Appendix A. Basic examples

A.1. Spreading on a cycle

Consider the example of SI spreading with power law weights on the cycle C_n with n vertices, with $\alpha \in (1/2, 1)$. The process on the cycle can be understood via the process on the integer line $\mathbb{Z}$: the time T_k of the infection of the k'th vertex has the same distribution on $\mathbb{Z}$ as on C_n, for every $k \leqslant n$. On $\mathbb{Z}$, we will denote by X_i the power law distributed random weight on the edge $(i-1, i)$ if $i > 0$, and on the edge $(i, i+1)$ if $i < 0$.

Proposition A.1. *For the SI spreading process $(T_k)_{k=1}^{\infty}$ on $\mathbb{Z}$ with power law weights $\alpha \in (1/2, 1)$, the expected time to infect k vertices satisfies*

$$\mathbb{E}(T_k) \asymp k^{1/\alpha},$$

where the constant factors depend only on α.

Proof. Let $S_k = \sum_{i=1}^{k} X_i$ and $S_k^* = \sum_{i=-1}^{-k} X_i$. Note that

$$\min\{S_{\lfloor k/2 \rfloor}, S^*_{\lfloor k/2 \rfloor}\} \leqslant T_{k+1} \leqslant \min\{S_k, S_k^*\}.$$

Then it is enough to prove that $\mathbb{E}(\min\{S_k, S_k^*\}) \asymp k^{1/\alpha}$. It is well-known (see [22, Theorem 3.7.2]) that the sum S_k as $k \to \infty$ is in the domain of attraction of the stable law Y with the same parameter α:

$$\mathbb{P}(S_k/k^{1/\alpha} > t) \xrightarrow[k\to\infty]{} \mathbb{P}(Y > t).$$

Denote $\overline{S}_k = S_k/k^{1/\alpha}$. The convergence is given via the convergence of characteristic functions, where the limit characteristic function is given by [22]:

$$\phi_Y(t) = \lim_{k\to\infty} \phi_{\overline{S}_k}(t) = C_{\mathrm{sgn}(t)} \exp(-b|t|^{\alpha}), \tag{A.1}$$

where the constants, $C_{-1} = \overline{C}_1$ and $b > 0$, depend on α. Hence, in the bounded interval $|t| < 1$ the convergence in (A.1) is uniform in t, thus we can write

$$\phi_{\overline{S}_k}(t) = C_{\mathrm{sgn}(t)} \exp(-b|t|^{\alpha})(1 + o(1)),$$

where $o(1) \to 0$ as $k \to \infty$ uniformly in $|t| < 1$. Using the relation between the tail distribution and the characteristic function, given by the following inequality [22, Eq. (3.3.1)]:

$$\mathbb{P}(|X| > 2/u) \leqslant \frac{1}{u} \int_{-u}^{u} (1 - \phi_X(t))\, dt,$$

where X is a random variable with characteristic function $\phi_X(t)$, we derive that when t is sufficiently large then for all k,

$$\mathbb{P}(\overline{S}_k > t) \leqslant t \int_{-2/t}^{2/t} 1 - (1 + o(1))\, C_{\mathrm{sgn}(t)} \exp\big(-b|x|^{\alpha}\big)\, dx < t \int_{-2/t}^{2/t} C_2 |x|^{\alpha} dx = C_3 t^{-\alpha}, \tag{A.2}$$

where $C_3 > 0$ is constant that depends on α. Thus we have for sufficiently large t:

$$\mathbb{P}(\min\{S_k, S_k^*\}/k^{1/\alpha} > t) \leqslant C_4 t^{-2\alpha},$$

where $C_4 > 0$ is constant that depends on α. Since S_k is positive then we can find a random variable Z with power law tail with exponent 2α such that $|\min\{S_k, S_k^*\}/k^{1/\alpha}| < Z$ a.s. for all $k > 0$, and thus by Dominated Convergence Theorem for $\alpha > 1/2$ we have convergence of expectations

$$\mathbb{E}(\min\{S_k, S_k^*\}/k^{1/\alpha}) \xrightarrow[k\to\infty]{} \mathbb{E}(\min\{Y, Y^*\}).$$

