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SUPPLEMENTARY MATERIAL: ON SPARSITY, POWER-LAW AND CLUSTERING PROPERTIES OF GRAPHEX PROCESSES

FRANÇOIS CARON,* ** *University of Oxford*

FRANCESCA PANERO,*** *London School of Economics and Political Science*

JUDITH ROUSSEAU,* **** *University of Oxford*

The supplementary material is organised as follows. Section A contains proofs of asymptotic bounds on the variances of the number of nodes, number of nodes of a given degree, and number of triangles of nodes with a given degree, as well as the proof of a secondary proposition for the local clustering coefficient. Section B contains proofs of secondary propositions for the central limit theorem. For the sake of simplicity, all Sections, Equations, Lemmas, etc., in the Supplementary material here are denoted with prefixes S and T, to differentiate them from the Sections, Equations, Lemmas, etc., of the main text [1].

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* Postal address: Department of Statistics, 24-29 St Giles', Oxford OX1 3LB, United Kingdom

** Email address: caron@stats.ox.ac.uk

*** Postal address: Department of Statistics, 69 Aldwych, London WC2B 4RR, United Kingdom. Email: f.panero@lse.ac.uk

**** Email address: judith.rousseau@stats.ox.ac.uk

Appendix A. Proofs of secondary propositions for the variances and clustering coefficients

A.1. Proof of Proposition ?? on var(N_α)

An application of the Slivnyak-Mecke and Campbell theorems gives

$$\begin{aligned}\operatorname{var}(N_\alpha) =& E(N_\alpha) + 2\alpha^2 \int_0^\infty \mu(x)(1-W(x,x))e^{-\alpha\mu(x)}dx \\ &+ \alpha^2 \int_{\mathbb{R}_+^2} (e^{\alpha\nu(x,y)} - 1 + W(x,y))(1-W(x,x))(1-W(y,y)) \\ &\times (1-W(x,y))e^{-\alpha\mu(x)-\alpha\mu(y)}dxdy.\end{aligned}$$

Using the inequality $e^x - 1 \leq xe^x$,

$$\begin{aligned}\operatorname{var}(N_\alpha) &\leq E(N_\alpha) + 2\alpha^2 \int_{\mathbb{R}_+} \mu(x)e^{-\alpha\mu(x)}dx \\ &\quad + \alpha^2 \int_{\mathbb{R}_+^2} e^{-\alpha\mu(x)-\alpha\mu(y)} \left\{ e^{\alpha\nu(x,y)} - 1 + W(x,y) \right\} dxdy \\ &\leq E(N_\alpha) + 2\alpha^2 \int_{\mathbb{R}_+} \mu(x)e^{-\alpha\mu(x)}dx \\ &\quad + \alpha^2 \int_{\mathbb{R}_+^2} e^{-\alpha\mu(x)-\alpha\mu(y)} \left\{ \alpha\nu(x,y)e^{\alpha\nu(x,y)} + W(x,y) \right\} dxdy \\ &= E(N_\alpha) + 2\alpha^2 \int_{\mathbb{R}_+} \mu(x)e^{-\alpha\mu(x)}dx \\ &\quad + \alpha^2 \int_{\mathbb{R}_+^2} e^{-\alpha\mu(x)-\alpha\mu(y)} W(x,y)dxdy \\ &\quad + \sum_{k=1}^\infty \alpha^{2+k} \int_{\mathbb{R}_+^2} \nu(x,y)^k e^{-\alpha\mu(x)-\alpha\mu(y)}dxdy.\end{aligned}$$

Now, using Lemmas **??** and **??**

$$\int_{\mathbb{R}_+^2} W(x,y)e^{-\alpha\mu(x)-\alpha\mu(y)}dxdy \leq \int_{\mathbb{R}_+} \mu(x)e^{-\alpha\mu(x)}dx = O\left(\alpha^{\sigma-1}\ell_\sigma(\alpha)\right).$$

$$\int_{\mathbb{R}_+^2} \nu(x,y)e^{-\alpha\mu(x)-\alpha\mu(y)+\alpha\nu(x,y)}dxdy = O\left(\alpha^{2\sigma-2a}\ell_\sigma^2(\alpha)\right).$$

It follows that $\operatorname{var}(N_\alpha) = O(\alpha^{3+2\sigma-2a}\ell_\sigma(\alpha)^2)$.

Assume Assumption **??** and **??** are satisfied, with $a = 1$. From the first part of Proposition **??**, we have the upper bound $\operatorname{var}(N_\alpha) = O\left(\alpha^{1+2\sigma}\ell_\sigma^2(\alpha)\right)$.

We now derive a lower bound. If $\sigma = 0$, $\mathrm{var}(N_\alpha) \geq E(N_\alpha) \gtrsim \alpha \ell_0(\alpha)$, hence $\mathrm{var}(N_\alpha) \asymp \alpha\ell(\alpha)$. Consider now the case $\sigma > 0$. We have

$$\mathrm{var}(N_\alpha) \geq \alpha^2 \int_{\mathbb{R}_+^2} (e^{\alpha\nu(x,y)} - 1)(1 - W(x,x))(1 - W(y,y))(1 - W(x,y)) \times e^{-\alpha\mu(x) - \alpha\mu(y)} dxdy$$

and using the inequality $e^x - 1 \geq x$ and Assumption **??**

$$\begin{aligned}\mathrm{var}(N_\alpha) &\geq \alpha^3 \int_{\mathbb{R}_+^2} \nu(x,y)(1 - W(x,x))(1 - W(y,y))(1 - W(x,y)) e^{-\alpha\mu(x) - \alpha\mu(y)} dxdy \\ &\geq C_0 \alpha^3 \int_{x_0}^\infty \int_{x_0}^\infty \mu(x)\mu(y)(1 - W(x,x))(1 - W(y,y))(1 - W(x,y)) \\ &\qquad \times e^{-\alpha\mu(x) - \alpha\mu(y)} dxdy\end{aligned}$$

Using Lemmas **??** and **??**, we have

$$\begin{aligned}&\int_{x_0}^\infty \int_{x_0}^\infty \mu(x)\mu(y)(1 - W(x,x))(1 - W(y,y))(1 - W(x,y)) e^{-\alpha\mu(x) - \alpha\mu(y)} dxdy \\ &\sim \int_0^\infty \int_0^\infty \mu(x)\mu(y) e^{-\alpha\mu(x) - \alpha\mu(y)} dxdy = \left(\int_{\mathbb{R}_+} \mu(x) e^{-\alpha\mu(x)} dx \right)^2 \sim \alpha^{2\sigma - 2} \ell_\sigma^2(\alpha).\end{aligned}$$

It follows that, for $\sigma > 0$, $\mathrm{var}(N_\alpha) \gtrsim \alpha^{1+2\sigma} \ell_\sigma^2(\alpha)$. Combining this with the upper bound gives, for all $\sigma \in [0,1]$ $\mathrm{var}(N_\alpha) \asymp \alpha^{1+2\sigma} \ell_\sigma^2(\alpha)$.

A.2. Proof of proposition ?? on $\mathrm{var}(N_{\alpha,j})$

We have,

$$\begin{aligned}&E(N_{\alpha,j}^2 \mid M) - E(N_{\alpha,j} \mid M) \\ &= \sum_{i_1 \neq i_2} \mathbb{1}_{\theta_{i_1} \leq \alpha} \mathbb{1}_{\theta_{i_2} \leq \alpha} \mathbb{P}\left\{ \sum_k \mathbb{1}_{\theta_k \leq \alpha} Z_{i_1 k} = j \text{ and } \sum_k \mathbb{1}_{\theta_k \leq \alpha} Z_{i_2,k} = j \mid M \right\}. \\ &= \sum_{b \in \{0,1\}^3} \sum_{j_1=0}^{j} \sum_{i_1 \neq i_2} \mathbb{1}_{\theta_{i_1} \leq \alpha} \mathbb{1}_{\theta_{i_2} \leq \alpha} \\ &\quad \times \mathbb{P}\left\{ \sum_k \mathbb{1}_{\theta_k \leq \alpha} Z_{i_1 k} = j \text{ and } \sum_k \mathbb{1}_{\theta_k \leq \alpha} Z_{i_2,k} = j \text{ and } \sum_k \mathbb{1}_{\theta_k \leq \alpha} Z_{i_1 k} Z_{i_2 k} = j - j_1 \right. \\ &\qquad\qquad \left. \text{and } Z_{i_1 i_1} = b_{11}, Z_{i_1 i_2} = b_{12}, Z_{i_2 i_2} = b_{22} \mid M \right\}\end{aligned}$$

where $b=(b_{11},b_{12},b_{22})\in\{0,1\}^3$. Let A_1,A_2,A_{12} be disjoint subsets of $\mathbb{N}\backslash\{i_1,i_2\}$ such that $|A_{12}|+b_{12}=j-j_1$, $|A_1|+|A_{1,2}|+b_{11}+b_{12}=|A_2|+|A_{1,2}|+b_{22}+b_{12}=j$ respectively corresponding to the indices of nodes only connected to node i_1, only to node i_2, or to both nodes (i_1,i_2). Let $A=\{i_1,i_2\}\cup A_1\cup A_2\cup A_{12}$. We have

$$\begin{aligned}
&\mathbb{P}\Bigg\{\sum_k \mathbb{1}_{\theta_k\leq\alpha}Z_{i_1k}=j,\sum_k \mathbb{1}_{\theta_k\leq\alpha}Z_{i_2,k}=j,\sum_k \mathbb{1}_{\theta_k\leq\alpha}Z_{i_1k}Z_{i_2k}=j-j_1,\\
&\qquad(Z_{i_1i_1},Z_{i_1i_2},Z_{i_2i_2})=b\mid M\Big\}\\
&=\sum_{A_1,A_2,A_{12}}\frac{\mathbb{1}_{\theta_{i_1}\leq\alpha}\mathbb{1}_{\theta_{i_2}\leq\alpha}}{(j-j_1-b_{12})!(j_1-b_{11})!(j_1-b_{22})!}W(\vartheta_{i_1},\vartheta_{i_1})^{b_{11}}W(\vartheta_{i_2},\vartheta_{i_2})^{b_{22}}W(\vartheta_{i_1},\vartheta_{i_2})^{b_{12}}\\
&\quad\times\{1-W(\vartheta_{i_1},\vartheta_{i_1})\}^{1-b_{11}}\{1-W(\vartheta_{i_2},\vartheta_{i_2})\}^{1-b_{22}}\{1-W(\vartheta_{i_1},\vartheta_{i_2})\}^{1-b_{12}}\\
&\quad\times\left[\prod_{k\in A_1}\mathbb{1}_{\theta_k\leq\alpha}W(\vartheta_{i_1},\vartheta_{i_k})\{1-W(\vartheta_{i_2},\vartheta_{i_k})\}\right]\left[\prod_{k\in A_2}\mathbb{1}_{\theta_k\leq\alpha}\{1-W(\vartheta_{i_1},\vartheta_{i_k})\}W(\vartheta_{i_2},\vartheta_{i_k})\right]\\
&\quad\left[\prod_{k\in A_{12}}\mathbb{1}_{\theta_k\leq\alpha}W(\vartheta_{i_1},\vartheta_{i_k})W(\vartheta_{i_2},\vartheta_{i_k})\right]\exp\left[-\sum_{k\in\mathbb{N}\backslash A}\{g_{\alpha,\vartheta_{i_1}}(\theta_k,\vartheta_k)+g_{\alpha,\vartheta_{i_2}}(\theta_k,\vartheta_k)\}\right]
\end{aligned}$$

Using the extended Slivnyak-Mecke theorem,

$$\begin{aligned}
&E(N_{\alpha,j}^2)-E(N_{\alpha,j})\\
&=\sum_{b\in\{0,1\}^3}\sum_{j_1=0}^{j}\frac{\alpha^{2+j+j_1-b_{11}-b_{12}-b_{22}}}{(j-j_1-b_{12})!(j_1-b_{11})!(j_1-b_{22})!}\mathbb{1}_{j_1\geq b_{11}}\mathbb{1}_{j_1\geq b_{22}}\mathbb{1}_{j_1\leq j-b_{12}}\\
&\qquad\times\int_{\mathbb{R}_+^2}\{\mu(x)-\nu(x,y)\}^{j_1-b_{11}}\{\mu(y)-\nu(x,y)\}^{j_1-b_{22}}\nu(x,y)^{j-j_1-b_{12}}\\
&\qquad\qquad\times e^{-\alpha\mu(x)-\alpha\mu(y)+\alpha\nu(x,y)}W(x,x)^{b_{11}}W(y,y)^{b_{22}}W(x,y)^{b_{12}}\{1-W(x,x)\}^{1-b_{11}}\\
&\qquad\qquad\times\{1-W(y,y)\}^{1-b_{22}}\{1-W(x,y)\}^{1-b_{12}}dxdy\\
&\leq\sum_{b\in\{0,1\}^3}\sum_{j_1=0}^{j}\frac{\alpha^{2+j+j_1-b_{11}-b_{12}-b_{22}}}{(j-j_1-b_{12})!(j_1-b_{11})!(j_1-b_{22})!}\mathbb{1}_{j_1\geq b_{11}}\mathbb{1}_{j_1\geq b_{22}}\mathbb{1}_{j_1\leq j-b_{12}}\\
&\qquad\times\int_{\mathbb{R}_+^2}\mu(x)^{j_1-b_{11}}\mu(y)^{j_1-b_{22}}\nu(x,y)^{j-j_1-b_{12}}e^{-\alpha\mu(x)-\alpha\mu(y)+\alpha\nu(x,y)}\\
&\qquad\qquad\times W(x,x)^{b_{11}}W(y,y)^{b_{22}}W(x,y)^{b_{12}}\{1-W(x,x)\}^{1-b_{11}}\{1-W(y,y)\}^{1-b_{22}}
\end{aligned}$$

$$\times \{1 - W(x,y)\}^{1-b_{12}} dxdy$$

We will need the following lemma.

