

Online appendix for “Combining decision-level data from multiple experiments: what is the pooled estimator doing?”

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An approximate weighting for logit marginal effects

The log-likelihood function is:

Let $\Lambda(x) = (1 + \exp(-x))^{-1}$ be the inverse logit function. It follows that $\Lambda'(x) = \Lambda(x)(1 - \Lambda(x))$ and $\Lambda''(x) = \Lambda'(x)(1 - 2\Lambda(x))$.

We can write the first-order condition of the bivariate logit as:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \sum_{\forall e} \sum_{\forall i, t \in e} \left[y_{e,i,t} - \Lambda(\hat{\beta}_0 + \hat{\beta}_1 x_{e,i,t}) \right] \begin{pmatrix} 1 \\ x_{e,i,t} \end{pmatrix}$$

Taking a 1st-order approximation around:

$$\begin{aligned} \hat{\beta}_0^e + \hat{\beta}_1^e x_e^* &= \log(\bar{y}_e) - \log(1 - \bar{y}_e) \\ x_e^* &= \frac{\text{logit}(\bar{y}_e) - \hat{\beta}_0^e}{\hat{\beta}_1^e} \end{aligned}$$

The first element of the first-order condition is approximately:

$$\begin{aligned} 0 &= \sum_{\forall i, t \in e} \left[y_{e,i,t} - \Lambda(\hat{\beta}_0^e + \hat{\beta}_1^e x_{e,i,t}) \right] \\ &\approx \sum_{\forall i, t \in e} \left[y_{e,i,t} - \bar{y}_e - \bar{y}_e(1 - \bar{y}_e) \hat{\beta}_1^e (x_{e,i,t} - x_e^*) \right] \\ &= \sum_{\forall i, t \in e} \left[y_{e,i,t} - \bar{y}_e - \bar{y}_e(1 - \bar{y}_e) \hat{\beta}_1^e \left(x_{e,i,t} - \frac{\text{logit}(\bar{y}_e) - \hat{\beta}_0^e}{\hat{\beta}_1^e} \right) \right] \\ &= \sum_{\forall i, t \in e} \left[y_{e,i,t} - \bar{y}_e - \bar{y}_e(1 - \bar{y}_e) \left(\hat{\beta}_1^e x_{e,i,t} - \text{logit}(\bar{y}_e) + \hat{\beta}_0^e \right) \right] \\ 0 &\approx \sum_{\forall i, t \in e} \left[\bar{y}_e(1 - \bar{y}_e) \left(\hat{\beta}_1^e x_{e,i,t} - \text{logit}(\bar{y}_e) + \hat{\beta}_0^e \right) \right] \\ 0 &\approx \sum_{\forall i, t \in e} \left[\left(\hat{\beta}_1^e x_{e,i,t} - \text{logit}(\bar{y}_e) + \hat{\beta}_0^e \right) \right] \\ \implies \hat{\beta}_0^e &\approx \text{logit}(\bar{y}_e) - \hat{\beta}_1^e \bar{x}_e \end{aligned}$$

Now for the 2nd element of the first-order condition:

$$\begin{aligned}
0 &= \sum_{\forall i,t \in e} \left[y_{e,i,t} - \Lambda(\hat{\beta}_0^e + \hat{\beta}_1^e x_{e,i,t}) \right] (x_{e,i,t} - \bar{x}_e) \\
&\approx \sum_{\forall i,t \in e} \left[y_{e,i,t} - \bar{y}_e - \bar{y}_e(1 - \bar{y}_e) \left(\hat{\beta}_1^e x_{e,i,t} - \text{logit}(\bar{y}_e) + \hat{\beta}_0^e \right) \right] (x_{e,i,t} - \bar{x}_e) \\
&\approx \sum_{\forall i,t \in e} \left[y_{e,i,t} - \bar{y}_e - \bar{y}_e(1 - \bar{y}_e) \left(\hat{\beta}_1^e x_{e,i,t} - \text{logit}(\bar{y}_e) + \text{logit}(\bar{y}_e) - \hat{\beta}_1^e \bar{x}_e \right) \right] (x_{e,i,t} - \bar{x}_e) \\
&= \sum_{\forall i,t \in e} \left[y_{e,i,t} - \bar{y}_e - \bar{y}_e(1 - \bar{y}_e) \hat{\beta}_1^e (x_{e,i,t} - \bar{x}_e) \right] (x_{e,i,t} - \bar{x}_e) \\
\hat{\beta}_1^e &\approx \frac{\sum_{\forall i,t \in e} (y_{e,i,t} - \bar{y}_e)(x_{e,i,t} - \bar{x}_e)}{\bar{y}_e(1 - \bar{y}_e) \sum_{i,t \in e} (x_{e,i,t} - \bar{x}_e)^2}
\end{aligned}$$

