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MARKOV CHAIN APPROXIMATION OF ONE-DIMENSIONAL STICKY DIFFUSIONS: SUPPLEMENTARY MATERIAL

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B. Additional proofs

Proof of Theorem 1: Suppose that the unique weak solution to (2.1) and (2.2) for $\rho = \infty$ is given by (X^1, B^1). Let

$$\begin{aligned}
\phi_t &= t + \frac{1}{2\rho} L_t^l \left(X^1\right), \\
T_t &= \phi_t^{-1}, \\
X_t &= X_{T_t}^1, \\
B_t &= B_{T_t}^1 + \int_0^t I\left(X_s = l\right) dB_s^0,
\end{aligned}$$

where B^0 is a Brownian motion, defined on an extended probability space if needed, that is independent of B^1. The local time process is continuous and non-decreasing; hence ϕ_t is strictly increasing and continuous (see [1, Chapter II.13]). This implies that T_t is also strictly increasing and continuous. Then B_t

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is a continuous local martingale and

$$\langle B\rangle_t = \langle B^1\rangle_{T_t} + \int_0^t I\left(X_s = l\right) ds = T_t + \int_0^t I\left(X_s = l\right) ds.$$

It also holds that

$$\begin{aligned} T_t &= \int_0^{T_t} ds - \int_0^{T_t} I\left(X_s^1 = 0\right) ds = \int_0^{T_t} I\left(X_s^1 > l\right) ds \\ &= \int_0^{T_t} I\left(X_s^1 > l\right)\left(ds + \frac{1}{2\rho} dL_s^l\left(X^1\right)\right) \\ &= \int_0^t I\left(X_{T_s}^1 > l\right) d\phi_{T_s} = \int_0^t I\left(X_s > l\right) ds, \end{aligned}$$

where we use that $dL_s^l(X)$ only increases for $X_s = l$, and we apply the change-of-variable formula. Therefore, $\langle B\rangle_t = t$, and by Lévy's characterization, B is a standard Brownian motion. Moreover,

$$\begin{aligned} X_t &= X_{T_t}^1 \\ &= \int_0^{T_t} \mu\left(X_s^1\right) I\left(X_s^1 > l\right) ds + \int_0^{T_t} \sigma\left(X_s^1\right) I\left(X_s^1 > l\right) dB_s^1 + \frac{1}{2}\int_0^{T_t} dL_s^l\left(X^1\right) \\ &= \int_0^{T_t} \mu\left(X_s^1\right) I\left(X_s^1 > l\right)\left(ds + \frac{1}{2\rho} dL_s^l\left(X^1\right)\right) \\ &\qquad + \int_0^t \sigma\left(X_{T_s}^1\right) I\left(X_{T_s}^1 > l\right) dB_{T_s}^1 + \frac{1}{2} L_{T_t}^l\left(X^1\right) \\ &= \int_0^t \mu\left(X_{T_s}^1\right) I\left(X_{T_s}^1 > l\right) d\phi_{T_s} + \int_0^t \sigma\left(X_s\right) I\left(X_s > l\right) dB_s + \frac{1}{2} L_{T_t}^l\left(X^1\right) \\ &= \int_0^t \mu\left(X_s\right) I\left(X_s > l\right) ds + \int_0^t \sigma\left(X_s\right) I\left(X_s > l\right) dB_s + \frac{1}{2} L_t^l\left(X\right), \end{aligned}$$

because $X_s = X_{T_s}^1$,

$$d\phi_{T_s} = ds + \frac{1}{2\rho} dL_s^l\left(X^1\right) = ds$$

as X^1 is the unique weak solution of the reflecting case, and $L_{T_t}^l(X^1) = L_t^l(X)$. This shows that (X, B) solves (2.1). Furthermore,

$$\begin{aligned} \int_0^t I\left(X_s = l\right) ds &= \int_0^t I\left(X_{T_s}^1 = l\right) d\phi_{T_s} = \int_0^{T_t} I\left(X_s^1 = l\right) d\phi_s \\ &= \int_0^{T_t} I\left(X_s^1 = l\right)\left(ds + \frac{1}{2\rho} dL_s^l\left(X^1\right)\right) = \frac{1}{2\rho} L_t^l\left(X\right), \end{aligned}$$

where the first term vanishes because for X^1 it holds that $I(X_s^1 = l)ds = 0$. The continuity of X follows from the continuity of X^1 and T. Hence, (X, B) also solves (2.2).

The next step is to show the uniqueness in law of the solution X. We reset the notation and suppose that (X, B) solves (2.1) and (2.2). Define

$$T_t = \int_0^t I\left(X_s > l\right) ds$$

for $t \geq 0$. Then T_t is continuous and strictly increasing almost surely. This can be shown by contradiction. Assume T_t is not strictly increasing; then there exists a set

$$\Gamma = \left\{\omega \in \Omega : T_{t_1} = T_{t_2} \text{ for some } 0 < t_1 < t_2\right\},$$

with $\mathbb{P}(\Gamma) > 0$ and t_1, t_2 depending on ω. Now $T_{t_1} = T_{t_2}$ implies that the process stays at the boundary for all $s \in [t_1, t_2]$, and so

$$\Gamma \subset \left\{\omega : \int_{t_1}^{t_2} dL_s^l\left(X\right) = L_{t_2}^l\left(X\right) - L_{t_1}^l\left(X\right) > 0 \text{ for some } 0 < t_1 < t_2\right\};$$

i.e., the local time increases between t_1 and t_2. On this set, $I(X_s > l) = 0$ for all $s \in [t_1, t_2]$, and hence

$$\Gamma \subset \left\{\omega : X_{t_2} = X_{t_1} + L_{t_2}^l\left(X\right) - L_{t_1}^l\left(X\right) > X_{t_1} \text{ for some } 0 < t_1 < t_2\right\},$$

as the drift and volatility vanish. This is a contradiction to $I(X_s > l) = 0$, and so, in summary, T_t is strictly increasing almost surely. The inverse of T_t, given by

$$\phi_t = \inf\left\{s \geq 0 : T_s > t\right\},$$

is therefore also continuous and almost surely finite. As X and ϕ are continuous, it follows that X is constant on every interval $[\phi_{t-}, \phi_t]$, and so ϕ is in synchronization with X (see [4, Definition 10.13], which refers to this as adaptedness of X to the time change ϕ).

