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SUPPLIMENTARY MATERIAL: EXPONENTIAL AND GAMMA FORM FOR TAIL EXPANSIONS OF FIRST-PASSAGE DISTRIBUTIONS IN SEMI-MARKOV PROCESSES

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7.1. Proof of Proposition 1

Suppose the state space $\mathbb{I}_M$ consists of relevant states in $\mathbb{I}_m$ and non-relevant states in $\mathbb{I}_{M-m} = \{m+1, \ldots, M\}$. Expression (4) can be applied to state space $\mathbb{I}_M$ using the $M \times M$ transmittance matrix $\mathbf{T}_M(s) = \mathbf{P}_M \odot \mathbf{M}_M(s)$. In block form, write

$$\mathbf{I}_M - \mathbf{T}_M(s) = \begin{pmatrix} \mathbf{I}_m - \mathbf{T}(s) & -\mathbf{T}_{\mathbb{I}_m \mathbb{I}_{M-m}}(s) \\ \mathbf{0} & \mathbf{I}_{M-m} - \mathbf{T}_{\mathbb{I}_{M-m}\mathbb{I}_{M-m}}(s) \end{pmatrix},$$

where $-\mathbf{T}_{\mathbb{I}_{M-m}\mathbb{I}_m}(s) = \mathbf{0}$ is a matrix of zeros, since the states in $\mathbb{I}_{M-m}$ are not relevant. Applying (4), then

$$\begin{aligned} f_{1m}\mathcal{F}_{1m}(s) &= \frac{(m,1) \text{ cofactor of } \mathbf{I}_M - \mathbf{T}_M(s)}{(m,m) \text{ cofactor of } \mathbf{I}_M - \mathbf{T}_M(s)} \\ &= \frac{(m,1) \text{ cofactor of } \mathbf{I}_m - \mathbf{T}(s)}{(m,m) \text{ cofactor of } \mathbf{I}_m - \mathbf{T}(s)} \times \frac{\left|\mathbf{I}_{M-m} - \mathbf{T}_{\mathbb{I}_{M-m}\mathbb{I}_{M-m}}(s)\right|}{\left|\mathbf{I}_{M-m} - \mathbf{T}_{\mathbb{I}_{M-m}\mathbb{I}_{M-m}}(s)\right|}. \end{aligned} \quad (1)$$

The first factor in (1) is defined on at least $\{\mathrm{Re}(s) \leq 0\}$. The second factor is 1 apart from at $s = 0$ where it takes the form 0/0. The determinant is 0 due to the factor that $\mathbf{T}_{\mathbb{I}_{M-m}\mathbb{I}_{M-m}}(0)\mathbf{1} = \mathbf{1}$, with $\mathbf{1} = (1, \ldots, 1)^T$. The order of the removable singularity is the algebraic multiplicity of eigenvalue 1 for the matrix $\mathbf{T}_{\mathbb{I}_{M-m}\mathbb{I}_{M-m}}(0) = \mathbf{P}_{\mathbb{I}_{M-m}\mathbb{I}_{M-m}}$ which counts the number of irreducible and absorbing subchains in $\mathbb{I}_{M-m}$.

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7.2. Convergence domain

Assuming $\mathcal{R}_{1\to m}$, there are a countable number of pathways from $1 \to m$ and all states in $\mathbb{I}_{m-1}$ are represented in at least one pathway. If all states in $\mathbb{I}_{m-1}$ are progressive, then the (m,m) cofactor of $\mathbf{I}_m - \mathbf{P} \odot \mathbf{Z}$ in the denominator of $\mathcal{F}_{1m}(s)$ is 1 as seen in (17). Thus, $\mathcal{F}_{1m}(s)$ has converge domain based only on convergence of the the $(m,1)$ cofactor of $\mathbf{I}_m - \mathbf{P} \odot \mathbf{Z}$ so that $b = \min\{b_{ij} : i \in \mathbb{I}_{m-1}, j \in \mathbb{I}_m \backslash \{1\}\}$.

Now suppose that not all states are progressive so there is at least one irreducible class in $\mathbb{I}_{m-1}$. This may be as limited as a single state i for which $p_{ii} > 0$ with all other states progressive. More generally, the communication equivalence relation partitions $\mathbb{I}_{m-1}$ into classes $\mathcal{C}_1, \ldots, \mathcal{C}_r$ which are either irreducible or else single progressive states with $\mathcal{K} \subseteq \{1, \ldots, r\}$ indexing the irreducible classes. We may order these r classes so that $\mathcal{C}_i \to \mathcal{C}_j$ can occur if $i < j$ but not when $j < i$. With this ordering, the matrix $(\mathbf{P} \odot \mathbf{Z})_{m;m}$ has block diagonals $\{(\mathbf{P} \odot \mathbf{Z})_{\mathcal{C}_i\mathcal{C}_i}\}$ for the classes $\{\mathcal{C}_i\}$ and consists of zero blocks below the diagonal. For each block $\mathcal{C}_i$ representing a progressive state, $(\mathbf{P} \odot \mathbf{Z})_{\mathcal{C}_i\mathcal{C}_i} = 0$ is 1×1. For an irreducible block $\mathcal{C}_i$ of $\mathbf{P} \odot \mathbf{Z}$, let $\lambda_1\{(\mathbf{P} \odot \mathbf{Z})_{\mathcal{C}_i\mathcal{C}_i}\}$ denote its Perron-Frobenius eigenvalue. Then, the Perron-Frobenius eigenvalue for $(\mathbf{P} \odot \mathbf{Z})_{m;m}$ is the largest such eigenvalue given by

$$\lambda_1\{(\mathbf{P} \odot \mathbf{Z})_{m;m}\} = \max_{i \in \mathcal{K}} \lambda_1\{(\mathbf{P} \odot \mathbf{Z})_{\mathcal{C}_i\mathcal{C}_i}\}. \tag{2}$$

When all entries of $\mathbf{Z}$ are 1, then $\lambda_1\{\mathbf{P}_{\mathcal{C}_i\mathcal{C}_i}\} < 1$ for each $i \in \mathcal{K}$ since each $\mathbf{P}_{\mathcal{C}_i\mathcal{C}_i}$ has a row sum that is < 1. This follows from the Perron-Frobenius theorem for irreducible matrices in Seneta (2006, thm. 1.5(e)). Thus $\mathfrak{O} \supset \{\mathbf{Z}_{m;} \in [0,1]^{(m-1)\times m}\}$. This same theorem ensures that the eigenvalues in (2) are all monotone increasing in z_{ij} for each $(i,j) \in \bar{\mathbb{Z}}^2$ (such that $p_{ij} > 0$) and $i, j \in \mathcal{C}_k$ for $k \in \mathcal{K}$. Hence the convergence domain for $\mathcal{F}_{1m}(s)$ must take the form $\{s < b\}$ for some $b > 0$.

7.3. Proof of Proposition 2

Let $U = \sum_{i\in\mathbb{I}_{m-1}}\sum_{j\in\mathbb{I}_m} N_{ij}$ count the number of steps required for first passage from $1\to m$. In block form, define

$$\tilde{\mathbf{P}} = \begin{pmatrix} \mathbf{P}_{m;m} & \mathbf{p}_m \\ \mathbf{0}^T & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{Z}} = \begin{pmatrix} \mathbf{Z}_{m;m} & \mathbf{z}_m \\ \mathbf{0}^T & 0 \end{pmatrix},$$

where $m;$ indicates removing the mth row and $;m$ indicates removing the mth column. Then,

$$\tilde{\mathbf{P}}^n = \begin{pmatrix} \mathbf{P}^n_{m;m} & \mathbf{P}^{n-1}_{m;m}\mathbf{p}_m \\ \mathbf{0}^T & 0 \end{pmatrix} \tag{3}$$

$$(\tilde{\mathbf{Z}}\odot\tilde{\mathbf{P}})^n = \begin{pmatrix} (\mathbf{Z}_{m;m}\odot\mathbf{P}_{m;m})^n & (\mathbf{Z}_{m;m}\odot\mathbf{P}_{m;m})^{n-1}(\mathbf{z}_m\odot\mathbf{p}_m) \\ \mathbf{0}^T & 0 \end{pmatrix}. \tag{4}$$

Take $\xi^T_{1\backslash}$ as the $(m-1)\times 1$ indicator vector for state 1. Then, by (3),

$$\mathbb{P}(U=n) = \xi^T_{1\backslash}\mathbf{P}^{n-1}_{m;m}\mathbf{p}_m = \xi^T_1\tilde{\mathbf{P}}^n\xi_m$$

where ξ_1 and ξ_m are $m\times 1$ indicators for states 1 and m. By the same argument using (4),

$$\begin{aligned}\mathbb{E}\left(\prod_{i\in\mathbb{I}_{m-1}}\prod_{j\in\mathbb{I}_m} z_{ij}^{N_{ij}} 1_{\{U=n\}}\right) &= \xi^T_{1\backslash}(\mathbf{Z}_{m;m}\odot\mathbf{P}_{m;m})^{n-1}(\mathbf{z}_m\odot\mathbf{p}_m) \\ &= \xi^T_1(\tilde{\mathbf{P}}\odot\mathbf{Z})^n\xi_m. \end{aligned}\tag{5}$$

Thus, if $\mathfrak{P}^{\le N}$ consists of all distinct pathways from $1\to m$ requiring at most N steps, then from (5),

$$\begin{aligned}\sum_{\mathfrak{p}\in\mathfrak{P}^{\le N}}\mathcal{T}_{\mathfrak{p}}(s) &= \sum_{n=1}^N \xi^T_1\{\tilde{\mathbf{P}}\odot\mathbf{M}(s)\}^n\xi_m \\ &= \xi^T_1\left(\left\{\mathbf{I}_m - \tilde{\mathbf{P}}\odot\mathbf{M}(s)\right\}^{-1}\left[\mathbf{I}_m - \{\tilde{\mathbf{P}}\odot\mathbf{M}(s)\}^{N+1}\right]\right)\xi_m \end{aligned}\tag{6}$$

when $\mathbf{I}_m - \tilde{\mathbf{P}}\odot\mathbf{M}(s)$ is invertible, i.e. on $\mathfrak{C} = \{s\in\mathbb{C} : ||\lambda_1\{\tilde{\mathbf{P}}\odot\mathbf{M}(s)\}|| < 1\}$.

For $s \in \mathfrak{C}$, $\{\tilde{\mathbf{P}} \odot \mathbf{M}(s)\}^{N+1} \to \mathbf{0}$ as $N \to \infty$ for the following reason. Complex matrix $\tilde{\mathbf{P}} \odot \mathbf{M}(s)$ is similar to its Jordan form matrix $\mathbf{J}(s)$ hence its Nth power is similar to $\mathbf{J}(s)^N$. By similarity the largest modulus term in $\{\tilde{\mathbf{P}} \odot \mathbf{M}(s)\}^N$ has the same asymptotic order as the largest modulus component of $\mathbf{J}(s)^N$ which is $O\left(N^m \|\lambda_1\{\tilde{\mathbf{P}} \odot \mathbf{M}(s)\}\|^N\right) \to 0$ as $N \to \infty$.

Thus, the right-hand side of (6) has the finite limit

$$\sum_{\mathfrak{p} \in \mathfrak{P}} \mathcal{T}_{\mathfrak{p}}(s) := \lim_{N \to \infty} \sum_{\mathfrak{p} \in \mathfrak{P}^{\leq N}} \mathcal{T}_{\mathfrak{p}}(s) = \xi_1^T \left\{ \mathbf{I}_m - \tilde{\mathbf{P}} \odot \mathbf{M}(s) \right\}^{-1} \xi_m \tag{7}$$
$$= \frac{(-1)^{m+1} |\mathbf{\Psi}_{m;1}(s)|}{|\mathbf{I}_m - \tilde{\mathbf{P}} \odot \mathbf{M}(s)|} = \frac{(-1)^{m+1} |\mathbf{\Psi}_{m;1}(s)|}{|\mathbf{\Psi}_{m;m}(s)|} \qquad s \in \mathfrak{C}.$$

Cramer's rule has been used in the third equality and the last equality follows from the fact that the last row of $\mathbf{I}_m - \tilde{\mathbf{P}} \odot \mathbf{M}(s)$ is ξ_m^T. The left side of (7) is therefore well-defined and converges to the cofactor ratio for $s \in \mathfrak{C} \subset \mathbb{C}$. Note that the values of s which lead to an eigenvalue of 1 for $\tilde{\mathbf{P}} \odot \mathbf{M}(s)$ are also zeros of $|\mathbf{\Psi}_{m;m}(s)|$ and vice versa. Under conditions $\mathcal{R}_{1 \to m}$ and $\mathcal{CD}_{1 \to m}$, $\mathfrak{C} \supset \{\mathrm{Re}(s) < b\}$ where b is the smallest positive zero of $|\mathbf{\Psi}_{m;m}(s)|$. Conditions $\mathcal{R}_{1 \to m}$ and $\mathcal{CD}_{1 \to m}$ are needed to ensure that $\mathbf{\Psi}_{m;1}(s)$ and $\mathbf{\Psi}_{m;m}(s)$ are complex analytic in a neighbourhood of 0 and that $|\mathbf{\Psi}_{m;m}(0)| > 0$. Since $|\mathbf{\Psi}_{m;m}(s)|$ is monotone decreasing in $s \geq 0$, a smallest positive zero for $|\mathbf{\Psi}_{m;m}(s)|$ must exist.

The identity in (7) continues to hold outside of $\{\mathrm{Re}(s) < b\}$ and in $\mathfrak{R} \subset \mathbb{C}$ as given in (10). For all $s \in \mathfrak{R}$, all elements of $\mathbf{M}_{m;}(s)$ with $p_{ij} > 0$ have modulus < 1 which places matrix $\mathbf{M}_{m;}(s)$ in the convergence domain of $\mathcal{P}(\mathbf{Z}_{m;}|X < \infty)$.

7.4. Proof of Proposition 3

First: Presuming a convergence domain of $\{\mathrm{Re}(s) < b\}$, we show $b < b_{\mathcal{I}}$. With this bound and $\mathcal{FS}_{\to m}$, then $b < b_{\min}$.