Lemma A.1. *Let $r \geq 1$, $j_1, j_2 \geq 0$. Define*

$$I_r := \int_{\mathbb{R}^2} [\alpha\mu(x)]^{j_1} [\alpha\mu(y)]^{j_2} (\alpha\nu(x,y))^r e^{-\alpha\mu(x)-\alpha\mu(y)+\alpha\nu(x,y)} dxdy.$$

Under Assumptions 1 and 2, we have for all $r \geq 1$

$$I_r = O(\alpha^{r-2ar+2\sigma}\ell_\sigma^2(\alpha)).$$

Proof. We have, using Assumption 2, that

$$I_r \leq \alpha^r \int_{\mathbb{R}_+^2} [\alpha\mu(x)]^{j_1} [\alpha\mu(y)]^{j_2} \nu(x,y)^r e^{-\alpha\{\mu(x)+\mu(y)\}/2} dxdy$$
$$\leq C_1^r \alpha^{r-2ar} \int_{\mathbb{R}_+} (\alpha\mu(x))^{j_1+ar} e^{-\alpha\mu(x)/2} dx \int_{\mathbb{R}_+} (\alpha\mu(x))^{j_2+ar} e^{-\alpha\mu(x)/2} dx + o(\alpha^{-p}).$$

for any $p > 0$. Assumption 1 and Lemmas **??** ($\sigma = 1$) and **??** ($\sigma \in [0,1)$) imply that for all $r \geq 1$

$$I_r = O(\alpha^{r-2ar+2\sigma}\ell_\sigma^2(\alpha)).$$

□

It follows

$$E(N_{\alpha,j}^2) - E(N_{\alpha,j}) \lesssim \sum_{b\in\{0,1\}^3} \frac{\alpha^{2+2j-b_{11}-b_{22}-2b_{12}}}{(j-b_{12}-b_{11})!(j-b_{12}-b_{22})!} \mathbb{1}_{j\geq b_{11}+b_{12}} \mathbb{1}_{j\geq b_{22}+b_{12}}$$
$$\times \int_{\mathbb{R}_+^2} \mu(x)^{j-b_{12}-b_{11}} \mu(y)^{j-b_{12}-b_{22}} e^{-\alpha\mu(x)-\alpha\mu(y)+\alpha\nu(x,y)} W(x,x)^{b_{11}} W(y,y)^{b_{22}} W(x,y)^{b_{12}}$$
$$\times \{1-W(x,x)\}^{1-b_{11}} \{1-W(y,y)\}^{1-b_{22}} \{1-W(x,y)\}^{1-b_{12}} dxdy + O\{\alpha^{2+2\sigma+1-2a}\ell_\sigma^2(\alpha)\}.$$

Let V_0 and V_1 respectively denote the sum of terms such that $b_{12} = 0$ and $b_{12} = 1$ in the above sum. Using the inequality $e^x \leq 1 + xe^x$,

$$V_0 = \sum_{b_{11},b_{22}\in\{0,1\}^2} \frac{\alpha^{2+2j-b_{11}-b_{22}}}{(j-b_{11})!(j-b_{22})!} \int_{\mathbb{R}_+^2} \mu(x)^{j-b_{11}} \mu(y)^{j-b_{12}} e^{-\alpha\mu(x)-\alpha\mu(y)+\alpha\nu(x,y)}$$
$$\times W(x,x)^{b_{11}} W(y,y)^{b_{22}} \{1-W(x,x)\}^{1-b_{11}} \{1-W(y,y)\}^{1-b_{22}} \{1-W(x,y)\} dxdy$$

$$
\begin{aligned}
&= \sum_{b_{11},b_{22}} \frac{\alpha^{2+2j-b_{11}-b_{22}}}{(j-b_{11})!(j-b_{22})!} \int_{\mathbb{R}_+^2} \mu(x)^{j-b_{11}} \mu(y)^{j-b_{12}} e^{-\alpha\mu(x)-\alpha\mu(y)} \\
&\quad \times W(x,x)^{b_{11}} W(y,y)^{b_{22}} \{1-W(x,x)\}^{1-b_{11}} \{1-W(y,y)\}^{1-b_{22}} dxdy \\
&\quad + O\left\{ \sum_{b_{11},b_{22}} \frac{\alpha^{3+2j-b_{11}-b_{22}}}{(j-b_{11})!(j-b_{22})!} \int_{\mathbb{R}_+^2} \mu(x)^{j-b_{11}} \mu(y)^{j-b_{12}} \nu(x,y) \right. \\
&\qquad \left. \times e^{-\alpha\mu(x)-\alpha\mu(y)+\alpha\nu(x,y)} dxdy \right\} \\
&= \sum_{b_{11},b_{22}} \left\{ \frac{\alpha^{1+j-b_{11}}}{(j-b_{11})!} \int_{\mathbb{R}_+} \mu(x)^{j-b_{11}} W(x,x)^{b_{11}} \{1-W(x,x)\}^{1-b_{11}} e^{-\mu(x)} dx \right\} \\
&\qquad \times \left\{ \frac{\alpha^{1+j-b_{22}}}{(j-b_{22})!} \int_{\mathbb{R}_+} \mu(y)^{j-b_{22}} W(y,y)^{b_{22}} \{1-W(y,y)\}^{1-b_{22}} e^{-\mu(y)} dy \right\} \\
&\quad + O\{\alpha^{2+2\sigma+1-2a} \ell_\sigma^2(\alpha)\} = E(N_{\alpha,j})^2 + O\{\alpha^{2+2\sigma+1-2a} \ell_\sigma^2(\alpha)\}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
V_1 &\leq \sum_{b_{11},b_{22}} \frac{\alpha^{2j-b_{11}-b_{22}} \mathbb{1}_{j\geq 1+b_{11}} \mathbb{1}_{j\geq 1+b_{22}}}{(j-1-b_{11})!(j-1-b_{22})!} \int_{\mathbb{R}_+^2} \mu(x)^{j-1-b_{11}} \mu(y)^{j-1-b_{12}} \\
&\qquad e^{\alpha\nu(x,y)-\alpha\mu(x)-\alpha\mu(y)} W(x,y) dxdy \\
&\leq \sum_{b_{11},b_{22}} \frac{\alpha^{2j-b_{11}-b_{22}} \mathbb{1}_{j\geq 1+b_{11}} \mathbb{1}_{j\geq 1+b_{22}}}{(j-1-b_{11})!(j-1-b_{22})!} \int_{\mathbb{R}_+^2} \mu(x)^{j-1-b_{11}} \mu(y)^{j-1-b_{12}} \\
&\qquad e^{-\alpha\mu(x)-\alpha\mu(y)} W(x,y) dxdy + O\{\alpha^{2+2\sigma+1-2a} \ell_\sigma^2(\alpha)\}
\end{aligned}
$$

For $j_1 \geq 1$ and $j_2 \geq 1$, using Cauchy-Schwarz and Lemma **??**,

$$
\begin{aligned}
&\int W(x,y) \mu(x)^{j_1} \mu(y)^{j_2} e^{-\alpha\mu(x)-\alpha\mu(y)} dxdy \\
&\leq \int_{\mathbb{R}_+} \mu(x)^{j_1} e^{-\alpha\mu(x)} \left\{ \int W(x,y) \mu(y)^{2j_2} e^{-2\alpha\mu(y)} dy \right\}^{1/2} \mu(x)^{1/2} dx \\
&\leq \left\{ \int \mu(x)^{j_1+1/2} e^{-\alpha\mu(x)} dx \right\} \left\{ \int \mu(y)^{2j_2} e^{-2\alpha\mu(y)} dy \right\}^{1/2} \\
&= O\left\{ \alpha^{3\sigma/2-j_1-j_2-1/2} \ell_\sigma^{3/2}(\alpha) \right\}
\end{aligned}
$$

and for $j_1 \geq 0$

$$
\int_{\mathbb{R}_+^2} \mu(x)^{j_1} e^{-\alpha\mu(x)-\alpha\mu(y)} W(x,y) dxdy \leq \int_{\mathbb{R}_+^2} \mu(x)^{j_1} e^{-\alpha\mu(x)} W(x,y) dxdy
$$

$$= \int_{\mathbb{R}_+} \mu(x)^{j_1+1} e^{-\alpha\mu(x)} dx = O\left\{\alpha^{\sigma-j_1-1}\ell_\sigma(\alpha)\right\}$$

It follows that $V_1 = O\{\alpha^{2+3\sigma/2-1/2}\ell_\sigma^{3/2}(\alpha)\} + O\{\alpha^{1+\sigma}\ell_\sigma^2(\alpha)\} + O\{\alpha^{2+2\sigma+1-2a}\ell_\sigma^2(\alpha)\}$. Combining the upper bounds on V_0 and V_1, we obtain $\mathrm{var}(N_{\alpha,j}) = O(\alpha^{3-2a+2\sigma}\ell_\sigma^2(\alpha))$ and this terminates the proof. In the case $\sigma = 0$ and $a = 1$, one can use Lemma **??** instead of Lemma **??** and replace big O by little o in the above bounds, together with the fact that $E(N_{\alpha,j}) = o(\alpha\ell(\alpha))$ and $\ell(t) = O(\ell^2(t))$ if $\sigma = 0$.

A.3. Proof of Proposition ??

We first prove the first equality. The proof is similar to that of Proposition **??**, given in Section A.2. For any $j \geq 2$,

$$2R_{\alpha,j} = 2\sum_i T_{\alpha i} \mathbb{1}_{D_{\alpha i}=j} \mathbb{1}_{\theta_i \leq \alpha} = \sum_{i \neq k \neq l} Z_{ik} Z_{il} Z_{kl} \mathbb{1}_{D_{\alpha i}=j} \mathbb{1}_{\theta_i \leq \alpha}.$$

Let $S_{\alpha j} := 4R_{\alpha j}^2$. We have

$$\begin{aligned} S_{\alpha j} &= \left(\sum_{i \neq k \neq l} Z_{ik} Z_{il} Z_{kl} \mathbb{1}_{D_{\alpha i}=j} \mathbb{1}_{\theta_i \leq \alpha} \right)^2 \qquad \text{(A.2)} \\ &= \sum_{i_1 \neq k_1 \neq l_1 \neq i_2 \neq k_2 \neq l_2} Z_{i_1k_1} Z_{i_1l_1} Z_{k_1l_1} Z_{i_2k_2} Z_{i_2l_2} Z_{k_2l_2} \mathbb{1}_{D_{\alpha i_1}=j} \mathbb{1}_{D_{\alpha i_2}=j} \mathbb{1}_{\theta_{i_1} \leq \alpha} \mathbb{1}_{\theta_{i_2} \leq \alpha} \\ &\quad + 2 \sum_{i_1 \neq k_1 \neq l_1 \neq i_2 \neq k_2} Z_{i_1k_1} Z_{i_1l_1} Z_{k_1l_1} Z_{i_2k_2} Z_{i_2l_1} Z_{k_2l_1} \mathbb{1}_{D_{\alpha i_1}=j} \mathbb{1}_{D_{\alpha i_2}=j} \mathbb{1}_{\theta_{i_1} \leq \alpha} \mathbb{1}_{\theta_{i_2} \leq \alpha} \\ &\quad + 2 \sum_{i_1 \neq k_1 \neq l_1 \neq i_2} Z_{i_1k_1} Z_{i_1l_1} Z_{k_1l_1} Z_{i_2k_1} Z_{i_2l_1} \mathbb{1}_{D_{\alpha i_1}=j} \mathbb{1}_{D_{\alpha i_2}=j} \mathbb{1}_{\theta_{i_1} \leq \alpha} \mathbb{1}_{\theta_{i_2} \leq \alpha} \\ &\quad + \sum_{i_1 \neq k_1 \neq l_1 \neq k_2 \neq l_2} Z_{i_1k_1} Z_{i_1l_1} Z_{k_1l_1} Z_{i_1k_2} Z_{i_1l_2} Z_{k_2l_2} \mathbb{1}_{D_{\alpha i_1}=j} \mathbb{1}_{\theta_{i_1} \leq \alpha} \\ &\quad + 2 \sum_{i_1 \neq k_1 \neq l_1 \neq k_2} Z_{i_1k_1} Z_{i_1l_1} Z_{k_1l_1} Z_{i_1k_2} Z_{k_2l_1} \mathbb{1}_{D_{\alpha i_1}=j} \mathbb{1}_{\theta_{i_1} \leq \alpha} \\ &\quad + 2 \sum_{i_1 \neq k_1 \neq l_1} Z_{i_1k_1} Z_{i_1l_1} Z_{k_1l_1} \mathbb{1}_{D_{\alpha i_1}=j} \mathbb{1}_{\theta_{i_1} \leq \alpha} \end{aligned}$$

Note that some of the terms above are equal to 0 if $j \leq 4$. First note that for any $j_0 \leq j$:

$$\sum_{i \neq k_1 \neq \ldots \neq k_{j_0}} \left(\prod_{l=1}^{j_0} Z_{il} \right) \mathbb{1}_{D_{\alpha i} = j} \mathbb{1}_{\theta_i \leq \alpha} \leq \binom{j}{j_0} N_{\alpha j} \tag{A.4}$$

Hence the last three terms of the right-handside of (A.2) are upper bounded by $C_j N_{\alpha,j}$, for some constant C_j that does not depend on α. Consider now

$$\begin{aligned} S_{\alpha,j,1} &= \sum_{i_1 \neq k_1 \neq l_1 \neq i_2 \neq k_2 \neq l_2} Z_{i_1 k_1} Z_{i_1 l_1} Z_{k_1 l_1} Z_{i_2 k_2} Z_{i_2 l_2} Z_{k_2 l_2} \mathbb{1}_{D_{\alpha i_1} = j} \mathbb{1}_{D_{\alpha i_2} = j} \mathbb{1}_{\theta_{i_1} \leq \alpha} \mathbb{1}_{\theta_{i_2} \leq \alpha} \\ &= \sum_{j_1=2}^{j} S_{\alpha,j,1,j_1} \end{aligned}$$

where, for $j_1 = 2, \ldots, j$

$$\begin{aligned} S_{\alpha,j,1,j_1} &= \sum_{\substack{i_1 \neq k_1 \neq l_1 \\ \neq i_2 \neq k_2 \neq l_2}} Z_{i_1 k_1} Z_{i_1 l_1} Z_{k_1 l_1} Z_{i_2 k_2} Z_{i_2 l_2} Z_{k_2 l_2} \mathbb{1}_{D_{\alpha i_1} = j} \mathbb{1}_{D_{\alpha i_2} = j} \\ &\quad \times \mathbb{1}_{\sum_k Z_{i_1 k} Z_{i_2 k} \mathbb{1}_{\theta_k \leq \alpha} = j - j_1} \mathbb{1}_{\theta_{i_1} \leq \alpha} \mathbb{1}_{\theta_{i_2} \leq \alpha} \\ &= \sum_{b \in \{0,1\}^3} \sum_{\substack{i_1 \neq k_1 \neq l_1 \\ \neq i_2 \neq k_2 \neq l_2}} Z_{i_1 k_1} Z_{i_1 l_1} Z_{k_1 l_1} Z_{i_2 k_2} Z_{i_2 l_2} Z_{k_2 l_2} \mathbb{1}_{D_{\alpha i_1} = j} \mathbb{1}_{D_{\alpha i_2} = j} \\ &\quad \times \mathbb{1}_{\sum_k Z_{i_1 k} Z_{i_2 k} \mathbb{1}_{\theta_k \leq \alpha} = j - j_1} \mathbb{1}_{Z_{i_1 i_1} = b_{11}} \mathbb{1}_{Z_{i_1 i_2} = b_{12}} \mathbb{1}_{Z_{i_2 i_2} = b_{22}} \mathbb{1}_{\theta_{i_1} \leq \alpha} \mathbb{1}_{\theta_{i_2} \leq \alpha} \end{aligned}$$

where we introduce $b = (b_{11}, b_{12}, b_{22}) \in \{0,1\}^3$ as in Section A.2. Using the extended Slivnyak-Mecke theorem, for $j_1 = 2, \ldots, j$,

$$\begin{aligned} &E(S_{\alpha,j,1,j_1}) \qquad \text{(A.5)} \\ &= \sum_{b \in \{0,1\}^3} \frac{\alpha^{2+j+j_1-b_{11}-b_{12}-b_{22}}}{(j - j_1 - b_{12})!(j_1 - b_{11} - 2)!(j_1 - b_{22} - 2)!} \mathbb{1}_{j_1 \leq j - b_{12}} \mathbb{1}_{j_1 \geq b_{11}} \mathbb{1}_{j_1 \geq b_{22}} \\ &\quad \times \int_{\mathbb{R}_+^6} \{\mu(x_1) - \nu(x_1, x_2)\}^{j_1 - 2 - b_{11}} \{\mu(x_2) - \nu(x_1, x_2)\}^{j_1 - 2 - b_{22}} \nu(x_1, x_2)^{j - j_1 - b_{12}} \\ &\quad \times e^{-\alpha\mu(x_1) - \alpha\mu(x_2) + \alpha\nu(x_1, x_2)} W(x_1, y_1) W(x_1, z_1) W(y_1, z_1) W(x_2, y_2) W(x_2, z_2) W(y_2, z_2) \\ &\quad \times W(x, x)^{b_{11}} W(y, y)^{b_{22}} W(x, y)^{b_{12}} \\ &\quad \times \{1 - W(x, x)\}^{1 - b_{11}} \{1 - W(y, y)\}^{1 - b_{22}} \{1 - W(x, y)\}^{1 - b_{12}} dx_1 dy_1 dz_1 dx_2 dy_2 dz_2 \\ &\leq \sum_{b \in \{0,1\}^3} \frac{\alpha^{2+j+j_1-b_{11}-b_{12}-b_{22}}}{(j - j_1 - b_{12})!(j_1 - b_{11} - 2)!(j_1 - b_{22} - 2)!} \mathbb{1}_{j_1 \leq j - b_{12}} \mathbb{1}_{j_1 \geq b_{11}} \mathbb{1}_{j_1 \geq b_{22}} \end{aligned}$$

$$\times \int_{\mathbb{R}_+^6} \mu(x_1)^{j_1-2-b_{11}} \mu(x_2)^{j_1-2-b_{22}} \nu(x_1,x_2)^{j-j_1-b_{12}} e^{-\alpha\mu(x)-\alpha\mu(y)+\alpha\nu(x_1,x_2)}$$
$$\times W(x_1,y_1)W(x_1,z_1)W(y_1,z_1)W(x_2,y_2)W(x_2,z_2)W(y_2,z_2)$$
$$\times W(x_1,x_1)^{b_{11}} W(x_2,x_2)^{b_{22}} W(x_1,x_2)^{b_{12}}$$
$$\times \{1-W(x_1,x_1)\}^{1-b_{11}} \{1-W(x_2,x_2)\}^{1-b_{22}} \{1-W(x_1,x_2)\}^{1-b_{12}} dx_1 dy_1 dz_1 dx_2 dy_2 dz_2 \tag{A.6}$$