Now set $X_t^1 = X_{\phi_t}$. Then X^1 is a continuous semimartingale (see Corollary

10.12 and Lemma 10.15 in [4]), and by (2.2),

$$t = T_{\phi_t} = \int_0^{\phi_t} I\left(X_s > l\right) ds = \phi_t - \int_0^{\phi_t} I\left(X_s = l\right) ds$$
$$= \phi_t - \frac{1}{2\rho} L^l_{\phi_t}\left(X\right) = \phi_t - \frac{1}{2\rho} L^l_t\left(X^1\right),$$

so

$$\phi_t = t + \frac{1}{2\rho} L^l_t\left(X^1\right),$$

which shows that ϕ is also strictly increasing. Let $B^1_t = \int_0^{\phi_t} I(X_s > l) dB_s$. Then B^1_t is a continuous local martingale with

$$\langle B^1 \rangle_t = \int_0^{\phi_t} I\left(X_s > l\right) ds = T_{\phi_t} = t,$$

and hence B^1 is a Brownian motion by Lévy's criterion. Furthermore, by (2.1) it follows that

$$X^1_t = X_0 + \int_0^{\phi_t} \mu\left(X_s\right) I\left(X_s > l\right) ds + \int_0^{\phi_t} \sigma\left(X_s\right) I\left(X_s > l\right) dB_s + \frac{1}{2} L^l_{\phi_t}\left(X\right)$$
$$= X_0 + \int_0^{t} \mu\left(X^1_s\right) I\left(X^1_s > l\right) d\phi_s + \int_0^{t} \sigma\left(X^1_s\right) I\left(X^1_s > l\right) dB^1_s + \frac{1}{2} L^l_t\left(X^1\right),$$

by the change of variables formula, and

$$dX^1_t = \mu\left(X^1_t\right) I\left(X^1_t > l\right) d\phi_t + \sigma\left(X^1_t\right) I\left(X^1_t > l\right) dB^1_t + \frac{1}{2} dL^l_t\left(X^1\right)$$
$$= \mu\left(X^1_t\right) I\left(X^1_t > l\right) \left(dt + \frac{1}{2\rho} dL^l_t\left(X^1\right)\right) + \sigma\left(X^1_t\right) I\left(X^1_t > l\right) dB^1_t + \frac{1}{2} dL^l_t\left(X^1\right)$$
$$= \mu\left(X^1_t\right) I\left(X^1_t > l\right) dt + \sigma\left(X^1_t\right) I\left(X^1_t > l\right) dB^1_t + \frac{1}{2} dL^l_t\left(X^1\right).$$

Moreover, we have

$$\int_0^t I\left(X^1_s = l\right) ds = \int_0^t I\left(X_{\phi_s} = l\right) dT_{\phi_s} = \int_0^{\phi_t} I\left(X_s = l\right) dT_s = 0.$$

The last two equations show that (X^1, B^1) is a unique weak solution to the system of SDEs (2.1) and (2.2) for $\rho = \infty$. Since X^1 is the unique solution to the reflecting SDE and $X_t = X_{\phi_{T_t}} = X^1_{T_t}$, the law of X is also unique. Theorem 3.1 in [2] states that uniqueness in law for X implies joint uniqueness in law for (X, B). □

We will restate some results from [8] and [9] without proof but with adjustments to incorporate the sticky boundary behavior at the left boundary.

Lemma 1. *For any $f, g : \mathbb{S}_n \to \mathbb{R}$ with $g(x_{n+1}) = 0$, we have*

$$\sum_{x\in\mathbb{S}_n^\circ} g\left(x\right)\delta^- x\nabla^- f\left(x\right) = -\sum_{x\in\mathbb{S}_n^-} f\left(x\right)\delta^+ x\nabla^+ g\left(x\right) - g\left(x_0\right)f\left(x_0\right). \tag{B.1}$$

Under Assumption 2, it can be seen that

$$\sup_{x,y\in(l,r)}\left|\frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial y^j}p\left(t,x,y\right)\right| < \infty \tag{B.2}$$

for $i, j = 0, 1, 2$ still holds, because of results from Sturm–Liouville theory and the proof of [8, Lemma 2]. The use of (B.2) is to prove claims of the form $|g(x)| \leq Ch_n^\beta$ for $\beta = 0, 1, 2$ such that the constant $C > 0$ is independent of x and n. The application of this result will not be mentioned explicitly below.