So similarly for $\hat{\beta}_1^{\text{pooled}}$:

$$\begin{aligned}
\hat{\beta}_1^{\text{pooled}} &\approx \frac{\sum_{\forall e} \sum_{\forall i,t \in e} (y_{e,i,t} - \bar{y})(x_{e,i,t} - \bar{x})}{\bar{y}(1 - \bar{y}) \sum_{\forall e} \sum_{\forall i,t \in e} (x_{e,i,t} - \bar{x})^2} \\
&= \frac{\sum_{\forall e} \sum_{\forall i,t \in e} (y_{e,i,t} - \bar{y}_e + \bar{y}_e - \bar{y})(x_{e,i,t} - \bar{x}_e + \bar{x}_e - \bar{x})}{\bar{y}(1 - \bar{y}) \sum_{\forall e} \sum_{\forall i,t \in e} (x_{e,i,t} - \bar{x}_e + \bar{x}_e - \bar{x})^2} \\
&= \frac{\sum_{\forall e} \sum_{\forall i,t \in e} (y_{e,i,t} - \bar{y}_e)(x_{e,i,t} - \bar{x}_e) + \sum_{\forall e} \sum_{\forall i,t \in e} (\bar{y}_e - \bar{y})(\bar{x}_e - \bar{x})}{\bar{y}(1 - \bar{y}) \left[\sum_{\forall e} \sum_{\forall i,t \in e} (x_{e,i,t} - \bar{x}_e)^2 + \sum_{\forall e} \sum_{\forall i,t \in e} (\bar{x}_e - \bar{x})^2 \right]}
\end{aligned}$$

Note that:

$$\begin{aligned}
&\sum_{\forall e} \sum_{\forall i,t \in e} (\bar{y}_e - \bar{y})(\bar{x}_e - \bar{x}) \\
&= \sum_{\forall e} \sum_{\forall i,t \in e} (\bar{y}_e - y_{e,i,t} + y_{e,i,t} - \bar{y})(\bar{x}_e - \bar{x}) \\
&= \sum_{\forall e} \sum_{\forall i,t \in e} (y_{e,i,t} - \bar{y})(\bar{x}_e - \bar{x}) + \sum_{\forall e} \sum_{\forall i,t \in e} (\bar{y}_e - y_{e,i,t})(\bar{x}_e - \bar{x}) \\
&= \sum_{\forall e} \sum_{\forall i,t \in e} (y_{e,i,t} - \bar{y})(\bar{x}_e - \bar{x}) + \sum_{\forall e} (\bar{x}_e - \bar{x}) \underbrace{\sum_{\forall i,t \in e} (\bar{y}_e - y_{e,i,t})}_{=0} \\
&= \sum_{\forall e} \sum_{\forall i,t \in e} (y_{e,i,t} - \bar{y})(\bar{x}_e - \bar{x}) \\
&\approx \tilde{\beta}_1 \bar{y}(1 - \bar{y}) \sum_{\forall e} \sum_{\forall i,t \in e} (\bar{x}_e - \bar{x})^2
\end{aligned}$$

where $\tilde{\beta}_1$ is the pooled estimator replacing all $x_{e,i,t}$ s with $\bar{x}_e$.

We can therefore write:

$$\hat{\beta}_1^{\text{pooled}} \approx \frac{\sum_{\forall e} \hat{\beta}_1^e \bar{y}_e(1 - \bar{y}_e) \sum_{\forall i,t \in e} (x_{e,i,t} - \bar{x}_e)^2 + \tilde{\beta}_1 \bar{y}(1 - \bar{y}) \sum_{\forall e} \sum_{\forall i,t \in e} (\bar{x}_e - \bar{x})^2}{\bar{y}(1 - \bar{y}) \left[\sum_{\forall e} \sum_{\forall i,t \in e} (x_{e,i,t} - \bar{x}_e)^2 + \sum_{\forall e} \sum_{\forall i,t \in e} (\bar{x}_e - \bar{x})^2 \right]}$$

Note here that the weights will not add up to one. This is because the weights for the within-experiment estimators are pre-multiplied by $\bar{y}_e(1 - \bar{y}_e)$, and not $\bar{y}(1 - \bar{y})$.