For $b_{12} \neq 0$ or $j \neq j_1$, we can bound the terms in the above sum by

$$\frac{\alpha^{2+j+j_1-b_{11}-b_{12}-b_{22}}}{(j-j_1-b_{12})!(j_1-b_{11}-2)!(j_1-b_{22}-2)!} \mathbb{1}_{j_1 \leq j-b_{12}} \mathbb{1}_{j_1 \geq b_{11}} \mathbb{1}_{j_1 \geq b_{22}}$$
$$\times \int_{\mathbb{R}_+^4} \mu(x_1)^{j_1-b_{11}} \mu(x_2)^{j_1-b_{22}} \nu(x_1,x_2)^{j-j_1-b_{12}} e^{-\alpha\mu(x)-\alpha\mu(y)+\alpha\nu(x_1,x_2)}$$
$$\times W(x_1,x_1)^{b_{11}} W(x_2,x_2)^{b_{22}} W(x_1,x_2)^{b_{12}}$$
$$\times \{1-W(x_1,x_1)\}^{1-b_{11}} \{1-W(x_2,x_2)\}^{1-b_{22}} \{1-W(x_1,x_2)\}^{1-b_{12}} dx_1 dx_2$$
$$= O(\alpha^{3+2\sigma-2a} \ell_\sigma(\alpha)^2) \tag{A.7}$$

using the intermediate results of the proof in Section A.2.

Consider now the sum of terms such that $b_{12} = 0$ and $j = j_1$ in (A.6). Using the inequality $e^x \leq 1 + xe^x$, this sum is upper bounded by

$$\sum_{b_{11},b_{12}} \frac{\alpha^{2+2j-b_{11}-b_{22}}}{(j-b_{11}-2)!(j-b_{22}-2)!} \int_{\mathbb{R}_+^6} \mu(x_1)^{j-2-b_{11}} \mu(x_2)^{j-2-b_{22}} e^{-\alpha\mu(x)-\alpha\mu(y)+\alpha\nu(x_1,x_2)}$$
$$\times W(x_1,y_1)W(x_1,z_1)W(y_1,z_1)W(x_2,y_2)W(x_2,z_2)W(y_2,z_2)$$
$$\times W(x_1,x_1)^{b_{11}} W(x_2,x_2)^{b_{22}} \{1-W(x_1,x_1)\}^{1-b_{11}} \{1-W(x_2,x_2)\}^{1-b_{22}} dx_1 dy_1 dz_1 dx_2 dy_2 dz_2$$
$$\leq \sum_{b_{11},b_{12}} \frac{\alpha^{2+2j-b_{11}-b_{22}}}{(j-b_{11}-2)!(j-b_{22}-2)!} \int_{\mathbb{R}_+^6} \mu(x_1)^{j-2-b_{11}} \mu(x_2)^{j-2-b_{22}} e^{-\alpha\mu(x)-\alpha\mu(y)}$$
$$\times W(x_1,y_1)W(x_1,z_1)W(y_1,z_1)W(x_2,y_2)W(x_2,z_2)W(y_2,z_2)$$
$$\times W(x_1,x_1)^{b_{11}} W(x_2,x_2)^{b_{22}} \{1-W(x_1,x_1)\}^{1-b_{11}} \{1-W(x_2,x_2)\}^{1-b_{22}} dx_1 dy_1 dz_1 dx_2 dy_2 dz_2$$
$$+ \sum_{b_{11},b_{12}} \frac{\alpha^{3+2j-b_{11}-b_{22}}}{(j-b_{11}-2)!(j-b_{22}-2)!} \int_{\mathbb{R}_+^6} \mu(x_1)^{j-2-b_{11}} \mu(x_2)^{j-2-b_{22}} \nu(x,y) e^{-\alpha\mu(x)-\alpha\mu(y)+\alpha\nu(x,y)}$$
$$\times W(x_1,y_1)W(x_1,z_1)W(y_1,z_1)W(x_2,y_2)W(x_2,z_2)W(y_2,z_2)$$

$$\times W(x_1,x_1)^{b_{11}}W(x_2,x_2)^{b_{22}}\times\{1-W(x_1,x_1)\}^{1-b_{11}}\{1-W(x_2,x_2)\}^{1-b_{22}}dx_1dy_1dz_1dx_2dy_2dz_2$$

$$\leq 4E(R_{\alpha,j})^2$$

$$+\sum_{b_{11},b_{12}}\frac{\alpha^{3+2j-b_{11}-b_{22}}}{(j-b_{11}-2)!(j-b_{22}-2)!}\int_{\mathbb{R}_+^2}\mu(x_1)^{j-b_{11}}\mu(x_2)^{j-b_{22}}\nu(x,y)e^{-\alpha\mu(x)-\alpha\mu(y)+\alpha\nu(x,y)}$$

$$\times W(x_1,x_1)^{b_{11}}W(x_2,x_2)^{b_{22}}\times\{1-W(x_1,x_1)\}^{1-b_{11}}\{1-W(x_2,x_2)\}^{1-b_{22}}dx_1dx_2$$

$$=4E(R_{\alpha,j})^2+O(\alpha^{3+2\sigma-2a}\ell_\sigma(\alpha)^2)$$

using Lemma A.1 in Section A.2. It follows that

$$E(S_{\alpha,j,1})=E(4R_{\alpha,j})^2+O(\alpha^{3+2\sigma-2a}\ell_\sigma(\alpha)^2) \tag{A.8}$$

Consider now

$$S_{\alpha,j,2}=\sum_{i_1\neq k_1\neq l_1\neq i_2\neq k_2}Z_{i_1k_1}Z_{i_1l_1}Z_{k_1l_1}Z_{i_2k_2}Z_{i_2l_1}Z_{k_2l_1}\mathbb{1}_{D_{\alpha i_1}=j}\mathbb{1}_{D_{\alpha i_2}=j}\mathbb{1}_{\theta_{i_1}\leq\alpha}\mathbb{1}_{\theta_{i_2}\leq\alpha}$$

We have similarly

$$E(S_{\alpha,j,2})=O(\alpha^{3+2\sigma-2a}\ell_\sigma(\alpha)^2) \tag{A.9}$$

using Lemma A.1. Similarly, using Lemma A.1, $E(S_{\alpha,j,3})=O(\alpha^{3+2\sigma-2a}\ell_\sigma(\alpha)^2)$. Combining the above bound with (A.8) and (A.9), we obtain $\mathrm{var}(R_{\alpha,j})=O(\alpha^{3+2\sigma-2a}\ell_\sigma(\alpha)^2)$. We now consider the second bound in Proposition **??**. Consider an increasing sequence $\alpha_n\to\infty$ such that $\alpha_{n+1}-\alpha_n=o(\alpha_n)$ as $n\to\infty$. Let $I_{\alpha_n}=\{i,\,\theta_i\leq\alpha_n\}$ and $I_n^c=I_{\alpha_{n+1}}\backslash I_{\alpha_n}$.

For any $j\geq 1$, let

$$\widetilde{R}_{nj}^{(1)}:=\sum_{i\in I_{\alpha_n}}T_{\alpha_{n+1}i}\mathbb{1}_{D_{\alpha_n i}=j}\mathbb{1}_{\sum_{i'\in I_n^c}Z_{ii'}=1}.$$

We have, similarly to Equation (A.2)

$$(\widetilde{R}_{nj}^{(1)})^2=\sum_{i_1,i_2}^{I_{\alpha_n}}\sum_{k_1\neq l_1\neq i_1}^{I_{\alpha_{n+1}}}\sum_{i_2\neq k_2\neq l_2}^{I_{\alpha_{n+1}}}Z_{i_1k_1}Z_{i_1l_1}Z_{k_1l_1}Z_{i_2k_2}Z_{i_2l_2}Z_{k_2l_2}$$

$$\times\mathbb{1}_{D_{\alpha_n i_1}=j}\mathbb{1}_{D_{\alpha_n i_2}=j}\mathbb{1}_{\sum_{i_1'\in I_n^c}Z_{i_1'i_1}=1}\mathbb{1}_{\sum_{i_2'\in I_n^c}Z_{i_2'i_2}=1}.$$

Hence using the same decomposition as (A.2) together with the fact that $\mathbb{1}_{\sum_{i' \in I_n^c} Z_{i'i}=1} \leq 1$, we derive the same bounds as (A.4), (A.7) and (A.9) so that

$$E((\widetilde{R}_{nj}^{(1)})^2) \lesssim \alpha_n^{\sigma+1}\ell_\sigma(\alpha_n) + \alpha_n^{3+2\sigma-2a}\ell_\sigma(\alpha_n)^2 + E(\widetilde{S}_{\alpha_n,j,1,j})$$

where, writing $b = (b_{11}, b_{22})$,

$$\widetilde{S}_{\alpha_n,j,1,j} = \sum_{b\in\{0,1\}^2} \sum_{i_1 \neq i_2}^{\in I_{\alpha_n}} \sum_{\substack{k_1 \neq l_1 \\ \neq k_2 \neq l_2}}^{\in I_{\alpha_{n+1}} \setminus \{i_1,i_2\}} Z_{i_1k_1}Z_{i_1l_1}Z_{k_1l_1}Z_{i_2k_2}Z_{i_2l_2}Z_{k_2l_2}\mathbb{1}_{D_{\alpha_n i_1}=j}\mathbb{1}_{D_{\alpha_n i_2}=j}$$

$$\times \mathbb{1}_{\sum_{k\in I_{\alpha_{n+1}}} Z_{i_1k}Z_{i_2k}=0}\mathbb{1}_{Z_{i_1i_1}=b_{11}}\mathbb{1}_{Z_{i_2i_2}=b_{22}}\mathbb{1}_{\sum_{i'\in I_n^c} Z_{i'i_1}=1}\mathbb{1}_{\sum_{i'\in I_n^c} Z_{i'i_2}=1},$$

so that $E(\widetilde{S}_{\alpha_n,j,1,j}) = E(\widetilde{R}_{nj}^{(1)})^2$. We thus obtain that $\mathrm{var}(\widetilde{R}_{nj}^{(1)}) = O(\alpha_n^{3+2\sigma-2a}\ell_\sigma(\alpha_n)^2)$.

A.4. Proof of Lemma ??

Let $I_n^c = I_{\alpha_{n+1}} \setminus I_{\alpha_n} = \{i \mid \theta_i \in (\alpha_n, \alpha_{n+1}]\}$. First note that $\widetilde{R}_{nj} = \sum_{r=1}^{j} \widetilde{R}_{nr}^{(1)} + \widetilde{R}_{nr}^{(2)} + \widetilde{R}_{nr}^{(3)}$ where

$$\widetilde{R}_{nr}^{(1)} = \sum_{i\in I_{\alpha_n}} T_{\alpha_{n+1}i}\mathbb{1}_{D_{\alpha_n i}=r}\mathbb{1}_{\sum_{i'\in I_n^c} Z_{ii'}=1},$$

$$\widetilde{R}_{nr}^{(2)} = \sum_{i\in I_{\alpha_{n+1}}} T_{\alpha_{n+1}i}\mathbb{1}_{D_{\alpha_n i}=r}\mathbb{1}_{\sum_{i'\in I_n^c} Z_{ii'}\geq 2},$$

$$\widetilde{R}_{nr}^{(3)} = \sum_{i\in I_n^c} T_{\alpha_{n+1}i}\mathbb{1}_{D_{\alpha_n i}=r}\mathbb{1}_{\sum_{i'\in I_n^c} Z_{ii'}=1}.$$

For any $r \leq j$

$$E\left(\widetilde{R}_{nr}^{(2)}\middle| M\right) \leq \sum_{i\in I_{\alpha_{n+1}}} \sum_{l\neq k}^{\in I_{\alpha_n},\neq i} W(\vartheta_i,\vartheta_l)W(\vartheta_i,\vartheta_k)W(\vartheta_l,\vartheta_k)J_n(i,r-2)\mathbb{P}\left(\sum_{i'\in I_n^c} Z_{ii'} \geq 2|M\right)$$

$$+ 2\sum_{i\in I_{\alpha_{n+1}}} \sum_{l\neq i}^{\in I_{\alpha_n}} \sum_{k\neq i}^{\in I_n^c} W(\vartheta_i,\vartheta_l)W(\vartheta_i,\vartheta_k)W(\vartheta_l,\vartheta_k)J_n(i,r-1)\mathbb{P}\left(\sum_{i'\in I_n^c} Z_{ii'} \geq 1|M\right)$$

$$+ \sum_{i\in I_{\alpha_{n+1}}} \sum_{l\neq k}^{\in I_n^c} W(\vartheta_i,\vartheta_l)W(\vartheta_i,\vartheta_k)W(\vartheta_l,\vartheta_k)J_n(i,r)$$

where, recalling the definition of $g_{\alpha,x}$ in Equation (**??**),

$$J_n(i,r) = \sum_{i_1 \neq i_2 \ldots \neq i_r \neq l \neq k}^{\in I_{\alpha_n}} \left[\prod_{s=1}^{r} W(\vartheta_i, \vartheta_{i_s}) \right] e^{-\sum_{s \neq l,k,i_1,\ldots,i_r}^{I_{\alpha_n}} g_{\alpha,\vartheta_i}(\theta_s,\vartheta_s)}.$$

Note that

$$\mathbb{P}\left(\sum_{i' \in I_n^c} Z_{ii'} \geq 2 | M \right) = 1 - e^{-\sum_{s \in I_n^c} g_{\alpha_{n+1}-\alpha_n,\vartheta_i}(\theta_s,\vartheta_s)} - \sum_{i' \in I_n^c} W(\vartheta_i, \vartheta_{i'}) e^{-\sum_{s \neq i'}^{I_n^c} g_{\alpha_{n+1}-\alpha_n,\vartheta_i}(\theta_s,\vartheta_s)}.$$

Using the Slivnyak-Mecke theorem, the inequality $1 - e^{-y} - ye^{-y} \leq y^2$ for $y \geq 0$, the condition (**??**) and Lemma **??**, we obtain

$$E\left(\widetilde{R}_{nr}^{(2)}\right) \lesssim \alpha_{n+1} \alpha_n^r (\alpha_{n+1} - \alpha_n)^2 \int L_2(x) \mu(x)^{r+2} e^{-\alpha_n \mu(x)} dx \lesssim \alpha_{n+1} \alpha_n^{\sigma-2} (\alpha_{n+1} - \alpha_n)^2 \ell(\alpha_n),$$

where $L_2(x)$ converges to $b \geq 0$ at infinity. Noting that $(\alpha_{n+1} - \alpha_n)/\alpha_n = O(1/n)$, we obtain $E\left(\widetilde{R}_{nr}^{(2)}\right) \lesssim \alpha_n^{\sigma+1} \ell(\alpha_n)/n^2$. This implies that $\sum_n E\left(\widetilde{R}_{nr}^{(2)}\right) / (\alpha_n^{\sigma+1} \ell(\alpha_n)) < +\infty$ so that, by Markov inequality and Borel-Cantelli lemma, $\widetilde{R}_{nr}^{(2)} = o(\alpha_n^{\sigma+1} \ell(\alpha_n))$ almost surely as n tends to infinity.