where Y, Y^* are stable with parameter α. The minimum of Y, Y^* has power law tail with exponent 2α thus has finite expectation and we have:

$$\mathbb{E}(\min\{S_k, S_k^*\}/k^{1/\alpha}) \asymp 1,$$

for all $k > 0$, which implies the statement of the proposition. □

A.2. Spreading on a star

The star graph St_n consists of a distinguished root vertex 0 and vertices $\{1, 2, \dots, n-1\}$ attached to it. On St_n, we consider the SI spreading process $T = (T_k)_{k=1}^n$ started from the root, with power law distributed random weights with $\alpha \in (1/2, 1)$, denoted as $X_1, X_2, \dots, X_{n-1}$.

Proposition A.2. *On the graph St_n with $n \geqslant 2$, the expected time to infect k vertices, for $k \leqslant n-1$, is bounded by*

$$\mathbb{E}(T_k) \leqslant C_\alpha k^{1/\alpha},$$

where $C_\alpha > 0$ is a constant that depends only on α. For $k = n-1$, in order to infect all but one vertices, we have

$$\mathbb{E}(T_{n-1}) \asymp n^{1/\alpha},$$

with the implicit constant factors depending on α.

Proof. Denote by $X_{(1)}^{n-1} < \cdots < X_{(n-1)}^{n-1}$ the order statistics of $X_1, \dots, X_{n-1}$. Then we have $T_{k+1} = X_{(k)}^{n-1}$, and it is obvious that

$$X_{(k)}^{n-1} \preceq X_{(k)}^{k+1},$$

for $k \leqslant n-2$, hence the second statement of the theorem implies the first one.

It is straightforward to calculate the tail distribution of $X_{(k)}^{k+1}$:

$$\begin{aligned}\mathbb{P}\big(X_{(k)}^{k+1} > t\big) &= 1 - \mathbb{P}(X_{(k)}^{k+1} < t) \\ &= 1 - (k+1)\,\mathbb{P}(X_1, \dots, X_k < t, X_{k+1} > t) - \mathbb{P}(X_1, \dots, X_{k+1} < t) \\ &= 1 - (k+1)(1 - t^{-\alpha})^k t^{-\alpha} - (1 - t^{-\alpha})^{k+1}.\end{aligned} \tag{A.3}$$

To get an upper bound on $\mathbb{E}(X_{(k)}^{k+1})$, we integrate (A.3) over $t > 0$, using the bound $(1 - t^{-\alpha})^k > 1 - kt^{-\alpha}$ for $t > k^{1/\alpha}$, and the bound $\mathbb{P}(X_{(k)}^{k+1} > t) \leqslant 1$ for $t < k^{1/\alpha}$.

After cancellations,

$$\begin{aligned}\mathbb{E}\big(X_{(k)}^{k+1}\big) &\leqslant \int\limits_{0}^{k^{1/\alpha}} 1\,dt + (k+1)k \int\limits_{k^{1/\alpha}}^{\infty} t^{-2\alpha} dt \\ &= k^{1/\alpha} + (k+1)k\,\frac{k^{1/\alpha-2}}{2\alpha-1} \leqslant C_\alpha k^{1/\alpha}.\end{aligned}$$

In order to get a lower bound, we use the bound $(1-t^{-\alpha})^k < 1 - kt^{-\alpha} + \frac{1}{2}k^2 t^{-2\alpha}$ for $t > k^{1/\alpha}$, and the bound $\mathbb{P}(X_{(k)}^{k+1} > t) \geq 0$ for $t < k^{1/\alpha}$:

$$\begin{aligned}\mathbb{E}\big(X_{(k)}^{k+1}\big) &\geqslant \left(k(k+1) - \frac{(k+1)^2}{2}\right) \int\limits_{k^{1/\alpha}}^{\infty} t^{-2\alpha} dt - \frac{(k+1)k^2}{2} \int\limits_{k^{1/\alpha}}^{\infty} t^{-3\alpha} dt \\ &\sim \frac{1}{2}\left(\frac{1}{2\alpha-1} - \frac{1}{3\alpha-1}\right) k^{1/\alpha} \geqslant c_\alpha k^{1/\alpha},\end{aligned}$$