So how does this go into the marginal effects? We need to multiply this by the average derivative of the inverse logit function:

$$\begin{aligned}
AME_e &= \frac{1}{T_e} \hat{\beta}_1^e \sum_{\forall i, t \in e} \Lambda'(\hat{\beta}_0^e + \hat{\beta}_1^e x_{e,i,t}) \\
&\approx \hat{\beta}_1^e \sum_{\forall i, t \in e} \left(\bar{y}_e(1 - \bar{y}_e) + \bar{y}_e(1 - \bar{y}_e)(1 - 2\bar{y}_e) \hat{\beta}_1^e \left(x_{e,i,t} - \frac{\text{logit}(\bar{y}_e) - \hat{\beta}_0^e}{\hat{\beta}_1^e} \right) \right) \\
&\approx \frac{1}{T_e} \hat{\beta}_1^e \sum_{\forall i, t \in e} \left(\bar{y}_e(1 - \bar{y}_e) + \bar{y}_e(1 - \bar{y}_e)(1 - 2\bar{y}_e) \underbrace{\left(\hat{\beta}_1^e x_{e,i,t} - \text{logit}(\bar{y}_e) - \hat{\beta}_0^e \right)}_{\text{sums to zero}} \right) \\
&= \hat{\beta}_1^e \bar{y}_e(1 - \bar{y}_e)
\end{aligned}$$

Similarly, we can write the AME for the pooled estimator as approximately:

$$\begin{aligned}
AME^{\text{pooled}} &\approx \hat{\beta}_1^{\text{pooled}} \bar{y}(1 - \bar{y}) \\
&\approx \bar{y}(1 - \bar{y}) \frac{\sum_{\forall e} \hat{\beta}_1^e \bar{y}_e(1 - \bar{y}_e) \sum_{\forall i, t \in e} (x_{e,i,t} - x_e)^2 + \tilde{\beta}_1 \bar{y}(1 - \bar{y}) \sum_{\forall e} \sum_{\forall i, t \in e} (\bar{x}_e - \bar{x})^2}{\bar{y}(1 - \bar{y}) \left[\sum_{\forall e} \sum_{\forall i, t \in e} (x_{e,i,t} - \bar{x}_e)^2 + \sum_{\forall e} \sum_{\forall i, t \in e} (\bar{x}_e - \bar{x})^2 \right]} \\
&= \frac{\sum_{\forall e} \hat{\beta}_1^e \bar{y}_e(1 - \bar{y}_e) \sum_{\forall i, t \in e} (x_{e,i,t} - x_e)^2 + \tilde{\beta}_1 \bar{y}(1 - \bar{y}) \sum_{\forall e} \sum_{\forall i, t \in e} (\bar{x}_e - \bar{x})^2}{\sum_{\forall e} \sum_{\forall i, t \in e} (x_{e,i,t} - \bar{x}_e)^2 + \sum_{\forall e} \sum_{\forall i, t \in e} (\bar{x}_e - \bar{x})^2} \\
&\approx \frac{\sum_{\forall e} AME_e \sum_{\forall i, t \in e} (x_{e,i,t} - x_e)^2 + \widetilde{AME} \sum_{\forall e} \sum_{\forall i, t \in e} (\bar{x}_e - \bar{x})^2}{\sum_{\forall e} \sum_{\forall i, t \in e} (x_{e,i,t} - \bar{x}_e)^2 + \sum_{\forall e} \sum_{\forall i, t \in e} (\bar{x}_e - \bar{x})^2} \\
&= \sum_{\forall e} w_e AME_e + \left(1 - \sum_{\forall e} w_e \right) \widetilde{AME}
\end{aligned}$$

where $\widetilde{AME}$ is the average marginal effect of the between estimator (i.e. replacing all $x_{e,i,t}$ s with their experiment-specific means).