We now study

$$\widetilde{R}_{nr}^{(3)} := \sum_{i \in I_n^c} T_{\alpha_{n+1} i} \mathbb{1}_{D_{\alpha_n i} = r} \mathbb{1}_{\sum_{i' \in I_n^c} Z_{ii'} = 1}.$$

Similarly to before

$$\begin{aligned} E\left(\widetilde{R}_{nr}^{(3)} \mid M\right) \leq & \sum_{i \in I_n^c} \sum_{l \neq k}^{\in I_{\alpha_n}} W(\vartheta_i, \vartheta_l) W(\vartheta_i, \vartheta_k) W(\vartheta_l, \vartheta_k) J_n(i, r-2) \mathbb{P}\left(\sum_{i' \in I_n^c} Z_{ii'} = 1 | M \right) \\ & + 2 \sum_{l \in I_n^c}^{l \neq i} \sum_{k \in I_{\alpha_n}} W(\vartheta_i, \vartheta_l) W(\vartheta_i, \vartheta_k) W(\vartheta_l, \vartheta_k) J_n(i, r-1) \end{aligned}$$

so that

$$E\left(\widetilde{R}_{nr}^{(3)}\right) \lesssim (\alpha_{n+1} - \alpha_n)^2 \alpha_n^r \int L_3(x) \mu(x)^{r+1} e^{-\alpha_n \mu(x)} dx \lesssim \frac{\alpha_n^{\sigma+1} \ell(\alpha_n)}{n^2},$$

where $L_3(x)$ converges to b and $\widetilde{R}_{nr}^{(3)} = o(\alpha_n^{\sigma+1} \ell(\alpha_n))$ almost surely as n tends to infinity. Finally, we have

$$
\begin{aligned}
E\left(\widetilde{R}_{nr}^{(1)}|M\right) \lesssim & \sum_{i\neq l\neq k}^{\in I_{\alpha_n}} W(\vartheta_i,\vartheta_l)W(\vartheta_i,\vartheta_k)W(\vartheta_l,\vartheta_k)J_n(i,r-2)\sum_{i'\in I_n^c} W(\vartheta_i,\vartheta_{i'}) \\
& + 2\sum_{i\neq l}^{\in I_{\alpha_n}}\sum_{k\in I_n^c} W(\vartheta_i,\vartheta_l)W(\vartheta_i,\vartheta_k)W(\vartheta_l,\vartheta_k)J_n(i,r-1)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
E\left(\frac{\widetilde{R}_{nr}^{(1)}}{\alpha_n^{\sigma+1}\ell_\sigma(\alpha_n)}\right) & \lesssim \frac{\alpha_n^{r+1}(\alpha_{n+1}-\alpha_n)}{\alpha_n^{\sigma+1}\ell_\sigma(\alpha_n)}\int_{\mathbb{R}_+^4} L_1(x)\mu(x)^{r+1}e^{-\alpha_n\mu(x)}dx \\
& = O\left(\frac{\alpha_{n+1}-\alpha_n}{\alpha_n}\right) = o(1).
\end{aligned}
$$

where $L_1(x)$ converges to b. Moreover, from Proposition **??**

$$
\operatorname{var}\left(\frac{\widetilde{R}_{nr}^{(1)}}{\alpha_n^{\sigma+1}\ell_\sigma(\alpha_n)}\right) = O\left(\alpha_n^{1-2a}\right)
$$

so that, $\widetilde{R}_{nr}^{(1)} = o(\alpha_n^{1+\sigma}\ell(\alpha_n))$ almost surely. It finally follows that, for any $j\geq 1$, $\widetilde{R}_{nj} = o(\alpha_n^{1+\sigma}\ell(\alpha_n))$ almost surely as n tends to infinity.

Appendix B. Proof of secondary propositions for the Central Limit Theorem

B.1. Proof of Proposition ??

Let

$$
Z_\alpha := N_\alpha - E(N_\alpha \mid M) = \sum_i \mathbb{1}_{\theta_i\leq\alpha}(\mathbb{1}_{D_{\alpha,i}\geq 1} - (1-e^{-M(g_{\alpha,\vartheta_i})}))
$$

where we recall that $g_{\alpha,x}(\theta,\vartheta) = -\log(1-W(x,\vartheta))\mathbb{1}_{\theta\leq\alpha}$ and

$$
e^{-M(g_{\alpha,\vartheta_i})} = e^{-\sum_j -\log(1-W(\vartheta_i,\vartheta_j))\mathbb{1}_{\theta_j\leq\alpha}} = \prod_j (1-W(\vartheta_i,\vartheta_j))^{\mathbb{1}_{\theta_j\leq\alpha}}
$$

We have $E(Z_\alpha \mid M) = 0$ hence $\operatorname{var}(Z_\alpha) = E(Z_\alpha^2)$. Note that

$$
Z_\alpha^2 = Z_\alpha + \sum_{i_1\neq i_2} \mathbb{1}_{\theta_{i_1}\leq\alpha}(\mathbb{1}_{D_{\alpha,i_1}\geq 1} - (1-e^{-M(g_{\alpha,\vartheta_{i_1}})}))\mathbb{1}_{\theta_{i_2}\leq\alpha}(\mathbb{1}_{D_{\alpha,i_2}\geq 1} - (1-e^{-M(g_{\alpha,\vartheta_{i_2}})}))
$$

$$= Z_\alpha + \sum_{i_1\neq i_2} \mathbb{1}_{\theta_{i_1}\leq\alpha}\mathbb{1}_{D_{\alpha,i_1}\geq 1}\mathbb{1}_{\theta_{i_2}\leq\alpha}\mathbb{1}_{D_{\alpha,i_2}\geq 1} - \sum_{i_1\neq i_2} \mathbb{1}_{\theta_{i_1}\leq\alpha}\mathbb{1}_{D_{\alpha,i_1}\geq 1}\mathbb{1}_{\theta_{i_2}\leq\alpha}(1-e^{-M(g_{\alpha,\vartheta_{i_2}})})$$
$$- \sum_{i_1\neq i_2} \mathbb{1}_{\theta_{i_1}\leq\alpha}(1-e^{-M(g_{\alpha,\vartheta_{i_1}})}))\mathbb{1}_{\theta_{i_2}\leq\alpha}\mathbb{1}_{D_{\alpha,i_2}\geq 1} + \sum_{i_1\neq i_2} \mathbb{1}_{\theta_{i_1}\leq\alpha}(1-e^{-M(g_{\alpha,\vartheta_{i_1}})})\mathbb{1}_{\theta_{i_2}\leq\alpha}(1-e^{-M(g_{\alpha,\vartheta_{i_2}})})$$

We have

$$E\left(Z_\alpha^2 \mid M\right) = \sum_{i_1\neq i_2} \mathbb{1}_{\theta_{i_1}\leq\alpha}\mathbb{1}_{\theta_{i_2}\leq\alpha} e^{-M(g_{\alpha,\vartheta_{i_1}})-M(g_{\alpha,\vartheta_{i_2}})}(e^{g_{\alpha,\vartheta_{i_1}}(\theta_{i_2},\vartheta_{i_2})}-1)$$

Applying the extended Slivnyak-Mecke theorem

$$E\left(Z_\alpha^2\right) = \alpha^2\int_{\mathbb{R}_+^2}(1-W(x,x))(1-W(y,y))(1-W(x,y))e^{-\alpha\mu(x)-\alpha\mu(y)+\alpha\nu(x,y)}(1-(1-W(x,y)))dxdy$$
$$\leq \alpha^2\int_{\mathbb{R}_+^2} W(x,y)e^{-\alpha\mu(x)-\alpha\mu(y)+\alpha\nu(x,y)}dxdy.$$

Using Cauchy-Schwarz inequality, $\nu(x,y)\leq\sqrt{\mu(x)\mu(y)}\leq\frac{1}{2}(\mu(x)+\mu(y))$, and Lemma **??**, we obtain

$$E\left(Z_\alpha^2\right) = \alpha^2\int_0^\infty \mu(x)e^{-\alpha/2\mu(x)}dx \asymp \alpha^{1+\sigma}\ell(\alpha) = \begin{cases} O(\alpha^{1+\sigma}\ell_\sigma(\alpha)) & \sigma\in[0,1) \\ o(\alpha\ell(\alpha)) & \sigma=0 \end{cases}.$$

It follows from Markov's inequality that, in probability

$$Z_\alpha = \begin{cases} O(\alpha^{1/2+\sigma/2}\ell_\sigma^{1/2}(\alpha)) & \sigma\in[0,1) \\ o(\alpha^{1/2}\ell^{1/2}(\alpha)) & \sigma=0 \end{cases}.$$

B.2. Proof of Proposition ??

Define $M(h_\alpha)=\sum_i \widetilde{Z}_i$ where $\widetilde{Z}_i = h_\alpha(\theta_i,\vartheta_i) = \mathbb{1}_{\theta_i\leq\alpha}\left[1-(1-W(\vartheta_i,\vartheta_i))e^{-\alpha\mu(\vartheta_i)}\right]$.

Using Campbell's formula

$$E\left(\sum_i \widetilde{Z}_i\right) = \alpha\int_0^\infty(1-(1-W(x,x))e^{-\alpha\mu(x)})dx = E(N_\alpha)$$
$$\operatorname{var}\left(\sum_i \widetilde{Z}_i\right) = \alpha\int_0^\infty\left[1-(1-W(x,x))e^{-\alpha\mu(x)}\right]^2dx \leq E(N_\alpha)$$

Noting that $E(N_\alpha)\sim\alpha^{1+\sigma}\Gamma(1-\sigma)\ell(\alpha)$, it follows from Chebyshev's inequality that, in probability,

$$\sum_i \widetilde{Z}_i - E(N_\alpha) = O\left(\sqrt{\operatorname{var}\left(\sum_i \widetilde{Z}_i\right)}\right) = O\left(\alpha^{1/2+\sigma/2}\ell_\sigma^{1/2}(\alpha)\right)$$

If μ has an unbounded support then, under Assumption **??**, either $\sigma > 0$ or $\sigma = 0$ and $\ell(t) \to \infty$. In both cases, in probability, $\sum_i \widetilde{Z}_i - E(N_\alpha) = o\left(\alpha^{1/2+\sigma}\ell_\sigma(\alpha)\right)$.

B.3. Proof of Proposition ??

Let

$$f_\alpha(M) = \sum_i \mathbb{1}_{\theta_i \leq \alpha}\left[(1 - W(\vartheta_i, \vartheta_i))e^{-\alpha\mu(\vartheta_i)} - e^{-M(g_{\alpha,\vartheta_i})}\right]$$

The idea is to use Theorem 1.1 from [2]. To do so, define

$$F_\alpha = \frac{f_\alpha(M)}{\sqrt{v_\alpha}} \tag{B.1}$$

where $v_\alpha = \mathrm{var}(f_\alpha(M)) \sim \mathrm{var}(N_\alpha) \asymp \alpha^{1+2\sigma}\ell_\sigma^2(\alpha)$. Note that $E(F_\alpha) = 0$ and $\mathrm{var}(F_\alpha) = 1$. Consider the difference operator $D_z F_\alpha$ defined by

$$D_z F_\alpha = \frac{1}{\sqrt{v_\alpha}}(f_\alpha(M + \delta_z) - f_\alpha(M))$$

Also

$$\begin{aligned} D^2_{z_1,z_2}F_\alpha &= D_{z_2}(D_{z_1}F_\alpha) = D_{z_2}\left(\frac{1}{\sqrt{v_\alpha}}(f_\alpha(M + \delta_{z_1}) - f_\alpha(M))\right) \\ &= \frac{1}{\sqrt{v_\alpha}}\left(f_\alpha(M + \delta_{z_1} + \delta_{z_2}) - f_\alpha(M + \delta_{z_1}) - f_\alpha(M + \delta_{z_2}) + f_\alpha(M)\right). \end{aligned}$$

Define

$$\gamma_{\alpha,1} := 2\left(\int_{\mathbb{R}_+^6} \sqrt{\mathbb{E}(D_{z_1}F_\alpha)^2(D_{z_2}F_\alpha)^2}\sqrt{\mathbb{E}(D^2_{z_1,z_3}F_\alpha)^2(D^2_{z_2,z_3}F_\alpha)^2}dz_1dz_2dz_3\right)^{1/2}$$

$$\gamma_{\alpha,2} := \left(\int_{\mathbb{R}_+^6} \mathbb{E}\left[(D^2_{z_1,z_3}F_\alpha)^2(D^2_{z_2,z_3}F_\alpha)^2\right]dz_1dz_2dz_3\right)^{1/2}$$

$$\gamma_{\alpha,3} := \int_{\mathbb{R}_+^2} \mathbb{E}|D_zF_\alpha|^3 dz$$

We state a corollary of Theorem 1.1 from [2].

Corollary B.1. *[2, Theorem 1.1] If $\gamma_{\alpha,1}, \gamma_{\alpha,2}, \gamma_{\alpha,3} \to 0$, then*

$$F_\alpha = \frac{f_\alpha(M)}{\sqrt{v_\alpha}} \to \mathcal{N}(0,1).$$

The rest of the proof aims to show that $\gamma_{\alpha,1}, \gamma_{\alpha,2}, \gamma_{\alpha,3} \to 0$. The proof is rather lengthy and therefore split in different subsections. We first state a few notations and lemmas that will be useful in the following.

B.3.1. *Definitions and lemmas* The following lemma, obtained with Hölder's inequality, will be used multiple times.

Lemma B.1. *For any $d \geq 1$ and any $z_1, \ldots, z_d > 0$,*

$$E\left(\prod_{k=1}^{d} e^{-M(g_{\alpha,z_k})}\right) \leq e^{-\frac{\alpha}{d}\sum_{k=1}^{d}\mu(z_k)}.$$

Proof. Using Hölder's inequality, for any $d \geq 1$

$$\begin{aligned}
E\left(\prod_{k=1}^{d} e^{-M(g_{\alpha,z_k})}\right) &\leq \prod_{k=1}^{d} E\left(e^{-dM(g_{\alpha,z_k})}\right)^{1/d} = \prod_{k=1}^{d} e^{-\frac{\alpha}{d}\int_0^\infty (1-[1-W(z_k,y)]^d)dy} \\
&\leq \prod_{k=1}^{d} e^{-\frac{\alpha}{d}\int_0^\infty (1-[1-W(z_k,y)])dy} = e^{-\frac{\alpha}{d}\sum_{k=1}^{d}\mu(z_k)}
\end{aligned}$$

□

For $i, j \geq 0$, let

$$H_{i,j}(x_1,x_2) = \int_{\mathbb{R}_+^2} W(x_1,y)^i W(x_2,y)^j e^{-\frac{\alpha}{4}\mu(y)}dy, \quad H_i(x) = H_{i,0}(x,x). \quad \text{(B.2)}$$

The following lemma compiles various useful bounds.

