ONLINE APPENDIX FOR: Leadership with Trustworthy Associates

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Equilibrium beliefs. In our model a politicians' equilibrium updating is based on the standard Beta-binomial model. Suppose that the leader j holds $d+1$ bits of information, i.e. she holds the private signal s_j and d politicians truthfully reveal their signals to her. The probability that l out of such $d+1$ signals equal one, conditional on θ is

$$f\left(l|\theta, d+1\right) = \frac{(d+1)!}{l!\left(d+1-l\right)!}\theta^l\left(1-\theta\right)^{(d+1-l)}.$$

Hence, politician j's posterior on θ is

$$f\left(\theta|l, d+1\right) = \frac{(d+2)!}{l!\left(d+1-l\right)!}\theta^l\left(1-\theta\right)^{(d+1-l)},$$

the expected value of θ is

$$E\left(\theta|l, d+1\right) = \frac{l+1}{d+3},$$

and the variance is

$$V(\theta|l, d+1) = \frac{\left(l+1\right)\left(d+2-l\right)}{\left(d+3\right)^2\left(d+4\right)}.$$

■

Derivation of Expression (2). Fix a leader j, consider a communication strategy profile $\mathbf{m}_{-j}$ and suppose that it is an equilibrium together with the strategy y_j in expression (1). Let $C_j(\mathbf{m}_{-j})$ be the set of players truthfully communicating with the leader j in the equilibrium. The equilibrium information of j is thus $d_j(\mathbf{m}_{-j})+1 = |C_j(\mathbf{m}_{-j})|+1$, the cardinality of $C_j(\mathbf{m}_{-j})$ plus j's signal s_j. Consider any player $i \in C_j(\mathbf{m}_{-j})$. Let $\mathbf{s}_{-i}(\mathbf{m}_{-j})$ be any vector containing s_j and the (truthful) messages of all players in $C_j(\mathbf{m}_{-j})$ except i. Let also $y_j(s_i, \mathbf{s}_{-i}(\mathbf{m_{-j}}))$ be the action that j takes if she has information $\mathbf{s}_{-i}(\mathbf{m}_{-j})$ and believes in the signal s_i sent from player i, analogously, $y_j(1-s_i, \mathbf{s}_{-i}(\mathbf{m}_{-j}))$ is the action that j takes if she has information $\mathbf{s}_{-i}$ and believes in the signal $1-s_i$ sent from player i. Simplifying notation, player i does not deviate from reporting truthfully signal s_i to the leader j if and only if

$$-\int_0^1 \sum_{\mathbf{s}_{-i}\in\{0,1\}^{d_j}} \left[(y_j(s_i, \mathbf{s}_{-i}) - \theta - b_i)^2 - (y_j(1-s_i, \mathbf{s}_{-i}) - \theta - b_i)^2\right] f(\theta, \mathbf{s}_{-i}|s_i)d\theta \geq 0.$$

Simplifying, we obtain:

$$-\int_0^1 \sum_{\mathbf{s}_{-i}\in\{0,1\}^{d_j}} \left(y_j(s_i,\mathbf{s}_{-i}) - y_j(1-s_i,\mathbf{s}_{-i})\right)\left[\frac{y_j(s_i,\mathbf{s}_{-i}) + y_j(1-s_i,\mathbf{s}_{-i})}{2} - (\theta+b_i)\right] f(\theta,\mathbf{s}_{-i}|s_i)d\theta \geq 0.$$

Next, observing that

$$y_j(s_i,\mathbf{s}_{-i}) = b_j + E\left[\theta|s_i,\mathbf{s}_{-i}\right],$$

we obtain

$$-\int_0^1 \sum_{\mathbf{s}_{-i}\in\{0,1\}^{d_j}} \left(E\left[\theta|s_i,\mathbf{s}_{-i}\right] - E\left[\theta|1-s_i,\mathbf{s}_{-i}\right]\right)\cdot$$

$$\cdot \quad \left[b_j + \frac{E\left[\theta|s_i,\mathbf{s}_{-i}\right] + E\left[\theta|1-s_i,\mathbf{s}_{-i}\right]}{2} - \theta - b_i)\right] f(\theta,\mathbf{s}_{-i}|s_i)d\theta \geq 0.$$