Corollary 1. *Under Assumption 2, for n sufficiently large, there exist constants $C_1, C_2 > 0$, independent of n and $x \in \mathbb{S}_n, y \in \mathbb{S}_n^\circ$, such that*

$$C_1 \leq s_n\left(x\right) \leq C_2, \quad C_1 \leq m_n\left(y\right) \leq C_2, \quad C_1 \leq M_n\left(x_0\right) \leq C_2.$$

Lemma 2. *Under Assumptions 2 and 3, there exists a constant $C > 0$, independent of k and n, such that for $h_n \in (0, \delta)$, where δ is small enough, the following holds:*

$$\lambda_k^n \leq Ck^2. \tag{B.3}$$

Proof. Let the matrix $\mathbb{M}_n$ again be a diagonal matrix with entries $\mathbb{M}_{n,i,i} = M_n(x_i)$ for $i = 0, \ldots, n$. Calculation of $\mathbb{M}_n\mathbb{G}_n$ and the choice of $M_n(x)$ as stated in Section 3.3 implies that

$$\begin{aligned}
\frac{\rho}{\delta^+ x_0}\beta M_n\left(x_0\right) &= \frac{-\mu\left(x_1\right)\delta^+ x_1 + \sigma^2\left(x_1\right)}{2\delta^- x_1 \delta x_1}M_n\left(x_1\right),\\
\frac{\mu\left(x\right)\delta^- x + \sigma^2\left(x\right)}{2\delta^+ x\delta x}M_n\left(x\right) &= \frac{-\mu\left(x^+\right)\delta^+ x^+ + \sigma^2\left(x^+\right)}{2\delta^- x^+ \delta x^+}M_n\left(x^+\right)
\end{aligned}$$

for $x = x_1, \dots, x_{n-1}$. Hence $\mathbb{M}_n\mathbb{G}_n$ is symmetric, and therefore $\mathbb{M}_n^{1/2}\mathbb{G}_n\mathbb{M}_n^{-1/2}$ is also symmetric. Furthermore, it can be seen that $\mathbb{M}_n^{1/2}\mathbb{G}_n\mathbb{M}_n^{-1/2}$ is similar to $\mathbb{G}_n$, and so both matrices have the same eigenvalues. The min-max principle derived in [9, Section 3] shows that

$$\lambda_k^n = \min_{U_k} \max_{f \in U_k} \frac{-f^T \mathbb{M}_n^{1/2}\mathbb{G}_n\mathbb{M}_n^{-1/2} f}{f^T f} = \min_{U_k} \max_{f \in U_k} \frac{(f, -G_n f)_n}{(f, f)_n}, \qquad \text{(B.4)}$$

where U_k denotes a k-dimensional subspace of functions defined on $\mathbb{S}_n$ with boundary condition $f(x_{n+1}) = 0$.

The upper boundary for λ_k^n can be derived in the following way. First,

$$\begin{aligned}
(f, -G_n f)_n &= -\sum_{x \in \mathbb{S}_n^-} f(x)\left(G_n f(x)\right) M_n(x) \\
&= -\sum_{x \in \mathbb{S}_n^\circ} f(x) \frac{1}{m_n(x)} \frac{\delta^- x}{\delta x} \nabla^- \left(\frac{1}{s_n(x)} \nabla^+ f(x) \right) m_n(x)\, \delta x \\
&\quad + \sum_{x \in \mathbb{S}_n^-} k(x) f(x)^2 M_n(x) - \rho\beta f(x_0) \nabla^+ f(x_0) M_n(x_0) \\
&= -\sum_{x \in \mathbb{S}_n^\circ} f(x)\, \delta^- x \nabla^- \left(\frac{1}{s_n(x)} \nabla^+ f(x) \right) + \sum_{x \in \mathbb{S}_n^-} k(x) f(x)^2 M_n(x) \\
&\quad - \rho\beta f(x_0) \nabla^+ f(x_0) M_n(x_0) \\
&= \sum_{x \in \mathbb{S}_n^-} \frac{\delta^+ x}{s_n(x)} \left(\nabla^+ f(x) \right)^2 + \sum_{x \in \mathbb{S}_n^-} k(x) f(x)^2 M_n(x) + \frac{1-\beta}{s_n(x_0)} f(x_0) \nabla^+ f(x_0),
\end{aligned}$$

using Lemma 1 and $1/s_n(x_0) = \rho M_n(x_0)$. Furthermore, by noting that for

Scheme 1, $1-\beta=0$, and for Scheme 2, $1-\beta=O(h_n)$, we derive

$$\begin{aligned}
&\sum_{x\in\mathbb{S}_n^-}\frac{\delta^+x}{s_n\left(x\right)}\left(\nabla^+f\left(x\right)\right)^2+\frac{1-\beta}{s_n\left(x_0\right)}f\left(x_0\right)\nabla^+f\left(x_0\right)\\
&\qquad\le\frac{C_1}{h_n}\sum_{x\in\mathbb{S}_n^-}\left(f\left(x^+\right)-f\left(x\right)\right)^2+\frac{C_2}{h_n}\left|1-\beta\right|\left|f\left(x_0\right)\right|\left|f\left(x_1\right)-f\left(x_0\right)\right|\\
&\qquad\le\frac{C_1}{h_n}\sum_{x\in\mathbb{S}_n^-}\left(f\left(x^+\right)-f\left(x\right)\right)^2+C_3\left|f\left(x_0\right)\right|\left|f\left(x_1\right)-f\left(x_0\right)\right|\\
&\qquad\le\frac{C_1}{h_n}\sum_{x\in\mathbb{S}_n^-}\left(f\left(x^+\right)-f\left(x\right)\right)^2+C_4\left(f\left(x_0\right)^2h_n+\frac{\left(f\left(x_0\right)-f\left(x_1\right)\right)^2}{h_n}\right)\\
&\qquad\le\frac{C_5}{h_n}\sum_{x\in\mathbb{S}_n^-}\left(f\left(x^+\right)-f\left(x\right)\right)^2+C_4f\left(x_0\right)^2h_n,
\end{aligned}$$

