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SUPPLEMENTARY MATERIAL

Supplementary material to “An extension of Thwaites’ method for turbulent boundary layers"

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1. Assessment of the validity of $\frac{m}{Re_\theta} \ll 1$

For the flows considered in this work, the validity of the assumption of treating $m/Re_\theta \ll 1$ is assessed. It is emphasized that the non-dimensional group m/Re_θ was considered a “small" parameter that allowed

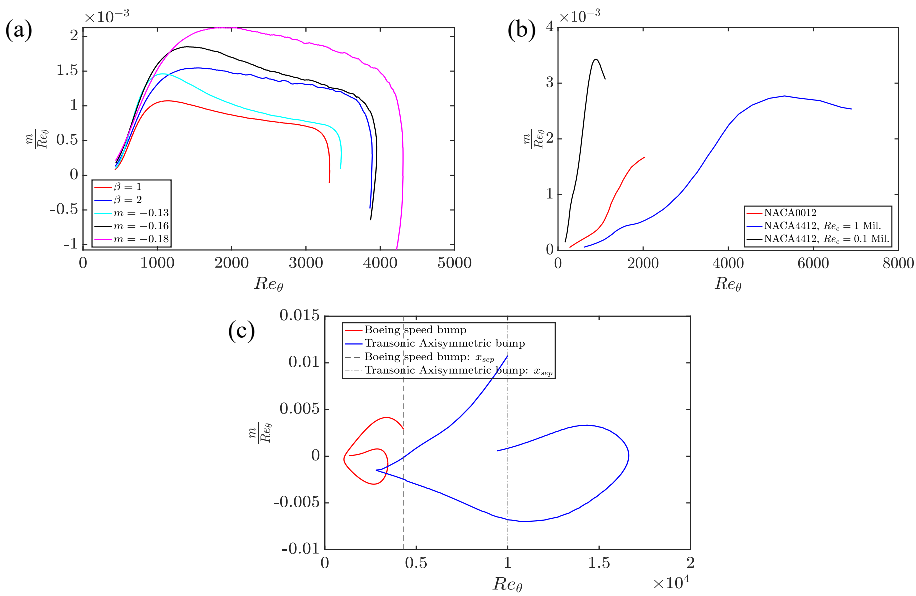

Figure 1. *The assessment of the assumption $\frac{m}{Re_\theta} \ll 1$ for the considered flows in this work. Subfigure (a) corresponds to the five flat-plate, adverse pressure gradient boundary layers (Bobke* et al.*, 2017), subfigure (b) denotes the two-dimensional wing (NACA airfoils) flows (Vinuesa* et al.*, 2017; Tanarro* et al.*, 2020) and subfigure (c) denotes the two cases of separating flows, the flow over the Boeing speed bump (Uzun & Malik, 2022) and the transonic flow over the Bachalo-Johnson bump (Uzun & Malik, 2019).*

the expansions in this work. Figure 1 shows the variation in $\frac{m}{Re_\theta}$ with Re_θ for all the non-equilibrium flows considered in this work. It is clear that, at the largest, $m/Re_\theta \approx 0.01$ and hence this assumption is justified.

2. Relation between of δ, δ^*, and θ

For laminar flows, the solution to Thwaites method for θ and the universal correlation between m and the shape factor H provides a δ^* which is useful for iteratively updating the "inviscid geometry" that is used for computing the freestream profiles. The proposed model in this work provides a good fit for θ, but a fit for δ^* is needed as well for iterative deployment with a potential flow solver. The analytical expression for the displacement thickness can be derived from the continuity equation as,

$$\frac{d\delta^*}{ds} = \frac{V_e}{U_e} + \frac{1}{U_e^2}\frac{dU_e}{ds}\int_0^\delta U dn \tag{1}$$

where δ is the thickness of the boundary layer. For general flows, the integral in Equation 1 is the measure of the mass flow rate inside the boundary layer, and is dependent on the local flow conditions (such as Re_τ and pressure gradient $\frac{dP_e}{ds}$) and is unknown "a priori". For given values of δ and the flow variables at the edge of the boundary layer U_e, V_e, and P_e, the growth rate of the displacement thickness is given as

$$\frac{d\delta^*}{ds} = \frac{V_e}{U_e} - \frac{1}{U_e^2}\frac{dU_e}{ds}\int_{s_0}^{s} \frac{V_e(r)}{U_e(r)} U_e(r) dr, \tag{2}$$

where s_0 is the streamwise location at which the flow can be first considered fully turbulent, within the boundary layer, and s is the location of interest. From geometrical arguments, the ratio V_e/U_e, can be approximately related to the growth of the boundary layer thickness as

$$\frac{V_e}{U_e} \approx \frac{d\delta}{ds} \tag{3}$$

Thus, the growth of the boundary layer affects the growth rate of the displacement thickness linearly as follows,

$$\frac{d\delta^*}{ds} = \frac{d\delta}{ds} - \frac{1}{U_e^2}\frac{dU_e}{ds}\int_{s_0}^{s} \frac{d\delta}{dr} U_e(r) dr \tag{4}$$