Denoting

$$\Delta\left(s_i,\mathbf{s}_{-i}\right) = E\left[\theta|s_i,\mathbf{s}_{-i}\right] - E\left[\theta|1-s_i,\mathbf{s}_{-i}\right],$$

observing that:

$$f(\theta,\mathbf{s}_{-i}|s_i) = f(\theta|\mathbf{s}_{-i},s_i)\Pr(\mathbf{s}_{-i}|s_i),$$

and simplifying, we get:

$$-\sum_{\mathbf{s}_{-i}\in\{0,1\}^{d_j}} \int_0^1 \Delta\left(s_i,\mathbf{s}_{-i}\right)\left(\frac{E\left[\theta|s_i,\mathbf{s}_{-i}\right] + E\left[\theta|1-s_i,\mathbf{s}_{-i}\right]}{2} + b_j - b_i - \theta\right) f(\theta|\mathbf{s}_{-i},s_i)\Pr(\mathbf{s}_{-i}|s_i) \geq 0.$$

Furthermore, using

$$\int_0^1 \theta f(\theta|\mathbf{s}_{-i},s_i)d\theta = E\left[\theta|s_i,\mathbf{s}_{-i}\right],$$

we obtain:

$$-\sum_{\mathbf{s}_{-i}\in\{0,1\}^{d_j}} \int_0^1 \Delta\left(s_i,\mathbf{s}_{-i}\right)\left(\frac{E\left[\theta|s_i,\mathbf{s}_{-i}\right] + E\left[\theta|1-s_i,\mathbf{s}_{-i}\right]}{2} + b_j - b_i - E\left[\theta|s_i,\mathbf{s}_{-i}\right]\right) f(\theta|\mathbf{s}_{-i},s_i)\Pr(\mathbf{s}_{-i}|s_i)$$

$$= -\sum_{\mathbf{s}_{-i}\in\{0,1\}^{d_j}} \int_0^1 \Delta\left(s_i,\mathbf{s}_{-i}\right)\left(-\frac{\Delta\left(s_i,\mathbf{s}_{-i}\right)}{2} + b_j - b_i\right) f(\theta|\mathbf{s}_{-i},s_i)\Pr(\mathbf{s}_{-i}|s_i) \geq 0.$$

Now, note that, for any number $l = 0, ..., d_j$ of digits equal to one in $\mathbf{s}_{-i}(\mathbf{m}_{-j})$,

$$\begin{aligned}
\Delta\left(s_i, \mathbf{s}_{-i}\right) &= E\left[\theta | s_i, \mathbf{s}_{-i}(\mathbf{m}_{-j})\right] - E\left[\theta | 1 - s_i, \mathbf{s}_{-i}(\mathbf{m})\right] \\
&= E\left[\theta | l + s_i, d_j(\mathbf{m}_{-j}) + 1\right] - E\left[\theta | l + 1 - s_i, d_j(\mathbf{m}_{-j}) + 1\right] \\
&= \left(l + 1 + s_i\right) / \left(d_j(\mathbf{m}_{-j}) + 3\right) - \left(l + 2 - s_i\right) / \left(d_j(\mathbf{m}_{-j}) + 3\right) \\
&= \begin{cases} -1/\left(d_j(\mathbf{m}_{-j}) + 3\right) \text{ if } s_i = 0 \\ 1/\left(d_j(\mathbf{m}_{-j}) + 3\right) \text{ if } s_i = 1. \end{cases}
\end{aligned}$$

We obtain that player i communicates truthfully the signal $s_i = 0$ to player j if and only if:

$$-\left(\frac{-1}{d_j(\mathbf{m}_{-j}) + 3}\right)\left(-\frac{-1}{2(d_j(\mathbf{m}_{-j}) + 3)} + b_j - b_i\right) \geq 0,$$

or

$$b_i - b_j \leq \frac{1}{2\left(d_j(\mathbf{m}) + 3\right)},$$

and note that this condition is redundant if $b_i - b_j < 0$.

Likewise, i communicates truthfully the signal $s_i = 1$ to player j if and only if:

$$-\left(\frac{1}{d_j(\mathbf{m}_{-j}) + 3}\right)\left(-\frac{1}{2(d_j(\mathbf{m}_{-j}) + 3)} + b_j - b_i\right) \geq 0,$$

or

$$b_i - b_j \geq -\frac{1}{2\left(d_j(\mathbf{m}_{-j}) + 3\right)},$$

and note that this condition is redundant if $b_i - b_j > 0$.