The proof amounts to finding the domain on which the expression $\mathcal{T}_{\mathfrak{P}}(s) = \sum_{\mathfrak{p} \in \mathfrak{P}} \mathcal{T}_{\mathfrak{p}}(s)$ is convergent. The class of all finite-step pathways $\mathfrak{P}$ is partitioned into two subsets, $\mathfrak{P}_0$ and $\mathfrak{P}_0^C$, so that

$$f_{1m} \mathcal{F}_{1m}(s) = \mathcal{T}_{\mathfrak{P}}(s) = \mathcal{T}_{\mathfrak{P}_0}(s) + \mathcal{T}_{\mathfrak{P}_0^C}(s),$$

where $\mathcal{T}_{\mathfrak{P}_0}(s)$ and $\mathcal{T}_{\mathfrak{P}_0^C}(s)$ denote sums of transmittances for the distinct pathways in $\mathfrak{P}_0$ and $\mathfrak{P}_0^C$ respectively. If $\mathcal{T}_{\mathfrak{P}}(s) < \infty$ for some $s \in (0, b)$, then $\mathcal{T}_{\mathfrak{P}_0}(s) < \mathcal{T}_{\mathfrak{P}}(s) < \infty$. Hence, if $\mathcal{T}_{\mathfrak{P}_0}(s)$ has a pole at s_0, then $\mathcal{T}_{\mathfrak{P}}(s)$ cannot be convergent at s_0 and $b < s_0$. Subset $\mathfrak{P}_0$ will be constructed in such a way that $\mathcal{T}_{\mathfrak{P}_0}$ has an infinity in $(0, b_{\mathcal{I}})$ so the convergence domain of $\mathcal{T}_{\mathfrak{P}_0}(s)$ is a proper subset of $\{\text{Re}(s) < b_{\mathcal{I}}\}$. Hence we conclude that $b < b_{\mathcal{I}}$.

For simplicity of argument, suppose $(2, 3) \in \mathcal{L}$ so that $b_{23} = b_{\mathcal{I}}$ and $\mathcal{M}_{23}(s)$ converges on $\{\text{Re}(s) < b_{\mathcal{I}}\}$ (the proof does not depend upon this particular choice). To get $\mathfrak{P}_0$, construct a sojourn that passes from state 2 to 3, loops back to 2, and then is absorbed into m. For definiteness, suppose this sequence is $1 \to 6 \to 2 \to 3 \to 2 \to m$ and call it pathway $\mathfrak{p}_0$ with feedback loop $2 \to 3 \to 2$ denoted as $\mathfrak{f}_0$ (again the proof does not depend upon these choices). Such a pathway exists because 2 is a relevant state and we can return to 2 from another state since all states in $\mathbb{I}_{m-1}$ communicate. The transmittance of this pathway $\mathfrak{p}_0$ is

$$\mathcal{T}_{\mathfrak{p}_0}(s) = \mathcal{T}_{16}(s)\mathcal{T}_{62}(s)\mathcal{T}_{\mathfrak{f}_0}(s)\mathcal{T}_{2m}(s) \qquad \text{Re}(s) < b_{23} = b_{\mathcal{I}}, \tag{8}$$

where $\mathcal{T}_{\mathfrak{f}_0}(s) = \mathcal{T}_{23}(s)\mathcal{T}_{32}(s)$. The existence of such a pathway $\mathfrak{p}_0$ for which $\mathcal{T}_{\mathfrak{p}_0}(0) > 0$ relies on the assumption of an irreducible class. Now let $\mathfrak{P}_0$ consist of the additional distinct pathways in which loop $\mathfrak{f}_0$ is allowed to repeat in sequence an arbitrary number of times. The pathway taking k such loops in sequence has transmittance

$$\mathcal{T}_{\mathfrak{p}_0,k}(s) = \mathcal{T}_{16}(s)\mathcal{T}_{62}(s)\mathcal{T}_{\mathfrak{f}_0}^k(s)\mathcal{T}_{2m}(s) \qquad k = 0, 1, \ldots \tag{9}$$

Let $\mathfrak{P}_0$ be the collection of pathways in (9). The summation is

$$\mathcal{T}_{\mathfrak{P}_0}(s) = \sum_{k=0}^{\infty} \mathcal{T}_{\mathfrak{p}_0,k}(s) = \frac{\mathcal{T}_{16}(s)\mathcal{T}_{62}(s)\mathcal{T}_{2m}(s)}{1 - \mathcal{T}_{\mathfrak{f}_0}(s)}. \tag{10}$$

The convergence domain for (10) is limited by zeros of $1 - \mathcal{T}_{\mathfrak{f}_0}(s)$. Since this function is monotonic decreasing over $s \in [0, b_{\mathcal{I}})$, ranging from $1 - \mathcal{T}_{\mathfrak{f}_0}(0) > 0$

to $1-\mathcal{T}_{\mathfrak{f}_0}(b_{\mathcal{I}})=-\infty$ (since $\mathcal{M}_{23}(b_{\mathcal{I}})=\infty$), $\mathcal{T}_{\mathfrak{P}_0}(s)$ has a simple pole at some $c\in(0,b_{\mathcal{I}})$. Thus, the convergence domain of $\mathcal{T}_{\mathfrak{P}_0}(s)$ and hence $\mathcal{F}_{1m}(s)$ is a subset of $\{\mathrm{Re}(s)<c\}$, where $c<b_{\mathcal{I}}$.

Second: From Proposition 1, b is the smallest positive zero of $|\mathbf{\Psi}_{m;m}(s)|$ and we show that it is a simple zero. This follows using the Perron-Frobenius theory in Seneta (2006).

To show the smallest zero b is a simple zero, consider the set of irreducible nonnegative matrices $\{\mathbf{T}_{m;m}(s):|s-b|<\varepsilon,\ s\in\mathbb{R}\}$ with ε sufficiently small. All these matrices have the same pattern of zeros and positive entries given by $\mathbf{T}_{m;m}(0)=\mathbf{P}_{m;m}$. Denote the $m-1$ eigenvalues of $\mathbf{T}_{m;m}(s)$ as $\lambda_1(s),\ldots,\lambda_{m-1}(s)$ so that

$$|\mathbf{\Psi}_{m;m}(s)|=\{1-\lambda_1(s)\}\prod_{j\geq 2}\{1-\lambda_j(s)\}, \tag{11}$$

where $\lambda_1(s)$ is the dominant real Perron-Frobenius eigenvalue.

If $\mathbf{P}_{m;m}$ is aperiodic then both $\mathbf{P}_{m;m}$ and $\mathbf{T}_{m;m}(s)$ are primitive (Seneta, 2006, thm. 1.4, p. 21). Then, by Theorem 1.1(a) and (c) of Seneta (2006, p. 3–4), $0<\lambda_1(s)>\|\lambda_j(s)\|$ for $j\geq 2$. If b is the smallest positive zero of $|\mathbf{\Psi}_{m;m}(s)|$, then it is also a zero of $1-\lambda_1(s)$. Since the components of $\mathbf{T}_{m;m}(s)$ are all real analytic in a neighbourhood of $\{|s-b|<\varepsilon\}\subset\mathbb{R}$, then so also is eigenvalue $\lambda_1(s)$. Therefore, as $s\to b$, $1-\lambda_1(s)\to 0$ and it has the expansion

$$1-\lambda_1(s)=-\lambda_1'(b)(s-b)+O(s-b)^2. \tag{12}$$

The value of $\lambda_1'(b)$ can be determined by direct implicit differentiation using the chain rule. Take $\mathbf{C}_{m;m}(s):=\lambda_1(s)\mathbf{I}_{m-1}-\mathbf{T}_{m;m}(s)$ and differentiate $0\equiv|\mathbf{C}_{m;m}(s)|$. Using the fact that $d|\mathbf{A}|=\mathrm{tr}\{\mathrm{adj}(\mathbf{A})d\mathbf{A}\}$, where $\mathrm{adj}(\mathbf{A})$ is the adjoint or adjugate of $\mathbf{A}$, then

$$0=d|\mathbf{C}_{m;m}(s)|=\mathrm{tr}\left[\mathrm{adj}\{\mathbf{C}_{m;m}(s)\}\left\{\lambda_1'(s)\mathbf{I}_{m-1}-\mathbf{T}_{m;m}'(s)\right\}\right]ds, \tag{13}$$

where $\mathbf{T}'_{m;m}(s) = \partial \mathbf{T}_{m;m}(s)/\partial s$. At $s = b$ we solve (13) to get

$$\lambda_1'(b) = \frac{\operatorname{tr}\left[\operatorname{adj}\{\mathbf{C}_{m;m}(b)\}\mathbf{T}'_{m;m}(b)\right]}{\operatorname{tr}\left[\operatorname{adj}\{\mathbf{C}_{m;m}(b)\}\right]} > 0. \tag{14}$$

To see that the ratio in (14) is positive, first note $\lambda_1(b) = 1$ is the Perron-Frobenius eigenvalue of $\mathbf{T}_{m;m}(b)$. Hence, all the cofactors of $\mathbf{I}_{m-1} - \mathbf{T}_{m;m}(b)$ which comprise $\operatorname{adj}\{\mathbf{C}_{m;m}(b)\}$ are positive; see the proof of theorem 1.1 part (f) on p. 7 in Seneta (2006). Since all components of $\mathbf{T}'_{m;m}(b)$ are nonnegative and at least one is positive, then both the numerator and denominator of (14) are positive. Substituting (12) into (11) gives

$$|\mathbf{\Psi}_{m;m}(s)| = -\lambda_1'(b)(s-b)\prod_{j\geq 2}\{1-\lambda_j(b)\} + O(s-b)^2 \tag{15}$$

as $s \to b$. The factor $\prod_{j\geq 2}\{1-\lambda_j(b)\} > 0$ because (a) it is a product of complex conjugate pairs and (b) none of the terms are zero due to $\|\lambda_j(b)\| < \lambda_1(b) = 1$ for $j \geq 2$. Thus, b is a simple zero.

When $\mathbf{P}_{m;m}$ is d-periodic, then matrix $\mathbf{T}_{m;m}(s)$ has d eigenvalues of magnitude $\lambda_1(s) > 0$ of the form $\lambda_1(s)\alpha_j$ where $\{\alpha_j = \exp(i2\pi j/d) : j = 0, \ldots, d-1\}$ as specified in Theorem 1.7 of Seneta (2006, p. 23). The value $\lambda_1'(b)$ is given in (14) and is positive since all entries of $\operatorname{adj}\{\mathbf{C}_{m;m}(b)\}$ are positive. Proof of this latter result follows from Theorem 1.5 of Seneta (2006, p. 22) and is based on replicating the argument used to prove Theorem 1.1 part (f) on p. 7 of Seneta (2006) for the periodic case as the argument does not rely on aperiodicity. The factor $\prod_{j\geq 2}\{1-\lambda_j(b)\} > 0$ by the same logic as in the aperiodic case. Note also that

$$1 - \lambda_j(b) = 1 - \lambda_1(b)\alpha_j = 1 - \alpha_j \neq 0 \qquad j = 1, \ldots, d-1.$$

Thus b is a simple zero of $|\mathbf{\Psi}_{m;m}(s)|$ when $\mathbf{P}_{m;m}$ is irreducible regardless of whether it is aperiodic or periodic.

Third: We show $(-1)^{m+1}|\mathbf{\Psi}_{m;1}(b)| > 0$. Let the cofactors of $\mathbf{\Psi}_{m;m}(b)$ be denoted as $\{C_{ij} : i, j = 1, \ldots, m-1\}$. They must all be positive according to the Perron-Frobenius theory mentioned in the second part of the proof. Then $|\mathbf{\Psi}_{m;m}(b)|$ has a cofactor expansion down its first column as

$$0 = |\mathbf{\Psi}_{m;m}(b)| = \{1 - \mathcal{T}_{11}(b)\}C_{11} - \sum_{j=2}^{m-1} \mathcal{T}_{j1}(b)C_{j1}. \tag{16}$$

Since the last term of (16) is strictly negative and $C_{11} > 0$, then necessarily $1 - \mathcal{T}_{11}(b) > 0$.

A cofactor expansion of $\mathbf{\Psi}_{m;1}(b)$ can be expressed in terms of the same cofactors. Take the $(m,1)$ minor of $\mathbf{I}_m - \mathbf{T}(b)$, move its last column $\{-\mathcal{T}_{1m}(b), \ldots, -\mathcal{T}_{m-1,m}(b)\}^T$ forward so it becomes the leading column of the resulting matrix. This leads to a matrix denoted as $\mathbf{\Psi}_{m;m;1\leftarrow m}(b)$ which is the (m,m) minor of $\mathbf{I}_m - \mathbf{T}(b)$ but with its first column replaced by the last column of the $(m,1)$ minor. The $m-2$ steps used to move the column changes the sign of $|\mathbf{\Psi}_{m;1}(b)|$ by a factor of $(-1)^{m-2}$ so

$$(-1)^{m+1}|\mathbf{\Psi}_{m;1}(b)| = (-1)^{m+1}(-1)^{m-2}|\mathbf{\Psi}_{m;m;1\leftarrow m}(b)| = -|\mathbf{\Psi}_{m;m;1\leftarrow m}(b)|.$$

Taking the cofactor expansion of $|\mathbf{\Psi}_{m;m;1\leftarrow m}(b)|$ down the first column yields

$$(-1)^{m+1}|\mathbf{\Psi}_{m;1}(b)| = -\sum_{j=1}^{m-1} -\mathcal{T}_{jm}(b)C_{j1} > 0. \tag{17}$$

7.5. Proof of Propositions 4 and 5

We consider the convergence domain of $\mathcal{P}(\mathbf{Z}_{m;}|\, X < \infty)$ for $\mathbf{Z}_{m;} \in \mathbb{C}^{(m-1)\times m}$ and show it is convergent for $\|\mathbf{Z}\|_{m;} = \{\, \|z_{ij}\|\}_{m;} \in \mathbb{R}^{(m-1)\times m}$ in the open $(m-1)m$-dimensional rectangle $(\mathbf{0}, \mathbf{M}_{m;}(b)) \subset \mathbb{R}^{(m-1)\times m}$ when $\mathbf{P}_{m;} > \mathbf{0}$ and all its components are > 0. For settings where $\mathbb{Z}^2 = \{(i,j) \in \mathbb{I}_{m-1} \times \mathbb{I}_m : p_{ij} = 0\} \neq \emptyset$, then the statement of convergence means convergent for $\mathbf{Z}_{m;}$ with arbitrary values of $\{z_{ij} : (i,j) \in \mathbb{Z}^2\}$ and with values $||z_{ij}|| < \mathcal{M}_{ij}(b)$ for all $(i,j) \in \bar{\mathbb{Z}}^2 = \mathbb{I}_{m-1} \times \mathbb{I}_m \backslash \mathbb{Z}^2$.