Finally, a linear relationship between V_e/U_e, L and m is fitted from the simulation data as

$$L(m, Re) = \frac{U_e}{\nu}\frac{d\theta^2}{ds} \approx 5 + 8m - 200\frac{d\delta}{ds} + 200\frac{d\delta zpg, corr}{ds} \tag{5}$$

which implies that,

$$\frac{d\delta}{ds} \approx \frac{5}{200} + \frac{8}{200}m - \frac{U_e}{200\nu}\frac{d\theta^2}{ds} + \frac{d\delta zpg, corr}{ds} \tag{6}$$

The quality of this fit is verified in Figure 2 by comparing the exact values of $L(m, Re)$ obtained from the datasets considered in this work, and from Equation 6. The fit for $L(m, Re)$ is reasonable for all cases considered, with some discrepancies observed in the low Reynolds number flow over the NACA 4412 airfoil ($Re_\tau \sim O(100)$ for most of the flow). The ordinate of subfigure (c) in Figure 2 is nearly zero as the boundary layer growth of a zero pressure gradient boundary layer is explicitly accounted, using a high Reynolds number fit, in Equation 6. With the two proposed fits in this work, the displacement thickness can be determined along the streamwise coordinate using Equations 2 and 6.

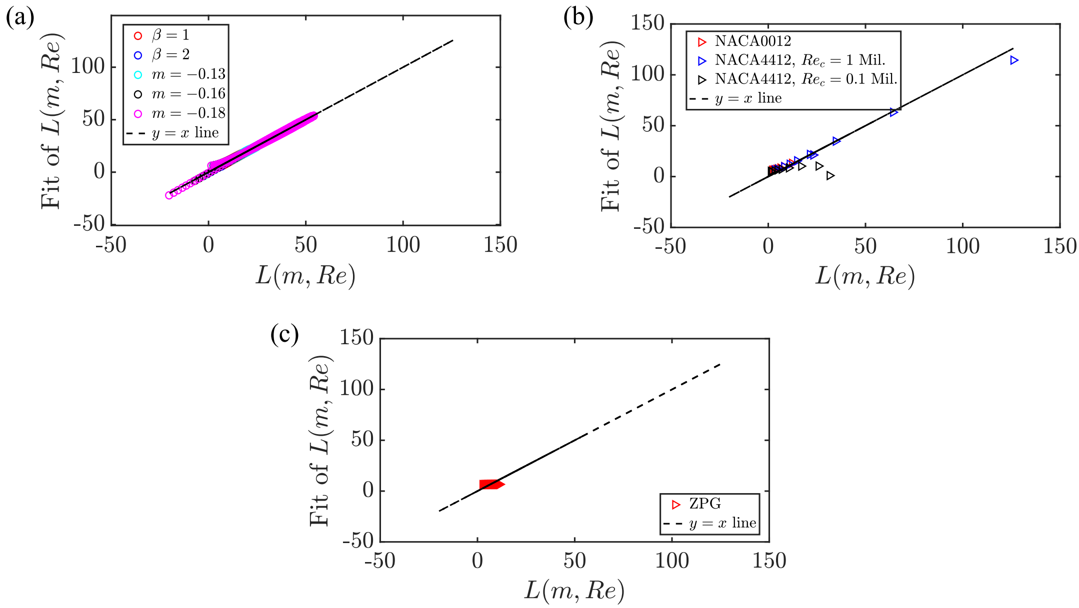

Figure 2. *The quality of the fit between the exact value of $L(m, Re)$ and that obtained from the proposed Equation 6. Subfigure (a) contains data from the five adverse pressure gradient boundary layers of Bobke* et al. *(2017), subfigure (b) contains the three boundary layers from the NACA airfoils (Vinuesa* et al.*, 2017; Tanarro* et al.*, 2020) and subfigure (c) contains the data from the zero pressure gradient boundary layer of Eitel-Amor* et al. *(2014).*

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