Collecting the two conditions yields expression (2). ■

Derivation of equilibrium welfare, expression (4). We consider any equilibrium $(\mathbf{m}_{-j}, y_j)$. The ex-ante expected utility of each player i is:

$$\begin{aligned}
Eu_i(\mathbf{m}_{-j}, y_j) &= -E\left[\left(y_j(s_j, \hat{\mathbf{m}}_{-j}) - \theta - b_i\right)^2\right] \\
&= -E\left[\left(b_j + E\left[\theta | s_j, \hat{\mathbf{m}}_{-j}\right] - \theta - b_i\right)^2\right].
\end{aligned}$$

Hence

$$\begin{aligned} Eu_i(\mathbf{m}_{-j}, y_j) &= -E\left[(b_j - b_i)^2 + (E[\theta|s_j, \hat{\mathbf{m}}_{-j}] - \theta)^2 - 2(b_j - b_i)(E[\theta|s_j, \hat{\mathbf{m}}_{-j}] - \theta)\right] \\ &= -\left[(b_j - b_i)^2 + E\left[(E[\theta|s_j, \hat{\mathbf{m}}_{-j}] - \theta)^2\right] - 2(b_j - b_i)(E[E[\theta|s_j, \hat{\mathbf{m}}_{-j}]] - E[\theta])\right], \end{aligned}$$

by the law of iterated expectations, $E[E[\theta|s_j, \hat{\mathbf{m}}_{-j}]] = E[\theta]$, so we obtain

$$Eu_i(\mathbf{m}_{-j}, y_j) = -(b_j - b_i)^2 - E\left[(E[\theta|s_j, \hat{\mathbf{m}}_{-j}] - \theta)^2\right].$$

Letting l be the number of digits equal to one in the $(d_j(\mathbf{m_{-j}}) + 1)$-digit leader's information vector $(s_j, \hat{\mathbf{m}}_{-j})$,

$$\begin{aligned} E\left[(E[\theta|s_j, \hat{\mathbf{m}}_{-j}] - \theta)^2\right] &= \int_0^1 \sum_{l=0}^{d_j(\mathbf{m_{-j}})+1} (E[\theta|l, d_j(\mathbf{m}_{-j}) + 1] - \theta)^2 f(l|d_j(\mathbf{m}_{-j}) + 1, \theta) d\theta \\ &= \int_0^1 \sum_{l=0}^{d_j(\mathbf{m}_{-j})+1} (E[\theta|l, d_j(\mathbf{m}_{-j}) + 1] - \theta)^2 \frac{f(\theta|l, d_j(\mathbf{m}_{-j}) + 1)}{d_j(\mathbf{m}_{-j}) + 2} d\theta, \end{aligned}$$

where the second equality follows from $f(l|d_j(\mathbf{m}_{-j}) + 1, \theta) = f(\theta|l, d_j(\mathbf{m}_{-j}) + 1)/(d_j(\mathbf{m}_{-j}) + 2)$.

Because the variance of a beta distribution of parameters l and $d + 1$, is

$$V(\theta|l, d+1) = \frac{(l+1)(d+2-l)}{(d+3)^2 (d+4)},$$

we obtain:

$$\begin{aligned} E\left[(E[\theta|s_j, \hat{\mathbf{m}}_{-j}] - \theta)^2\right] &= \frac{1}{d_j(\mathbf{m}_{-j}) + 2}\left[\sum_{l=0}^{d_j(\mathbf{m}_{-j})+1} V(\theta|l, d_j(\mathbf{m_{-j}}) + 1)\right] \\ &= \sum_{l=0}^{d_j(\mathbf{m}_{-j})+1} \frac{(l+1)(d_j(\mathbf{m}_{-j}) + 2 - l)}{(d_j(\mathbf{m}_{-j}) + 2)(d_j(\mathbf{m_{-j}}) + 3)^2 (d_j(\mathbf{m}_{-j}) + 4)} \\ &= \frac{1}{6(d_j(\mathbf{m}_{-j}) + 3)}. \end{aligned}$$