Assume $\mathbf{P}_{m;} > \mathbf{0}$ and take $\mathbf{Z}_{m;} \in \mathbb{C}^{(m-1)\times m}$ such that $\|\mathbf{Z}\|_{m;} \in \mathfrak{O} \subset$

$\mathbb{R}^{(m-1)\times m}$, where $\mathfrak{O}$ is in (8). Then,

$$\|\mathcal{P}(\mathbf{Z}_{m;}|X<\infty)\|= \left\| \mathbb{E}\left(\prod_{i\in\mathbb{I}_{m-1}} \prod_{j\in\mathbb{I}_m} z_{ij}^{N_{ij}} \right) \right\| \leq \mathbb{E}\left(\prod_{i\in\mathbb{I}_{m-1}} \prod_{j\in\mathbb{I}_m} \|z_{ij}\|^{N_{ij}} \right)$$
$$= \mathcal{P}(\, \|\mathbf{Z}\|_{m;} \, |X<\infty) < \infty. \tag{18}$$

For real-valued $\mathbf{0} \leq \mathbf{Z}_{m;} \in \mathfrak{O}$, generating function $\mathcal{P}(\mathbf{Z}_{m;}|X<\infty)$ is strictly increasing in each component of $\mathbf{Z}_{m;}$. As $s \in [0,b)$ increases, then each component of $\mathbf{M}_{m;}(s)$ increases forming a one-dimensional path through $\mathfrak{O} \cap [1,\infty)^{(m-1)\times m}$ with tangent direction $\mathbf{M}'_{m;}(s)$ such that $\mathbf{M}'_{m;}(s) > \mathbf{0}$ componentwise for each $s \in [0,b)$. Since $b < b_{\min}$, the path crosses the boundary of $\mathfrak{O} \cap (0,\infty)^{(m-1)\times m}$ at $\mathbf{M}_{m;}(b) \in (1,\infty)^{(m-1)\times m}$ so that $\mathcal{P}(\mathbf{Z}_{m;}|X<\infty)$ converges in the open $(m-1)m$-dimensional rectangle $\mathbf{Z}_{m;} \in (\mathbf{0}, \mathbf{M}_{m;}(b)) \subset \mathbb{R}^{(m-1)\times m}$. For $\mathbf{Z}_{m;} \in \mathbb{C}^{(m-1)\times m}$, $\mathcal{P}(\mathbf{Z}_{m;}|X<\infty)$ is convergent for $\|\mathbf{Z}\|_{m;} \in (\mathbf{0}, \mathbf{M}_{m;}(b))$.

More generally, some components of $\mathbf{P}_{m;}$ are zero for branches in $\mathbb{Z}^2$. In this setting, the same argument used above applies but with the geometry of $\mathfrak{O} \cap [1,\infty)^{(m-1)\times m}$ restricted to the subspace of branches in complementary set $\bar{\mathbb{Z}}^2 = \{(i,j) \in \mathbb{I}_{m-1} \times \mathbb{I}_m : p_{ij} > 0\}$. Since the value of $\mathcal{P}(\mathbf{Z}_{m;}|X<\infty)$ does not depend on values of z_{ij} when $(i,j) \in \mathbb{Z}^2$, we may conclude generally that $\mathcal{P}(\mathbf{Z}_{m;}|X<\infty)$ is convergent for $\mathbf{Z}_{m;}$ with arbitrary values of $\{z_{ij} : (i,j) \in \mathbb{Z}^2\}$ and with values $||z_{ij}|| < \mathcal{M}_{ij}(b)$ for all $(i,j) \in \bar{\mathbb{Z}}^2$. In the remaining proofs of §7.5.1, we assume $\mathbb{Z}^2 = \emptyset$ with all $p_{ij} > 0$ for $(i,j) \in \mathbb{I}_{m-1} \times \mathbb{I}_m$. When this is not the case, then the same proofs hold when modified to consider only components $(i,j) \in \bar{\mathbb{Z}}^2$.

7.5.1. *Continuous-time processes* Since $b < b_{\min}$ and b is a pole of $\mathcal{F}_{1m}(s)$, then b must be a zero of $|\mathbf{\Psi}_{m;m}(s)|$. Furthermore, there must exist a $\eta_1 > 0$ such that all components of $\mathbf{\Psi}_{m;m}(s)$ and $\mathbf{\Psi}_{m;1}(s)$ are analytic on $\{s \in \mathbb{C} : \mathrm{Re}(s) \leq b+\eta_1\}$. Thus, there exists $Y > 0$ such that

$$\max_{(i,j)\in\mathbb{I}_{m-1}\times\mathbb{I}_m} \max_{0\leq x\leq b+\eta_1} \|\mathcal{M}_{ij}(x+iy)\| < 1 \qquad \forall |y| > Y.$$

This ensures for any $s \in A := \{s \in \mathbb{C} : 0 \leq \mathrm{Re}(s) \leq b + \eta_1 \cap \mathrm{Im}(s) > Y\} \subset \mathbb{C}$ that

$$\|\mathbf{M}_{m;}(s)\| := \{\|\mathcal{M}_{ij}(s)\|\}_{m;} \in \left\{\mathbf{Z}_{m;} \in \Re^{(m-1)\times m} : \|z_{ij}\| \leq 1 \text{ for } (i,j) \in \mathbb{I}_{m-1} \times \mathbb{I}_m\right\} \subset \mathfrak{O}.$$

Thus, the compound MGF $\mathcal{F}_{1m}(s) = \mathcal{P}\{\mathbf{M}_{m;}(s)|X < \infty\}/f_{1m}$ is analytic on both $A \subset \mathbb{C}$ and its the complex conjugate region $\bar{A} \subset \mathbb{C}$ where $A \cup \bar{A}$ includes the distant regions away from the real axis in both its convergence domain and in a η_1-strip of its analytic continuation. Thus, analytic continuation holds on $\{s \in \mathbb{C} : \mathrm{Re}(s) < b + \eta_1, \, |\,\mathrm{Im}(s)| > Y\}$.

Now consider analytic continuation when $\{s \in \mathbb{C} : |\,\mathrm{Im}(s)| \leq Y\}$. Since b is an isolated pole of $\mathcal{F}_{1m}(s)$, there exists $\eta_2 > 0$ such that $\mathcal{F}_{1m}(s)$ is analytic in $D(b, \eta_2)\backslash\{b\}$, an open punctured disk of radius η_2 centred at b.

Now consider points on the line $\{b + iy : \eta_2/2 \leq y \leq Y + 1\}$. For every $y \in [\eta_2/2, Y + 1]$, we show there exists disk $D(b + iy, \eta_y)$ with $\eta_y > 0$ such that $\mathcal{F}_{1m}(s)$ is analytic on $D(b + iy, \eta_y)$. The argument for this uses the fact that

$$\|\mathcal{M}_{ij}(b + iy)\| < \mathcal{M}_{ij}(b) < \infty \qquad y \neq 0; \, (i,j) \in \mathbb{I}_{m-1} \times \mathbb{I}_m \tag{19}$$

as was shown by Daniels (1954, p. 632) for absolutely continuous distributions. That (19) holds with non-strict inequality $\leq$ follows directly from the triangle inequality for integrals; that it holds with strict inequality is what Daniels showed.

From (19), there exists a sufficiently small $\eta_y > 0$ such that

$$\sup_{s \in D(b+iy,\eta_y)} \|\mathbf{M}_{m;}(s)\| < \mathbf{M}_{m;}(b), \tag{20}$$

where the sup is performed individually over each component $(i,j) \in \bar{\mathbb{Z}}^2$. Applying inequalities (18) and (20) for any $s \in D(b + iy, \eta_y)$, then

$$\|\mathcal{P}\{\mathbf{M}_{m;}(s)|X < \infty\}\| \leq \mathcal{P}\{\,\|\mathbf{M}_{m;}(s)\|\,|X < \infty\} < \mathcal{P}\{\mathbf{M}_{m;}(b)|X < \infty\} = \infty$$

so that $\mathcal{F}_{1m}(s)$ is analytic in $D(b+iy, \eta_y)$. Thus, $\mathcal{F}_{1m}(s)$ is analytic on the open cover $\cup_{y \in [\eta_2/2, Y+1]} D(b + iy, \varepsilon_y)$ for compact set $\{b + iy : \eta_2/2 \leq y \leq Y + 1\}$.

Compactness guarantees a finite subcover $\cup_{j=1}^{n} D(b+iy_j,\varepsilon_{y_j})$ with $y_1 < y_2 < \cdots < y_n$. The two circular neighbourhoods associated with contiguous points (b, iy_j) and (b, iy_{j+1}) create a rectangle $[b, b+\lambda_j] \times [iy_j, iy_{j+1}]$ with $\lambda_j > 0$ on which $\mathcal{F}_{1m}(s)$ is analytic. Thus, if ε_0 is taken to be $\varepsilon_0 = \min\{\eta_2, \lambda_1, \ldots \lambda_{n-1}\}$, then $\mathcal{F}_{1m}(s)$ is analytic on $\{0 \le \mathrm{Re}(s) \le b+\varepsilon_0\}\backslash\{b\}$.

7.5.2. *Integer-time processes* The arguments in this setting are the same as those for the continuous-time setting and only concern $\mathcal{F}_{1m}(s)$ defined within $\{s : \mathrm{Re}(s) \le b+\eta_1$ and $-\pi < \mathrm{Im}(s\} \le \pi\}$, where $\{-\pi < \mathrm{Im}(s\} \le \pi\}$ restricts consideration to the principal convergence domain in the integer-lattice case. For points on the boundary $\{b+iy : \eta_2/2 < y \le \pi\}$, we can find an open cover $\cup_{y\in[\eta_2/2,\pi]} D(b+iy,\varepsilon_y)$ on which $\mathcal{F}_{1m}(s)$ is analytic if (19) can be shown to hold for $y \in (-\pi,\pi]\backslash\{0\}$. The required strict inequality in (19) does not hold if the mass function for $\mathcal{M}_{ij}$ is either periodic or degenerate at a single integer. In the former case the maximum modulus $\mathcal{M}_{ij}(b)$ repeats itself for $y \in (-\pi,\pi]$ according to the periodicity while in the latter case the inequality becomes an identity. Apart from these two settings, strict inequality holds as formalised in Lemma 1 below. Thus, the remainder of the argument in continuous time applies to give $\mathcal{F}_{1m}$ an analytic continuation $\{s : \mathrm{Re}(s) \le b+\varepsilon_0$ and $-\pi < \mathrm{Im}(s\} \le \pi\}\backslash\{b\}$ for some $\varepsilon_0 > 0$.

Lemma 1. *In the integer-time setting, condition $\mathcal{ND}$-$\mathcal{A}_{1\to m}$ ensures that the strict inequality in (19) holds for $y \in (-\pi,\pi]\backslash\{0\}$.*

Proof. The proof is by contradiction. Suppose $\mathcal{ND}$-$\mathcal{A}_{1\to m}$ holds but (19) does not. Then there must exist $y \in (-\pi,\pi]\backslash\{0\}$ such that $\mathcal{M}_{ij}(b+iy) = \mathcal{M}_{ij}(b)e^{\alpha i}$ for some α. Since $b+iy$ and b are in the convergence domain of $\mathcal{M}_{ij}$,

$$\begin{aligned} 0+0i = \mathcal{M}_{ij}(b+iy) - \mathcal{M}_{ij}(b)e^{i\alpha} &= \sum_{n\ge 0} e^{bn} p(n) \left[\{\cos(yn) - \cos\alpha\} + i\{\sin(yn) - \sin\alpha\}\right] \\ &= A + iB, \end{aligned}$$

where p is the mass function for $\mathcal{M}_{ij}$. Thus,

$$0 = A\cos\alpha + B\sin\alpha = \sum_{n\geq 0} e^{bn} p(n)\{\cos(yn-\alpha) - 1\}. \tag{21}$$

For equality (21) to hold, we require that $yn - \alpha \in \{0, \pm 2\pi, \ldots\}$ for a.e. n, i.e. for $\{n \geq 0 : p(n) > 0\}$. We now show this cannot hold, hence we reach a contradiction, if $\mathcal{ND}$-$\mathcal{A}_{1\to m}$ holds. There are two aperiodic settings to consider.

Suppose the support for p is two points. It can only be aperiodic in this case if there is an n_0 for which $p(n_0) > 0 < p(n_0+1)$. Then a solution in $y \in (-\pi, \pi]\backslash\{0\}$ to

$$\begin{aligned} y(n_0+1) - \alpha &= 2m_0\pi \qquad \exists m_0 \in \mathbb{I} \\ yn_0 - \alpha &= 2m_1\pi \qquad \exists m_1 \in \mathbb{I},\ m_1 \neq m_0 \end{aligned}$$

is $y = 2(m_0 - m_1)\pi$ which is not in $(-\pi, \pi]\backslash\{0\}$. Thus we reach a contradiction.

Suppose the support for p is three or more points $\{n_i : i = 0-2\}$ for which $n_1 - n_0$ and $n_2 - n_1$ have no common prime factors. A solution to

$$yn_i - \alpha = 2m_i\pi \qquad \exists_{m_i\in\mathbb{I}} \text{ for } i = 0-2$$

can be obtained by taking first differences to get

$$\begin{aligned} y(n_1 - n_0) &= 2(m_1 - m_0)\pi \\ y(n_2 - n_1) &= 2(m_2 - m_1)\pi. \end{aligned} \tag{22}$$

In this solution, $n_1 - n_0$ cannot divide $m_1 - m_0$ since if it did, then $y = 2m_3\pi$ for some integer m_3 and $y \notin (-\pi, \pi]\backslash\{0\}$. Thus

$$2\pi\frac{m_2 - m_1}{n_2 - n_1} = y = 2\pi\frac{m_1 - m_0}{n_1 - n_0}. \tag{23}$$

The $\{m_i\}$ must be distinct since otherwise $y = 0$ by (22). Therefore

$$m_2 - m_1 = \frac{1}{n_1 - n_0}(n_2 - n_1)(m_1 - m_0).$$

If $n_1 - n_0 = 1$, then the argument from the first case provides a contradiction. If $n_1 - n_0 \geq 2$, then its prime factors must all cancel with prime factors of

$(n_2 - n_1)(m_1 - m_0)$ for $m_2 - m_1$ to be an integer. By assumption $\mathcal{ND}$-$\mathcal{A}_{1\to m}$, none cancel with $n_2 - n_1$ so all must cancel with $m_1 - m_0$ and hence $n_1 - n_0$ must divide $m_1 - m_0$. Hence by (23), $y = 2m_4\pi$ for some integer $m_4 \neq 0$ so $y \notin (-\pi, \pi]\backslash\{0\}$ and a contradiction is reached. □

7.6. Proof of Theorem 1

The integer-time results follow directly from Butler (2020, thm. 2). The continuous-time results require the arguments below which show that the conditions of Theorems 1 and 2 in Butler (2019) are satisfied and therefore apply.

Continuous-time survival expansion. The derivation of (14) uses Butler (2019, thm. 2). The results of Proposition 4 ensure that condition $\mathcal{AC}$ of Theorem 2 in Butler (2019) holds.