Lemma B.2. *Assume Assumptions 1 and 5. Then*

- *For all $j \geq 1$ and all $x_1, \ldots, x_{j-1} > 0$, $y_1 > 0$ and $p_1, \ldots, p_j \geq 1$, $\alpha > 0$*

$$\begin{aligned}
&\int_0^\infty \left(\int_0^\infty W(y_1,x_j)^{p_j}\left[\prod_{k=1}^{j-1} W(y_1,x_k)^{p_k}\right] e^{-\alpha\mu(y_1)}dy_1\right)^{1/2} W(y_2,x_j)dx_j \\
&\leq L(y_2)\mu(y_2)\left(\int L(y_1)^{p_j}\mu(y_1)^{p_j}\left[\prod_{k=1}^{j-1} W(y_1,x_k)^{p_k}\right] e^{-\alpha\mu(y_1)}dy_1\right)^{1/2}
\end{aligned}$$

- *For any $x_2 > 0$,*

$$\int H_{1,1}(x_1,x_2)^2 dx_1 \leq \int L(y_1)L(y_2)\mu(y_1)\mu(y_2)W(x_2,y_1)W(x_2,y_2)e^{-\frac{\alpha}{4}(\mu(y_1)+\mu(y_2))}dy_1dy_2 \quad \text{(B.3)}$$

- *For any* $q = 1, 2, \dots$

$$\int_0^\infty H_1(x)^q dx = \int \prod_{i=1}^q \int W(x, y_i) e^{-\frac{\alpha}{4}\mu(y_i)} dy_i dx \quad = O(\alpha^{q\sigma - q} \ell_\sigma(\alpha)^q) \tag{B.4}$$

- *For any* $q \geq 1$*,* $p \geq 1$*,*

$$\int (\alpha H_1(x_1))^{q/2} L(x_1) \mu(x_1)^{p/2} e^{-\frac{\alpha}{4}\mu(x_1)} dx_1 = O(\alpha^{(q+1)\sigma/2 - p/2} \ell_\sigma(\alpha)^{(q+1)/2}) \tag{B.5}$$

- *If* $q \leq 3$*,*

$$\int L(x_1)\mu(x_1)^p W(x_1, x_2)^q e^{-\frac{\alpha}{4}\mu(x_1)} dx_1 \leq \mu(x_2)^q L(x_2)^q \left(\int L(x_1)^2 \mu(x_1)^{2p} e^{-\frac{\alpha}{2}\mu(x_1)} dx_1 \right)^2. \tag{B.6}$$

Proof. The first inequality comes from Hölder's inequality, together with Assumptions **??** and **??**. Also (B.3) is a consequence of

$$\int H_{1,1}(x_1, x_2)^2 dx_1 = \int \nu(y_1, y_2) W(x_2, y_1) W(x_2, y_2) e^{-\frac{\alpha}{4}(\mu(y_1) + \mu(y_2))} dy_1 dy_2$$

Under Assumption **??**, For any $q = 1, 2, \dots$, using Assumption **??**, **??**, and Lemma **??**,

$$\begin{aligned} \int_0^\infty H_1(x)^q dx &= \int \prod_{i=1}^q \int W(x, y_i) e^{-\frac{\alpha}{4}\mu(y_i)} dy_i dx \\ &\leq \prod_{i=1}^q \int L(y_i)\mu(y_i) e^{-\frac{\alpha}{4}\mu(y_i)} dy_i = O(\alpha^{q\sigma - q} \ell_\sigma(\alpha)^q). \end{aligned}$$

Using Hölder's inequality,

$$\begin{aligned} \int (\alpha H_1(x_1))^{q/2} L(x_1) \mu(x_1)^{p/2} e^{-\frac{\alpha}{4}\mu(x_1)} dx_1 &\leq \left(\int (\alpha H_1(x_1))^q dx_1 \int L(x_1)^2 \mu(x_1)^p e^{-\frac{\alpha}{2}\mu(x_1)} dx_1 \right)^{1/2} \\ &= O(\alpha^{(q+1)\sigma/2 - p/2} \ell_\sigma(\alpha)^{(q+1)/2}) \end{aligned}$$

which proves (B.5). Finally recall that from Lemma **??** that for any $p \geq 1$,

$$\int L(x_1) \mu(x_1)^p e^{-\frac{\alpha}{4}\mu(x_1)} dx_1 = O(\alpha^{\sigma - p} \ell_\sigma(\alpha)).$$

so that using Hölder and Assumption **??**, for any $q \leq 3$

$$\int L(x_1)\mu(x_1)^p W(x_1,x_2)^q e^{-\frac{\alpha}{4}\mu(x_1)}dx_1 \leq \mu(x_2)^q L(x_2)^q \left(\int L(x_1)^2\mu(x_1)^{2p}e^{-\frac{\alpha}{2}\mu(x_1)}dx_1\right)^2$$

□

B.3.2. *General bounds* Let $z=(t,x)$. Recall that $g_{\alpha,x}(\theta,\vartheta) = -\log(1-W(x,\vartheta))\mathbb{1}_{\theta\leq\alpha}$.

$$\sqrt{v_\alpha}\times D_zF_\alpha = \mathbb{1}_{t\leq\alpha}(1-W(x,x))\left[e^{-\alpha\mu(x)} - e^{-M(g_{\alpha,x})}\right] + \mathbb{1}_{t\leq\alpha}\sum_i \mathbb{1}_{\theta_i\leq\alpha}W(\vartheta_i,x)e^{-M(g_{\alpha,\vartheta_i})}.$$

We have

$$\sqrt{v_\alpha}\,|D_zF_\alpha| \leq \mathbb{1}_{t\leq\alpha}\left(\left|e^{-\alpha\mu(x)} - e^{-M(g_{\alpha,x})}\right| + \sum_i \mathbb{1}_{\theta_i\leq\alpha}W(\vartheta_i,x)e^{-M(g_{\alpha,\vartheta_i})}\right) \tag{B.7}$$

Similarly,

$$\begin{aligned}\sqrt{v_\alpha}D^2_{z_1,z_2}(F_\alpha) &= \mathbb{1}_{t_1,t_2\leq\alpha}W(x_1,x_2)\left[(1-W(x_1,x_1))e^{-M(g_{\alpha,x_1})} + (1-W(x_2,x_2))e^{-M(g_{\alpha,x_2})}\right]\\ &\quad - \mathbb{1}_{t_1,t_2\leq\alpha}\sum_i \mathbb{1}_{\theta_i\leq\alpha}W(\vartheta_i,x_1)W(\vartheta_i,x_2)e^{-M(g_{\alpha,\vartheta_i})}\end{aligned}$$

Note that the above is equal to 0 if $t_1>\alpha$ or $t_2>\alpha$. For $t_1,t_2\leq\alpha$

$$\begin{aligned}|\sqrt{v_\alpha}\times D^2_{z_1,z_2}F_\alpha|^2 &\leq 2W(x_1,x_2)^2(e^{-M(g_{\alpha,x_1})} + e^{-M(g_{\alpha,x_2})})^2\\ &\quad + 2\left(\sum_i \mathbb{1}_{\theta_i\leq\alpha}W(\vartheta_i,x_1)W(\vartheta_i,x_2)e^{-M(g_{\alpha,\vartheta_i})}\right)^2\\ &\leq 4W(x_1,x_2)^2(e^{-M(g_{\alpha,x_1})} + e^{-M(g_{\alpha,x_2})}) + 2\sum_i \mathbb{1}_{\theta_i\leq\alpha}W(\vartheta_i,x_1)^2W(\vartheta_i,x_2)^2e^{-2M(g_{\alpha,\vartheta_i})}\\ &\quad + 2\sum_{i\neq j}\mathbb{1}_{\theta_i\leq\alpha}\mathbb{1}_{\theta_j\leq\alpha}W(\vartheta_i,x_1)W(\vartheta_i,x_2)W(\vartheta_j,x_1)W(\vartheta_j,x_2)e^{-M(g_{\alpha,\vartheta_i})-M(g_{\alpha,\vartheta_j})}\end{aligned}$$

$$\begin{aligned}&v_\alpha^2(D^2_{z_1,z_3}F_\alpha)^2(D^2_{z_2,z_3}F_\alpha)^2\\ &\leq 16W(x_1,x_3)^2W(x_2,x_3)^2(e^{-M(g_{\alpha,x_1})} + e^{-M(g_{\alpha,x_3})})(e^{-M(g_{\alpha,x_2})} + e^{-M(g_{\alpha,x_3})})\\ &\quad + 8W(x_1,x_3)^2(e^{-M(g_{\alpha,x_1})} + e^{-M(g_{\alpha,x_3})})\left(\sum_i \mathbb{1}_{\theta_i\leq\alpha}W(\vartheta_i,x_2)W(\vartheta_i,x_3)e^{-M(g_{\alpha,\vartheta_i})}\right)^2\end{aligned}$$

$$+ 8W(x_2,x_3)^2(e^{-M(g_{\alpha,x_2})}+e^{-M(g_{\alpha,x_3})})\left(\sum_i \mathbb{1}_{\theta_i\leq\alpha}W(\vartheta_i,x_1)W(\vartheta_i,x_3)e^{-M(g_{\alpha,\vartheta_i})}\right)^2$$

$$+ 4\sum_{i_1,i_2,i_3,i_4}\mathbb{1}_{\theta_{i_1}\leq\alpha}\mathbb{1}_{\theta_{i_2}\leq\alpha}\mathbb{1}_{\theta_{i_3}\leq\alpha}\mathbb{1}_{\theta_{i_4}\leq\alpha}W(\vartheta_{i_1},x_1)W(\vartheta_{i_1},x_3)W(\vartheta_{i_2},x_1)W(\vartheta_{i_2},x_3)$$

$$\times W(\vartheta_{i_3},x_2)W(\vartheta_{i_3},x_3)W(\vartheta_{i_4},x_2)W(\vartheta_{i_4},x_3)e^{-\sum_{k=1}^4 M(g_{\alpha,\vartheta_{i_k}})}$$

We obtain, using the inequality (B.3)

$$\begin{aligned}
&E\left(v_\alpha^2(D^2_{z_1,z_3}F)^2(D^2_{z_2,z_3}F)^2\right)\\
&\leq C\times\Big(W(x_1,x_3)^2W(x_2,x_3)^2(e^{-\alpha/2\mu(x_1))}+e^{-\alpha/2\mu(x_3)})(e^{-\alpha/2\mu(x_2)}+e^{-\alpha/2\mu(x_3)})\\
&+(\alpha^2H_{1,1}(x_2,x_3)^2+\alpha H_{2,2}(x_2,x_3))W(x_1,x_3)^2(e^{-\alpha/3\mu(x_1)}+e^{-\alpha/3\mu(x_3))})\\
&+(\alpha^2H_{1,1}(x_1,x_3)^2+\alpha H_{2,2}(x_1,x_3))W(x_2,x_3)^2(e^{-\alpha/3\mu(x_2)}+e^{-\alpha/3\mu(x_3))})\\
&+A_2(x_1,x_2,x_3)\Big)
\end{aligned}\tag{B.8}$$

for some constant $C>0$, where

$$\begin{aligned}
&A_2(x_1,x_2,x_3)\\
&=E\left(\sum_{i_1,i_2,i_3,i_4}\prod_{\ell=1}^4 W(\vartheta_{i_\ell},x_3)\mathbb{1}_{\theta_{i_\ell}\leq\alpha}W(\vartheta_{i_1},x_1)W(\vartheta_{i_2},x_1)W(\vartheta_{i_3},x_2)W(\vartheta_{i_4},x_2)e^{-\sum_{k=1}^4 M(g_{\alpha,\vartheta_{i_k}})}\right)\\
&=\alpha^4\int\prod_{\ell=1}^4 W(y_\ell,x_3)W(y_1,x_1)W(y_2,x_1)W(y_2,x_3)W(y_3,x_2)W(y_4,x_2)e^{-\alpha/4(\sum_{i=1}^4\mu(y_i))}dy_{1:4}\\
&\quad+\alpha^3\int(W(y_1,x_1)^2W(y_1,x_3)^2W(y_2,x_2)W(y_2,x_3)W(y_3,x_2)W(y_3,x_3)\\
&\qquad+W(y_1,x_2)^2W(y_1,x_3)^2W(y_2,x_1)W(y_2,x_3)W(y_3,x_1)W(y_3,x_3))e^{-\alpha/3(\sum_{i=1}^3\mu(y_i))}dy_{1:3}\\
&\quad+\alpha^2\int(W(y_1,x_1)^2W(y_1,x_3)^2W(y_2,x_2)^2W(y_2,x_3)^2\\
&\qquad+W(y_1,x_2)W(y_1,x_1)W(y_1,x_3)^2W(y_2,x_1)W(y_2,x_2)W(y_2,x_3)^2)e^{-\alpha/2(\sum_{i=1}^2\mu(y_i))}dy_{1:2}\\
&\quad+\alpha\int W(y_1,x_1)^2W(y_1,x_2)^2W(y_1,x_3)^4e^{-\alpha\mu(y_1)}dy_1\Bigg)
\end{aligned}$$

B.3.3. *Proof that* $\gamma_{\alpha,3}\to 0$ We show here that $\gamma_{\alpha,3}\to 0$, or equivalently $\int E\left|\sqrt{v_\alpha}D_zF\right|^3dz = o(\alpha^{3/2+3\sigma}\ell_\sigma^3(\alpha))$. From Equation (B.7) and using the inequality $(a+b)^3\leq$

$4(a^3+b^3)$ for any $a,b\geq 0$, a sufficient condition is

$$\int_0^\infty E\left[\left|e^{-\alpha\mu(x)}-e^{-M(g_{\alpha,x})}\right|^3+\left(\sum_i \mathbb{1}_{\theta_i\leq\alpha}W(\vartheta_i,x)e^{-M(g_{\alpha,\vartheta_i})}\right)^3\right]dx=o(\alpha^{1/2+3\sigma}\ell_\sigma^3(\alpha)).$$

We have

$$\begin{aligned}\int_0^\infty \mathbb{E}\left[\left|e^{-\alpha\mu(x)}-e^{-M(g_{\alpha,x})}\right|^3\right]dx &\leq 2\int E\left(\left(e^{-\alpha\mu(x)}-e^{-M(g_{\alpha,x})}\right)^2\right)dx\\ &\leq \int(1-e^{-2\alpha\mu(x)})dx=O(\alpha^\sigma\ell_\sigma(\alpha))\end{aligned}$$

Also under Assumptions **??** and **??**, using Lemma B.1

$$\int_0^\infty E\left[\left(\sum_i \mathbb{1}_{\theta_i\leq\alpha}W(\vartheta_i,x)e^{-M(g_{\alpha,\vartheta_i})}\right)^3\right]dx\leq\int_0^\infty E\left[\sum_{i_1,i_2,i_3}\prod_{\ell=1}^3\mathbb{1}_{\theta_{i_\ell}\leq\alpha}W(\vartheta_{i_\ell},x)e^{-M(g_{\alpha,\vartheta_{i_\ell}})}\right]dx$$

$$\leq\alpha^3\left(\int L(y)\mu(y)e^{-\alpha\mu(y)/3}dy\right)^3+3\alpha^2\left(\int L(y)^2\mu(y)^2e^{-\alpha\mu(y)/3}dy\right)\left(\int L(y)\mu(y)e^{-\alpha\mu(y)/3}dy\right)$$

$$+\alpha\int L(y)^3\mu(y)^3e^{-\alpha\mu(y)/3}dy=O(\alpha^{3\sigma}\ell_\sigma(\alpha)^3).$$

It follows that $\gamma_{\alpha,3}\to 0$ as $\alpha\to\infty$.