■

Proof of Lemma 1. We note that

$$\begin{aligned}U_i^*(j) &= -(b_i-b_j)^2-\left[6\left(d_j^*+3\right)\right]^{-1} = -(b_i-b_{i'}+b_{i'}-b_j)^2-\left[6\left(d_j^*+3\right)\right]^{-1}\\ &= -(b_i-b_{i'})^2-(b_{i'}-b_j)^2-2(b_i-b_{i'})(b_{i'}-b_j)-\left[6\left(d_j^*+3\right)\right]^{-1}\\ &= -(b_i-b_{i'})\left((b_i-b_{i'})+2(b_{i'}-b_j)\right)+U_{i'}^*(j)\\ &= -(b_i-b_{i'})(b_i+b_{i'}-2b_j)+U_{i'}^*(j)\end{aligned}$$

and

$$U_i^*(j') = -(b_i-b_{i'})\left(b_i+b_{i'}-2b_{j'}\right)+U_{i'}^*(j').$$

If $i<i'$, $j<j'$ and $U_{i'}^*(j)>U_{i'}^*(j)$, then $U_i^*(j)>U_i^*(j')$ is implied by

$$-(b_i-b_{i'})(b_i+b_{i'}-2b_j)\geq -(b_i-b_{i'})\left(b_i+b_{i'}-2b_{j'}\right)$$

or, because $i<i'$, by

$$b_i+b_{i'}-2b_j\geq b_i+b_{i'}-2b_{j'}$$

which is implied by $j<j'$. ■

Proof of Proposition 3. Suppose that there is a constant $\beta>0$ such that $b_{i+1}-b_i=\beta$ for all $i=1,...,n-1$. Then, for any real number $b>0$, the size of ideological neighborhood $N_j(b)$ is constant in j for all players j such that the number of politicians $i<j$ who belong to $N_j(b)$ is the same as the number of politicians $i>j$ who belong to $N_j(b)$. Formally, letting $\bar{\imath}_j(b)=\max\{i\in N:|b_i-b_j|\leq b\}$ and $\underline{i}_j(b)=\min\{i\in N:|b_i-b_j|\leq b\}$, we have that $N_j(b)=2\lfloor b/\beta\rfloor+1$, for any j such that $\bar{\imath}_j(b)-j=j-\underline{i}_j(b)$, where the notation $\lfloor b/\beta\rfloor$ denotes the largest integer smaller than b/β.

The remaining players j are constrained by the boundaries of the ideology spectrum b_1 and b_n in the size of their ideological neighborhood $N_j(b)$, so that it is either the case that $\bar{\imath}_j=n$, in which case $N_j(b)=\lfloor b/\beta\rfloor+1+\bar{\imath}_j(b)-j$, or that $\underline{i}_j=1$, in which case $N_j(b)=\lfloor b/\beta\rfloor+1+j-\underline{i}_j(b)$; and in both cases $N_j(b)<2\lfloor b/\beta\rfloor+1$.

Because $m=(n+1)/2$, by construction $N_m(b)=2\lfloor b/\beta\rfloor+1$ for all values of b, and hence $N_m(b)\geq N_j(b)$ for all other politician j and values of b. We note that $N_j(b)$ weakly increases in b and $\frac{1}{2(d+3)}$ decreases in d, and hence d_j^* is maximal for the index(es) j that maximize the function $N_j(\cdot)$. That is to say, when there is a constant $\beta > 0$ such that $b_{i+1} - b_i = \beta$ for all $i = 1, ..., n-1$, the median politician m weakly dominates all other politicians in terms of judgement, and should always be selected as group leader. ■

Analysis of the 5 Player Case in Section 6, Proof of Lemma 2 and of Proposition 4. We calculate all the parameter regions in which $d_2^* > d_3^*$. We first note that $d_3^* = 0$ if $\beta_2 > 1/8$ and $\beta_3 > 1/8$; so that $d_2^* \leq 1$ as 3 will never be truthful to 2, and $d_2^* = 1$ if $\beta_1 \leq 1/8$. We then see that $d_3^* = 1$ if $\beta_2 \leq 1/8$ and $\beta_3 > 1/10$; so that $d_2^* \leq 2$ as 4 will never be truthful to 2, and $d_2^* = 2$ if $\beta_1 \leq 1/10$ and $\beta_2 \leq 1/10$. Also, we see that $d_3^* = 1$ if $\beta_2 > 1/10$ and $\beta_3 \leq 1/8$; so that $d_2^* \leq 1$ as 3 will never be truthful to 2. Then, we note that $d_3^* = 2$ if $\beta_2 \leq 1/10$, $\beta_3 \leq 1/10$, $\beta_1 + \beta > 1/12$ and $\beta_3 + \beta_4 > 1/12$; so that $d_2^* \leq 3$ as 5 will never be truthful to 2, and $d_2^* = 3$ if $\beta_2 + \beta_3 \leq 1/12$ and $\beta_1 \leq 1/12$. Further, we note that $d_3^* = 3$ if $\beta_1 + \beta_2 \leq 1/12$, $\beta_3 \leq 1/12$ and $\beta_3 + \beta_4 > 1/14$; so that $d_2^* \leq 3$ as 5 will never be truthful to 2. Finally we see that $d_3^* = 3$ if $\beta_1 + \beta_2 > 1/14$, $\beta_2 \leq 1/12$ and $\beta_3 + \beta_4 \leq 1/12$; so that $d_2^* \leq 4$, and $d_2^* = 4$ if $\beta_2 + \beta_3 + \beta_4 \leq 1/16$ and $\beta_1 \leq 1/16$.