where the constants $C_1,\dots,C_5>0$ are independent of n and f. Note that with $f(x_{n+1})=0$, the following holds:

$$\begin{aligned}
\sum_{x\in\mathbb{S}_n^-}\left(f\left(x^+\right)-f\left(x\right)\right)^2&=\sum_{x\in\mathbb{S}_n^-}f\left(x^+\right)^2-2\sum_{x\in\mathbb{S}_n^-}f\left(x^+\right)f\left(x\right)+\sum_{x\in\mathbb{S}_n^-}f\left(x\right)^2\\
&=2\sum_{x\in\mathbb{S}_n^\circ}f\left(x\right)^2-\sum_{x\in\mathbb{S}_n^\circ}f\left(x\right)f\left(x^-\right)-\sum_{x\in\mathbb{S}_n^\circ}f\left(x\right)f\left(x^+\right)\\
&\qquad+f\left(x_0\right)^2-f\left(x_0\right)f\left(x_1\right)\\
&=\sum_{x\in\mathbb{S}_n^\circ}f\left(x\right)\left(-f\left(x^-\right)+2f\left(x\right)-f\left(x^+\right)\right)\\
&\qquad+f\left(x_0\right)^2-f\left(x_0\right)f\left(x_1\right).
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{x\in\mathbb{S}_n^-}\frac{\delta^+x}{s_n\left(x\right)}\left(\nabla^+f\left(x\right)\right)^2+\frac{1-\beta}{s_n\left(x_0\right)}f\left(x_0\right)\nabla^+f\left(x_0\right)\\
&\quad\le\frac{C_5}{h_n}\sum_{x\in\mathbb{S}_n^\circ}f\left(x\right)\left(-f\left(x^-\right)+2f\left(x\right)-f\left(x^+\right)\right)+\frac{C_5}{h_n}f\left(x_0\right)\left(f\left(x_0\right)-f\left(x_1\right)\right)\\
&\qquad\qquad+C_4f\left(x_0\right)^2h_n\\
&\quad=C_5\frac{1}{h_n}f_n^TAf_n+C_4f\left(x_0\right)^2h_n,
\end{aligned}$$

where $f_n=(f(x_0),f(x_1),\dots,f(x_n))^T\in\mathbb{R}^{n+1}$, and A is an $(n+1)\times(n+1)$ tridiagonal matrix with diagonal elements 2 (the first diagonal entry is equal to

1) and off-diagonal elements -1. It can easily be seen that

$$0 \le \frac{\sum_{x\in\mathbb{S}_n^-} k\left(x\right) f\left(x\right)^2 M_n\left(x\right)}{\left(f,f\right)_n} \le C_6 \frac{\sum_{x\in\mathbb{S}_n^-} f\left(x\right)^2 M_n\left(x\right)}{\sum_{x\in\mathbb{S}_n^-} f\left(x\right)^2 M_n\left(x\right)} \le C_6$$

for some constant $C_6 > 0$ independent of n and f, as all terms are positive and $k(x)$ is bounded. Lastly,

$$\begin{aligned}\left(f,f\right)_n &= \sum_{x\in\mathbb{S}_n^-} f\left(x\right)^2 M_n\left(x\right)\\ &\ge \sum_{x\in\mathbb{S}_n^\circ} f\left(x\right)^2 m_n\left(x\right)\delta x + f\left(x_0\right)^2 M_n\left(x_0\right) h_n\\ &\ge C_7 h_n f_n^T f_n\end{aligned}$$

for a constant $C_7 > 0$ independent of n and f, as $h_n \in (0,\delta)$ with δ small enough. Hence,

$$\frac{\left(f,-G_n f\right)_n}{\left(f,f\right)_n} \le \frac{C_5 \frac{1}{h_n} f_n^T A f_n}{C_7 h_n f_n^T f_n} + \frac{C_4 h_n f\left(x_0\right)^2}{C_7 h_n f_n^T f_n} + C_6.$$

Putting these results into (B.4), one obtains

$$\lambda_k^n \le \frac{C_5}{C_7 h_n^2} \min_{U_k} \max_{f\in U_k} \frac{f_n^T A f_n}{f_n^T f_n} + C_8.$$

As

$$\min_{U_k} \max_{f\in U_k} \frac{f_n^T A f_n}{f_n^T f_n}$$

is the kth eigenvalue of the matrix A, one can use the results of [6, Table 2] or alternatively [7, Theorem 2] to obtain that

$$\lambda_k(A) = 4\sin^2 \frac{(2k-1)\pi}{4n+6} \le \frac{4k^2\pi^2}{\left(n+1\right)^2}, \qquad k = 1,2,\ldots,n+1.$$

This now shows that

$$\lambda_k^n \le \frac{C_5}{C_7 h_n^2} \frac{4k^2\pi^2}{\left(n+1\right)^2} + C_8 \le \frac{4\pi^2 C_5}{C_7 C_9^2} k^2 + C_8 \le C_{10} k^2,$$

as by Assumption 3, we have $C_9 \le h_n(n+1)$. □

Lemma 3. *Consider a grid such that $h_n \in (0, \delta)$ with δ small enough; then there exists a constant $C > 0$ such that for any $1 \le k \le h_n^{-1/4}$,*

$$\lambda_k^n - \lambda_k \ge -Ch_n^{1/4},$$

where C is independent of k and n.

