}{2}\sum_{n=0}^{N}(1+(-1)^n)c_n = 0, \text{ and } r_1 = \frac{1}{2}\sum_{n=0}^{N}(1-(-1)^n)c_n = 0. \tag{S4-33}$$

Or, equivalently, the both the sum of the even coefficients and the sum of the odd coefficients need to be zero.

S4.6 - Computation of the Integral (S4-13)

Combining equations (S4-13) and (S4-21) we have the following integral

$$I = -A\int_{-1}^{1}\frac{\sum_{n=0}^{N} c_n T_n(\zeta)}{\sqrt{1-\zeta^2}(\zeta - z)}\ d\zeta = -A\sum_{n=0}^{N} c_n \int_{-1}^{1}\frac{T_n(\zeta)}{\sqrt{1-\zeta^2}(\zeta-z)}\ d\zeta. \tag{S4-34}$$

We use equation (9.22a) from Mason and Handscomb to evaluate the integral:

$$\int_{-1}^{1}\frac{T_n(\zeta)}{\sqrt{1-\zeta^2}(\zeta-z)}\ dy = \pi U_{n-1}(z). \tag{S4-35}$$

Note that this equations also holds for $n = 0$ if we define $U_{-1}(z) = 0$, because

$$\int_{-1}^{1}\frac{1}{\sqrt{1-\zeta^2}(\zeta-z)}\ dy = 0, \tag{S4-36}$$

see notes on Lemma 8.3 from Mason and Handscomb.

Combining equations (S4-34) and (S4-35) we obtain

$$I = -A\pi\sum_{n=0}^{N} c_n U_{n-1}(z). \tag{S4-37}$$

S4.7 - Summary of the Problem and Numerical Workflow

Step 1: choose a value of p.
Step 2: choose values of Δy and $\bar{\bar{y}}$ and compute $y = \frac{\Delta y}{2} z + \bar{\bar{y}}$.
Step 3: use expressions for stresses from the paper and the definitions of R and W

$$R(y,p) = \Sigma_{\|}(y,p) - \mu_{st}\Sigma'_{\perp}(y,p) \tag{S4-38}$$

$$W(y,p) = \Sigma'_{\perp}(y,p)\left(\mu_{st} - \mu_{dyn}\right)/\delta_c. \tag{S4-39}$$

to compute

$$r(z;p,\Delta y,\bar{\bar{y}}) = R(y = y(z;\Delta y,\bar{\bar{y}}),p). \tag{S4-40}$$

and

$$w(z;p,\Delta y,\bar{\bar{y}}) = W(y = y(z;\Delta y,\bar{\bar{y}}),p)\cdot \Delta y/2, \tag{S4-41}$$

note the additional factor of $\Delta y/2$.
Step 4: solve the problem

$$r(z) + w(z)\left(-c_0 \arccos(z) - \sum_{n=1}^{N} c_n\sqrt{1-z^2}\,U_{n-1}(z)\right) = -A\pi\sum_{n=1}^{N} c_n U_{n-1}(z). \tag{S4-42}$$

This can be done in two ways. The first approach, an "elegant" approach, could be to take the inner product with U_m, $m = 0..N$ on both sides; the RHS is simplified greatly, however the LHS is still a mess due to the presence of the functions $w(z)$ and the $\sqrt{1-z^2}$.
The second approach is to evaluate equation (S4-42) at $N+1$ points z_i, $i = 0,\ldots,N$. This yields $N+1$ coupled linear equations for the coefficients c_n, $n = 0,\ldots,N$, i.e., we obtain a linear system of the form

$$A\mathbf{x} = \mathbf{b}, \tag{S4-43}$$

where the vector $\mathbf{x}$ contains the unknown coefficients c_n, $n = 0,\ldots,N$

$$\mathbf{x} = \begin{bmatrix} c_0 \\ \vdots \\ c_N \end{bmatrix} \tag{S4-44}$$

and where A is an $N+1$ by $N+1$ matrix and $\mathbf{b} \in \mathbb{R}^{N+1}$;

$$A = [a_{ij}],\ \mathbf{b} = [b_i]. \tag{S4-45}$$

The coefficients a_{ij} are given by

$$a_{i0} = -w(z_i)\arccos z_i,\ 0 \leqslant i \leqslant N \tag{S4-46}$$

and

$$a_{ij} = -\sqrt{1-z_i^2}\,w(z_i)U_{j-1}(z_i) + A\pi U_{j-1}(z_i),\ 0 \leqslant i \leqslant N,\ 1 \leqslant j \leqslant N \tag{S4-47}$$

and the coefficients b_i are given by

$$b_i = -r(z_i),\ 0 \leqslant i \leqslant N \tag{S4-48}$$

Step 5: compute the errors (take N even)

$$r_0 = \sum_{n=0}^{N/2} c_{2n},\ \text{and}\ r_1 = \sum_{n=1}^{N/2} c_{2n-1}. \tag{S4-49}$$

Step 6: repeat Steps 2 -5 and minimize r_0 and r_1 until they are below a given threshold.

S4.8 - Remarks

- The scaling introduced in Section 4.2 is a partial version compared to the scaling used in the paper where all variables were made dimensionless. This is not of relevance to the results.
- We implemented the "forward simulation" approach as described in this Section in the same Matlab code that was used for simulation using the "analytical inverse" approach as described in the paper. The two approaches resulted mostly in identical values for the slip patch boundaries but, at least in our implementation, the "analytical inverse" approach appeared to be more robust than the "forward simulation" approach in that the latter sometimes produced spurious results during the iterative procedure to find the patch boundaries.
- In this section we expanded the scaled slip gradient ∇d in terms of first-kind Chebyshev polynomials T_n, such that the scaled Coulomb stresses are expressed in terms of second kind Chebyshev polynomials U_{n-1} (shifted over one position). Instead, in the paper we expanded the scaled Coulomb stresses in terms of T_n such that the scaled slip gradient was expressed in terms of U_{n-1}. Yet another choice could have been to expand the scaled slip d in terms of T_n. We did not analyze the consequences of these choices.

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