We consider the case in which $W^*(2) > W^*(4), U_3^*(2) > U_3^*(4)$, $\beta_1 \leq 1/10, \beta_2 \leq 1/10, \beta_3 > 1/10$ and hence $\delta = \beta_4 - \beta_1 + 2\beta_3 > 1/10$, $d_2^* = 2, d_1^* = 1$. Using expression (4), we can calculate the aggregate expected payoffs for selecting either politician 2 or 3 as the leader:

$$\begin{aligned} W^*(2) &= -\beta_1^2 - \beta_2^2 - (\beta_2 + \beta_3)^2 - (\beta_2 + \beta_3 + \beta_4)^2 - 5\frac{1}{6(2+3)}, \\ W^*(3) &= -(\beta_1 + \beta_2)^2 - \beta_2^2 - \beta_3^2 - (\beta_3 + \beta_4)^2 - 5\frac{1}{6(1+3)}. \end{aligned}$$

The centre-left politician 3 is optimally selected as the leader whenever

$$W^*(2) - W^*(3) = -2\delta\beta_{23} - \beta_2^2 + \frac{1}{24} > 0 \text{ or } \beta_2 < \tau(\delta) \equiv \sqrt{\delta^2 + 1/24} - \delta$$

It is easy to verify that the threshold $\tau(\delta)$ is strictly decreasing in δ, with $\tau(1/10) \approx 0.1273 > 1/10$, that $\tau(\delta)$ is strictly positive for any δ and equals zero only in the limit as δ approaches infinity.

In sum, we conclude that, whenever β_2 is sufficiently small — i.e., smaller than $1/10$ and than $\tau(\delta)$, $\beta_1 \leq 1/10$ and $\beta_3 > 1/10$, then the centre-left politician 2 should be optimally selected as the leader in lieu of the most moderate candidate, politician 3. This is because 2 has better judgement, as it can count on two trustworthy associates, whereas 3 has only one; and 2 is not too much more extremist than 3, as β_2 is small.

Turning to studying the election of the leader by majority vote, we first calculate player 3's payoffs for selecting politician 2 or 3 as the leader, using expression (5):

$$U_3^*(2) = -\beta_2^2 - \frac{1}{6(1+3)} \text{ and } U_3^*(3) = -\frac{1}{6(3)},$$

the median politician 3 will delegate leadership to player 2 whenever

$$U_3^*(2) - U_3^*(3) = \frac{1 - 120\beta_{23}^2}{120} > 0 \text{ or } \beta_2 < \frac{1}{2\sqrt{30}} \approx 0.0913.$$

In light of Proposition 2, we obtain that, whenever β_2 is smaller than $\frac{1}{2\sqrt{30}}$, $\beta_1 \leq 1/10$ and $\beta_3 > 1/10$, the politician 2 is the Condorcet winner of the election game. Again, this is because 2 can count on two ideologically close trustworthy associates, whereas 3 has only one, and because 2 does not hold views too different from the ones of 3.

It is interesting to compare this situation with the equidistant case in which $b_{i+1} - b_i$ is constant for all $i = 1, ..., 4$ and smaller than $\frac{1}{2\sqrt{30}}$. Suppose that the centre-right politician 4 extremizes her ideology b_4 away from the median b_3, so as to increase β_3 beyond $1/10$. Paradoxically, by doing so, she will make the elected leader's ideology move in the opposite direction, as the centre-left politician 2 will gain better judgement than the median politician 3, and win the election. Equivalently, suppose that, initially $b_{i+1} - b_i = \beta > 1/10$ for all $i = 1, ..., 4$. If the leftist politicians 1 and 2 moderate their views, so that β_2 becomes smaller than $\frac{1}{2\sqrt{30}}$ and β_1 becomes smaller than $1/10$, then they move the elected leader's decision towards their views, by making the centre-left politician 2 the leader, in lieu of the median politician 3.

