FROM ARROW'S THEOREM TO "DARK MATTER:" SUPPLEMENTARY NOTES

DONALD G. SAARI

These notes clarify and expand upon comments made in the paper (denoted by AT-DM).

1. Sen's theorem

Sen's Theorem[1] is an interesting illustration of Arrow's result. As Arrow permits selecting any decision rule for each pair, Sen's choice is motivated by individual rights; it is where each of at least two decisive agents determines the societal outcome for at least one assigned pair. For other pairs, even an unanimous vote could be required.[2] Sen shows, by use of examples, that cycles can occur. In other words, his result shows that "individual rights," which can arise in society, can define a particular reductionist approach; thus his theorem further illustrates the reductionist methodology's failure.

The source of Sen's cycles is clear: Thm. 1 in AT-DM proves that Sen's cycles can be caused only by ranking wheel configurations embedded within the list of voter preferences. But because Sen models actual practices, knowing what causes problems is not sufficient. The challenge is to resolve Sen's reductionist difficulties in a way that respects individual rights while explaining why these personal prerogatives generate unexpected consequences.

Suppose a Sen society has two agents and four alternatives, where the first agent determines $\{A, C\}$ and $\{B, D\}$ societal outcomes; the second determines $\{A, B\}$ and $\{C, D\}$

[1] Sen 1970

[2] Sen has a slightly relaxed "no-cycle" societal condition. I know of no Sen-example with indifference in the outcomes, so Sen's "no-cycle" becomes the same as Arrow's requirement of transitive outcomes. Indeed, the only general proof of Sen's result (Li and Saari 2008) closely mimics a proof of Arrow's Theorem.

outcomes. Their rankings are, respectively,

$$C \succ A \succ D \succ B \text{ and } B \succ A \succ D \succ C. \tag{1}$$

As both agents prefer $A \succ D$, that is the societal outcome. Consistent with their personal rights and preferences, the first and second agents designate, respectively, $D \succ B$ and $B \succ A$ as societal rankings to define a $A \succ D, D \succ B, B \succ A$ Sen-cycle.

Because ranking wheel links cause this cycle (Thm. 1, AT-DM), it is reasonable to search this example for unexpected features. What emerges is the subtle consequence that, rather than having a minimal impact on others, *each agent's personal decision imposes a strong negative externality on the other agent!* The first agent's $D \succ B$ societal selection strongly conflicts with the second agent's $B \succ D$ preference, where "strong" means that the ranking of these alternatives is separated by at least another choice (here, A). Similarly, the second agent's choice of $B \succ A$ imposes a strong negative externality on the first agent's strong $A \succ B$ ranking (with D being the separating alternative).

Instead of harmless personal decisions, each agent inflicts a strong negative impact on the other agent! This is not an isolated peculiarity; the ranking wheel structure ensures that dissatisfaction accompanies *all possible Sen cycles.*[3] These guaranteed strong negative externalities are counterparts of the landslide victories in the AT-DM Eq. 2 cycle.

Theorem 1. [4] *With any number of agents and with any Sen cycle, each agent suffers a strong negative externality.*

This reinterpretation of Sen's theorem makes his assertion understandable; rather than involving innocuous individual decisions, his cycles more accurately reflect a dysfunctional society where personal decisions cause *everyone*—even non-decisive agents—to suffer strong

[3]Saari 2008

[4]Li and Saari 2008; Saari 2008, Chap. 2

discontent. Examples include uncomfortable transitions between societal norms,[5] such as where one group insists on the personal right to smoke in a restaurant while another group strenuously objects. While this unexpected dysfunctionality introduces novel twists for Sen's conditions (which have not been adequately explored), a first challenge is to find ways to attenuate the negative consequences.