We show that $\mathcal{F}_{1m}(s)$ satisfies condition $\mathcal{X}$ of Theorem 2 in Butler (2019). Since $\{b \leq \text{Re}(s) \leq b + \varepsilon_0\}$ lies within the convergence domain for all MGFs in the first $m-1$ rows of matrix $\mathbf{T}(s)$, then

$$\max_{0 \leq x \leq \varepsilon} |\mathcal{M}_{ij}(b + x + iN)| \to 0$$

as $N \to \infty$ for all $(i,j) \in \mathbb{I}_{m-1} \times \mathbb{I}_m$. Thus, for the denominator of $\mathcal{F}_{1m}$, $|\mathbf{\Psi}_{m;m}(b + x + iN)| \to 1$ uniformly over all $x \in [0, \varepsilon]$. For the numerator, $|\mathbf{\Psi}_{m;1}(b + x + iN)| \to 0$ uniformly in $x \in [0, \varepsilon]$ as $N \to \infty$. Thus, $\mathcal{F}_{1m}(b + x + iN) \to 0$ uniformly for all $x \in [0, \varepsilon]$ and condition $\mathcal{X}$ holds.

The remaining condition $\mathcal{UI}^S$ is to show that the principal-value integral

$$\int_{-\infty}^{\infty} \frac{\mathcal{F}_{1m}(b^+ + iy)}{b^+ + iy} e^{-ity} dy \tag{24}$$

is uniformly integrable for $t > T_0$, for some T_0. Break the integral into the range $[-N, N]$ and $(-\infty, -N] \cup [N, \infty)$ where $N > Y$ and Y is chosen sufficiently large that

$$\max_{(i,j)\in\mathbb{I}_{m-1}\times\mathbb{I}_m} \sup_{|y|\geq Y} \left\| \mathcal{M}_{ij}(b^+ + iy) \right\| < 1. \tag{25}$$

Using the Riemann-Lebesgue lemma for arbitrary $\eta > 0$ and N,

$$\int_{-N}^{N} \frac{\mathcal{F}_{1m}(b^+ + iy)}{b^+ + iy} e^{-ity} dy < \eta/2 \qquad t \geq T_1(N). \tag{26}$$

For the remaining portion of the integral, use identity (9) of Proposition 2 which holds in the analytic continuation due to (25) and $N > Y$. This allows us to consider this integral separately over each distinct pathway from $1 \to m$ and then add them up. For each pathway transmittance $\mathcal{T}_{\mathfrak{p}}$ and with $N > Y$,

$$\left(\int_N^\infty + \int_{-\infty}^{-N} \right) \frac{\mathcal{T}_{\mathfrak{p}}(b^+ + iy)}{b^+ + iy} e^{-ity} dy \leq 2p_{\mathfrak{p}} \int_N^\infty \left| \frac{\mathcal{M}_{\mathfrak{b}(\mathfrak{p})}(b^+ + iy)}{b^+ + iy} \right| dy, \tag{27}$$

where $p_{\mathfrak{p}}$ is the probability of pathway $\mathfrak{p}$ and $\mathfrak{b}(\mathfrak{p}) \in \mathfrak{p} \cap \mathcal{B}$ denotes a branch of blockade set $\mathcal{B}$ in pathway $\mathfrak{p}$. Using Hölder's inequality, then (27) is bounded above by

$$2p_{\mathfrak{p}} \left\{ \int_N^\infty \left\| \mathcal{M}_{\mathfrak{b}(\mathfrak{p})}(b^+ + iy) \right\|^q dy \right\}^{1/q} \left\{ \int_N^\infty \frac{1}{\|b^+ + iy\|^r} dy \right\}^{1/r} \tag{28}$$

where $q = q\{\mathfrak{b}(\mathfrak{p})\}$ and $1/q + 1/r = 1$. The product in (28) can be made $< \eta p_{\mathfrak{p}}/2$ for $N \geq N_1\{\mathfrak{b}(\mathfrak{p})\} > Y$. Thus, adding up all these upper bounds over all distinct pathways and bound (26), the value of (24) is $< \eta$ for $t \geq T_0 = T_1 \left[\max_{(i.j) \in \mathcal{B}} N_1\{(i,j)\} \right]$ and uniform integrability holds.

Continuous-time density expansion. The derivation of (16) uses Theorem 1 in Butler (2019).

We first must show that the unknown density $f(t)$ is locally of bounded variation. To do this, invert the summation in (9) term-by-term to give

$$f(t) = \sum_{\mathfrak{p} \in \mathfrak{P}} p_{\mathfrak{p}} f_{\mathfrak{p}}(t) \qquad \text{a.e. } t$$

where $f_{\mathfrak{p}}$ is the convolution of densities for pathway $\mathfrak{p}$. A formal proof that the inversion of a countably infinite summation may be done in this manner is given in Doetsch (1974, thm. 30.1). The total variation of f is therefore

$$V(f) \leq \sum_{\mathfrak{p} \in \mathfrak{P}} p_{\mathfrak{p}} V(f_{\mathfrak{p}}). \tag{29}$$

By Lemma 3 in Butler (2019, §5.2.1), $V(f_{\mathfrak{p}}) \leq \min_{(i,j)\in\mathfrak{p}} V(g_{ij})$. For each pathway $\mathfrak{p}$, at least one of its branches $\mathfrak{b}(\mathfrak{p}) \in \mathcal{B}$. Therefore, by assumption $\mathcal{BTV}_{1\to m}$,

$$V(f_{\mathfrak{p}}) \leq V(g_{\mathfrak{b}(\mathfrak{p})}) \leq \max_{(i,j)\in\mathcal{B}} V(g_{ij}) = V_{\max} < \infty.$$

Hence (29) is bounded above by $V_{\max}$ and f has bounded total variation.

The last remaining condition $\mathcal{UI}$ in Butler (2019, thm. 1) is uniform integrability of $\int_{-\infty}^{\infty} \mathcal{F}_{1m}(b^{+}+iy)e^{-ity}dy$ for $t > T_0$. By the Riemann-Lebesgue lemma,

$$\int_{-N}^{N} \mathcal{F}_{1m}(b^{+}+iy)e^{-ity}dy < \eta/2 \qquad t > T_1(N).$$

Suppose $\mathfrak{P}^{\geq}$ consists of all pathways in which at least $q(i,j)$ transitions from $i \to j$ occur for at least one branch $(i,j) \in \mathcal{B}_U \subseteq \mathbb{I}_{m-1} \times \mathbb{I}_{m-1}$ in the pathway. Let $\mathfrak{P}^{<} = \mathfrak{P}\backslash\mathfrak{P}^{\geq}$. In the integration over $(-\infty,-N]\cup[N,\infty)$ for $N > Y$, express $\mathcal{F}_{1m}$ as the summation over all pathways as in Proposition 2 and consider sets $\mathfrak{P}^{\geq}$ and $\mathfrak{P}^{<}$ separately. For pathway $\mathfrak{p} \in \mathfrak{P}^{\geq}$, denote $ij(\mathfrak{p}) \in \mathcal{B}_U$ as the branch of pathway $\mathfrak{p} \in \mathfrak{P}^{\geq}$ for which there are more than $q\{ij(\mathfrak{p})\}$ transitions. Also let $N_3(i,j)$ be such that

$$\int_{N_3(i,j)}^{\infty} \left\| \mathcal{M}_{ij(\mathfrak{p})}(b^{+}+iy) \right\|^{q\{ij(\mathfrak{p})\}} dy < \eta/4.$$

For infinite set $\mathfrak{P}^{\geq}$,

$$\begin{aligned}\sum_{\mathfrak{p}\in\mathfrak{P}^{\geq}} \left(\int_{-\infty}^{-N} + \int_{N}^{\infty} \right) \mathcal{T}_{\mathfrak{p}}(b^{+}+iy)e^{-ity}dy &\leq 2 \sum_{\mathfrak{p}\in\mathfrak{P}^{\geq}} p_{\mathfrak{p}} \int_{N}^{\infty} \left\| \mathcal{M}_{ij(\mathfrak{p})}(b^{+}+iy) \right\|^{q\{ij(\mathfrak{p})\}} dy \\ &< \eta/2 \sum_{\mathfrak{p}\in\mathfrak{P}^{\geq}} p_{\mathfrak{p}}\end{aligned}$$

for $N > N_1 := \max_{(i,j)\in\mathcal{B}_U} N_3(i,j) > Y$.

For finite set $\mathfrak{P}^{<}$, note $\{\mathrm{Re}(s) = b^{+}\}$ is in the convergence domain for $\mathcal{T}_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathfrak{P}^{<}$. Therefore, since $\mathcal{T}_{\mathfrak{p}}(b^{+}+iy)$ is the transform for $p_{\mathfrak{p}} e^{b^{+}t} f_{\mathfrak{p}}(t)$,

$$\begin{aligned}
&\sum_{\mathfrak{p}\in\mathfrak{P}^{<}} \left(\int_{-\infty}^{-N} + \int_{N}^{\infty} \right) \mathcal{T}_{\mathfrak{p}}(b^{+}+iy) e^{-ity} dy \\
&= \sum_{\mathfrak{p}\in\mathfrak{P}^{<}} \left\{ 2\pi p_{\mathfrak{p}} e^{b^{+}t} f_{\mathfrak{p}}(t) - \int_{-N}^{N} \mathcal{T}_{\mathfrak{p}}(b^{+}+iy) e^{-ity} dy \right\} \\
&< 2\pi \sum_{\mathfrak{p}\in\mathfrak{P}^{<}} p_{\mathfrak{p}} e^{b^{+}t} f_{\mathfrak{p}}(t) < \eta/2 \sum_{\mathfrak{p}\in\mathfrak{P}^{<}} p_{\mathfrak{p}} \qquad (30)
\end{aligned}$$

for $t > T_2$ where the last inequality needs to be justified. For this we need the following lemma.

Lemma 2. *For a finite number of densities $g_1, \ldots, g_n$, if $e^{b^{+}t} g_i(t) \to 0$ as $t \to \infty$ and $\mathcal{M}_{g_i}(b^{+}) < \infty$ for $i = 1, \ldots, n$, then the tilted convolution $e^{b^{+}t} \left(\circledast_{i=1}^{n} g_i \right)(t) \to 0$ as $t \to \infty$.*

Proof. The proof is by induction. For arbitrary $\eta_2 > 0$, let T_3 be such that $e^{b^{+}t/2} \max\{g_1(t/2), g_2(t/2)\} < \eta_2 / \{\mathcal{M}_{g_1}(b^{+}) + \mathcal{M}_{g_2}(b^{+})\}$ for $t > T_3$. Then,

$$e^{b^{+}t}(g_1 * g_2)(t) = \left(\int_{0}^{t/2} + \int_{t/2}^{t} \right) e^{b^{+}(t-u)} g_1(t-u) e^{b^{+}u} g_2(u) du. \qquad (31)$$

The choice of T_3 allows us to exploit that $e^{b^{+}(t-u)} g_1(t-u)$ is bounded above for $u \in [0, t/2]$ and $t > T_3$ and also $e^{b^{+}u} g_2(u)$ is bounded above for $u \in [t/2, t]$ and $t > T_3$. Thus, an upper bound for (31) is

$$\frac{\eta_2}{\mathcal{M}_{g_2}(b^{+}) + \mathcal{M}_{g_1}(b^{+})} \left(\int_{0}^{t/2} e^{b^{+}u} g_2(u) du + \int_{t/2}^{t} e^{b^{+}(t-u)} g_1(t-u) du \right) \leq \eta_2$$

for $t > T_3$. The remainder of the induction proof is the same argument. $\square$

By assumption $\mathcal{ZD}_{1\to m}$ and Lemma 2, $e^{b^{+}t} f_{\mathfrak{p}}(t) \to 0$ as $t \to \infty$ for each pathway $\mathfrak{p} \in \mathfrak{P}^{<}$. Hence (30) holds for $t > T_2$. Overall the value of T_0 required is $T_0 = \max\{T_2, T_1(N_1)\}$.

7.6.1. *Residue computation* The residue is

$$\beta_{-1} = \mathrm{Res}\{\mathcal{F}_{1m}(s); b\} = \frac{1}{f_{1m}} \lim_{s\to b}\{(s-b)\mathcal{F}_{1m}(s)\}$$
$$= \frac{\boldsymbol{\Psi}_{m;m}(0)\,|}{(-1)^{m+1}\,|\boldsymbol{\Psi}_{m;1}(0)|}\frac{(-1)^{m+1}\,|\boldsymbol{\Psi}_{m;1}(b)|}{\partial|\boldsymbol{\Psi}_{m;m}(s)\,|/\partial s|_{s=b}},$$

which is (11) since $\partial|\boldsymbol{\Psi}_{m;m}(s)\,|/\partial s|_{s=b} = \mathrm{tr}\left[\mathrm{adj}\{\boldsymbol{\Psi}_{m;m}(b)\}\boldsymbol{\Psi}'_{m;m}(b)\right]$.

7.6.2. *Proof of Corollary 1* The two conditions $\mathcal{ZD}_{1\to m}$ and $\mathcal{UB}_{1\to m}$ are used to deal with the integral over $(-\infty,-N]\cup[N,\infty)$ for showing uniform integrability. With a minimum of q steps for first passage, however, a generalised Hölder inequality can be used instead to deal with this integral. For example, with $q=3$ then take $N>Y$ so the identity in Proposition 2 holds. Then,

$$\sum_{\mathfrak{p}\in\mathfrak{P}}\left(\int_N^{\infty}+\int_{-\infty}^{-N}\right)\mathcal{T}_{\mathfrak{p}}(b^{+}+iy)e^{-ity}dy$$
$$\le 2\sum_{\mathfrak{p}\in\mathfrak{P}} p_{\mathfrak{p}}\int_N^{\infty}\left\|\mathcal{M}_{\mathfrak{b}_1}(b^{+}+iy)\right\|\left\|\mathcal{M}_{\mathfrak{b}_2}(b^{+}+iy)\right\|\left\|\mathcal{M}_{\mathfrak{b}_3}(b^{+}+iy)\right\|dy, \tag{32}$$

where $\mathfrak{b}_1,\mathfrak{b}_2,\mathfrak{b}_3\in\mathfrak{p}\cap(\mathbb{I}_{m-1}\times\mathbb{I}_m)$. Using a generalised Hölder inequality, an upper bound on (32) is

$$2\sum_{\mathfrak{p}\in\mathfrak{P}} p_{\mathfrak{p}}\{I(\mathfrak{b}_1)I(\mathfrak{b}_2)I(\mathfrak{b}_3)\}^{1/3} < \eta/2\sum_{\mathfrak{p}\in\mathfrak{P}} p_{\mathfrak{p}},$$

where $I(\mathfrak{b}_j)=\int_N^{\infty}\left\|\mathcal{M}_{\mathfrak{b}_j}(b^{+}+iy)\right\|^3 dy$ for $j=1{-}3$ and the last inequality holds for $N>\max_{\mathfrak{b}_1,\mathfrak{b}_2,\mathfrak{b}_3\in\mathbb{I}_{m-1}\times\mathbb{I}_m} N_3(\mathfrak{b}_1,\mathfrak{b}_2,\mathfrak{b}_3)$, where $N>N_3(\mathfrak{b}_1,\mathfrak{b}_2,\mathfrak{b}_3)$ ensures that $\max_{j=1-3} I(\mathfrak{b}_j)<\eta/4$.