We now compare election and selection of the leader. Because $\tau(\delta)$ is strictly decreasing in δ, $\tau(1/10) > 1/10$ and $\tau(\delta) \to 0$ as $\delta \to \infty$, it is immediate to see that there is a unique threshold $\bar{\delta} > 1/10$ such that $\tau(\delta) > \frac{1}{2\sqrt{30}}$ for all $\delta < \bar{\delta}$ and $\tau(\delta) < \frac{1}{2\sqrt{30}}$ for all $\delta > \bar{\delta}$. This implies that, whenever $\delta < \bar{\delta}$, there exists an interval $\left(1/[2\sqrt{30}], 1/10\right)$ of the parameter β_2 such that the centre-left politician 2 should be optimally selected as leader but the median politician 3 is the Condorcet winner of the election game. A surprising result occurs when $\delta > \bar{\delta}$, so that

$b_5 - b_3$ and $b_4 - b_3$ are sufficiently large relative to $b_2 - b_1$. For values of β_2 larger than $\tau(\delta)$ but smaller than $\frac{1}{2\sqrt{30}}$, the Condorcet winner is the centre-left politician 2 despite the fact that optimal leader is the median politician 3. In the election game, the median politician 3 delegates leadership to a less moderate politician, 2, despite the fact that it would be optimal for the group if she retained leadership for herself. ■

Analysis of the 6 Player Example in Section 7, Proof of Proposition 5. Suppose that there are 6 politicians, with ideologies such that $b_{i+1} - b_i = \beta$ for all $i = 1, ..., 5$, arranged symmetrically around the median ideology zero, so that $b_3 = -\beta/2$ and $b_4 = \beta/2$. Politicians 1, 2, 3 belong to party A, and politicians $4, 5, 6$ to party B. Unless politicians 2 and 5 can count of more trustworthy advisers than 3 and 4, the latter will be selected by their parties and tie the general election, in equilibrium. Because of symmetry of $\mathbf{b}$, let us now just focus on the selection of party A candidates. Candidate 1 will never be selected, so we consider 2 and 3. Because 3 can rely on 2, if 3 communicates to 2 in equilibrium, it follows that the only case in which 2 has better judgement than 3 is when $d_2^* = 2$ and $d_3^* = 1$, which requires that $\beta \leq 1/10$ and that $2\beta > 1/10$.

Because of symmetry of $\mathbf{b}$, if $U_0(2) > U_0(3)$, then there cannot be an equilibrium in which party A selects 3 as its candidate in the general election; if they did, in fact, party B would select 5 as candidate and win the election. When $U_0(2) > U_0(3)$, the unique equilibrium of the game has candidates 2 and 5 tie the general election. Simplifying this condition, we obtain:

$$U_0(2) - U_0(3) = -(\beta + \beta/2)^2 - \frac{1}{6(2+3)} - \left[-(\beta/2)^2 - \frac{1}{6(1+3)}\right] = \frac{1}{120}\left(1 - 240\beta^2\right) > 0.$$

Because the last inequality holds if and only if $\beta < \frac{1}{4\sqrt{15}}$, we conclude that when $1/20 < \beta < \frac{1}{4\sqrt{15}}$, the winners of the general election are not the most moderate politicians 3 and 4, despite the fact that the politicians' ideologies are evenly distributed on the line. ■

Analysis of the 5 Player Example in Section 7 and Proof of Proposition 6 Suppose that there are 5 politicians, with $b_2 < 0 < b_3$. Politicians $1, 2$ belong to party A and $3, 4, 5$ belong to party B, and we assume that $b_3 < -b_2$. Party B has more informed politicians, and it can also select a candidate, player 3, whose views are closer to the median voter. If there were no communication to the winner of the general election, party B would always win by selecting politician 3. However, politician 2 wins the general election if she has better judgement than player 3. As there is only another informed politician in party A, this may only happen if $d_2^* = 1 > d_3^* = 0$, and this requires $\beta_1 \le 1/8$, $\beta_3 > 1/8$ and $\beta_4 > 1/8$. Party A is more ideologically cohesive, and can express a candidate, 2, with a larger network of trustworthy associates than any candidate available to party B. The median voter turns out to prefer to elect politician 2 than politician 3 whenever

$$U_0(2) - U_0(3) = -b_2^2 - \frac{1}{6(1+3)} - \left[-b_3^2 - \frac{1}{6(3)}\right] = \frac{1}{72} - (b_2^2 - b_3^2) > 0,$$

i.e., $b_2^2 - b_3^2 < 1/72$.