with some $c_\alpha > 0$. Thus,

$$\mathbb{E}\big(X_{(k)}^{k+1}\big) \asymp k^{1/\alpha},$$

which finishes proof of the theorem. ∎

Appendix B. Preliminary lemmas for general deterministic graphs

Lemma B.1. *Let $(b_n)_{n=1}^{\infty}$ be a positive sequence that satisfies the following recursive inequality for some $C > 0$ and $0 < \alpha < 1$:*

$$b_{n+1} \leqslant b_n + C b_n^{1-\alpha}. \tag{B.1}$$

Then

$$b_n \leqslant d n^{1/\alpha},$$

with $d = \max\{b_1, (\alpha C)^{1/\alpha}\}$.

Proof. We prove the statement by induction. By definition, the statement holds for $b_1 \leqslant d$. Suppose the statement holds for some $n > 1$: for any k with $1 \leqslant k \leqslant n$, we have $b_k \leqslant d k^{1/\alpha}$. Now rewrite (B.1) as

$$b_{k+1} - b_k \leqslant C b_k^{1-\alpha}.$$

Making a telescopic sum, then using the induction hypothesis and bounding the sum with an integral, we obtain

$$\begin{aligned}b_{n+1} - b_1 &\leqslant \sum_{k=1}^{n} C b_k^{1-\alpha} \leqslant \sum_{k=1}^{n} C d^{1-\alpha} k^{1/\alpha-1} \leqslant \int\limits_{1}^{n+1} C d^{1-\alpha} x^{1/\alpha-1} dx \\ &= \alpha C d^{1-\alpha}\left((n+1)^{1/\alpha} - 1\right) \leqslant d\left((n+1)^{1/\alpha} - 1\right).\end{aligned} \tag{B.2}$$

Adding $b_1 \leqslant d$ to this inequality, we arrive at $b_{n+1} \leqslant d(n+1)^{1/\alpha}$, as desired. ∎

Lemma B.2. *Let X and Y be i.i.d. power law distributed random variables with $\alpha \in (1/2, 1)$. Then, for any $t > 1$:*

$$\mathbb{E}(\min\{X, Y - t\} | Y > t) \asymp t^{1-\alpha},$$

with the constant factors depending on α.

Proof. The conditional tail distribution of the minimum is the following:

$$\mathbb{P}\big(\min\{X, Y-t\} > s \mid Y > t\big) = \frac{\mathbb{P}(X > s, Y > t+s)}{\mathbb{P}(Y > t)} = $$
$$= \begin{cases} s^{-\alpha}\left(1 + \dfrac{s}{t}\right)^{-\alpha}, & s > 1; \\ \left(1 + \dfrac{s}{t}\right)^{-\alpha}, & 0 < s < 1. \end{cases}$$

Using the substitution $u = \dfrac{s}{t}$ we write the expected value as follows:

$$\begin{aligned} \mathbb{E}(\min\{X, Y-t\} | Y > t) &= \int_0^\infty \mathbb{P}\big(\min\{X, Y-t\} > s | Y > t\big) ds \\ &= \int_0^1 \left(1 + \frac{s}{t}\right)^{-\alpha} ds + t^{-\alpha} \int_1^\infty \left(\frac{s}{t}\left(1 + \frac{s}{t}\right)\right)^{-\alpha} ds \\ &= t \int_0^{1/t} (1+u)^{-\alpha} du + t^{1-\alpha} \int_{1/t}^\infty (u(1+u))^{-\alpha} du. \end{aligned}$$

In the first integral, $1 \leqslant 1 + u \leqslant 2$, hence that integral is $\asymp 1/t$ and the term is altogether $\asymp 1$. To calculate the second term, we split the interval of integration into two parts again:

$$t^{1-\alpha} \int_{1/t}^\infty (u(1+u))^{-\alpha} du = t^{1-\alpha} \left[\int_{1/t}^1 (u(1+u))^{-\alpha} du + \int_1^\infty (u(1+u))^{-\alpha} du \right]. \tag{B.3}$$