B.3.4. *Proof that* $\gamma_{\alpha,2}\to 0$ We now need to show that the integral of the right hand-side of Equation (B.8) with respect to x_1,x_2,x_3 is $o(\alpha^{-3}v_\alpha^2)=o(\alpha^{-1+4\sigma}\ell_\sigma^2(\alpha))$. For the first term in the right hand-side of the inequality (B.8), we have

$$\int W(x_1,x_3)^2W(x_2,x_3)^2(e^{-\frac{\alpha}{2}\mu(x_1))}+e^{-\frac{\alpha}{2}\mu(x_3)})(e^{-\frac{\alpha}{2}\mu(x_2)}+e^{-\frac{\alpha}{2}\mu(x_3)})dx_1dx_2dx_3$$

$$\leq 3\int W(x_1,x_3)W(x_2,x_3)(e^{-\frac{\alpha}{2}(\mu(x_1)+\mu(x_2))}+e^{-\frac{\alpha}{2}\mu(x_3)})dx_1dx_2dx_3$$

$$\leq 3\int\nu(x_1,x_2)e^{-\frac{\alpha}{2}(\mu(x_1)+\mu(x_2))}dx_1dx_2+3\int\mu(x_3)^2e^{-\frac{\alpha}{2}\mu(x_3)}dx_3$$

$$=O(\alpha^{2\sigma-2}\ell_\sigma^2(\alpha))+O(\alpha^{\sigma-2}\ell_\sigma(\alpha))$$

For the second line (and similarly for the third line) in the RHS of Equation (B.8), we have, noting that $H_{2,2}(x_2,x_3)\leq H_{1,1}(x_2,x_3)$

$$\int(\alpha^2H_{1,1}(x_2,x_3)^2+\alpha H_{1,1}(x_2,x_3))W(x_1,x_3)^2(e^{-\alpha/3\mu(x_1)}+e^{-\alpha/3\mu(x_3))})dx_1dx_2dx_3$$

$$\leq \alpha^2 \int W(x_1,x_3)^2 W(y_1,x_2)W(y_1,x_3)W(y_2,x_2)W(y_2,x_3)$$
$$\times (e^{-\alpha/3(\mu(y_1)+\mu(y_2)+\mu(x_1))} + e^{-\alpha/3(\mu(y_1)+\mu(y_2)+\mu(x_3))})dx_1dx_2dx_3dy_1dy_2$$
$$+\alpha \int W(x_1,x_3)^2 W(y_1,x_2)W(y_1,x_3)(e^{-\alpha/3(\mu(y_1)+\mu(x_1))} + e^{-\alpha/3(\mu(y_1)+\mu(x_3))})dx_1dx_2dx_3dy_1$$
$$= \alpha^2 \int W(x_1,x_3)^2 \nu(y_1,y_2)W(y_1,x_3)W(y_2,x_3)$$
$$\times (e^{-\alpha/3(\mu(y_1)+\mu(y_2)+\mu(x_1))} + e^{-\alpha/3(\mu(y_1)+\mu(y_2)+\mu(x_3))})dx_1dx_3dy_1dy_2$$
$$+\alpha \int W(x_1,x_3)^2 \mu(y_1)W(y_1,x_3)(e^{-\alpha/3(\mu(y_1)+\mu(x_1))} + e^{-\alpha/3(\mu(y_1)+\mu(x_3))})dx_1dx_3dy_1$$
$$\leq 2\alpha^2 \int L(x_1)^2 L(y_1)L(y_2)\mu(x_1)^2\mu(y_1)\mu(y_2)e^{-\alpha/3(\mu(y_1)+\mu(y_2)+\mu(x_1))}dx_1dy_1dy_2$$
$$+\alpha \int \mu(y_1)^2 L(y_1)L(x_1)^2\mu(x_1)^2 e^{-\alpha/3(\mu(y_1)+\mu(x_1))}dx_1dy_1 + \alpha \int \mu(x_3)^2 L(x_3)^2 \mu(y_1)e^{-\alpha/3(\mu(y_1)+\mu(x_3))}dx_3dy_1$$
$$= O(\alpha^{3\sigma-2}\ell_\sigma(\alpha)^3)$$

using Assumption **??** and Lemma **??**. For the third term in the right-handside of Equation (B.8), we obtain

$$\int A_2(x_1,x_2,x_3)dx_1dx_2dx_3$$
$$\leq \alpha^4 \left(\int L(y)^2\mu(y)^2 e^{-\alpha/4\mu(y)}dy\right)^4 + \alpha^3 \int L(y_1)^4\mu(y_1)^4 e^{-\alpha/3\mu(y_1)}dy_1 \left(\int L(y)^2\mu(y)^2 e^{-\alpha/3\mu(y)}dy\right)^2$$
$$+\alpha^2 \left(\int L(y)^4\mu(y)^4 e^{-\alpha/2\mu(y)}dy\right)^2 + \alpha \int L(y)^8\mu(y)^8 e^{-\alpha\mu(y)}dy = O(\alpha^{4\sigma-4})\ell_\sigma^4(\alpha)$$

It follows that $\gamma_{\alpha,2} \to 0$ as $\alpha \to \infty$.

B.3.5. *Proof that* $\gamma_{\alpha,1} \to 0$ For any $x > 0$ and any unit-rate Poisson point measure M on $\mathbb{R}_+^2$, denote

$$r_\alpha(x, M) = e^{-\alpha\mu(x)} + e^{-M(g_{\alpha,x})} \tag{B.9}$$

For any $z_1 = (t_1, x_1), z_2 = (t_2, x_2)$, if $t_1 > \alpha$ or $t_2 > \alpha$, then $|D_{z_1}(F_\alpha)|^2 |D_{z_2}(F_\alpha)|^2 = 0$. Otherwise, if $t_1, t_2 \leq \alpha$, we have from Equation (B.7),

$$v_\alpha |D_{z_1}(F_\alpha)|^2 |D_{z_2}(F_\alpha)|^2$$

$$\leq \left(4r_\alpha(x_1,M)+2\sum_{i,j}\mathbb{1}_{\theta_i\leq\alpha}\mathbb{1}_{\theta_j\leq\alpha}W(\vartheta_i,x_1)W(\vartheta_j,x_1)e^{-M(g_{\alpha,\vartheta_i})-M(g_{\alpha,\vartheta_j})}\right)$$
$$\times\left(4r_\alpha(x_2,M)+2\sum_{i,j}\mathbb{1}_{\theta_i\leq\alpha}\mathbb{1}_{\theta_j\leq\alpha}W(\vartheta_i,x_2)W(\vartheta_j,x_2)e^{-M(g_{\alpha,\vartheta_i})-M(g_{\alpha,\vartheta_j})}\right)$$
$$\leq 16r_\alpha(x_1,M)r_\alpha(x_2,M)$$
$$+8\sum_{i,j}\mathbb{1}_{\theta_i,\theta_j\leq\alpha}\left\{W(\vartheta_i,x_1)W(\vartheta_j,x_1)r_\alpha(x_2,M)+W(\vartheta_i,x_2)W(\vartheta_j,x_2)r_\alpha(x_1,M)\right)e^{-M(g_{\alpha,\vartheta_i})-M(g_{\alpha,\vartheta_j})}$$
$$+4\sum_{i_1,i_2,i_3,i_4}W(\vartheta_{i_1},x_1)W(\vartheta_{i_2},x_1)W(\vartheta_{i_3},x_2)W(\vartheta_{i_4},x_2)\prod_{k=1}^4\left(\mathbb{1}_{\theta_{i_k}\leq\alpha}e^{-M(g_{\alpha,\vartheta_{i_k}})}\right).$$

Note that, using Campbell theorem, together with Lemma B.1

$$E(r_\alpha(x_1,M)r_\alpha(x_2,M))\leq e^{-\alpha(\mu(x_1)+\mu(x_2))/2}$$

It follows that, using the extended Slivnyak-Mecke theorem

$$v_\alpha^2E\left(|D_{z_1}(F_\alpha)|^2\,|D_{z_2}(F_\alpha)|^2\right)$$
$$\leq C\left(e^{-\frac{\alpha}{2}(\mu(x_1)+\mu(x_2))}+\alpha\int_0^\infty(W(y,x_1)e^{-\frac{\alpha}{2}\mu(x_2)}+W(y,x_2)e^{-\frac{\alpha}{2}\mu(x_1)})e^{-\frac{\alpha}{2}\mu(y)}dy\right.$$
$$+\alpha^2\int_{\mathbb{R}_+^2}\left\{(W(y_1,x_1)W(y_2,x_1)e^{-\frac{\alpha}{3}\mu(x_2)}+W(y_1,x_2)W(y_2,x_2)e^{-\frac{\alpha}{3}\mu(x_1)}\right\}e^{-\alpha\mu(y_1)/3-\alpha\mu(y_2)/3}dy_1dy_2$$
$$+\alpha^3\int_{\mathbb{R}_+^3}(W(y_1,x_1)^2W(y_2,x_2)W(y_3,x_2)+W(y_1,x_2)^2W(y_2,x_1)W(y_3,x_1))e^{-\alpha\sum_{k=1}^3\mu(y_k)/3}dy_1dy_2dy_3$$
$$\left.+\alpha^4\int_{\mathbb{R}_+^4}W(y_1,x_1)W(y_2,x_1)W(y_3,x_2)W(y_4,x_2)e^{-\alpha\sum_{k=1}^4\mu(y_k)/4}dy_1dy_2dy_3dy_4\right)$$
$$\leq C\left(e^{-\frac{\alpha}{3}(\mu(x_1)+\mu(x_2))}+\alpha(H_1(x_1)e^{-\frac{\alpha}{3}\mu(x_2)}+H_1(x_2)e^{-\frac{\alpha}{3}\mu(x_1)})+\alpha^2(H_1(x_1)^2e^{-\frac{\alpha}{3}\mu(x_2)}+H_1(x_2)^2e^{-\frac{\alpha}{3}\mu(x_1)})\right.$$
$$\left.+\alpha^3(H_1(x_1)^2H_{2,0}(x_2)+H_{2,0}(x_1)H_1(x_2)^2)+\alpha^4H_1(x_1)^2H_1(x_2)^2\right)$$

where $H_{i,j}$ are defined in Equation (B.2); therefore, for any $t_1,t_2\leq\alpha$, using the fact that $H_{2,0}\leq H_1$ and that $\sqrt{\sum_{i=1}^p a_i}\leq\sqrt{p}\sum_{i=1}^p\sqrt{a_i}$

$$v_\alpha\sqrt{\mathbb{E}\,|D_{z_1}F_\alpha|^2\,|D_{z_2}F_\alpha|^2}\leq\sqrt{6C}\sum_{q_1=0}^2\sum_{q_2=0}^2(\alpha H_1(x_1))^{q_1/2}(\alpha H_1(x_2))^{q_2/2}e^{-\frac{\alpha}{6}(\mu(x_1)\mathbb{1}_{q_1=0}+\mu(x_2)\mathbb{1}_{q_2=0})}$$

Additionally, from Equation (B.8), we have

$$v_\alpha \int \sqrt{E\left((D^2_{z_1,z_3}F_\alpha)^2(D^2_{z_2,z_3}F_\alpha)^2\right)}dx_3$$

$$\le C \times \Bigg(\underbrace{\int W(x_1,x_3)W(x_2,x_3)(e^{-\frac{\alpha}{4}(\mu(x_1)+\mu(x_2))} + e^{-\alpha\mu(x_3)/4})dx_3}_{B_{\alpha,1}(x_1,x_2)}$$

$$+ \underbrace{\int (\alpha H_{1,1}(x_2,x_3) + \sqrt{\alpha H_{2,2}(x_2,x_3)})W(x_1,x_3)(e^{-\alpha/6\mu(x_1)} + e^{-\alpha/6\mu(x_3)})dx_3}_{B_{\alpha,2}(x_1,x_2)}$$

$$+ \int (\alpha H_{1,1}(x_1,x_3) + \sqrt{\alpha H_{2,2}(x_1,x_3)})W(x_2,x_3)(e^{-\alpha/6\mu(x_2)} + e^{-\alpha/6\mu(x_3)})dx_3$$

$$+ \underbrace{\int \sqrt{A_2(x_1,x_2,x_3)}dx_3}_{B_{\alpha,3}(x_1,x_2)} \Bigg)$$

for some constant C. To show that $\gamma_{\alpha,1} \to 0$, we aim to show that, for any $q_1, q_2 \in \{0,1,2\}$, and any $k = 1,2,3$

$$I_{\alpha,k}(q_1,q_2) := \int (\alpha H_1(x_1))^{q_1/2}(\alpha H_1(x_2))^{q_2/2} e^{-\frac{\alpha}{6}(\mu(x_1)\mathbb{1}_{q_1=0}+\mu(x_2)\mathbb{1}_{q_2=0})} B_{\alpha,k}(x_1,x_2)dx_1dx_2$$
$$= o(\alpha^{4\sigma-1}\ell_\sigma(\alpha)^4)$$

Consider first

$$I_{\alpha,1}(q_1,q_2) = \int (\alpha H_1(x_1))^{q_1/2}(\alpha H_1(x_2))^{q_2/2} e^{-\frac{\alpha}{6}(\mu(x_1)\mathbb{1}_{q_1=0}+\mu(x_2)\mathbb{1}_{q_2=0})}$$
$$\times W(x_1,x_3)W(x_2,x_3)(e^{-\frac{\alpha}{4}(\mu(x_1)+\mu(x_2))} + e^{-\alpha\mu(x_3)/4})dx_1dx_2dx_3$$
$$\le \int (\alpha H_1(x_1))^{q_1/2} L(x_1)\mu(x_1)e^{-\frac{\alpha}{4}\mu(x_1)}dx_1 \int (\alpha H_1(x_2))^{q_2/2} L(x_2)\mu(x_2)e^{-\frac{\alpha}{4}\mu(x_2)}dx_2$$
$$+ \int (\alpha H_1(x_1))^{q_1/2}(\alpha H_1(x_2))^{q_2/2} W(x_1,x_3)W(x_2,x_3)e^{-\alpha\mu(x_3)/4}dx_1dx_2dx_3$$

For $q_1, q_2 \ge 1$, using Hölder's inequality and Assumptions **??** and **??**,

$$\int (\alpha H_1(x_1))^{q_1/2}(\alpha H_1(x_2))^{q_2/2} W(x_1,x_3)W(x_2,x_3)e^{-\alpha/4\mu(x_3)}dx_1dx_2dx_3$$
$$\le \Bigg(\int (\alpha H_1(x_1))^{q_1}dx_1 \int L(x_3)^2\mu(x_3)^2 e^{-\alpha\mu(x_3)/4}dx_3$$
$$\times \int (\alpha H_1(x_2))^{q_2}dx_2 \int L(x_3)^2\mu(x_3)^2 e^{-\alpha\mu(x_3)/4}dx_3 \Bigg)^{1/2}$$

$$= O(\alpha^{(q_1/2+q_2/2+1)\sigma-2}\ell_\sigma(\alpha)^{q_1/2+q_2/2+1})$$

Similarly,

$$\int (\alpha H_1(x_1))^{q_1/2} W(x_1,x_3)W(x_2,x_3)e^{-\alpha\mu(x_3)/4}dx_1dx_2dx_3 = O(\alpha^{(q_1/2+1/2)\sigma-2}\ell_\sigma(\alpha)^{q_1/2+1/2})$$

$$\int W(x_1,x_3)W(x_2,x_3)e^{-\alpha\mu(x_3)/4}dx_1dx_2dx_3 = O(\alpha^{\sigma-2}\ell_\sigma(\alpha))$$

and it follows that for any $q_1, q_2 \in \{0,1,2\}$, $I_{\alpha,1}(q_1,q_2) = O(\alpha^{3\sigma-2}\ell_\sigma(\alpha)^3) = o(\alpha^{-1+4\sigma}\ell_\sigma(\alpha)^2)$ as required. Consider now

$$I_{\alpha,2}(q_1,q_2) = \int (\alpha H_1(x_1))^{q_1/2}(\alpha H_1(x_2))^{q_2/2} e^{-\frac{\alpha}{6}(\mu(x_1)\mathbb{1}_{q_1=0}+\mu(x_2)\mathbb{1}_{q_2=0})}$$
$$\times (\alpha H_{1,1}(x_2,x_3) + \sqrt{\alpha H_{2,2}(x_2,x_3)})W(x_1,x_3)(e^{-\alpha/6\mu(x_1)} + e^{-\alpha/6\mu(x_3)})dx_1dx_2dx_3.$$