To prove the claim that candidate 2 can lose the election by moving closer to the median voter, suppose that we start from an ideology profile $\mathbf{b}$ such that β_1 is smaller than but close to $1/8$. If politician 2 moves ideologically closer to the median voter (i.e., $-b_2$ decreases), then the condition $\beta_1 \le 1/8$ will not be satisfied anymore, candidate 2 will lose the truthful advice of party fellow 1, in turn losing the informational advantage over 3, and the general election. ■

Shared leadership. Consider a group of politicians $i = 1, ..., n$. Suppose that, instead of electing a single leader j, it is possible to select a vector α of shares of leadership α_j for $j = 1, ..., n$ such that $\alpha_j \ge 0$ for all j, and $\sum_{j=1}^n \alpha_j = 1$. For every vector α, its support $L_\alpha \equiv \{j : \alpha_j > 0\}$ denotes the associated set of leaders. The communication by each player i to the leaders L_α may be *private* (hence, the message $\hat{m}_{ij} \in \{0, 1\}$ sent by i to j may differ across $j \in L_\alpha$), or *public* (and then $\hat{m}_{ij}$ must be the same for all $j \in L_\alpha$).

A vector of authority shares α determines the mixture over outcomes:

$$y(\mathbf{s}, \mathbf{m}; \alpha) = \sum_{j=1}^{n} \alpha_j \left[b_j + E\left[\theta | s_j, \mathbf{m}_{-j}\right]\right],$$

given the signals $\mathrm{s} = (s_j)_{j=1}^n$ and the equilibrium communication strategies $\mathbf{m} = (\mathbf{m}_{-j})_{j=1}^n$. And this yields each player i expected utility:

$$U_i(\mathbf{s}, \mathbf{m}; \alpha) = -\sum_{j=1}^{n} \alpha_j (b_i - b_j)^2 - \sum_{j=1}^{n} \alpha_j \frac{1}{6\left(d_j^*(\mathbf{m}_{-j}) + 3\right)}.$$

In terms of optimal choice, the possibility of choosing α optimally improves utilitarian welfare over single leadership weakly by definition, in our model. It is easy to find examples where it improves utilitarian welfare strictly—see Example 1 in Dewan et al. (2015), for instance.

Let us consider now the majority choice among share of leadership vectors α. The space of vectors α can be linearly ordered according to the mixture over biases $\bar{b}(\alpha) = \sum_{j=1}^{n} \alpha_j b_j$. It is immediate to then extend the proof of Lemma 1 to this environment. As a consequence, the set of Condorcet winning share of leadership vectors α coincides with the set of vectors α that maximize the expected payoff of the median player m.

The same kinds of inefficiency described in Lemma 2 and Proposition 4 extends to this richer environment. As we now demonstrate, there are examples, parametrized by the bias vector b, in which the optimal share of leadership vector α differs from the majority choice.

We consider the 5-player case studied in Section 6.3, and so assume $\beta_1 \leq 1/10$, $\beta_2 \leq 1/10$, $\beta_3 > 1/10$, and hence $\delta > 1/10$. Suppose that $\tau(\delta) < \frac{1}{2\sqrt{30}}$, and that $\tau(\delta) < \beta_2 < \frac{1}{2\sqrt{30}}$. As shown in Lemma 2, the optimal leader is 3, but 3 delegates to 2 who is better informed, because $d_2^* = 2$ and $d_3^* = 1$. Allowing for shared leadership, it would be possible to get 4 to communicate truthfully to 3 only if including 5 in the set of leaders L_α, and considering public communication. With private communication, 4 would not be truthful to 3 in equilibrium, as it would wish to distort the decision y_3 regardless of the message $\hat{m}_{45}$ she sends to player 5. Using Lemma 1 of Dewan et al. (2015), there is an equilibrium in which players 2 and 4 are truthful to 3 and 5 if and only if:

