Appendices

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A Math Proofs

Proof of Proposition 1. For candidate L^A, she needs to maximize the following problem:

$$\max_{x_L^A} \pi_L^A \cdot (-|-1-x_L^A|) + (1-\pi_L^A) \cdot (-|-1-x_R^A|), \tag{1}$$

where π_j^k is a function of x_L^A, x_R^A, x_L^B, and x_R^B.

First, we assume that $0 < x_L^A \leq x_L^B < x^* < x_R^B \leq x_R^A < 1$ and $x^* \in [\mu-\alpha, \mu+\alpha]$, where x^* is the indifference position that the voter is indifferent between choosing two candidates in election A; we will prove this assumption is held in equilibrium later. This assumption implies

$$x^* = \frac{x_L^A + x_R^A + \gamma(x_L^B + x_R^B)}{2(1+\gamma)}.$$

Therefore, the probability that L^A wins is $\pi_L^A(x_L^A, x_R^A, x_L^B, x_R^B) = \Pr(x_m < x^*)$. Since x_m is uniformly distributed on $[\mu-\alpha, \mu+\alpha]$, it follows $\pi_L^A = \frac{1}{2\alpha}[x^* - (\mu-\alpha)]$. Plug π_L^A into Eq.(1), we have

$$\max_{x_L^A} \frac{1}{2\alpha}\left[\frac{x_L^A + x_R^A + \gamma(x_L^B + x_R^B)}{2(1+\gamma)} - \mu + \alpha\right] \cdot (x_R^A - x_L^A) \tag{2}$$

By the First Order Condition,

$$x_L^A = -\frac{\gamma}{2}(x_L^B + x_R^B) + (1+\gamma)(\mu-\alpha). \tag{3}$$

Similarly, following the same approach, we consider the maximization problem for candidate

R^A. It follows

$$x_R^A = -\frac{\gamma}{2}(x_L^B + x_R^B) + (1+\gamma)(\mu+\alpha). \tag{4}$$

In Election B, we also have

$$x_L^B = -\frac{\beta}{2}(x_L^A + x_R^A) + (1+\beta)(\mu-\alpha), \tag{5}$$

$$x_R^B = -\frac{\beta}{2}(x_L^A + x_R^A) + (1+\beta)(\mu+\alpha) \tag{6}$$

Then we can solve this system of linear equations, Eq.(3)-Eq(6), and get,

$$x_L^A = \mu - (1+\gamma)\alpha, \qquad x_R^A = \mu + (1+\gamma)\alpha,$$

$$x_L^B = \mu - (1+\beta)\alpha, \qquad x_R^B = \mu + (1+\beta)\alpha.$$

We can find that these results satisfy the assumption.

Now we show that all equilibria must satisfy the condition $0 < x_L^A \leq x_L^B < x_R^B \leq x_R^A < 1$. It implies the solution above is the unique equilibrium.

Consider the situation where an equilibrium result satisfies the condition $x_L^A < x_R^A < x_L^B < x_R^B$. In election A, there are two possible scenarios: the indifference position $x_A^* = \frac{x_L^A + x_R^A + \gamma(x_R^B - x_L^B)}{2}$, or $x_R^A - x_L^A = \gamma(x_R^B - x_L^B)$. Similarly, in election B, there are two possible cases: the indifference position $x_B^* = \frac{x_L^B + x_R^B - \beta(x_R^A - x_L^A)}{2}$ or $\beta(x_R^A - x_L^A) = x_R^B - x_L^B$.

In the case that $x_A^* = \frac{x_L^A + x_R^A + \gamma(x_R^B - x_L^B)}{2}$ and $x_B^* = \frac{x_L^B + x_R^B - \beta(x_R^A - x_L^A)}{2}$, we can find: $x_L^A = \mu - \alpha - \frac{\gamma}{2}(x_R^B - x_L^B)$ and $x_R^A = \mu + \alpha - \frac{\gamma}{2}(x_R^B - x_L^B)$. Similarly, $x_L^B = \mu - \alpha - \frac{\beta}{2}(x_R^A - x_L^A)$ and $x_R^B = \mu + \alpha - \frac{\beta}{2}(x_R^A - x_L^A)$. Thus, $x_L^A = \mu - \alpha - \alpha\gamma$, $x_R^A = \mu + \alpha - \alpha\gamma$, $x_L^A = \mu - \alpha + \alpha\beta$ and $x_L^A = \mu + \alpha + \alpha\beta$, which contradicts with $x_R^A < x_L^B$.