The first integral on the RHS of (B.3) is less than $\int_0^1 u^{-\alpha} du = \frac{1}{1-\alpha}$. The second integral can be bounded in the following way:

$$\int_1^\infty (1+u)^{-2\alpha} du \leqslant \int_1^\infty (u(1+u))^{-\alpha} du \leqslant \int_1^\infty u^{-2\alpha} du,$$

which is clearly $\asymp \frac{1}{2\alpha - 1}$. Summing up the three terms we have calculated, we have proved the lemma. □

Appendix C. Speed of convergence to the infinite tree

Proposition C.1. *Let $\mathcal{T}^N$ be a CGW tree conditioned on $Z_N > 0$ and $\mathcal{T}^\infty$ be an infinite CGW tree. Then, as $N \to \infty$, for any $\varepsilon > 0$ there exist $\delta > 0$ and a coupling between $\mathcal{T}^N$ and $\mathcal{T}^\infty$, such that*

$$\mathbb{P}(\mathcal{T}^N[\delta N] \neq \mathcal{T}^\infty[\delta N]) < \varepsilon.$$

Proof. In order to prove the statement of the proposition we show that the conditioned measure and the infinite measure are close in total variation distance. First we establish bounds on the conditioned measure. Consider a rooted tree T with height k, where $k \leqslant \delta N$ and $\delta > 0$ is small. Then by Bayes' formula,

$$\begin{aligned} \mathbb{P}(\mathcal{T}[k] = T | Z_N > 0) &= \frac{\mathbb{P}(Z_N > 0 | \mathcal{T}[k] = T)}{\mathbb{P}(Z_N > 0)} \mathbb{P}(\mathcal{T}[k] = T) \\ &= \frac{\mathbb{P}(Z_{N-k}^{(1)} > 0 \cup \cdots \cup Z_{N-k}^{(\#T_k)} > 0)}{\mathbb{P}(Z_N > 0)} \mathbb{P}(\mathcal{T}[k] = T), \end{aligned} \tag{C.1}$$

where $Z_{N-k}^{(i)}$ denotes the $(N-k)$'th generation in the copy of the CGW process $Z^{(i)}$, started from a vertex at level k. By Lemma 3.2, for a large enough N there exists $\varepsilon_0 > 0$ such that,

$$\frac{2}{\sigma^2 N}(1 - \varepsilon_0) < \mathbb{P}(Z_N > 0) < \frac{2}{\sigma^2 N}(1 + \varepsilon_0). \tag{C.2}$$

Also, when $N - k$ is large enough, there exists $\varepsilon_1 > 0$, such that

$$\frac{2}{\sigma^2 (N-k)}(1 - \varepsilon_1) < \mathbb{P}(Z_{N-k}^{(i)} > 0) < \frac{2}{\sigma^2 (N-k)}(1 + \varepsilon_1), \tag{C.3}$$

where $1 \leqslant i \leqslant \#T_k$. In order to simplify the further calculations we take common $\varepsilon_2 := \max(\varepsilon_0, \varepsilon_1)$ instead of ε_0 and ε_1 in (C.2) and (C.3), and the conditioned measure is denoted as $\mathbb{P}_N(\cdot) := \mathbb{P}(\cdot \mid Z_N > 0)$.

Upper bound. We use the union bound on the right-hand side of (C.1) and together with (C.2) and (C.3), we obtain

$$\begin{aligned} \mathbb{P}_N(\mathcal{T}[k] = T) &\leqslant \frac{\#T_k \mathbb{P}(Z_{N-k} > 0)}{\mathbb{P}(Z_N > 0)} \mathbb{P}(\mathcal{T}[k] = T) \\ &< \frac{N}{(N-k)} \#T_k \mathbb{P}(\mathcal{T}[k] = T) \frac{1 + \varepsilon_2}{1 - \varepsilon_2}. \end{aligned} \tag{C.4}$$

Therefore, we can write that for small enough k there exists $\varepsilon_3 > 0$, such that

$$\mathbb{P}_N(\mathcal{T}[k] = T) < \frac{N}{(N-k)} \#T_k \mathbb{P}(\mathcal{T}[k] = T)(1 + \varepsilon_3). \tag{C.5}$$