We have, using Lemma B.2

$$I_{\alpha,2,1}(q_1,q_2) := \int (\alpha H_1(x_1))^{q_1/2}(\alpha H_1(x_2))^{q_2/2} e^{-\frac{\alpha}{6}(\mu(x_1)\mathbb{1}_{q_1=0}+\mu(x_2)\mathbb{1}_{q_2=0})}$$
$$\times \alpha H_{1,1}(x_2,x_3)W(x_1,x_3)e^{-\alpha/6\mu(x_1)}dx_1dx_2dx_3$$
$$\leq \alpha\left(\int (\alpha H_1(x_1))^{q_1/2}e^{-\frac{\alpha}{6}\mu(x_1)\mathbb{1}_{q_1=0}}L(x_1)\mu(x_1)e^{-\alpha/6\mu(x_1)}dx_1\right)$$
$$\times \left(\int (\alpha H_1(x_2))^{q_2/2}e^{-\frac{\alpha}{6}\mu(x_2)\mathbb{1}_{q_2=0}}W(x_2,y)L(y)\mu(y)e^{-\frac{\alpha}{4}\mu(y)}dx_2dy\right)$$

If $q_1 = 0$,

$$\int e^{-\frac{\alpha}{3}\mu(x_1)}L(x_1)\mu(x_1)dx_1 = O(\alpha^{\sigma-1}\ell_\sigma(\alpha))$$

and if $q_1 \geq 1$, using the bound (B.5),

$$\int (\alpha H_1(x_1))^{q_1/2}L(x_1)\mu(x_1)e^{-\frac{\alpha}{6}\mu(x_1)}dx_1 = O(\alpha^{(q_1+1)\sigma/2-1}\ell_\alpha(\alpha)^{q_1/2+1/2}) \tag{B.10}$$

If $q_2 = 0$,

$$\int e^{-\frac{\alpha}{6}\mu(x_2)}W(x_2,y)L(y)\mu(y)e^{-\frac{\alpha}{4}\mu(y)}dx_2dy \leq \int L(y)\mu(y)^2e^{-\frac{\alpha}{4}\mu(y)}dy = O(\alpha^{\sigma-2}\ell_\sigma(\alpha))$$

If $q_2 \geq 1$, using Hölder's inequality, the bound (B.4) and Assumptions **??** and **??**,

$$\int (\alpha H_1(x_2))^{q_2/2}W(x_2,y)L(y)\mu(y)e^{-\frac{\alpha}{4}\mu(y)}dx_2dy$$

$$\leq \left(\int (\alpha H_1(x_2))^{q_2}\right)^{1/2} \int \left(\int W(x_2,y)^2 dx_2\right)^{1/2} L(y)\mu(y) e^{-\frac{\alpha}{4}\mu(y)} dy$$
$$\leq \left(\int (\alpha H_1(x_2))^{q_2}\right)^{1/2} \int L(y)^2 \mu(y)^2 e^{-\frac{\alpha}{4}\mu(y)} dy = O(\alpha^{(q_2/2+1)\sigma-2}\ell_\sigma(\alpha^{q_2/2+1}))$$

Hence $I_{\alpha,2,1}(q_1,q_2) = o(\alpha^{4\sigma-1}\ell_\sigma^2(\alpha))$. Consider now

$$I_{\alpha,2,2}(q_1,q_2) := \int (\alpha H_1(x_1))^{q_1/2}(\alpha H_1(x_2))^{q_2/2} e^{-\frac{\alpha}{6}(\mu(x_1)\mathbb{1}_{q_1=0}+\mu(x_2)\mathbb{1}_{q_2=0})}$$
$$\times \alpha H_{1,1}(x_2,x_3) W(x_1,x_3) e^{-\alpha/6\mu(x_3)} dx_1 dx_2 dx_3$$

If $q_1 = 0$, we obtain the following bound, using the same computations as above,

$$I_{\alpha,2,2}(q_1,q_2) \leq \alpha \int e^{-\frac{\alpha}{6}\mu(x_1)} L(x_1)\mu(x_1) dx_1 \int (\alpha H_1(x_2))^{q_2/2} e^{-\frac{\alpha}{6}\mu(x_2)\mathbb{1}_{q_2=0}} W(x_2,y) L(y)\mu(y) e^{-\frac{\alpha}{4}\mu(y)} dx_2 dy$$
$$= O(\alpha^{4\sigma-2}\ell_\sigma(\alpha)^{4\sigma})$$

If $q_1 > 0$, we have

$$I_{\alpha,2,2}(q_1,q_2) \leq \int \left(\int (\alpha H_1(x_1))^{q_1} dx_1\right)^{1/2} \left(\int W(x_1,x_3)^2 dx_1\right)^{1/2} (\alpha H_1(x_2))^{q_2/2} e^{-\frac{\alpha}{6}\mu(x_2)\mathbb{1}_{q_2=0}}$$
$$\times \alpha H_{1,1}(x_2,x_3) e^{-\alpha/6\mu(x_3)} dx_2 dx_3$$
$$\leq \left(\int (\alpha H_1(x_1))^{q_1} dx_1\right)^{1/2} \int L(x_3)\mu(x_3)(\alpha H_1(x_2))^{q_2/2} e^{-\frac{\alpha}{6}\mu(x_2)\mathbb{1}_{q_2=0}}$$
$$\times \alpha H_{1,1}(x_2,x_3) e^{-\alpha/6\mu(x_3)} dx_2 dx_3$$

If $q_2 = 0$, noting that $H_{1,1}(x_2,x_3) \leq L(x_2)\mu(x_2)L(x_3)\mu(x_3)$, we obtain

$$I_{\alpha,2,2}(q_1,q_2) \leq \alpha \left(\int (\alpha H_1(x_1))^{q_1} dx_1\right)^{1/2} \int L(x_2)\mu(x_2)L(x_3)^2\mu(x_3)^2 e^{-\frac{\alpha}{6}\mu(x_2)} e^{-\alpha/6\mu(x_3)} dx_2 dx_3$$
$$= O(\alpha^{(q_1/2+2)\sigma-2}\ell_\sigma(\alpha)^{q_1/2+2})$$

If $q_2 > 0$, using Hölder's inequality and the bound (B.6),

$$I_{\alpha,2,2} \leq \alpha \left(\int (\alpha H_1(x_1))^{q_1} dx_1\right)^{1/2} \left(\int (\alpha H_1(x_2))^{q_2} dx_2\right)^{1/2}$$
$$\times \left(\int L(x_3)^2\mu(x_3)^2 e^{-\alpha/3\mu(x_3)} dx_3\right)^{1/2} \int L(y)^2\mu(y)^2 e^{-\frac{\alpha}{4}\mu(y)} dy = O(\alpha^{7\sigma/2-3}\ell(\alpha)^{7\sigma/2})$$

Now, using Hölder and the fact that $H_{2,2} \leq H_{1,1}$, together with (B.5),

$$
\begin{aligned}
I_{\alpha,2,3}(q_1,q_2) := &\int (\alpha H_1(x_1))^{q_1/2}(\alpha H_1(x_2))^{q_2/2} e^{-\frac{\alpha}{6}(\mu(x_1)\mathbb{1}_{q_1=0}+\mu(x_2)\mathbb{1}_{q_2=0})} \\
&\times (\sqrt{\alpha H_{2,2}(x_2,x_3)}) W(x_1,x_3) e^{-\alpha/6\mu(x_1)} dx_1 dx_2 dx_3 \\
\leq &\sqrt{\alpha} \int (\alpha H_1(x_1))^{q_1/2} L(x_1)\mu(x_1) e^{-\alpha\mu(x_1)/6} dx_1 \\
&\times \int (\alpha H_1(x_2))^{q_2/2} \left(\int \mu(y) W(x_2,y) e^{-\frac{\alpha}{4}\mu(y)} dy \right)^{1/2} e^{-\frac{\alpha}{6}\mu(x_2)\mathbb{1}_{q_2=0}} dx_2 \\
= &O(\alpha^{(3+1/4)\sigma-3/2}\ell_\sigma(\alpha)^{3+1/4})
\end{aligned}
$$

Consider

$$
\begin{aligned}
I_{\alpha,2,4}(q_1,q_2) := &\int (\alpha H_1(x_1))^{q_1/2}(\alpha H_1(x_2))^{q_2/2} e^{-\frac{\alpha}{6}(\mu(x_1)\mathbb{1}_{q_1=0}+\mu(x_2)\mathbb{1}_{q_2=0})} \\
&\times \sqrt{\alpha H_{2,2}(x_2,x_3)} W(x_1,x_3) e^{-\alpha/6\mu(x_3)} dx_1 dx_2 dx_3
\end{aligned}
$$

If $q_1 = 0$, then, using the above computations,

$$
\begin{aligned}
I_{\alpha,2,4}(q_1,q_2) \leq &\int (\alpha H_1(x_2))^{q_2/2} e^{-\frac{\alpha}{6}(\mu(x_1)+\mu(x_2)\mathbb{1}_{q_2=0})} \\
&\times \sqrt{\alpha H_{1,1}(x_2,x_3)} W(x_1,x_3) dx_1 dx_2 dx_3 \\
= &O(\alpha^{(3+1/4)\sigma-3/2}\ell_\sigma(\alpha)^{3+1/4})
\end{aligned}
$$

If $q_1 > 0$ and $q_2 = 0$, noting that $H_{2,2}(x_2,x_3) \leq L(x_2)^2\mu(x_2)^2 L(x_3)^2\mu(x_3)^2$ and using Hölder's inequality and Assumptions **??** and **??**,

$$
\begin{aligned}
I_{\alpha,2,4}(q_1,q_2) &\leq \sqrt{\alpha} \left(\int (\alpha H_1(x_1))^{q_1} dx_1 \right)^{1/2} \int L(x_3)\mu(x_3) e^{-\frac{\alpha}{6}\mu(x_2)} \sqrt{H_{2,2}(x_2,x_3)} e^{-\alpha/6\mu(x_3)} dx_2 dx_3 \\
&\leq \sqrt{\alpha} \left(\int (\alpha H_1(x_1))^{q_1} dx_1 \right)^{1/2} \int L(x_3)^2\mu(x_3)^2 e^{-\alpha/6\mu(x_3)} dx_3 \int e^{-\frac{\alpha}{6}\mu(x_2)} \mu(x_2) L(x_2) dx_2 \\
&= O(\alpha^{3\sigma-3}\ell_\sigma(\alpha)^3)
\end{aligned}
$$

If $q_1, q_2 > 0$, noting that $\int H_{1,1}(x_2,x_3) = \int \mu(y)^2 e^{-\frac{\alpha}{4}\mu(y)} dy = O(\alpha^{\sigma-2}\ell_\sigma(\alpha))$,

$I_{\alpha,2,4}(q_1,q_2)$

$$\le \sqrt{\alpha}\left(\int (\alpha H_1(x_1))^{q_1} dx_1\right)^{1/2} \times \int L(x_3)\mu(x_3)(\alpha H_1(x_2))^{q_2/2}\sqrt{H_{1,1}(x_2,x_3)}e^{-\alpha/6\mu(x_3)}dx_2dx_3$$
$$\le \sqrt{\alpha}\left(\int (\alpha H_1(x_1))^{q_1} dx_1\right)^{1/2}\left(\int (\alpha H_1(x_2))^{q_2} dx_2\right)^{1/2}$$
$$\times\left(\int L(x_3)^2\mu(x_3)^2 e^{-\alpha/6\mu(x_3)}dx_3\right)^{1/2}\left(\int H_{1,1}(x_2,x_3)dx_3dx_2\right)^{1/2} = O(\alpha^{3\sigma-3/2}\ell_\sigma(\alpha^{3\sigma}))$$

It follows that

$$I_{\alpha,2}(q_1,q_2) = o(\alpha^{-3}v_\alpha^2). \tag{B.11}$$

Finally, consider

$$I_{\alpha,3}(q_1,q_2) = \int (\alpha H_1(x_1))^{q_1/2}(\alpha H_1(x_2))^{q_2/2}e^{-\frac{\alpha}{6}(\mu(x_1)\mathbb{1}_{q_1=0}+\mu(x_2)\mathbb{1}_{q_2=0})}\sqrt{A_2(x_1,x_2,x_3)}dx_1dx_2dx_3$$

where

$$\sqrt{A_2(x_1,x_2,x_3)}$$
$$\le \alpha^2 H_{1,1}(x_1,x_3)H_{1,1}(x_2,x_3) + \alpha^{3/2}\left(H_{1,1}(x_2,x_3)\sqrt{H_{2,2}(x_1,x_3)} + H_{1,1}(x_1,x_3)\sqrt{H_{2,2}(x_2,x_3)}\right)$$
$$+\alpha\left(\sqrt{H_{2,2}(x_1,x_3)}\sqrt{H_{2,2}(x_2,x_3)}\right.$$
$$\left.+\sqrt{W(y_1,x_2)W(y_1,x_1)W(y_1,x_3)^2W(y_2,x_1)W(y_2,x_2)W(y_2,x_3)^2)e^{-\alpha/2(\sum_{i=1}^2\mu(y_i))}dy_{1:2}}\right)$$
$$+\sqrt{\alpha}\left(\int W(y_1,x_1)^2W(y_1,x_2)^2W(y_1,x_3)^4e^{-\alpha\mu(y_1)}dy_1\right)^{1/2}$$

Using Assumption **??**,

$$\int H_{1,1}(x_1,x_3))H_{1,1}(x_2,x_3)dx_3 \quad\le \int W(x_1,y_1)L(y_1)\mu(y_1)e^{-\frac{\alpha}{4}\mu(y_1)}dy_1 \int W(x_2,y_2)L(y_2)\mu(y_2)e^{-\frac{\alpha}{4}\mu(y_2)}dy_2$$

Therefore, using the asymptotic bounds (B.5) and Assumption **??**,

$$I_{\alpha,3,1}(q_1,q_2)$$
$$:= \alpha^2\int (\alpha H_1(x_1))^{q_1/2}(\alpha H_1(x_2))^{q_2/2}e^{-\frac{\alpha}{6}(\mu(x_1)\mathbb{1}_{q_1=0}+\mu(x_2)\mathbb{1}_{q_2=0})}H_{1,1}(x_1,x_3)H_{1,1}(x_2,x_3)dx_1dx_2dx_3$$
$$\le \alpha^2\int (\alpha H_1(x_1))^{q_1/2}(\alpha H_1(x_2))^{q_2/2}H_{1,1}(x_1,x_3)H_{1,1}(x_2,x_3)dx_1dx_2dx_3$$
$$\le \alpha^2\int (\alpha H_1(x_1))^{q_1/2}L(y_1)\mu(y_1)W(x_1,y_1)e^{-\frac{\alpha}{4}\mu(y_1)}dy_1dx_1$$

$$\times \int (\alpha H_1(x_2))^{q_2/2} L(y_2)\mu(y_2) W(x_2,y_2) e^{-\frac{\alpha}{4}\mu(y_2)} dy_2 dx_2$$

for any $q_1, q_2 \leq 2$. If $q = 0$,

$$\int (\alpha H_1(x))^{q/2} L(y)\mu(y) W(x,y) e^{-\frac{\alpha}{4}\mu(y)} dy dx = \int L(y)\mu(y)^2 e^{-\frac{\alpha}{4}\mu(y)} dy = O(\alpha^{\sigma-2}\ell_\sigma(\alpha))$$