7.7. Theorem 5 and its proof

Theorem 5. (Progressive-state expansions). *Let all transient states in* $\mathbb{I}_{m-1}$ *be progressive as described in §3.2 and suppose* $\mathcal{R}_{1\to m}$ *and* $\mathcal{CD}_{1\to m}$ *hold along with the following condition.*

($\mathcal{AC}_{\mathcal{L}}$) *For all* $(i,j)\in\mathcal{L}$, $\mathcal{M}_{ij}(s)$ *can be analytically continued across its convergence bound* b *to* $\{b\le\mathrm{Re}(s)<b+\varepsilon_0\}$ *save from an* $\mathfrak{m}_{ij}$*-pole at* b *for some* $\varepsilon_0>0$. *Denote* $\mathfrak{m}=\max_{(i,j)\in\mathcal{L}}\mathfrak{m}_{ij}$.

Integer time. *The conditional first-passage survival and mass functions of $X|X<\infty$ have discretised Gamma (k, e^{-b}) tail expansions for $k=1,\ldots \mathfrak{m}$ as $n\to\infty$ given by*

$$S(n)=S_1(n)+R_1^S(n):=\sum_{k=1}^{\mathfrak{m}} S_{\mathrm{DG}(k,b)}(n)\frac{(-1)^k\beta_{-k}}{b^k}+R_1^S(n) \tag{33}$$

$$p(n)=p_1(n)+R_1(n):=e^{-bn}\sum_{k=1}^{\mathfrak{m}} n^{k-1}\frac{(-1)^k\beta_{-k}}{(k-1)!}+R_1(n), \tag{34}$$

where $R_1^S(n)=o(e^{-b^+n})=R_1(n)$ as $n\to\infty$, $b^+=b+\varepsilon$ for sufficiently small $\varepsilon>0$, and

$$S_{\mathrm{DG}(k,b)}(n):=\frac{b^k}{(k-1)!}\sum_{j=n}^{\infty}e^{-jb}j^{k-1}.$$

Here, $\{\beta_{-k}\}$ are Laurent coefficients from $\sum_{k=1}^{\mathfrak{m}}\beta_{-k}(s-b)^{-k}$, the principal part of the Laurent expansion of $\mathcal{F}_{1m}(s)$ about b. Explicit expressions for the errors $R_1(n)$ and $R_1^S(n)$ are given in Theorem 1.

Continuous time. *The conditional first-passage survival and density functions of $X|X<\infty$ have Gamma (k,b) tail expansions for $k=1,\ldots\mathfrak{m}$ as $n\to\infty$. With the additional condition $\mathcal{B}_{1\to m}$ from Theorem 1 and $\mathcal{ZM}_{\mathcal{L}}$ below, the survival function expansion is*

$$S(t)=S_1(t)+R_1^S(t):=\sum_{k=1}^{\mathfrak{m}} S_{\mathrm{G}(k,b)}(t)\frac{(-1)^k\beta_{-k}}{b^k}+R_1^S(t), \tag{35}$$

where $S_{\mathrm{G}(k,b)}$ is the survival function of a Gamma (k,b) distribution or

$$S_{\mathrm{G}(k,b)}(t)=e^{-bt}\sum_{j=1}^{k-1}\frac{(bt)^j}{j!},$$

$R_1^S(t)=o(e^{-(b+\varepsilon)t})$ as $t\to\infty$, andε satisfies $\mathcal{ZM}_{\mathcal{L}}$.

($\mathcal{ZM}_{\mathcal{L}}$) *For some* $b^+=b+\varepsilon$ with $\varepsilon\in(0,\varepsilon_0)$, $\max_{b\le x\le b^+}|\mathcal{M}_{ij}(x+iN)|\to 0$ as $N\to\infty$ *for all* $(i,j)\in\mathcal{L}$.

Assuming additional condition $\mathcal{ZM}_{\mathcal{L}}$ above along with $\mathcal{BTV}_{1\to m}$ from Theorem 1 and $\mathcal{ONE}_{1\to m}$ and $\mathcal{MIN}_{1\to m}$ from Corollary 1, the density function *expansion is*

$$f(t) = f_1(t) + R_1(t) := e^{-bt} \sum_{k=1}^{\mathfrak{m}} t^{k-1} \frac{(-1)^k \beta_{-k}}{(k-1)!} + R_1(t), \tag{36}$$

where $R_1(t) = o(e^{-b^+ t})$ *as* $t \to \infty$. *Error terms* $R_1^S(t)$ *and* $R_1(t)$ *are well-defined and given in (15) of Theorem 1 under the above conditions.*

Proof. The proof follows the same arguments used in Theorem 1 and Corollary 1. In the integer time setting, condition $\mathcal{AC}_{\mathcal{L}}$ is required in order to apply Cauchy's theorem as used in Butler (2020, thm. 2).

In continuous time, both $\mathcal{AC}_{\mathcal{L}}$ and $\mathcal{ZM}_{\mathcal{L}}$ are required to apply Cauchy's theorem as used in Butler (2019, thms. 1 and 2). The remainder of the arguments needed to show that Theorems 1 and 2 in Butler (2019) apply are exactly the same as those used in Theorem 1 and Corollary 1 in §7.6 of these Supplementary Materials. □

The expansion results in integer time could be expressed in terms of Negative Binomial (k, e^{-b}) tail expansions for $k = 1, \ldots \mathfrak{m}$ as $n \to \infty$ as presented in Butler (2020, thm. 1). These expansions are analytically the same as those given in (33) and (34) and are based on the Laurent coefficients of the first-passage PGF $\mathcal{F}_{1m}(\ln z)$ rather than those from MGF $\mathcal{F}_{1m}(s)$. See Butler (2020, §2) for further discussion.

The conditions in Theorem 5 are somewhat less eloquent than in Theorem 1 and this is due to the need to make assumptions about $\{\mathcal{M}_{ij}(s) : (i,j) \in \mathcal{L}\}$ in their analytic continuation $\{b \le \text{Re}(s) < b^+\}$. In continuous time, the main conditions are $\mathcal{MIN}_{1\to m}$ and $\mathcal{ONE}_{1\to m}$ of Corollary 1 which provide some of the weakest conditions that can be imposed to yield a general density expansion for progressive SMPs lacking feedback. For conservative CTMCs, such conditions automatically hold. The simpler condition that $\|\mathcal{M}_{ij}(b^+ + iy)\|$ is integrable in y for all $(i,j) \in \mathbb{I}_{m-1} \times \mathbb{I}_m$ ensures that $\mathcal{MIN}_{1\to m}$ and $\mathcal{ONE}_{1\to m}$ hold, however this condition excludes CTMCs.

7.8. Proof of Proposition 6

The proofs are, for the most part, identical in form to those used in Propositions 2–4. For example, the second part of the proof of Proposition 3 deals with the Perron-Frobenius theory of $\mathbf{T}_{m;m}(s)$; the comparable matrix relevant to the proof here is $\mathbf{T}_{\mathcal{I}\mathcal{I}}(s)$ as justified by (18).

However, the third part of the proof of Proposition 3 to show b is a simple pole requires proving that $(-1)^{m+1}|\mathbf{\Psi}_{m;1}(b)| > 0$. The argument for this is not the same as in Proposition 3 and is entirely non-trivial and is now given over the next 6 pages.

We must show $(-1)^{m+1}|\mathbf{\Psi}_{m;1}(b)| > 0$ for a SMP with $\mathbb{I}_{m-1} = \mathcal{P} \cup \mathcal{I}$. The proof begins with cofactor expansion (17) applied to this new setting with some additional progressive states. Stage 1 of the proof takes $\mathcal{P} = \mathcal{P}^1$ so there are no progressive states that can be entered after leaving $\mathcal{I}$. Stage 2 includes $\mathcal{P}^2$ states and we shall see that the proof with such states reduces to the proof in stage 1.

7.8.1. *Stage 1* We use general arguments later on but simplify the proof initially by working with $p = 2$ progressive states and $I = 3$ states in the irreducible class so $m = 6$. The structure of $\mathbf{I} - \mathbf{T}(s)$ has the form

$$\mathbf{I} - \mathbf{T}(b) = \begin{pmatrix} 1 & -\mathcal{T}_{12} & * & -\mathcal{T}_{16} \\ 0 & 1 & * & -\mathcal{T}_{26} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 - \mathbf{T}_{\mathcal{I}\mathcal{I}} & -\mathbf{T}_{\mathcal{I}6} \\ * & * & * & 1 - \mathcal{T}_{66} \end{pmatrix}$$

where $*$ indicates a number or block of numbers which are either negative or 0, and $\mathcal{T}_{ij} = \mathcal{T}_{ij}(b)$, etc.. Then, with $m = 6$, we take matrix $\mathbf{\Psi}_{m;1}(b)$ and move its last column forward to be its first column. This is a movement of 4 column exchanges. Thus

$$-|\mathbf{\Psi}_{m;1}(b)| = -|\mathbf{\Psi}_{m;m;1\leftarrow m}(b)| = -\begin{vmatrix} -\mathcal{T}_{16} & -\mathcal{T}_{12} & * \\ -\mathcal{T}_{26} & 1 & * \\ -\mathbf{T}_{\mathcal{I}6} & \mathbf{0} & \mathbf{I}_3 - \mathbf{T}_{\mathcal{I}\mathcal{I}} \end{vmatrix} = \begin{vmatrix} \mathcal{T}_{16} & -\mathcal{T}_{12} & * \\ \mathcal{T}_{26} & 1 & * \\ \mathbf{T}_{\mathcal{I}6} & \mathbf{0} & \mathbf{I}_3 - \mathbf{T}_{\mathcal{I}\mathcal{I}} \end{vmatrix}.$$

Taking the cofactor expansion down the first column, the first two terms are

$$\mathcal{T}_{16}\,|\mathbf{I}_3 - \mathbf{T}_{\mathcal{I}\mathcal{I}}| - \mathcal{T}_{26}(-\mathcal{T}_{12})\,|\mathbf{I}_3 - \mathbf{T}_{\mathcal{I}\mathcal{I}}| = 0$$

since $|\mathbf{I}_3 - \mathbf{T}_{\mathcal{I}\mathcal{I}}(b)| = 0$. The final three terms are

$$\mathcal{T}_{36}\begin{vmatrix} -\mathcal{T}_{12} & * & * & * \\ 1 & -\mathcal{T}_{23} & -\mathcal{T}_{24} & -\mathcal{T}_{25} \\ 0 & -\mathcal{T}_{43} & 1-\mathcal{T}_{44} & -\mathcal{T}_{45} \\ 0 & -\mathcal{T}_{53} & -\mathcal{T}_{54} & 1-\mathcal{T}_{55} \end{vmatrix} - \mathcal{T}_{46}\begin{vmatrix} -\mathcal{T}_{12} & * & * & * \\ 1 & * & * & * \\ 0 & 1-\mathcal{T}_{33} & -\mathcal{T}_{34} & -\mathcal{T}_{35} \\ 0 & -\mathcal{T}_{53} & -\mathcal{T}_{54} & 1-\mathcal{T}_{55} \end{vmatrix}$$

$$+\,\mathcal{T}_{56}\begin{vmatrix} -\mathcal{T}_{12} & * & * & * \\ 1 & * & * & * \\ 0 & 1-\mathcal{T}_{33} & -\mathcal{T}_{34} & -\mathcal{T}_{35} \\ 0 & -\mathcal{T}_{43} & 1-\mathcal{T}_{44} & -\mathcal{T}_{45} \end{vmatrix} = \mathcal{T}_{36}A - \mathcal{T}_{46}B + \mathcal{T}_{56}C.$$

It suffices to show that $A > 0$, $B < 0$, and $C > 0$. If it can be shown that the pattern in A leads to $A > 0$, then this implies $B < 0$ since the pattern in A is obtained by interchanging columns 2 and 3 of B. Likewise $A > 0$ implies $C > 0$ by interchanging columns 3 and 4 in C followed by columns 2 and 3.

A cofactor expansion of A down its first column yields

$$A = -\mathcal{T}_{12}\begin{vmatrix} -\mathcal{T}_{23} & -\mathcal{T}_{24} & -\mathcal{T}_{25} \\ -\mathcal{T}_{43} & 1-\mathcal{T}_{44} & -\mathcal{T}_{45} \\ -\mathcal{T}_{53} & -\mathcal{T}_{54} & 1-\mathcal{T}_{55} \end{vmatrix} - \begin{vmatrix} * & * & * \\ -\mathcal{T}_{43} & 1-\mathcal{T}_{44} & -\mathcal{T}_{45} \\ -\mathcal{T}_{53} & -\mathcal{T}_{54} & 1-\mathcal{T}_{55} \end{vmatrix} = -\mathcal{T}_{12}D - E$$

where D and E need to be negative but also note that they have the same structure. All matrices of the form given by D and E have negative determinant as shown in Lemma 4, which proves the result.

The case above has $p = 2$ progressive states and $I = 3$ states in irreducible class $\mathcal{I}$ but a more general argument is needed to determine the sign of

$$(-1)^{m+1}|\mathbf{\Psi}_{m;1}(a)| = (-1)^{m+1+m-2}|\mathbf{\Psi}_{m;m;1\leftarrow m}(a)| = -|\mathbf{\Psi}_{m;m;1\leftarrow m}(a)|.$$

With $\mathcal{P}$ as the first p states and $\mathcal{I} = \{p+1, \ldots, p+I = m-1\}$, we compute a cofactor expansion down the first column of $|\mathbf{\Psi}_{m;m;1\leftarrow m}(a)|$. The portion of this sum over only the progressive block of p terms leads to a summation of the form $\sum_{i=1}^{p}(-1)^{i+1}\mathcal{T}_{im}|\mathbf{U}_i|\,|\mathbf{I}_I - \mathbf{T}_{\mathcal{II}}| = 0$ in which $\mathbf{U}_i$ is a square matrix and $|\mathbf{I}_I - \mathbf{T}_{\mathcal{II}}| = 0$ is a factor in each term. The remaining block, with leading terms from $\mathcal{I}$, is considered for the cases $I = 1$ and $I \geq 2$ separately.