Problems are caused by ignored pairwise links, so answers must identify how interactions among agents can restore these links. To motivate an approach, the Eq. 1 rankings come from the game

$$(2)\qquad \begin{array}{c|cc|cc|} & \multicolumn{2}{c}{L} & \multicolumn{2}{c}{R} \\ \cline{2-5} T & 5 & 5 & -1 & 7 \\ \cline{2-5} B & 7 & -1 & 1 & 1 \\ \cline{2-5} \end{array} \quad \text{with the} \quad \begin{array}{|c|c|} \hline A & B \\ \hline C & D \\ \hline \end{array} \quad \text{labeling for the cells.}$$

Agent one is the row player (deciding the $\{A, C\}$ and $\{B, D\}$ outcomes); agent two is the column player (determining $\{A, B\}$ and $\{C, D\}$ outcomes). Both prefer 7 to 5 to 1 to -1, which defines the Eq. 1 rankings.

The dysfunctional societal nature (Thm. 1) of this Sen-cycle (Eq. 1) is manifested by being identified with the Prisoner's Dilemma (Eq. 2), where players could cooperate to achieve the common payoff of 5, but temptations of a personally preferred payoff of 7 threaten cooperative efforts—and induce the Eq. 1 Sen-cycle. Permitting punishment of those who deviate from the common payoff, as prescribed by tit-for-tat strategies, can force cooperation. The Eqs. 1-2 equivalence means that this same strategy (i.e., "as long as you create a strong negative externality for me, I will do the same for you") can generate cooperation to rectify the Eq. 1 Sen cycle.

Prisoner's Dilemma games always have a Sen-cycle representation. For such Sen-settings, tit-for-tat interactions (which affect ranking wheel terms) allow Sen's decisive agents to

[5]Saari 2008, Chap 2.

retain individual rights, but the pressure coming from others—the tit-for-tat punishment—can persuade them to adopt a societal cooperative choice that would drop the negative externalities that they impose on others.

Surprisingly, even for Sen-settings not associated with a Prisoner's Dilemma, the same tit-for-tat resolution[6] applies. This is because the dysfunctional nature of any Sen-society (Thm. 1) grants each decisive agent the power to inflict a strong negative externality punishment upon another decisive agent.

Rather than rare, such techniques are commonly adopted resolutions. For instance, a person over-fishing in a common pool of resources may be ostracized or have equipment damaged.[7] Similar resolutions arise in tragedy-of-the-commons settings.[8]

2. Extending results

As *all* paired comparison problems are caused by ranking wheel configurations, it is reasonable to expect that these configurations can be used to explain all of the results found in that large literature developed to prevent cycles. This includes the Black single peaked requirement,[9] , the so-called Nakamura number used with supermajority voting,[10] Greenberg's Theorem[11] ensuring the existence of a core in spatial voting, super-majority voting, as well as conditions yet to be discovered. A description of how this is done, along

[6] Saari 2008, Chap. 2

[7] Seihi and Somanathan 1996; Tavoni, Schlüter, Leven 2014

[8] Levin 2014.

[9] Black 1958

[10] Nakamura 1975, 1978

[11] Greenberg 1979

with several new conclusions, is in Saari.[12] Difficulties in non-parametric statistics can be similarly handled.[13]

If paired comparisons cause problems, then why not replace pairs with triplets, or quadruples, or ...? The same difficulties arise;[14] while ranking wheel configurations always cause problems, they are accompanied with new kinds of connecting strands of information that, again, can introduce difficulties when severed. With four alternatives and triplets, for instance, a new link (reflecting symmetries of a square) is

$$A \succ B \succ C \succ D,\ D \succ C \succ B \succ A,\ B \succ A \succ D \succ C,\ C \succ D \succ A \succ B. \tag{3}$$

Again, each alternative is in first, second, third, and fourth place once, so the natural outcome is a complete tie, which occurs with all positional outcomes (where each candidate is a assigned a specific weight depending on where she is positioned on a ballot). Each ranking is accompanied by its reversal, so all majority vote outcomes are ties.

It remains to determine the outcomes of triplets. But emphasizing triplets defines a reductionist program, so expect problems. Indeed, computing the outcomes for triplets severs the Eq. 3 information threads as reflected by the fact that, rather than ties, their plurality outcomes (where $X \sim Y$ is a tie) are

$$A \succ C \sim D,\ B \succ C \sim D,\ C \succ A \sim B,\ D \succ A \sim B$$

to create an unusual kind of cycle caused, again, by the reductionist method. Along with ranking wheel configurations, these Eq. 3 kinds of arrangements create all triplet complications. Again, the Borda Count handles these problems.