Proof. It should first be noted that as $\mathcal{G}$ is a self-adjoint operator, the min-max principle holds (see [3, Theorem 2.1]). In particular,

$$\lambda_k = \min_{L \subset \mathcal{D}, \dim L = k} \max_{\psi \in L, \psi \neq 0} \frac{(\psi, -\mathcal{G}\psi)}{(\psi, \psi)},$$

where L is a linear subspace of the domain of $\mathcal{G}$.

For $i = 1, \ldots, k$ define $\psi_i : \mathbb{S} \to \mathbb{R}$ as a linear interpolation of the approximate eigenfunction φ_i^n over the interval $\mathbb{S}$, which is given by

$$\psi_i(x) = \varphi_i^n\left(y^-\right) + \nabla^- \varphi_i^n(y)\left(x - y^-\right)$$

for $x \in [y^-, y]$ and $y \in \mathbb{S}_n^+$. Then $\{\psi_1, \ldots, \psi_k\}$ form a k-dimensional linear space. Furthermore, set $\psi_a(x) = \sum_{i=1}^k a_i \psi_i(x)$ where the a_i are normalized, i.e. $\sum_{i=1}^k a_i^2 = 1$. Using the min-max principle and integration by parts, we obtain

$$\begin{aligned}\lambda_k &\le \max_{a_1,\ldots,a_k : \sum_{i=1}^k a_i^2 = 1} \frac{(\psi_a, -\mathcal{G}\psi_a)}{(\psi_a, \psi_a)} \\ &= \max_{\sum_{i=1}^k a_i^2 = 1} \frac{\int_l^r \frac{\psi_a'(x)^2}{s(x)} dx + \int_l^r k(x)\, \psi_a(x)^2 M(dx)}{\int_l^r \psi_a(x)^2 M(dx)}. \end{aligned} \tag{B.5}$$

We will now estimate the different terms appearing in this equation. First, note that as $1 - \beta = 0$ for Scheme 1 and $|1 - \beta| \le C\delta^+ x_0$ for Scheme 2, it holds that

$$\begin{aligned}|1 - \beta| \left|\psi_a(x_0)\, \nabla^+ \psi_a(x_0)\right| &\le C_1 \psi_a(x_0)^2 \sqrt{\delta^+ x_0} + C_1 \left(\nabla^+ \psi_a(x_0)\right)^2 \left(\delta^+ x_0\right)^{3/2} \\ &\le C_2 \sqrt{h_n} + C_2 \sqrt{\delta} \sum_{x \in \mathbb{S}_n^-} \left(\nabla^+ \psi_a(x)\right)^2 \delta^+ x, \end{aligned} \tag{B.6}$$

with constants $C_1, C_2 > 0$ independent of a and n, because

$$\psi_a(x_0)^2 \leq \sum_{x \in \mathbb{S}_n^-} \psi_a(x_0) \frac{M_n(x)}{M_n(x)} \leq C_3 \sum_{i_1=1}^{k} \sum_{i_2=1}^{k} a_{i_1} a_{i_2} \sum_{x \in \mathbb{S}_n^-} \varphi_{i_1}^n(x) \varphi_{i_2}^n(x) M_n(x) = C_3,$$

as

$$\sum_{x \in \mathbb{S}_n^-} \varphi_{i_1}^n(x) \varphi_{i_2}^n(x) M_n(x) = (\varphi_{i_1}^n, \varphi_{i_2}^n)_n = \delta_{i_1, i_2},$$

and $C_3 > 0$ is independent of a and n. Using this result and the fact that ψ_a is a piecewise linear function, we obtain

$$\begin{aligned}
&\int_l^r \frac{1}{s(x)} \psi_a'(x)^2 \, dx - \sum_{x \in \mathbb{S}_n^-} \frac{1}{s_n(x)} \left(\nabla^+ \psi_a(x)\right)^2 \delta^+ x - \frac{1-\beta}{s_n(x_0)} \psi_a(x_0) \nabla^+ \psi_a(x_0) \\
&\quad \leq \sum_{x \in \mathbb{S}_n^-} \left(\nabla^+ \psi_a(x)\right)^2 \int_x^{x^+} \left| \frac{1}{s(y)} - \frac{1}{s_n(x)} \right| dy - \frac{1-\beta}{s_n(x_0)} \psi_a(x_0) \nabla^+ \psi_a(x_0) \\
&\quad \leq C_4 h_n \sum_{x \in \mathbb{S}_n^-} \left(\nabla^+ \psi_a(x)\right)^2 \delta^+ x + C_5 \left(\sqrt{h_n} + \sqrt{\delta} \sum_{x \in \mathbb{S}_n^-} \left(\nabla^+ \psi_a(x)\right)^2 \delta^+ x \right) \\
&\quad \leq C_6 h_n \sum_{x \in \mathbb{S}_n^-} \left(\nabla^+ \psi_a(x)\right)^2 \delta^+ x + C_5 \sqrt{h_n}, \qquad \text{(B.7)}
\end{aligned}$$

where $C_4, C_5, C_6 > 0$ are independent of a and n. The term appearing in (B.7) can be handled as follows:

$$\begin{aligned}
&\sum_{x\in\mathbb{S}_n^-}\left(\nabla^+\psi_a(x)\right)^2\delta^+x \\
&\quad\le -C_7\frac{\beta}{s_n(x_0)}\psi_a(x_0)\nabla^+\psi_a(x_0)+C_7\frac{\beta}{s_n(x_0)}\psi_a(x_0)\nabla^+\psi_a(x_0)\\
&\qquad + C_7\sum_{x\in\mathbb{S}_n^-}\frac{1}{s_n(x)}\left(\nabla^+\psi_a(x)\right)^2\delta^+x + C_7\sum_{x\in\mathbb{S}_n^-}k(x)M_n(x)\psi_a(x)^2\\
&\quad= -C_7\beta\rho M_n(x_0)\psi_a(x_0)\nabla^+\psi_a(x_0)+\frac{C_7\beta}{s_n(x_0)}\psi_a(x_0)\nabla^+\psi_a(x_0)\\
&\qquad - C_7\sum_{x\in\mathbb{S}_n^\circ}\psi_a(x)\delta^-x\nabla^-\left(\frac{1}{s_n(x)}\nabla^+\psi_a(x)\right)-\frac{C_7}{s_n(x_0)}\psi_a(x_0)\nabla^+\psi_a(x_0)\\
&\qquad + C_7\sum_{x\in\mathbb{S}_n^-}k(x)M_n(x)\psi_a(x)^2\\
&\quad= -C_7\sum_{x\in\mathbb{S}_n^-}\psi_a(x)M_n(x)G_n\psi_a(x)+\frac{C_7(\beta-1)}{s_n(x_0)}\psi_a(x_0)\nabla^+\psi_a(x_0)\\
&\quad\le C_7\sum_{i_1=1}^{k}\sum_{i_2=1}^{k}a_{i_1}a_{i_2}\lambda_{i_2}^n\sum_{x\in\mathbb{S}_n^-}\varphi_{i_1}^n(x)\varphi_{i_2}^n(x)M_n(x)\\
&\qquad + C_8\left(\sqrt{h_n}+\sqrt{\delta}\sum_{x\in\mathbb{S}_n^-}\left(\nabla^+\psi_a(x)\right)^2\delta^+x\right)\\
&\quad\le C_9\lambda_k^n + C_7\sqrt{\delta}\sum_{x\in\mathbb{S}_n^-}\left(\nabla^+\psi_a(x)\right)^2\delta^+x, \qquad \text{(B.8)}
\end{aligned}$$

where the last inequality follows from the fact that $0\le\lambda_1^n<\lambda_2^n<\cdots<\lambda_k^n$. The constants $C_7, C_8, C_9>0$ are independent of a and n. We can now choose δ small enough so that $1-C_7\sqrt{\delta}>0$; then

$$\sum_{x\in\mathbb{S}_n^-}\left(\nabla^+\psi_a(x)\right)^2\delta^+x\le\frac{C_9}{1-C_7\sqrt{\delta}}\lambda_k^n\le C_{10}\lambda_k^n.$$

Combining (B.7) and the previous results yields

$$\begin{aligned}
\int_l^r\frac{1}{s(x)}\psi_a'(x)\,dx &\le \frac{1-\beta}{s_n(x_0)}\psi_a(x_0)\nabla^+\psi_a(x_0)+\sum_{x\in\mathbb{S}_n^-}\frac{1}{s_n(x)}\left(\nabla^+\psi_a(x)\right)^2\delta^+x\\
&\qquad + C_{11}\lambda_k^n h_n + C_5 h_n^{1/2}.
\end{aligned}$$

The second term in (B.5) can be handled as follows:

$$\begin{aligned}
&\left|\int_l^r k(x)\,\psi_a(x)^2\,M(dx) - \sum_{x\in\mathbb{S}_n^-} k(x)\,\psi_a(x)^2\,M_n(x)\right| \\
&\quad\le \left|k(x_0)\,\psi_a(x_0)^2\,M(x_0) - k(x_0)\,\psi_a(x_0)^2\,M_n(x_0)\right| \\
&\qquad + \left|\int_l^r k(x)\,\psi_a(x)^2\,m(x)\,dx - \sum_{x\in\mathbb{S}_n^\circ} k(x)\,\psi_a(x)^2\,m_n(x)\,\delta x\right| \\
&\quad\le C_{12} h_n \psi_a(x_0)^2 + \frac{1}{2}\sum_{x\in\mathbb{S}_n^-}\int_x^{x^+}\left|k(y)\,\psi_a(y)^2\,m(y) - k(x)\,\psi_a(x)^2\,m_n(x)\right| dy \\
&\qquad + \frac{1}{2}\sum_{x\in\mathbb{S}_n^-}\int_x^{x^+}\left|k(y)\,\psi_a(y)^2\,m(y) - k\left(x^+\right)\psi_a\left(x^+\right)^2 m_n\left(x^+\right)\right| dy \\
&\quad\le C_{13}\left(\sqrt{\lambda_k^n} h_n + \lambda_k^n h_n^2\right),
\end{aligned}$$

where the last inequality follows in the same way as in the proof of [9, Lemma 4]. Using this result to bound the numerator, one obtains

$$\begin{aligned}
&\int_l^r \frac{1}{s(x)}\psi_a'(x)^2\,dx + \int_l^r k(x)\,\psi_a(x)^2\,M(dx) \\
&\quad\le -\sum_{x\in\mathbb{S}_n^-}\psi_a(x)\,M_n(x)\,G_n\psi_a(x) + C_{14}\left(\left(\sqrt{\lambda_k^n} + \lambda_k^n\right)h_n + \lambda_k^n h_n^2 + h_n^{1/2}\right) \\
&\quad\le \lambda_k^n + C_{14}\left(\left(\sqrt{\lambda_k^n} + \lambda_k^n\right)h_n + \lambda_k^n h_n^2 + h_n^{1/2}\right)
\end{aligned}$$

for some constant $C_{14} > 0$ independent of a, k, and n. The denominator can be estimated similarly as before by setting $k(x) = 1$:

$$\left|\int_l^r \psi_a(x)^2\,M(dx) - 1\right| \le C_{15}\left(\sqrt{\lambda_k^n} h_n + \lambda_k^n h_n^2\right),$$

where $C_{15} > 0$ is a constant independent of a, k and n. As all of the constants are independent of a, it follows that

$$\lambda_k \le \frac{\lambda_k^n + C_{14}\left(\left(\sqrt{\lambda_k^n} + \lambda_k^n\right)h_n + \lambda_k^n h_n^2 + h_n^{1/2}\right)}{1 - C_{15}\left(\sqrt{\lambda_k^n} h_n + \lambda_k^n h_n^2\right)}.$$