If $I = 1$, then the single cofactor term is $-\mathcal{T}_{p+1,m}(-1)^{p+2}|\mathbf{A}_{p+1}|$ where $\mathbf{A}_{p+1}$ is $p \times p$ and is a patterned matrix taking the following forms. Let $-$ stand for a negative entry. Then, $\mathbf{A}_2 = (-)$,

$$\mathbf{A}_3 = \begin{pmatrix} - & - \\ 1 & - \end{pmatrix}, \tag{37}$$

and, for $p \geq 3$,

$$\mathbf{A}_{p+1} = \begin{array}{c|ccccccc|} \#\mathbb{I}_{m-1} & 2 & 3 & 4 & \cdots & p-1 & p & \mathcal{I} \\ \hline 1 & -_a & - & - & \cdots & - & - & - \\ 2 & 1 & - & - & \cdots & - & - & - \\ 3 & 0 & 1 & - & \cdots & - & - & - \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ p-2 & 0 & 0 & 0 & \ddots & - & - & - \\ p-1 & 0 & 0 & 0 & \ddots & 1 & - & - \\ p & 0 & 0 & 0 & \cdots & 0 & 1 & - \\ \hline \end{array}, \tag{38}$$

where the $p \times p$ matrix is enclosed by the solid lines and the exit states and destination states are enumerated on the left and top. Lemma 3 below proves that $\operatorname{sgn}\{|\mathbf{A}_{p+1}|\} = (-1)^p$ for all $p \geq 1$. Therefore cofactor term $-\mathcal{T}_{p+1,m}(-1)^{p+2}|\mathbf{A}_{p+1}| < 0$ as required.

For the case in which $I \geq 2$, the cofactor expansion terms over $\mathcal{I}$ are

$$\sum_{i=1}^{I} -\mathcal{T}_{p+i,m}(-1)^{p+i+1}|\mathbf{B}_{p+i}| \tag{39}$$

where $\{\mathbf{B}_{p+i}\}$ is a sequence of $(p+I-1) \times (p+I-1)$ patterned matrices. Matrix $\mathbf{B}_{p+1}$ takes the form

$$\mathbf{B}_{p+1} = \begin{array}{c|cccccc|} \#\mathbb{I}_{m-1} & 2 & 3 & \cdots & p & p+1 & \mathcal{I}_{\setminus} \\ \hline 1 & -_a & - & \cdots & - & - & (-,\ldots,-) \\ 2 & 1 & - & \cdots & - & - & (-,\ldots,-) \\ 3 & 0 & 1 & \ddots & - & - & (-,\ldots,-) \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ p & 0 & 0 & \ddots & 1 & - & (-,\ldots,-) \\ \mathcal{I}_{\setminus} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{T}_{\mathcal{I}_{\setminus},p+1} & \mathbf{I}_{I-1} - \mathbf{T}_{\mathcal{I}_{\setminus},\mathcal{I}_{\setminus}} \\ \hline \end{array} \tag{40}$$

with $\mathcal{I}_{\setminus} = \mathcal{I}\setminus\{p+1\}$, i.e. $\mathcal{I}$ without its first member, $\mathbf{T}_{\mathcal{I}_{\setminus},p+1} = (\mathcal{T}_{p+2,p+1}, \ldots, \mathcal{T}_{m-1,p+1})^T$. For the moment, we assume that the determinant sign for patterned matrix $\mathbf{B}_{p+1}$ is $(-1)^p$ where the power $p = 1 + (p-1)$, where $p-1$ counts the number of columns with the value $\mathbf{0}$ in the bottom row block (This result is proved further below.). Thus the $i = 1$ term in (39) has

$$\text{sgn}\left\{-\mathcal{T}_{p+1,m}(-1)^{p+1+1}|\mathbf{B}_{p+1}|\right\} = \text{sgn}\left\{-\mathcal{T}_{p+1,m}(-1)^p(-1)^p\right\} = -1.$$

The rest of the patterns for $i \geq 2$ are such that $\mathbf{B}_{p+i}$ matches the pattern of $\mathbf{B}_{p+1}$ if the $(p+i-1)$st column (holding destination state $p+i$) is moved $i-1$ steps to the left so it occupies the pth column (holding destination state $p+1$). This results in $(-1)^{i-1}$ sign changes so that

$$\begin{aligned} \text{sgn}\left\{-\mathcal{T}_{p+i,m}(-1)^{p+i+1}|\mathbf{B}_{p+i}|\right\} &= \text{sgn}\left\{-\mathcal{T}_{p+i,m}(-1)^{p+i+1}(-1)^{i-1}|\mathbf{B}_{p+1}|\right\} \\ &= \text{sgn}\left\{-\mathcal{T}_{p+i,m}(-1)^p(-1)^p\right\} = -1. \end{aligned}$$

Thus what remains is to show that sgn$\{|\mathbf{B}_{p+1}| = (-1)^p$. Take the cofactor expansion of $\mathbf{B}_{p+1}$ down its first column. One sees that the two minor matrices involved have the same pattern as $\mathbf{B}_{p+1}$ but with one less column with $\mathbf{0}$ in the bottom row while their scalar factors are negative. Hence if $|\mathbf{C}_1|$ is the $(1,1)$ cofactor of $\mathbf{B}_{p+1}$, then $\operatorname{sgn}\{|\mathbf{B}_{p+1}|\} = -\operatorname{sgn}\{|\mathbf{C}_1|\}$. Continuing such cofactor expansions down the first column of $(p+I-2)\times(p+I-2)$ matrix $\mathbf{C}_1$, the same pattern emerges so that $\operatorname{sgn}\{|\mathbf{C}_1|\} = -\operatorname{sgn}\{|\mathbf{C}_2|\}$ where $|\mathbf{C}_2|$ is the $(1,1)$ cofactor of $\mathbf{C}_1$. This progression of cofactor expansions continues until $I\times I$ matrix $\mathbf{C}_{p-1}$ is reached. This matrix has the form in (42) given in Lemma 4 below so its determinant must be negative and $\operatorname{sgn}\{|\mathbf{C}_{p-1}|\} = -1$. Thus,

$$\operatorname{sgn}\{|\mathbf{B}_{p+1}|\} = -\operatorname{sgn}\{|\mathbf{C}_1|\} = \operatorname{sgn}\{|\mathbf{C}_2|\} = \cdots = (-1)^{p-1}\operatorname{sgn}\{|\mathbf{C}_{p-1}|\} = (-1)^p.$$

Thus, $-|\mathbf{\Psi}_{m;m;1\leftarrow m}(b)| > 0$ as required. □

Lemma 3. For matrices A_{p+1} given in (37) and (38), $\operatorname{sgn}\{|\mathbf{A}_{p+1}|\} = (-1)^p$ for all $p \geq 1$.

Proof. Proof is by induction. The result clearly holds for $\#\mathcal{I} = p = 1$ and 2. Assume it holds for general p. Then, taking the cofactor expansion down the first column,

$$|\mathbf{A}_{p+2}| = (-1)^2(-_a)|\mathbf{A}^1_{p+1}| + (-1)^{2+1}|\mathbf{A}^2_{p+1}| \tag{41}$$

where $\mathbf{A}^1_{p+1}$ and $\mathbf{A}^2_{p+1}$ are the same sort of patterned matrices for case p. Therefore,

$$\operatorname{sgn}\{|\mathbf{A}_{p+2}|\} = -\operatorname{sgn}\{|\mathbf{A}^1_{p+1}|\} = -(-1)^p = (-1)^{p+1}$$

and the lemma follows. □

Lemma 4. Let $\mathcal{J}\subset\mathcal{I}$ be a subset of the irreducible relevant set with $\#\mathcal{J} = n \leq I-1$ and with $\#\mathcal{I} = I$. Consider the $(n+1)\times(n+1)$ matrix

$$\begin{pmatrix} -_a & (-_{b1},\ldots -_{bn}) \\ (-_{c1},\ldots -_{cn})^T & \mathbf{I}_n - \mathbf{T}_{\mathcal{J}\mathcal{J}} \end{pmatrix}, \tag{42}$$

where all elements in the first row and column are negative and $\mathbf{T}_{\mathcal{J}\mathcal{J}} = \mathbf{T}_{\mathcal{J}\mathcal{J}}(b)$. Then the matrix in (42) has a negative determinant.

Proof. The proof is by induction. Consider the $n = 1$ case so $\#\mathcal{J} = 1$. The determinant is

$$(-_a)(1 - \mathcal{T}_{\mathcal{J}\mathcal{J}}) - (-_{c1})(-_{b1}) < 0,$$

since $1 - \mathcal{T}_{\mathcal{J}\mathcal{J}} > 0$; see the third part of the proof of Proposition 2. Now, suppose the result holds for all $\#\mathcal{J} = n$ cases. We must show it holds in all the $\#\mathcal{J} = n + 1$ cases. With n replaced by $n + 1$ in (42), take a cofactor expansion down its first column so the determinant is

$$(-_a)|\mathbf{I}_{n+1} - \mathbf{T}_{\mathcal{J}\mathcal{J}}| + \sum_{j=1}^{n+1} (-1)^j (-_{cj}) \begin{vmatrix} (-_{b1}, \ldots -_{b,n+1}) \\ (\mathbf{I}_{n+1} - \mathbf{T}_{\mathcal{J}\mathcal{J}})_{\setminus j} \end{vmatrix} \tag{43}$$

where $(\mathbf{I}_{n+1} - \mathbf{T}_{\mathcal{J}\mathcal{J}})_{\setminus j}$ is $\mathbf{I}_{n+1} - \mathbf{T}_{\mathcal{J}\mathcal{J}}$ without its jth row. The $j = 1$ term in (43) has the form in (42) with $\#\mathcal{J}_{\setminus 1} = n$ instead of $n + 1$. For $j \geq 2$, the jth determinant in (43) can be put into the same form by moving column j to the left so it is the first column. This entails $j - 1$ steps or $j - 1$ sign changes so (43) is

$$(-_a)|\mathbf{I}_{n+1} - \mathbf{T}_{\mathcal{J}\mathcal{J}}| + \sum_{j=1}^{n+1} (-1)^j (-_{cj})(-1)^{j-1} \begin{vmatrix} -_{bj} & -_{\mathbf{b}\setminus j} \\ (-_{d1j}, \ldots -_{dnj})^T & \mathbf{I}_n - \mathbf{T}_{\mathcal{J}_{\setminus j}\mathcal{J}_{\setminus j}} \end{vmatrix} \tag{44}$$

where $(-_{d1j}, \ldots -_{dnj})^T$ represents the jth column of $(\mathbf{I}_{n+1} - \mathbf{T}_{\mathcal{J}\mathcal{J}})_{\setminus j}$, $-_{\mathbf{b}\setminus j}$ is $(-_{b1}, \ldots -_{b,n+1})$ without its jth column, and $\mathcal{J}_{\setminus j} = \mathcal{J} \setminus \{j\}$. The determinant in the summation in (44) is for a matrix which has the form (42). Since $\{\mathcal{J}_{\setminus j}\}$ are size n, by assumption the determinants are negative so the last summation must be negative. The result follows if the leading term is also negative, or if $|\mathbf{I}_{n+1} - \mathbf{T}_{\mathcal{J}\mathcal{J}}| > 0$ for any $\mathcal{J} \subset \mathcal{I}$ of size $\leq I - 1$. This result is shown in the next lemma. □

Lemma 5. *If* $\mathcal{J} \subset \mathcal{I}$ and $\#\mathcal{J} = n \leq I - 1$, *then* $|I_n - \mathbf{T}_{\mathcal{J}\mathcal{J}}| > 0$ *where* $\mathbf{T}_{\mathcal{J}\mathcal{J}} = \mathbf{T}_{\mathcal{J}\mathcal{J}}(b)$ *and* b *is the asymptotic failure rate for irreducible class* $\mathcal{I}$.

Proof. Matrix $\mathbf{T}_{\mathcal{I}\mathcal{I}}(b)$ has Perron-Frobenius eigenvalue 1. After suitable permutation of the rows and columns of $\mathbf{T}_{\mathcal{I}\mathcal{I}}$, suppose $\mathbf{T}_{\mathcal{J}\mathcal{J}}$ forms the $n \times n$ block for $\mathcal{J}$ in the lower right of $\mathbf{T}_{\mathcal{I}\mathcal{I}}$. Then

$$\mathbf{0} \leq \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_{\mathcal{J}\mathcal{J}} \end{pmatrix} \leq \mathbf{T}_{\mathcal{I}\mathcal{I}} \tag{45}$$

componentwise with strict inequality occurring in some components other than those in the lower right. Let $e_{\mathcal{J}} \in \mathbb{C}$ be the eigenvalue of $\mathbf{T}_{\mathcal{J}\mathcal{J}}$ with largest modulus. Since $\mathcal{J}$ may not be irreducible we do not know if $e_{\mathcal{J}}$ is a Perron-Frobenius eigenvalue and hence whether it is real. Since $\mathbf{T}_{\mathcal{I}\mathcal{I}}$ has Perron-Frobenius eigenvalue 1, then using (45) and Theorem 1.5e of Seneta (2006, p. 22), $|e_{\mathcal{J}}| \leq 1$. However, if $|e_{\mathcal{J}}| = 1$, then equality must hold in (45) which is false. Thus $0 < |e_{\mathcal{J}}| < 1$. Denote the characteristic polynomial for $\mathbf{T}_{\mathcal{J}\mathcal{J}}$ as $\varphi(x) := |x\mathbf{I}_n - \mathbf{T}_{\mathcal{J}\mathcal{J}}|$ so that $\varphi(e_{\mathcal{J}}) = 0$. For $x > |e_{\mathcal{J}}|$, the characteristic polynomial cannot have further roots (this would contradict that $e_{\mathcal{J}}$ is the eigenvalue of largest modulus). Since $\varphi(x) \to \infty$ as $x \to \infty$, then $\varphi(x) > 0$ for $x > |e_{\mathcal{J}}|$. Therefore, since $1 > |e_{\mathcal{J}}|$ then

$$0 < \varphi(1) = |\mathbf{I}_n - \mathbf{T}_{\mathcal{J}\mathcal{J}}|.$$

7.8.2. *Stage 2* Assuming $\mathcal{P} = \mathcal{P}^1 \cup \mathcal{P}^2$, the numerator of (4) or $(-1)^{m+1}|\mathbf{\Psi}_{m;1}(b)|$ is

$$(-1)^{m+1}\left|\begin{pmatrix} \{\mathbf{I}_{p_1} - \mathbf{T}_{\mathcal{P}^1\mathcal{P}^1}\}_{;1} & -\mathbf{T}_{\mathcal{P}^1\mathcal{I}} & -\mathbf{T}_{\mathcal{P}^1\mathcal{P}^2} & -\mathbf{T}_{\mathcal{P}^1 m} \\ \mathbf{0} & \mathbf{I}_I - \mathbf{T}_{\mathcal{I}\mathcal{I}} & -\mathbf{T}_{\mathcal{I}\mathcal{P}^2}(s) & -\mathbf{T}_{\mathcal{I}m} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{p_2} - \mathbf{T}_{\mathcal{P}^2\mathcal{P}^2} & -\mathbf{T}_{\mathcal{P}^2 m} \end{pmatrix}\right|$$