---

[12]Saari 2014

[13]Bargagliotti and Saari 2010

[14]Saari 2000

An evolving pessimistic message about the reductionist approach is that difficulties appear to arise almost everywhere. Indeed, all positional election paradoxical outcomes can be completely explained in terms of the reductionist method. An insidious feature is how it may not be recognized that the true nature of complications (such as with Arrow's Theorem), puzzles, and perhaps incorrect answers reflect this methodology.

3. General problems with reductionist methods

The reader probably has developed a sense about how to create any number of "Arrow-type" extensions that apply to, well, almost everything! The idea is simple: The reductionist approach divides whatever is being considered into component parts. But the whole rarely is a helter-skelter collection of whatever happens with components; in general, consistency conditions (such as transitive outcomes for voting settings, satisfying dictates of the church for the example of registering for a church, or consistency with Newton's laws for the dark matter discussion) dictate how conclusions must interact to provide an acceptable description of the whole. These consistency conditions are what determine connecting links among components. What permits an Arrow-type conclusion is that, by separately concentrating on each component, the reductionist division ignores crucial linking information and introduces violations of consistency.

Rather than stating formal "impossibility theorems" (which are easy to pose and prove),[15] combining this prescription with lessons learned from Arrow-type results suffices to identify what to explore. In doing so, it is not necessary to understand what constitutes the whole—an accepted consistency requirement suffices. Anticipate difficulties in settings with an emphasis on individual components.

[15] Saari 2010

Of particular importance is the new theme that, by identifying what are the connecting links, it can be possible to find resolutions that preserve the spirit of the original assumptions. This was demonstrated for Arrow's and Sen's problems, the controversy between Borda and Condorcet rankings, and even the mystery of dark matter.

References

[1] Bargagliotti, A., and D. G. Saari, 2010, Symmetry of Nonparametric Statistical Tests on Three Samples, *Journal of Mathematics and Statistics* 6 (4), 395-408.

[2] Black, D., 1958, *The Theory of Committees and Elections*, Cambridge University Press, London.

[3] Greenberg, J., 1979, Consistent majority rule over compact sets of alternatives, *Econometrica* 47, 627-636.

[4] Levin, S., 2014, Public goods in relation to competition, cooperation, and spite, *Proceed. Nat. Acad. Sciences*, **111**, 10838.

[5] Li, L., and D. G. Saari, 2008, Sen's Theorem: Geometric Proof and New Interpretations, *Social Choice & Welfare*, 31, 393-434.

[6] Nakamura, K., 1975, The core of a simple game with ordinal preferences, *International Journal of Game Theory* 4, 95-104.

[7] Nakamura, K., 1978, The voters in a simple game with ordinal preferences, *International Journal of Game Theory* 8, 55-61.

[8] Saari, D. G., 2000, Mathematical structure of voting paradoxes 2: positional voting. *Econ. Theory* 15, 55-101

[9] Saari, D. G., 2008, *Disposing Dictators, Demystifying Voting Paradoxes*, Cambridge University Press, New York.

[10] Saari, D. G., 2010, Aggregation and multilevel design for systems: Finding guidelines, *Journal of Mechanical Design*, **132**, 081006-1 to 081006-9.

[11] Saari, D. G., 2014, Unifying voting theory from Nakamura's to Greenberg's Theorems, *Mathematical Social Sciences* 69, 1-11.

[12] Seihi, R., and E. Somanathan, 1996,The evolution of social norms in common property resource use, *American Econ. Review* 86, 766-788.

[13] 1970, Sen, A., The impossibility of a Paretian liberal, *Jour. Political Economy,* 78, 152-157.

[14] Tavoni, A., M. Schlüter, S. Levin, 2014,The survival of the conformist: Social pressure and renewal resource management, *Jour. Theor. Biol.* doi:10.1016/j.jtbi.2011.07.003.