Using Lemma 2, i.e., $\lambda_k^n \le C_{16} k^2 \le C_{16} h_n^{-1/2}$ for some $C_{16} > 0$ independent of

k and n, we have

$$\lambda_k - \lambda_k^n \leq \frac{C_{14}\left(\left(\sqrt{\lambda_k^n} + \lambda_k^n\right)h_n + \lambda_k^n h_n^2 + h_n^{1/2}\right) + C_{15}\lambda_k^n\left(\sqrt{\lambda_k^n} h_n + \lambda_k^n h_n^2\right)}{1 - C_{15}\left(\sqrt{\lambda_k^n} h_n + \lambda_k^n h_n^2\right)}$$
$$\leq \frac{C_{17}\left(h_n^{3/4} + h_n^{1/2} + h_n^{3/2} + h_n^{1/4} + h_n^{1/2}\right)}{1 - C_{15}\left(\delta^{3/4} + \delta^{3/2}\right)} \leq C_{18} h_n^{1/4}$$

for constants $C_{17}, C_{18} > 0$ independent of k and n, as long as δ is small enough so that $1 - C_{15}(\delta^{3/4} + \delta^{3/2}) > 0$. □

Lemma 4. *If $h_n \in (0, \delta)$ for δ small enough, there exists a constant $C > 0$ such that for any $1 \leq k \leq n$,*

$$\|\varphi_k^n\|_{n,\infty} \leq Ck,$$

where C is independent of k and n.

Proof. Note that for every $y \in \mathbb{S}_n^-$,

$$\varphi_k^n(y) = \sum_{y \leq x < x_{n+1}} \varphi_k^n(x) - \varphi_k^n\left(x^+\right) = -\sum_{y \leq x < x_{n+1}} \nabla^+ \varphi_k^n(x)\, \delta^+ x,$$

as $\varphi_k^n(x_{n+1}) = 0$. Then

$$|\varphi_k^n(y)| = \left| -\sum_{y \leq x < x_{n+1}} \nabla^+ \varphi_k^n(x)\, \delta^+ x \right| \leq \sum_{x \in \mathbb{S}_n^-} \left|\nabla^+ \varphi_k^n(x)\right| \delta^+ x$$
$$\leq \sqrt{\sum_{x \in \mathbb{S}_n^-} \left(\nabla^+ \varphi_k^n(x)\right)^2 \delta^+ x \sum_{x \in \mathbb{S}_n^-} \delta^+ x} \leq C_1 \sqrt{\lambda_k^n}$$

for a constant $C_1 > 0$ independent of k, n, and y, because of the same steps as shown in (B.8) with ψ_a replaced by φ_k^n. Furthermore, by Lemma 2, i.e., $\lambda_k^n \leq C_2 k^2$, it follows that

$$\|\varphi_k^n\|_{n,\infty} \leq C_1 \sqrt{\lambda_k^n} \leq C_3 k$$

for constants $C_2, C_3 > 0$ independent of k and n. □

Lemma 5. *It holds that*

$$\left| \sum_{x \in \mathbb{S}_n^-} f(x)\, M_n(x) - \int_{x_0}^{x_{n+1}} f(x)\, M(dx) \right| \leq C \max\{\|f\|_\infty, \|f''\|_\infty\} h_n^\gamma$$

for some constant $C > 0$ independent of n and f.

Proof. We can prove the following by using Proposition 2 and the trapezoidal rule:

$$
\begin{aligned}
&\sum_{x\in\mathbb{S}_n^-} f(x)\, M_n(x) - \int_{x_0}^{x_{n+1}} f(x)\, M(dx) \\
&\quad = f(x_0)\, M_n(x_0) - f(x_0)\, M(x_0) + \sum_{x\in\mathbb{S}_n^\circ} f(x)\, m_n(x)\,\delta x \\
&\qquad - \int_{x_0}^{x_{n+1}} f(x)\, m(x)\, dx \\
&\quad = f(x_0)\, \frac{\delta^+ x_0}{\sigma^2(x_0)} \left(M(x_0)\,\alpha - 1\right) + O\left(h_n^2\right) + \sum_{x\in\mathbb{S}_n^\circ} f(x)\, m_n(x)\,\delta x \\
&\qquad - \int_{x_0}^{x_{n+1}} f(x)\, m(x)\, dx \\
&\quad \le C_1 \left|M(x_0)\,\alpha - 1\right| \|f\|_\infty h_n + C_2 \left\|f''\right\|_\infty h_n^2 \\
&\quad \le \begin{cases} C_3 \|f\|_\infty h_n & \text{for } \alpha = \mu(x_0), \\ C_2 \|f''\|_\infty h_n^2 & \text{for } \alpha = \rho \quad (\text{as } M(x_0)\,\alpha = \frac{1}{\rho} \times \rho = 1) \end{cases} \\
&\quad \le C_4 \max\{\|f\|_\infty, \left\|f''\right\|_\infty\} h_n^\gamma,
\end{aligned}
$$

where $C_1, \ldots, C_4 > 0$ are independent of n and f. □

Corollary 2. *For $h_n \in (0, \delta)$, the following lower bound holds for every $1 \le k \le h_n^{-1/4}$:*

$$\lambda_k^n \ge C k^2,$$

if δ is sufficiently small and $C > 0$ is a constant independent of k and n.