$$= (-1)^{m+1}\left|\begin{pmatrix} \mathbb{I}_{m-1} & \mathcal{P}^1_{\setminus 1} & \mathcal{I} & \mathcal{P}^2 & \{m\} \\ \{1\} & -\mathbf{T}_{1\mathcal{P}^1_{\setminus 1}} & -\mathbf{T}_{1\mathcal{I}} & -\mathbf{T}_{1\mathcal{P}^2} & -\mathcal{T}_{1m} \\ \mathcal{P}^1_{\setminus 1} & \mathbf{I}_{p_1-1} - \mathbf{T}_{\mathcal{P}^1_{\setminus 1}\mathcal{P}^1_{\setminus 1}} & -\mathbf{T}_{\mathcal{P}^1_{\setminus 1}\mathcal{I}} & -\mathbf{T}_{\mathcal{P}^1_{\setminus 1}\mathcal{P}^2} & -\mathbf{T}_{\mathcal{P}^1_{\setminus 1}m} \\ \mathcal{I} & \mathbf{0} & \mathbf{I}_I - \mathbf{T}_{\mathcal{I}\mathcal{I}} & -\mathbf{T}_{\mathcal{I}\mathcal{P}^2} & -\mathbf{T}_{\mathcal{I}m} \\ \mathcal{P}^2 & \mathbf{0} & \mathbf{0} & \mathbf{I}_{p_2} - \mathbf{T}_{\mathcal{P}^2\mathcal{P}^2} & -\mathbf{T}_{\mathcal{P}^2 m} \end{pmatrix}\right|, \tag{46}$$

where $\{\mathbf{I}_{p_1} - \mathbf{T}_{\mathcal{P}^1\mathcal{P}^1}\}_{;1}$ is the matrix without its first column, $\mathbf{T}_{\mathcal{P}^1\mathcal{I}} = \mathbf{T}_{\mathcal{P}^1\mathcal{I}}(b)$, $\mathcal{P}^1_{\setminus 1} = \mathcal{P}^1 \setminus \{1\}$, and the matrix whose determinant is computed is enclosed in solid lines with exit and destination states noted in the left and top columns. The computation in (46), is facilitated by moving the last column so it becomes the first column. This is $p_2 + I + p_1 - 1 = m - 2$ interchanges resulting in a sign change of $(-1)^{m-2}$ so (46) is

$$
-\left|\begin{array}{c|c|cccc}
\mathbb{I}_{m-1} & \{m\} & \mathcal{P}^1_{\setminus 1} & \mathcal{I} & \mathcal{P}^2 \\
\hline
\{1\} & -\mathcal{T}_{1m} & -\mathbf{T}_{1\mathcal{P}^1_{\setminus 1}} & -\mathbf{T}_{1\mathcal{I}} & -\mathbf{T}_{1\mathcal{P}^2} \\
\mathcal{P}^1_{\setminus 1} & -\mathbf{T}_{\mathcal{P}^1_{\setminus 1}m} & \mathbf{I}_{p_1-1} - \mathbf{T}_{\mathcal{P}^1_{\setminus 1}\mathcal{P}^1_{\setminus 1}} & -\mathbf{T}_{\mathcal{P}^1_{\setminus 1}\mathcal{I}} & -\mathbf{T}_{\mathcal{P}^1_{\setminus 1}\mathcal{P}^2} \\
\mathcal{I} & -\mathbf{T}_{\mathcal{I}m} & \mathbf{0} & \mathbf{I}_I - \mathbf{T}_{\mathcal{I}\mathcal{I}} & -\mathbf{T}_{\mathcal{I}\mathcal{P}^2} \\
\mathcal{P}^2 & -\mathbf{T}_{\mathcal{P}^2m} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{p_2} - \mathbf{T}_{\mathcal{P}^2\mathcal{P}^2}
\end{array}\right|
$$

$$
= -\left(\sum_{i=1}^{p_1}(-1)^{i+1}(-\mathcal{T}_{im})|\mathbf{A}_i| + \sum_{i=p_1+1}^{p_1+I}(-1)^{i+1}(-\mathcal{T}_{im})|\mathbf{B}_i| + \sum_{i=p_1+I+1}^{p+I}(-1)^{i+1}(-\mathcal{T}_{im})|\mathbf{C}_i|\right),
$$

where the last step uses a cofactor expansion down the first column. For states in $\mathcal{P}^1$,

$$
\begin{aligned}
|\mathbf{A}_1| &= |\mathbf{I}_{p_1-1} - \mathbf{T}_{\mathcal{P}^1_{\setminus 1}\mathcal{P}^1_{\setminus 1}}| \times |\mathbf{I}_I - \mathbf{T}_{\mathcal{I}\mathcal{I}}| \times |\mathbf{I}_{p_2} - \mathbf{T}_{\mathcal{P}^2\mathcal{P}^2}| = 0 \\
|\mathbf{A}_i| &= |\mathbf{A}_{i1}| \times |\mathbf{I}_I - \mathbf{T}_{\mathcal{I}\mathcal{I}}| \times |\mathbf{I}_{p_2} - \mathbf{T}_{\mathcal{P}^2\mathcal{P}^2}| = 0 \qquad i = 2, \ldots, p_1,
\end{aligned}
$$

for some $(p_1 - 1) \times (p_1 - 1)$ matrices $\mathbf{A}_{i1}$. The cofactor terms $(-1)^{i+1}|\mathbf{B}_i|$ are exactly the same structure as in stage 1 except they include the extra factor $|\mathbf{I}_{p_2} - \mathbf{T}_{\mathcal{P}^2\mathcal{P}^2}| = 1$; hence they have the same sign as in stage 1 and contribute an overall negative sign.

Finally consider the cofactor terms from $\mathcal{P}^2$. The first term is $(-1)^{p_1+I}(-\mathcal{T}_{p_1+I+1,m})|\mathbf{C}_{p_1+I+1}|$ where, splitting $\mathcal{P}^1_{\setminus 1}$ into $\{2\}$ and $\mathcal{P}^1_{\setminus\{1,2\}}$ for clarity, we have

$$\mathbf{C}_{p_1+I+1} = \begin{array}{c} \mathbb{I}_{m-1} \\ \{1\} \\ \mathcal{P}^1_{\backslash 1} \\ \mathcal{I} \\ \mathcal{P}^2_{\backslash 1} \end{array} \begin{array}{ccccc} \{2\} & \mathcal{P}^1_{\backslash\{1,2\}} & \mathcal{I} & \{p_1+I+1\} & \mathcal{P}^2_{\backslash 1} \\ \hline -\mathcal{T}_{12} & - & - & - & - \\ \mathbf{I}_{p_1-1} - \mathbf{T}_{\mathcal{P}^1_{\backslash 1}\mathcal{P}^1_{\backslash 1}} & & - & - & - \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_I - \mathbf{T}_{\mathcal{I}\mathcal{I}} & - & - \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{p_2-1} - \mathbf{T}_{\mathcal{P}^2_{\backslash 1}\mathcal{P}^2_{\backslash 1}} \end{array}$$

where $\mathcal{P}^2_{\backslash 1} = \mathcal{P}^2 \backslash \{p_1 + I + 1\}$. Thus,

$$|\mathbf{C}_{p_1+I+1}| = \left| \begin{array}{c} \mathbb{I}_{m-1} \\ \{1\} \\ \mathcal{P}^1_{\backslash 1} \\ \mathcal{I} \end{array} \begin{array}{cccc} \{2\} & \mathcal{P}^1_{\backslash\{1,2\}} & \mathcal{I} & \{p_1+I+1\} \\ \hline -\mathcal{T}_{12} & - & - & - \\ \mathbf{I}_{p_1-1} - \mathbf{T}_{\mathcal{P}^1_{\backslash 1}\mathcal{P}^1_{\backslash 1}} & & - & - \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_I - \mathbf{T}_{\mathcal{I}\mathcal{I}} & - \end{array} \right| \tag{47}$$

The matrix in (47) takes the form of $\mathbf{B}_{p+1}$ in (40) when the last column is moved I steps to the left and inserted between columns $\mathcal{P}^1_{\backslash\{1,2\}}$ and $\mathcal{I}$; this gives a factor of $(-1)^I$. There are $p_1 - 1$ nested cofactor steps to determine the sign of $\mathbf{C}_{p_1+I+1}$ just as there were $p-1$ to determine the sign of $\mathbf{B}_{p+1}$ as $(-1)^p$. Therefore

$$\text{sgn}\{|\mathbf{C}_{p_1+I+1}|\} = (-1)^I(-1)^{p_1}$$

and

$$\text{sgn}\{(-1)^{p_1+I}(-\mathcal{T}_{p_1+I+1,m})|\mathbf{C}_{p_1+I+1}|\} = (-1)^{p_1+I+1+I+p_1} = -1$$

as required. For the remaining terms, matrix $\mathbf{C}_{p_1+I+j}$ has the same pattern as $\mathbf{C}_{p_1+I+1}$ if the jth column in the $\mathcal{P}^2$ block is moved leftward $j-1$ columns to replace the first column in the $\mathcal{P}^2$ block of columns. Then $|\mathbf{C}_{p_1+I+j}|$ is computed with the same pattern as given in (47) and

$$\text{sgn}\{|\mathbf{C}_{p_1+I+j}|\} = (-1)^{j-1}(-1)^I(-1)^{p_1}$$

so that

$$\sum_{i=p_1+I+1}^{p+I}(-1)^{i+1}(-\mathcal{T}_{im})\,\mathrm{sgn}\{|\mathbf{C}_i|\} = \sum_{j=1}^{p_2}(-1)^{p_1+I+1+j}(-\mathcal{T}_{p_1+I+j,m})(-1)^{j-1}(-1)^{I}(-1)^{p_1}$$
$$= \sum_{j=1}^{p_2}(-\mathcal{T}_{p_1+I+j,m}) < 0$$

as required. □

7.8.3. *Residue for Theorem 2* The residue in (19) follows by showing that $|\mathbf{\Psi}_{m;m}(0)| = |\mathbf{\Psi}_{\mathcal{I}\mathcal{I}}(0)|$ and

$$\frac{d}{ds}\left|\mathbf{\Psi}_{m;m}(s)\right|\Big|_{s=b} = \mathrm{tr}[\mathrm{adj}\{\mathbf{\Psi}_{\mathcal{I}\mathcal{I}}(b)\}\mathbf{\Psi}'_{\mathcal{I}\mathcal{I}}(b)]. \tag{48}$$

These results follow directly from the identity $|\mathbf{\Psi}_{m;m}(s)| = |\mathbf{\Psi}_{\mathcal{I}\mathcal{I}}(s)|$ as stated in (18).

7.9. Corollary 2

7.9.1. *Derivation of conditional phase-type distributions* The process starts in state 1 and is in state j at time t with probability $\xi_1^T \exp(\mathbf{Q}_{m;m}t)\xi_j$. In time dt transition $j \to m$ occurs w.p. $q_{jm}dt$ so that the unconditional density of passage time X is $\xi_1^T \exp(\mathbf{Q}_{m;m}t)\mathbf{q}_m$. The absorption probability into state m starting from state 1 is $-\xi_1^T\mathbf{Q}_{m;m}^{-1}\mathbf{q}_m$ so the conditional density of X given $X < \infty$ is

$$f(t) = \frac{\xi_1^T \exp(\mathbf{Q}_{m;m}t)\mathbf{q}_m}{-\xi_1^T\mathbf{Q}_{m;m}^{-1}\mathbf{q}_m}.$$

The conditional survival function is

$$S(t) = \int_t^\infty f(u)du = \frac{\xi_1^T \exp(\mathbf{Q}_{m;m}t)(-\mathbf{Q}_{m;m}^{-1})\mathbf{q}_m}{-\xi_1^T\mathbf{Q}_{m;m}^{-1}\mathbf{q}_m}$$

as given in (21).

7.9.2. *Proof of Corollary 2* In continuous time, the zeros of $|\mathbf{\Psi}_{m;m}(s)|$ are determined in terms of $\mathbf{Q}$, by writing

$$\mathbf{\Psi}_{m;m}(s) = \mathbf{I}_{m-1} - \mathbf{T}_{m;m}(s) = \begin{pmatrix} 1 & \frac{q_{12}}{s+q_{11}} & \cdots & \frac{q_{1,m-1}}{s+q_{11}} \\ \frac{q_{21}}{s+q_{22}} & 1 & & \frac{q_{2,m-1}}{s+q_{22}} \\ \vdots & & \ddots & \vdots \\ \frac{q_{m-1,1}}{s+q_{m-1,m-1}} & \frac{q_{m-1,2}}{s+q_{m-1,m-1}} & & 1 \end{pmatrix}.$$

Then, factoring out $\{(s+q_{ii})^{-1}\}$ from the rows, we see that

$$|\mathbf{\Psi}_{m;m}(s)| = |s\mathbf{I}_{m-1} + \mathbf{Q}_{m;m}| \prod_{i=1}^{m-1} (s+q_{ii})^{-1}.$$

To compute $(-1)^{m+1}|\mathbf{\Psi}_{m;1}(s)|$, define

$$\mathbf{A}_{m;1}(s) = \begin{pmatrix} q_{12} & q_{13} & \cdots & q_{1,m} \\ s+q_{22} & q_{23} & \cdots & q_{2,m} \\ & \ddots & \ddots & \vdots \\ q_{m-1,2} & & s+q_{m-1,m-1} & q_{m-1,m} \end{pmatrix}$$

so that

$$|\mathbf{\Psi}_{m;1}(s)| = |\mathbf{A}_{m;1}(s)| \prod_{i=1}^{m-1} (s+q_{ii})^{-1}$$

Then,

$$\mathcal{F}_{1m}(s) = \frac{(-1)^{m+1}|\mathbf{\Psi}_{m;1}(s)|}{|\mathbf{\Psi}_{m;m}(s)|} = \frac{(-1)^{m+1}\,|\mathbf{A}_{m;1}(s)|}{|s\mathbf{I}_{m-1} + \mathbf{Q}_{m;m}|}.$$

Let $I = \#\mathcal{I}$. Since

$$|s\mathbf{I}_{m-1} + \mathbf{Q}_{m;m}| = |s\mathbf{I}_{I} + \mathbf{Q}_{\mathcal{I}\mathcal{I}}| \times \prod_{i\in\mathcal{P}} (s+q_{ii}),$$

the zeros of $|\mathbf{\Psi}_{m;m}(s)|$ are either the eigenvalues of $-\mathbf{Q}_{\mathcal{I}\mathcal{I}}$ or values $\{-q_{ii}; i \in \mathcal{P}\}$. Since $\mathcal{I}$ is irreducible, the dominant eigenvalue of $-\mathbf{Q}_{\mathcal{I}\mathcal{I}}$ is $b > 0$ and for the other eigenvalues λ_j of $-\mathbf{Q}_{\mathcal{I}\mathcal{I}}$ for $j = 2, \ldots, I$, $\mathrm{Re}(\lambda_j) > b$. By assumption $b < \min_{i\in\mathcal{P}}(-q_{ii})$ so b is a simple pole for $\mathcal{F}_{1m}(s)$.