Lower bound. We rewrite (C.1) using (C.5) as follows:

$$\begin{aligned}\mathbb{P}_N(\mathcal{T}[k]=T) &= \frac{1}{\mathbb{P}(Z_N>0)}\left(1-\mathbb{P}(Z^{(1)}_{N-k}=0\cap\cdots\cap Z^{(\#T_k)}_{N-k}=0)\right)\mathbb{P}(\mathcal{T}[k]=T)\\ &= \frac{1}{\mathbb{P}(Z_N>0)}\left(1-(1-\mathbb{P}(Z_{N-k}>0))^{\#T_k}\right)\mathbb{P}(\mathcal{T}[k]=T)\\ &> \frac{1}{\mathbb{P}(Z_N>0)}\left(1-\left(1-\frac{2(1-\varepsilon_1)}{\sigma^2(N-k)}\right)^{\#T_k}\right)\mathbb{P}(\mathcal{T}[k]=T).\end{aligned} \tag{C.6}$$

Since for any x, where $x>0$, we have $1-x<\exp(-x)<1-x+x^2/2$, then for any $n\geqslant 1$

$$1-(1-x)^n > 1-\exp(-nx) > nx-\frac{(nx)^2}{2}. \tag{C.7}$$

We rewrite (C.6) using (C.7) for $x=\mathbb{P}(Z_{N-k}>0)$ and $n=\#T_k$ as follows:

$$\mathbb{P}_N(\mathcal{T}[k]=T) > \frac{\mathbb{P}(\mathcal{T}[k]=T)}{\mathbb{P}(Z_N>0)}\left(\#T_k\mathbb{P}(Z_{N-k}>0)-\frac{1}{2}\left(\#T_k\mathbb{P}(Z_{N-k}>0)\right)^2\right). \tag{C.8}$$

Now use (C.3) and we obtain:

$$\begin{aligned}\mathbb{P}_N(\mathcal{T}[k]=T) &> \frac{\mathbb{P}(\mathcal{T}[k]=T)}{\mathbb{P}(Z_N>0)}\left(\frac{2\#T_k(1-\varepsilon_2)}{\sigma^2(N-k)}-\frac{1}{2}\left(\frac{2\#T_k(1+\varepsilon_2)}{\sigma^2(N-k)}\right)^2\right)\\ &> \mathbb{P}(\mathcal{T}[k]=T)\#T_k\left(\frac{N}{N-k}\frac{1-\varepsilon_2}{1+\varepsilon_2}-\#T_k\frac{C_1N}{(N-k)^2}\right),\end{aligned} \tag{C.9}$$

where $C_1=\frac{(1+\varepsilon_2)}{2\sigma^2}<1/\sigma^2$. Therefore, we can write that for small enough k there exists $\varepsilon_4>0$ and a bounded $C_2>0$, that depends on σ and ε_2, such that

$$\mathbb{P}_N(\mathcal{T}[k]=T) > \mathbb{P}(\mathcal{T}[k]=T)\#T_k\left(\frac{N}{N-k}-\#T_k\frac{C_2N}{(N-k)^2}\right)(1-\varepsilon_4). \tag{C.10}$$

Combining the (C.5) and (C.10), and choosing $\varepsilon_5:=\max\{\varepsilon_3,\varepsilon_4\}$, we obtain the following bounds on the probability $\mathbb{P}_N(\mathcal{T}[k]=T)$:

$$\left(\frac{N}{N-k}-\#T_k\frac{C_2N}{(N-k)^2}\right)(1-\varepsilon_5)\leqslant\frac{\mathbb{P}_N(\mathcal{T}[k]=T)}{\#T_k\mathbb{P}(\mathcal{T}[k]=T)}\leqslant\frac{N}{(N-k)}(1+\varepsilon_5). \tag{C.11}$$