Proof. The proof is the same as the proof of [5, Corollary 3.7], using Proposition 3 and Lemma 2. □

C. Pseudocode

We provide pseudocode for our CTMC approximation algorithm for pricing of some payoff f under a diffusion model with sticky lower boundary. Symbols in bold are vectors or matrices. Round brackets show the index or range of indices (indices start at 0), and square brackets create a vector.

Algorithm 1: Pricing of some payoff f under a diffusion model with sticky lower boundary

def `CTMCPricing`$(l, r, \mu, \sigma, \rho, k, x, f, T, n, \mathsf{scheme})$**:**

  Data: Parameters of underlying diffusion and product to be priced; scheme indicates the scheme to be used.

  Result: Price of the product with payoff f.

  if $r == \infty$**:**

    $r \leftarrow r < \infty$

  $\boldsymbol{x} = (x_0, \dots, x_{n+1}) = \texttt{Linspace}(l, r, n+2)$

  $\boldsymbol{\delta}^+ \leftarrow \boldsymbol{x}(2:n) - \boldsymbol{x}(1:n-1),\ \boldsymbol{\delta}^- \leftarrow \boldsymbol{x}(1:n-1) - \boldsymbol{x}(0:n-2),\ \boldsymbol{\delta} \leftarrow \frac{1}{2}\left(\boldsymbol{\delta}^+ + \boldsymbol{\delta}^-\right)$

  $\boldsymbol{v}_l \leftarrow -\frac{\mu(\boldsymbol{x}(1:n-1)) * \boldsymbol{\delta}^+}{2\boldsymbol{\delta}^- * \boldsymbol{\delta}} + \frac{\sigma^2(\boldsymbol{x}(1:n-1))}{2\boldsymbol{\delta}^- * \boldsymbol{\delta}},\ \boldsymbol{v}_u \leftarrow \frac{\mu(\boldsymbol{x}(1:n-1)) * \boldsymbol{\delta}^+}{2\boldsymbol{\delta}^- * \boldsymbol{\delta}} + \frac{\sigma^2(\boldsymbol{x}(1:n-1))}{2\boldsymbol{\delta}^- * \boldsymbol{\delta}}$

  $\boldsymbol{v}_d \leftarrow [0, -\boldsymbol{v}_l - \boldsymbol{v}_u - k\left(\boldsymbol{x}(1:n-1)\right), 0],\ \boldsymbol{v}_u \leftarrow [0, \boldsymbol{v}_u(0:n-2), 0],\ \boldsymbol{v}_l \leftarrow [\boldsymbol{v}_l, 0]$

  if $\mathsf{scheme} == 1$**:**

    $\boldsymbol{v}_u(0) \leftarrow \rho / \left(\boldsymbol{x}(1) - \boldsymbol{x}(0)\right),\ \boldsymbol{v}_d(0) \leftarrow -\boldsymbol{v}_u(0) - k\left(\boldsymbol{x}(0)\right)$

  else:

    $\boldsymbol{v}_u(0) \leftarrow \frac{\rho}{\boldsymbol{x}(1) - \boldsymbol{x}(0) + \frac{\rho - \mu(\boldsymbol{x}(0))}{\sigma^2(\boldsymbol{x}(0))}\left(\boldsymbol{x}(1) - \boldsymbol{x}(0)\right)^2},\ \boldsymbol{v}_d(0) \leftarrow -\boldsymbol{v}_u(0) - k\left(\boldsymbol{x}(0)\right)$

  $\boldsymbol{G} \leftarrow \begin{pmatrix} \ddots & \ddots & \ddots & & \\ & \boldsymbol{v}_l & \boldsymbol{v}_d & \boldsymbol{v}_u & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$

  $\boldsymbol{f} \leftarrow f(\boldsymbol{x})$

  $\boldsymbol{P} = \texttt{MatrixExponential}(T\boldsymbol{G})\boldsymbol{f}$

  if $x \in \boldsymbol{x}$**:**

    $i = \texttt{FindIndex}(x, \boldsymbol{x})$

    return $P(i)$

  else:

    $P(\cdot) = \texttt{CubicSplineInterpolation}(\boldsymbol{x}, \boldsymbol{P})$

    return $P(x)$

In line 4, a uniform grid is generated by the function `Linspace`, but this can be replaced by a non-uniform grid. The operations in line 6 are understood to be componentwise. Line 12 generates the tridiagonal matrix based on the transition rates in $\boldsymbol{v}_l, \boldsymbol{v}_d$, and $\boldsymbol{v}_u$, and $\boldsymbol{f}$ is the vector of payoffs at each grid point. In line 14 the matrix exponential is calculated for $T\boldsymbol{G}$ and multiplied by the vector $\boldsymbol{f}$. This is a matrix–vector multiplication, and the result is again an $(n+1)$-dimensional vector $\boldsymbol{P}$. The calculation of the matrix exponential can be done using any one of the algorithms described in Section 3.2. The final step is to return the price corresponding to the starting point x. If the starting point is in the grid $\boldsymbol{x}$, then line 16 returns the index in the vector corresponding to x. In case x is not on the grid, we apply cubic spline interpolation to the price vector and obtain the price at x from the cubic spline function.

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