The residue computation is

$$\begin{aligned}
\mathrm{Res}\{\mathcal{F}_{1m}; b\} &= \frac{1}{\mathcal{F}_{1m}(0)} \lim_{s\to b} \{(s-b)\mathcal{F}_{1m}(s)\} \\
&= \frac{|\mathbf{Q}_{m;m}| \;\; |\mathbf{A}_{m;1}(b)|}{|\mathbf{A}_{m;1}(0)|} \prod_{i\in\mathcal{P}} (b+q_{ii})^{-1} \lim_{s\to b} \frac{s-b}{|s\mathbf{I}_I + \mathbf{Q}_{\mathcal{I}\mathcal{I}}|} \\
&= \frac{|\mathbf{Q}_{m;m}| \;\; |\mathbf{A}_{m;1}(b)|}{|\mathbf{A}_{m;1}(0)|} \prod_{i\in\mathcal{P}} (b+q_{ii})^{-1} \lim_{s\to b} \frac{s-b}{(s-b)\prod_{j=2}^{I}(s-\lambda_j)} \\
&= \frac{|\mathbf{Q}_{m;m}| \;\; |\mathbf{A}_{m;1}(b)|}{|\mathbf{A}_{m;1}(0)|} \prod_{i\in\mathcal{P}} (b+q_{ii})^{-1} \prod_{j=2}^{I} (b-\lambda_j)^{-1}. \qquad (49)
\end{aligned}$$

The exact expression for the conditional survival function is given in (20). In this expression, the eigenvalues of block $\mathbf{Q}_{\mathcal{I}\mathcal{I}}$ and their corresponding left/right eigenvectors of $\mathbf{Q}_{m;m}$ are denoted as $-b$, $\mathbf{u}_1^T$, and $\mathbf{v}_1$ for the dominant eigenvalue and $-\lambda_j$, $\mathbf{u}_j^T$, and $\mathbf{v}_j$ with $j = 2, \ldots, I$ for the others with $\mathrm{Re}(\lambda_j) > b$ (Seneta, thm. 2.6(c)). Associated with $\mathcal{P} = \mathcal{P}^1 \cup \mathcal{P}^2$ are eigenvalues q_{ii} for $i = 1, \ldots, p$ and corresponding left/right eigenvectors of $\mathbf{Q}_{m;m}$ denoted as $\mathbf{w}_i^T$, and $\mathbf{x}_i$. Assuming $\mathbf{Q}_{m;m}$ is diagonalisable (as occurs when the eigenvalues are distinct), we have the exponential expansion

$$\begin{aligned}
S(t) &= \frac{1}{\xi_1^T \mathbf{Q}_{m;m}^{-1} \mathbf{q}_m} \xi_1^T \left\{ \left(\frac{1}{-b} e^{-bt} \mathbf{u}_1 \mathbf{v}_1^T + \sum_{j=2}^{I} \frac{1}{-\lambda_j} e^{-\lambda_j t} \mathbf{u}_j \mathbf{v}_j^T \right) + \sum_{i\in\mathcal{P}} \frac{1}{q_{ii}} e^{q_{ii} t} \mathbf{w}_i \mathbf{x}_i^T \right\} \mathbf{q}_m \\
&= \frac{1}{b} e^{-bt} \frac{\xi_1^T \mathbf{u}_1 \mathbf{v}_1^T \mathbf{q}_m}{-\xi_1^T \mathbf{Q}_{m;m}^{-1} \mathbf{q}_m} + o(e^{-b^+ t})
\end{aligned}$$

where $b^+ < b_2 = \min\{\min_{j\geq 2} \mathrm{Re}(\lambda_j), \min_{i\in\mathcal{P}}(-q_{ii})\}$. The coefficient $(\xi_1^T \mathbf{u}_1)(\mathbf{v}_1^T \mathbf{q}_m)/\{-\xi_1^T \mathbf{Q}_{m;m}^{-1} \mathbf{q}_m\}$ of e^{-bt}/b in this expression is $-\mathrm{Res}\{\mathcal{F}_{1m}; b\}$ so

$$\beta_{-1} = \mathrm{Res}\{\mathcal{F}_{1m}; b\} = \frac{\xi_1^T \mathbf{u}_1 \mathbf{v}_1^T \mathbf{q}_m}{\xi_1^T \mathbf{Q}_{m;m}^{-1} \mathbf{q}_m}. \qquad (50)$$

If $\mathbf{Q}_{m;m}$ is not diagonalisable, then its Jordan form must be used and the same argument applies. □

7.10. Proof of Corollary 3

7.10.1. *Derivation of conditional phase-type distributions* The process starts in state 1 and arrives for the first time in state m at time n w.p. $\xi_1^T \mathbf{P}_{m;m}^{n-1} \mathbf{p}_m$. The probability of finite passage is $\sum_{n=1}^{\infty} \xi_1^T \mathbf{P}_{m;m}^{n-1} \mathbf{p}_m = \xi_1^T (\mathbf{I}_{m-1} - \mathbf{P}_{m;m})^{-1} \mathbf{p}_m$ so the conditional first passage mass function is

$$p(n) = \frac{\xi_1^T \mathbf{P}_{m;m}^{n-1} \mathbf{p}_m}{\xi_1^T (\mathbf{I}_{m-1} - \mathbf{P}_{m;m})^{-1} \mathbf{p}_m}.$$

The conditional survival function is

$$S(n) = \sum_{k=n}^{\infty} p(n) = \frac{\xi_1^T (\mathbf{I}_{m-1} - \mathbf{P}_{m;m})^{-1} \mathbf{P}_{m;m}^{n-1} \mathbf{p}_m}{\xi_1^T (\mathbf{I}_{m-1} - \mathbf{P}_{m;m})^{-1} \mathbf{p}_m}$$

as given in (23).

7.10.2. $\mathbf{P}_{\mathcal{I}\mathcal{I}}$ *is aperiodic* The transmittance matrix $\mathbf{I}_{m-1} - \mathbf{P}_{m;m}\mathbf{e}^s$ for states in $\mathbb{I}_{m-1}$ has block form with zero blocks below its diagonal blocks. Thus (22) holds. If $\mathbf{P}_{m;m}$ can be diagonalised, then the eigenvalue/eigenvector decomposition of $\mathbf{P}_{m;m}$ is

$$\mathbf{P}_{m;m} = \lambda_1 \mathbf{u}_1 \mathbf{v}_1^T + \sum_{j=2}^{I} \lambda_j \mathbf{u}_j \mathbf{v}_j^T + \sum_{\{i \in \mathcal{P}: p_{ii} > 0\}} p_{ii}^{-1} \mathbf{w}_i \mathbf{x}_i^T$$

so that

$$\mathbf{P}_{m;m}^{n-1} = \lambda_1^{n-1} \mathbf{u}_1 \mathbf{v}_1^T + \sum_{j=2}^{I} \lambda_j^{n-1} \mathbf{u}_j \mathbf{v}_j + \sum_{\{i \in \mathcal{P}: p_{ii} > 0\}} p_{ii}^{-(n-1)} \mathbf{w}_i \mathbf{x}_i^T.$$

With $b = -\ln \lambda_1$, and using (23), then

$$S(n) = \frac{\xi_1^T (\mathbf{I}_{m-1} - \mathbf{P}_{m;m})^{-1} \mathbf{P}_{m;m}^{n-1} \mathbf{p}_m}{\xi_1^T (\mathbf{I}_{m-1} - \mathbf{P}_{m;m})^{-1} \mathbf{p}_m} = e^{-b(n-1)} \frac{\xi_1^T (\mathbf{I}_{m-1} - \mathbf{P}_{m;m})^{-1} \mathbf{u}_1 \mathbf{v}_1^T \mathbf{p}_m}{\xi_1^T (\mathbf{I}_{m-1} - \mathbf{P}_{m;m})^{-1} \mathbf{p}_m} + o(e^{-b^+ n}) \tag{51}$$

$$= e^{-b(n-1)} \frac{\xi_1^T \mathbf{u}_1 \mathbf{v}_1^T \mathbf{p}_m}{(1 - e^{-b}) \xi_1^T (\mathbf{I}_{m-1} - \mathbf{P}_{m;m})^{-1} \mathbf{p}_m} + o(e^{-b^+ n})$$
$$= e^{-bn} \frac{-\beta_{-1}}{1 - e^{-b}} + o(e^{-b^+ n}),$$

where in the second equality $\mathbf{P}_{m;m}^{n-1} \sim \lambda_1^{n-1} \mathbf{u}_1 \mathbf{v}_1^T$, and in the third equality we use $(\mathbf{I}_{m-1} - \mathbf{P}_{m;m})^{-1} \mathbf{u}_1 = \mathbf{u}_1 / (1 - e^{-b})$. This leads to

$$\beta_{-1} = \text{Res}\{\mathcal{F}_{1m}(s); b\} = -e^b \frac{\xi_1^T \mathbf{u}_1 \mathbf{v}_1^T \mathbf{p}_m}{\xi_1^T (\mathbf{I}_{m-1} - \mathbf{P}_{m;m})^{-1} \mathbf{p}_m} \tag{52}$$

In addition,

$$p(n) = S(n) - S(n+1) = \left\{ e^{-b(n-1)} - e^{-bn} \right\} \frac{-\beta_{-1}}{(1-e^{-b})} + o(e^{-b^+ n}) = e^{-bn}(-\beta_{-1}) + o(e^{-b^+ n}). \tag{53}$$

If $\mathbf{P}_{m;m}$ cannot be diagonalised, then its Jordan decomposition leads to the same expansions as in (51) and (53).

7.10.3. $\mathbf{P}_{\mathcal{I}\mathcal{I}}$ *is periodic* For the periodic setting, the transmittance matrix $\mathbf{T}_{\mathcal{I}\mathcal{I}}(s) = \mathbf{P}_{\mathcal{I}\mathcal{I}} e^s$ is an irreducible periodic matrix and if $|\mathbf{I}_I - \mathbf{P}_{\mathcal{I}\mathcal{I}} e^b| = 0$, then $\mathbf{P}_{\mathcal{I}\mathcal{I}} e^b$ has d dominant eigenvalues $\{\alpha_l\}$ equally-spaced round the unit circle in $\mathbb{C}$ starting at $\alpha_0 = 0$ as specified in Theorem 1.7 of Seneta (2006, p. 23). Thus if α_l is any one of these, then

$$0 = |\alpha_l \mathbf{I}_I - \mathbf{P}_{\mathcal{I}\mathcal{I}} e^b| \alpha_l^{-I} = |\mathbf{I}_I - \mathbf{P}_{\mathcal{I}\mathcal{I}} e^{b - \ln \alpha_l}| = |\mathbf{\Psi}_{m;m}(b - \ln \alpha_l)|$$

and all values

$$b - \ln \alpha_l = b + i2\pi l/d \qquad l = \lceil -(d-1)/2 \rceil, \ldots, 0, \ldots, \lceil d/2 \rceil$$

are simple poles on the boundary of the principal convergence domain $\{s \in \mathbb{C} : \text{Re}(s) = b, -\pi < \text{Im}(s) \leq \pi\}$.

7.11. Proof of Theorems 3 and 4

7.11.1. *Theorem 3* The proof also entails showing that Propositions 2–5 apply to first-return from $1 \to 1$. To show Proposition 2, note expression (26) expresses $f_{11}\mathcal{F}_{11}(s)$ as a linear function of $\{f_{j1}\mathcal{F}_{j1}(s) : j \geq 2\}$. By Proposition 2, all members of $\{f_{j1}\mathcal{F}_{j1}(s) : j \geq 2\}$ admit infinite expansions summing over all distinct pathways from $j \to 1$ for $j \geq 2$ hence we have the same type of expansions, summing over all distinct pathways from $1 \to 1$, for $f_{11}\mathcal{F}_{11}(s)$ and Proposition 2 applies to first return to 1.

The proof of Proposition 3 for first return to 1 is almost entirely the same as passage from $1 \to m$. Once the process enters $\mathbb{I}_{m\backslash 1}$, exit from this irreducible class with asymptotic hazard rate $b > 0$ is the same idea as exit from $\mathbb{I}_{m-1}$ for first passage $1 \to m$. The only difference in the proof is the need to show that the simple zero of $|\Psi_{11}(s)|$ at b leads to a simple pole for $\mathcal{F}_{11}(s)$. This can be done by using (26) and noting that $\{(-1)^{j+1}|\mathbf{\Psi}_{1j}(b)| : j \geq 2\}$ are all positive by arguments from Proposition 3. Another more direct route, is to note that $|\mathbf{I}_m - \mathbf{T}(s)|$ is monotone decreasing in s, zero at $s = 0$, and therefore negative at $b > 0$.

The proof of Proposition 4 also uses (26). By Proposition 4, there exists $\varepsilon_0 > 0$ such that $\{\mathcal{F}_{j1}(s) : j \geq 2\}$ are analytic on $\{s \in \mathbb{C} : \text{Re}(s) < b + \varepsilon_0\}$ apart from them all having a simple pole at b. Thus by (26) the same is true for $\mathcal{F}_{11}(s)$.

The proof of Theorem 3 now follows using exactly the same arguments as in Theorem 1 and relying on results from Propositions 2–5 as they apply to first return to 1.

7.11.2. *Theorem 4* The proof requires extending first-return results with $\mathbb{I}_{m\backslash 1} = \mathcal{I}$ to the setting in which $\mathbb{I}_{m\backslash 1} = \mathcal{P}^1 \cup \mathcal{P}^2 \cup \mathcal{I}$. Proof of Proposition 6 for $f_{11}\mathcal{F}_{11}(s)$ makes use of the identity (26) in which $f_{11}\mathcal{F}_{11}(s)$ is linear in $\{f_{j1}\mathcal{F}_{j1}(s) : j \geq 2\}$. Since Proposition 6 applies to each $f_{j1}\mathcal{F}_{j1}(s)$, this may be extended to $f_{11}\mathcal{F}_{11}(s)$ using the same arguments as used to prove Theorem 3. Proving Theorem 4 from Proposition 6 as applied to first return to 1 uses the same arguments as used concerning first passage $1 \to m$.