Total variation distance. Now we bound the total variation distance between conditioned and infinite measures. From the upper bound in (C.11) we obtain that when k is small enough, the following inequality holds:

$$\begin{aligned}\mathbb{P}_N(\mathcal{T}[k]=T)-\mathbb{P}(\mathcal{T}^\infty[k]=T) &\leqslant \left(\frac{N}{N-k}(1+\varepsilon_5)-1\right)\#T_k\mathbb{P}(\mathcal{T}[k]=T)\\ &= \left(\left(\frac{N}{N-k}-1\right)+\frac{N}{N-k}\varepsilon_5\right)\#T_k\mathbb{P}(\mathcal{T}[k]=T),\end{aligned}$$

and, on the other hand, from the lower bound in (C.11) we obtain

$$\begin{aligned}\mathbb{P}(\mathcal{T}^\infty[k]=T)-\mathbb{P}_N(\mathcal{T}[k]=T) &\leqslant \left(1-\frac{N}{N-k}(1-\varepsilon_5)+\#T_k\frac{C_\varepsilon N}{(N-k)^2}(1-\varepsilon_5)\right)\cdot\\ &\quad\cdot\#T_k\mathbb{P}(\mathcal{T}[k]=T)\\ &=\left(\left(1-\frac{N}{N-k}\right)+\frac{N}{N-k}\varepsilon_5+\#T_k\frac{C_\varepsilon N}{(N-k)^2}(1-\varepsilon_5)\right)\cdot\\ &\quad\cdot\#T_k\mathbb{P}(\mathcal{T}[k]=T).\end{aligned}$$

Comparing both bounds we see that all summands are positive, except of $\left(1-\frac{N}{N-k}\right)$, thus we can inverse the sign it and derive the bound for an absolute value. Summing those bounds over all possible trees of height k, we obtain

$$\begin{aligned}\sum_T\Big|\mathbb{P}_N(\mathcal{T}[k]=T)-\mathbb{P}(\mathcal{T}^\infty[k]=T)\Big| &\leqslant\sum_T\left(\left(\frac{N}{N-k}-1\right)+\varepsilon_5\frac{N}{N-k}+\#T_k\frac{C_\varepsilon N}{(N-k)^2}(1-\varepsilon_5)\right)\cdot\\ &\quad\cdot\#T_k\mathbb{P}(\mathcal{T}[k]=T).\end{aligned}\tag{C.12}$$

From the fact that we have a measure on the set of infinite trees and from Lemma 3.4 we have

$$\sum_T\mathbb{P}(\mathcal{T}^\infty[k]=T)=\sum_T\#T_k\mathbb{P}(\mathcal{T}[k]=T)=1,$$

$$\mathbb{E}(Z_k\mid\mathcal{T}^\infty)=\sum_T(\#T_k)^2\mathbb{P}(\mathcal{T}[k]=T)=1+k\sigma^2.$$

Therefore we can rewrite (C.12) for $k=\delta N$, when $\delta>0$ is small, and obtain

$$\sum_T\Big|\mathbb{P}_N(\mathcal{T}[\delta N]=T)-\mathbb{P}(\mathcal{T}^\infty[\delta N]=T)\Big|\leqslant\frac{\delta}{1-\delta}+\frac{\varepsilon_5}{1-\delta}+C_2'\delta\frac{1-\varepsilon_5}{(1-\delta)^2}+\frac{C_2}{N}\frac{1-\varepsilon_5}{(1-\delta)^2},$$

where $C_2'=C_2\sigma^2<1$. Hence, for any $\varepsilon_6>0$ we can find large N and small $\delta>0$, such that

$$\sum_T\Big|\mathbb{P}_N(\mathcal{T}[\delta N]=T)-\mathbb{P}(\mathcal{T}^\infty[\delta N]=T)\Big|\leqslant\varepsilon_6.\tag{C.13}$$

Denote the projection of measures $\mathbb{P}_N$ and $\mathbb{P}_\infty$ onto the trees with common first δN layers $\mathcal{T}[\delta N]$ as $\mathbb{P}_N\restriction_{\delta N}$ and $\mathbb{P}_\infty\restriction_{\delta N}$ respectively. Then, by (C.13) and the definition of the total variation distance we have

$$d_{TV}\left(\mathbb{P}_N\restriction_{\delta N},\mathbb{P}_\infty\restriction_{\delta N}\right)\leqslant\frac{1}{2}\varepsilon_6.$$

Hence by Strassen's Theorem there exists a coupling of random variables $\mathcal{T}[\delta N]$ and $\mathcal{T}^\infty[\delta N]$, with the same d_{TV}. This finishes the proof of Proposition 1.1. □