$$|b_4 - (\gamma_3 b_3 + \gamma_5 b_5)| \leq \gamma_3 \frac{1}{2(d_3+2)} + \gamma_5 \frac{1}{2(d_5+2)} \tag{6}$$

$$|b_2 - (\gamma_3 b_3 + \gamma_5 b_5)| \leq \gamma_3 \frac{1}{2(d_3+2)} + \gamma_5 \frac{1}{2(d_5+2)} \tag{7}$$

where $\gamma_3 = \frac{\alpha_3/2(d_3+2)}{\alpha_3/2(d_3+2)+\alpha_5/2(d_5+2)}$ and $\gamma_5 = \frac{\alpha_5/2(d_5+2)}{\alpha_3/2(d_3+2)+\alpha_5/2(d_5+2)}$, and $\alpha_3 + \alpha_5 = 1$.

Here, because $b_5 - b_3 > b_4 - b_3 > 1/10$, player 3 and 5 cannot be truthful to each other, hence $d_3 = 2$ and $d_5 = 2$. Conditions (6) and (7) become:

$$|b_4 - (\alpha_3 b_3 + (1-\alpha_3) b_5)| = \alpha_3 \beta_3 - (1-\alpha_3)\beta_4 \leq \frac{1}{10}$$

$$|b_2 - (\alpha_3 b_3 + (1-\alpha_3) b_5)| = \alpha_3 \beta_2 + (1-\alpha_3)(\beta_2 + \beta_3 + \beta_4) \leq \frac{1}{10}.$$

Condition (6) is satisfied tightly for $\alpha_3 = \frac{\beta_4+1/10}{\beta_3+\beta_4}$, plugging this into condition (7), we obtain:

$$\frac{\beta_4+1/10}{\beta_3+\beta_4}(\beta_2-1/10)+\left(1-\frac{\beta_4+1/10}{\beta_3+\beta_4}\right)(\beta_2+\beta_3+\beta_4-1/10)=\beta_2+\beta_3-1/5\leq 0.$$

that is violated for $\beta_3 > 1/5 - \beta_2$, i.e., $\beta_3 > 1/10$, because $\beta_2 \leq 1/10$. We conclude that, for $0 < \beta_2 \leq 1/10$, $0 < \beta_4 \leq 1/10$ and $\beta_3 > 1/10$, it is not possible to get 2 and 4 to communicate truthfully to 3 in equilibrium with any shared leadership vector α. In other terms, $d_3^* \leq 1$ in equilibrium.

Suppose further that $\tau(\delta) < \frac{1}{2\sqrt{30}}$, noting that $\tau(\delta) = \sqrt{\delta^2 + 1/24} - \delta$ decreases in $\delta = \beta_4 + 2\beta_3 - \beta_1$, so that the condition $\tau(\delta) < \frac{1}{2\sqrt{30}}$ is satisfied for $\delta > \tau^{-1}\left(\frac{1}{2\sqrt{30}}\right) = 1/\sqrt{30} \approx 0.182\,57$, and does impose any upper bound on β_3. Consider any β_2 such that $\tau(\delta) < \beta_2 < \frac{1}{2\sqrt{30}}$, and note that $0 < \tau(\delta) < \beta_2 < \frac{1}{2\sqrt{30}} < 1/10$. The proof of Lemma 2 implies that, because $d_2^* = 2$ and $d_3^* = 1$, the optimal leader is 3, but 3 prefers to delegates to 2 who is better informed, and hence 2 is elected by majority voting. The same kind of inefficiency described in Lemma 2 and Proposition 4 extends to the environment that includes the possibility of shared leadership.

We conclude by noting that a different way to define shared leadership would be to fix a system α of sharing rules α_L for all possible sets of leaders $L \subseteq \{1, ..., n\}$, and restrict the optimal and majority choice only to the set of leaders L given the system α. For example, α could be an "egalitarian system" such that $\alpha_{jL} = 1/|L|$ for all sets of leaders L, and all $j \in L$. Regardless of the selected/elected set of leaders L, each leader $j \in L$ has equal share of power. Alternatively, the system α could include forms of seniority among politicians.

It is obvious that fixing the system α and selecting L optimally is a weak improvement upon optimal individual leadership, and that it is weakly dominated by optimal selection of a vector of shares α. Further, the extended example above demonstrates that the kinds of inefficiency described in Lemma 2 occur also in this environment. There are examples, parametrized by the bias vector b and leadership sharing system α, in which the optimal choice of L given α differs from the majority choice. ■