If $q > 0$, noting that, using Hölder's inequality and Assumption 3,

$$\int (\alpha H_1(x))^{q/2} W(x,y) dx \leq L(y)\mu(y) \left(\int (\alpha H_1(x))^q dx \right)^{1/2}$$

we have

$$\begin{aligned}\int (\alpha H_1(x))^{q/2} L(y)\mu(y) W(x,y) e^{-\frac{\alpha}{4}\mu(y)} dy dx &\leq \left(\int (\alpha H_1(x))^q dx \right)^{1/2} \left(\int L(y)^2 \mu(y)^2 e^{-\frac{\alpha}{4}\mu(y)} dy \right)^{1/2} \\ &= O(\alpha^{(q/2+1)\sigma - 2} \ell_\sigma(\alpha)^{q/2+1}).\end{aligned}$$

It follows that $I_{\alpha,3,1}(q_1,q_2) = O(\alpha^{4\sigma-2}\ell(\alpha)^{4\sigma})$ for any $q_1, q_2 \in \{0,1,2\}$. Consider now

$$\begin{aligned}I_{\alpha,3,2}(q_1,q_2) :=& \alpha^{3/2} \int (\alpha H_1(x_1))^{q_1/2} (\alpha H_1(x_2))^{q_2/2} e^{-\frac{\alpha}{6}(\mu(x_1)\mathbb{1}_{q_1=0} + \mu(x_2)\mathbb{1}_{q_2=0})} H_{1,1}(x_2,x_3) \\ &\times \left(\int W(y_1,x_1)^2 W(y_1,x_3)^2 e^{-\alpha/3\mu(y_1)} dy_1 \right)^{1/2} dx_1 dx_2 dx_3 \\ \leq& \alpha^{3/2} \int (\alpha H_1(x_1))^{q_1/2} (\alpha H_1(x_2))^{q_2/2} H_{1,1}(x_2,x_3) \\ &\times \left(\int W(y_1,x_1)^2 W(y_1,x_3)^2 e^{-\alpha/3\mu(y_1)} dy_1 \right)^{1/2} dx_1 dx_2 dx_3\end{aligned}$$

Using Lemma B.2,

$$\begin{aligned}&\int H_{1,1}(x_2,x_3) \left(\int W(y_1,x_1)^2 W(y_1,x_3)^2 e^{-\alpha/3\mu(y_1)} dy_1 \right)^{1/2} dx_3 \\ &\leq \int W(x_2,y_2) L(y_2)\mu(y_2) e^{-\frac{\alpha}{4}\mu(y_2)} dy_2 \times \left(\int L(y_1)^2 \mu(y_1)^2 W(y_1,x_1)^2 e^{-\frac{\alpha}{3}\mu(y_1)} dy_1 \right)^{1/2}\end{aligned}$$

Therefore

$$I_{\alpha,3,2}(q_1,q_2) \leq \alpha^{3/2} \int (\alpha H_1(x_2))^{q_1/2} W(x_2,y_2) L(y_2)\mu(y_2) e^{-\frac{\alpha}{4}\mu(y_2)} dy_2 dx_2$$

$$\times \int (\alpha H_1(x_1))^{q_2/2} \left(\int L(y_1)^2 \mu(y_1)^2 W(y_1, x_1)^2 e^{-\frac{\alpha}{3}\mu(y_1)} dy_1 \right)^{1/2} dx_1$$

For $q_1 = 0$,

$$\int L(y_2)\mu(y_2)^2 e^{-\frac{\alpha}{4}\mu(y_2)} dy_2 = O(\alpha^{\sigma-2}\ell_\sigma(\alpha)),$$

while for $q_1 \geq 1$,

$$\int (\alpha H_1(x_2))^{q_1/2} W(x_2, y_2) L(y_2)\mu(y_2) e^{-\frac{\alpha}{4}\mu(y_2)} dy_2 dx_2$$
$$\leq \left(\int (\alpha H_1(x_2))^{q_1} dx_2 \times \int W(x_2, y_2)^2 L(y_2)^2 \mu(y_2)^2 e^{-\frac{\alpha}{4}\mu(y_2)} dy_2 dx_2 \right)^{1/2}$$
$$\leq \left(\int (\alpha H_1(x_2))^{q_1} dx_2 \times \int L(y_2)^4 \mu(y_2)^4 e^{-\frac{\alpha}{4}\mu(y_2)} dy_2 dx_2 \right)^{1/2} = O(\alpha^{(q_1/2+1/2)\sigma-2}\ell_\sigma(\alpha)^{q_1+1/2}).$$

Additionally, for $q_2 = 0$, using Hölder's inequality and Assumptions **??** and **??**,

$$\int \left(\int L(y_1)^2 \mu(y_1)^2 W(y_1, x_1)^2 e^{-\frac{\alpha}{3}\mu(y_1)} dy_1 \right)^{1/2} dx_1$$
$$\leq \int \left(\int L(y_1)^4 \mu(y_1)^4 e^{-\frac{2\alpha}{3}\mu(y_1)} dy_1 \int W(y_1, x_1)^4 dy_1 \right)^{1/4} dx_1$$
$$\leq \int L(x_1)\mu(x_1) dx_1 \left(\int L(y_1)^4 \mu(y_1)^4 e^{-\frac{2\alpha}{3}\mu(y_1)} dy_1 \right)^{1/4} = O(\alpha^{\sigma/4-1}\ell_\sigma(\alpha)^{1/4})$$

while for $q_2 \geq 1$

$$\int (\alpha H_1(x_1))^{q_2/2} \left(\int L(y_1)^2 \mu(y_1)^2 W(y_1, x_1)^2 e^{-\frac{\alpha}{3}\mu(y_1)} dy_1 \right)^{1/2} dx_1$$
$$\leq \left(\int (\alpha H_1(x_1))^{q_2} dx_1 \int L(y_1)^2 \mu(y_1)^2 W(y_1, x_1)^2 e^{-\frac{\alpha}{3}\mu(y_1)} dy_1 dx_1 \right)^{1/2}$$
$$\leq \left(\int (\alpha H_1(x_1))^{q_2} dx_1 \int L(y_1)^4 \mu(y_1)^4 e^{-\frac{\alpha}{3}\mu(y_1)} dy_1 \right)^{1/2} = O(\alpha^{(q_2+1/2)\sigma-2}\ell_\sigma(\alpha)^{q_2/2+1/2}).$$

Hence, for any $q_1, q_2 \leq 2$, $I_{\alpha,3,2} = O(\alpha^{3\sigma-2}\ell_\sigma(\alpha)^3) = o(\alpha^{-3}v_\alpha^2)$. Consider now

$$I_{\alpha,3,3}(q_1, q_2) := \alpha \int (\alpha H_1(x_1))^{q_1/2} (\alpha H_1(x_2))^{q_2/2} e^{-\frac{\alpha}{6}(\mu(x_1)\mathbb{1}_{q_1=0} + \mu(x_2)\mathbb{1}_{q_2=0})}$$
$$\times \left[\left(\int W(y_1, x_1)^2 W(y_1, x_3)^2 W(y_2, x_2)^2 W(y_2, x_3)^2 e^{-\alpha/2(\sum_{i=1}^2 \mu(y_i))} dy_{1:2} \right)^{1/2} \right.$$
$$\left. + \left(\int W(y_1, x_2) W(y_1, x_1) W(y_1, x_3)^2 W(y_2, x_1) W(y_2, x_2) W(y_2, x_3)^2 e^{-\frac{\alpha}{2(\sum_{i=1}^2 \mu(y_i))}} dy_{1:2} \right)^{1/2} \right]$$

$$dx_1 dx_2 dx_3$$

Using Hölder's inequality,

$$\int \left(\int W(y_1,x_1)^2 W(y_1,x_3)^2 W(y_2,x_2)^2 W(y_2,x_3)^2 e^{-\alpha/2(\sum_{i=1}^2 \mu(y_i))} dy_{1:2} \right)^{1/2} dx_3$$
$$\leq \left(\int W(y_1,x_1)^2 W(y_1,x_3)^2 e^{-\frac{\alpha}{2}\mu(y_1)} dy_1 dx_3 \right)^{1/2} \left(\int W(y_2,x_2)^2 W(y_2,x_3)^2 e^{-\frac{\alpha}{2}\mu(y_2)} dy_2 dx_3 \right)^{1/2}$$

For $q = 0$, using Assumption **??** and **??**

$$\int e^{-\frac{\alpha}{6}\mu(x)} \left(\int H_{2,2}(x,x_3) dx_3 \right)^{1/2} dx \leq \int e^{-\frac{\alpha}{6}\mu(x)} \left(\int W(y,x)^2 W(y,x_3) dy dx_3 \right)^{1/2} dx$$
$$= O(\alpha^{\sigma-1}\ell_\sigma(\alpha))$$

Additionally,

$$\left(\int W(y_1,x_1)^2 W(y_1,x_3)^2 e^{-\frac{\alpha}{2}\mu(y_1)} dy_1 dx_3 \right)^{1/2} \left(\int W(y_2,x_2)^2 W(y_2,x_3)^2 e^{-\frac{\alpha}{2}\mu(y_2)} dy_2 dx_3 \right)^{1/2}$$
$$\leq \left(\int L(y_1)^2 \mu(y_1)^2 W(y_1,x_1)^2 e^{-\frac{\alpha}{2}\mu(y_1)} dy_1 \right)^{1/2} \left(\int L(y_2)^2 \mu(y_2)^2 W(y_2,x_2)^2 e^{-\frac{\alpha}{2}\mu(y_2)} dy_2 \right)^{1/2}$$

It follows that, for $q \geq 1$, using Hölder's inequality and Assumptions **??** and **??**

$$\int (\alpha H_1(x))^{q/2} \left(\int L(y)^2 \mu(y)^2 W(y,x)^2 e^{-\frac{\alpha}{2}\mu(y)} dy \right)^{1/2} dx$$
$$\leq \left(\int (\alpha H_1(x))^q dx \right)^{1/2} \left(\int L(y)^4 \mu(y)^4 e^{-\frac{\alpha}{2}\mu(y)} dy \right)^{1/2} = O\left(\alpha^{(q/2+1/2)\sigma-2} \ell_\sigma(\alpha)^{(q/2+1/2)} \right)$$

Similarly, for the second term of $I_{\alpha,3,3}(q_1,q_2)$

$$\int \left(\int W(y_1,x_2) W(y_1,x_1) W(y_1,x_3)^2 W(y_2,x_1) W(y_2,x_2) W(y_2,x_3)^2 e^{-\alpha/2(\sum_{i=1}^2 \mu(y_i))} dy_{1:2} \right)^{1/2} dx_3$$
$$\leq \left(\int W(y_1,x_1) W(y_1,x_2) W(y_1,x_3)^2 e^{-\frac{\alpha}{2}\mu(y_1)} dy_1 dx_3 \int W(y_2,x_1) W(y_2,x_2) W(y_2,x_3)^2 e^{-\frac{\alpha}{2}\mu(y_2)} dy_2 dx_3 \right)^{1/2}$$

and, using Hölder's inequality and (B.5),

$$\int (\alpha H_1(x_1))^{q_1/2} (\alpha H_1(x_2))^{q_2/2} e^{-\frac{\alpha}{6}(\mu(x_1)\mathbb{1}_{q_1=0} + \mu(x_2)\mathbb{1}_{q_2=0})}$$

$$
\begin{aligned}
&\left(\int W(y_1,x_1)W(y_1,x_2)W(y_1,x_3)^2 e^{-\frac{\alpha}{2}\mu(y_1)}dy_1dx_3\right)^{1/2}\\
&\quad\times\left(\int W(y_2,x_1)W(y_2,x_2)W(y_2,x_3)^2 e^{-\frac{\alpha}{2}\mu(y_2)}dy_2dx_3\right)^{1/2}dx_1dx_2\\
&\leq\left(\int(\alpha H_1(x_1))^{q_1}e^{-\frac{\alpha}{6}\mu(x_1)\mathbb{1}_{q_1=0}}W(y_1,x_1)\mu(y_1)^3L(y_1)^2e^{-\frac{\alpha}{2}\mu(y_1)}dy_1dx_1\right)^{1/2}\\
&\quad\times\left(\int(\alpha H_1(x_2))^{q_2}e^{-\frac{\alpha}{6}\mu(x_2)\mathbb{1}_{q_2=0}}W(y_2,x_2)\mu(y_2)^3L(y_2)^2e^{-\frac{\alpha}{2}\mu(y_2)}dy_2dx_2\right)^{1/2}
\end{aligned}
$$

For $q=0$,

$$
\left(\int e^{-\frac{\alpha}{6}\mu(x)}W(y,x)\mu(y)^3L(y)^2e^{-\frac{\alpha}{2}\mu(y)}dydx\right)^{1/2}=O(\alpha^{\sigma/2-2}\ell_\sigma(\alpha)^{1/2})
$$

while for $q\geq 1$,

$$
\left(\int(\alpha H_1(x))^qW(y,x)\mu(y)^3L(y)^2e^{-\frac{\alpha}{2}\mu(y)}dydx\right)^{1/2}=O(\alpha^{(q/2+1/2)\sigma-3/2}\ell_\sigma(\alpha)^{q/2+1/2})
$$

Combining the above results, we obtain, for any $q_1,q_2\in\{0,1,2\}$, $I_{\alpha,3,3}(q_1,q_2)=o(\alpha^{-3}v_\alpha^2)$. Consider finally

$$
\begin{aligned}
I_{\alpha,3,4}(q_1,q_2):=&\int(\alpha H_1(x_1))^{q_1/2}(\alpha H_1(x_2))^{q_2/2}e^{-\frac{\alpha}{6}(\mu(x_1)\mathbb{1}_{q_1=0}+\mu(x_2)\mathbb{1}_{q_2=0})}\\
&\times\sqrt{\alpha}\left(\int W(y_1,x_1)^2W(y_1,x_2)^2W(y_1,x_3)^4e^{-\alpha\mu(y_1)}dy_1\right)^{1/2}dx_1dx_2dx_3
\end{aligned}
$$

We have

$$
\begin{aligned}
&\int\left(\int W(y_1,x_1)^2W(y_1,x_2)^2W(y_1,x_3)^4e^{-\alpha\mu(y_1)}dy_1\right)^{1/2}dx_3\\
&\leq\left(\int L(x_3)\mu(x_3)dx_3\right)\left(\int W(y_1,x_1)^6e^{-\alpha\mu(y_1)}dy_1\right)^{1/6}\left(\int W(y_1,x_2)^6e^{-\alpha\mu(y_1)}dy_1\right)^{1/6}
\end{aligned}
$$

and, for $q_1\geq 1$, using Hölder's inequality,

$$
\int(\alpha H_1(x_1))^{q_1}\left(\int W(y_1,x_1)^6e^{-\alpha\mu(y_1)}dy_1\right)^{1/6}dx_1=O(\alpha^{(q_1+1/6)\sigma-1}\ell_\sigma(\alpha)^{q_1+1/6}).
$$

For $q_1=0$,

$$
\int e^{-\frac{\alpha}{6}\mu(x_1)}\left(\int W(y_1,x_1)^6e^{-\alpha\mu(y_1)}dy_1\right)^{1/6}dx_1=O(\alpha^{\sigma-1}\ell_\sigma(\alpha))
$$

It follows that, for $q_1, q_2 \leq 2$

$$I_{\alpha,3,4}(q_1, q_2) = O(\alpha^{(4+1/3)\sigma - 3/2} \ell_\sigma(\alpha)^{4+1/3}) = o(\alpha^{-3} v_\alpha^2) \tag{B.12}$$

$$I_{\alpha,3}(q_1, q_2) = O(\alpha^{(4+1/3)\sigma - 3/2} \ell_\sigma(\alpha)^{4+1/3}) = o(\alpha^{-3} v_\alpha^2) \tag{B.13}$$

and, combining the bounds (B.3.5), (B.11) and (B.13), we obtain $\gamma_{\alpha,1} \to 0$.

References

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