In the case that $x_R^A - x_L^A = \gamma(x_R^B - x_L^B)$ and $\beta(x_R^A - x_L^A) = x_R^B - x_L^B$, we have $\beta\gamma = 1$. Now consider election A, since $x_L^A < x_R^A < x_L^B < x_R^B$, then, any voter whose ideal point is less than x_R^A will vote for candidate L^A, and any voter whose ideal point is greater than x_L^B will vote for candidate R^A. Voters in interval $[x_R^A, x_L^B]$ are indifferent. These indifferent voters will flip a coin to make the decision, therefore half of them will support L^A and other half of them will support R^A. It is equivalent that voters whose ideal points are less than $\frac{x_R^A+x_L^B}{2}$ will support L^A and those whose ideal points are greater than $\frac{x_R^A+x_L^B}{2}$ will support R^A. Then, L^A's question becomes

$$\max_{x_L^A} \frac{1}{2\alpha}\left[\frac{x_R^A + x_L^B}{2} - \mu + \alpha\right] \cdot (x_R^A - x_L^A). \tag{7}$$

Thus, $x_L^A = -\infty$. Then, using the same approach we can find that $x_R^B = +\infty$. Therefore, it is impossible that $x_R^A - x_L^A = \gamma(x_R^B - x_L^B)$ because both sides are infinity.

In the case that $x_A^* = \frac{x_L^A + x_R^A + \gamma(x_R^B - x_L^B)}{2}$ and $\beta(x_R^A - x_L^A) = x_R^B - x_L^B$. We know this cannot be an equilibrium. In election B, every voter $x_i \in [x_R^A, x_L^B]$ is indifferent between x_L^B and x_R^B. It also means $x_i < x_R^A$ will prefer x_L^B and $x_i > x_L^B$ will prefer x_R^A. Now if candidate from Party L moves her policy a lightly left from x_L^B, then she will win all votes in $[x_R^A, x_L^B]$ and be closer to her ideal point -1. Therefore this case cannot be an equilibrium. Similarly, the case with $x_R^A - x_L^A = \gamma(x_R^B - x_L^B)$ and $x_B^* = \frac{x_L^B + x_R^B - \beta(x_R^A - x_L^A)}{2}$ cannot be an equilibrium. Therefore, any solution with $x_L^A < x_R^A < x_L^B < x_R^B$ cannot be an equilibrium. Using the similar approach, we can show that all other situations cannot be on the equilibrium path. □

To prove Proposition 2, we need the following lemma.

Lemma A.1. *When $x_L^A = \bar{\mu} - T$ and $x_R^A = \bar{\mu} + T$ with $T \leq (1+\gamma)\alpha$, then in election B^k, we have*

(1) $x_L^k = \bar{\mu} - (1+\beta)\alpha$ and $x_R^k = \bar{\mu} + (1+\beta)\alpha$ when $\mu^k = \bar{\mu}$.

(2) When $\bar{\mu} - T \leq \mu^k \leq \bar{\mu} + T$, and $\mu^k \neq \bar{\mu}$

$$x_L^k = (1+\beta)(\mu^k - \alpha) - \beta\bar{\mu}; \qquad x_R^k = (1+\beta)(\mu^k + \alpha) - \beta\bar{\mu},$$

(3) When $\mu^k < \bar{\mu} - T$

$$x_L^k = \mu^k - \alpha - \beta T; \qquad x_R^k = \mu^k + \alpha - \beta T.$$

(4) When $\mu^k > \bar{\mu} + T$,

$$x_L^k = \mu^k - \alpha + \beta T; \qquad x_R^k = \mu^k + \alpha + \beta T$$

Proof of Lemma A.1. For the first part, in the Nth district, the indifference point of election B is $x^{*N} = \frac{x_L^N + x_R^N + \beta(x_L^A + x_R^A)}{2(1+\beta)} = \frac{x_L^N + x_R^N}{2(1+\beta)} + \frac{\beta}{1+\beta}\bar{\mu}$, following the proof of Proposition 1, we have $x_L^k = \bar{\mu} - (1+\beta)\alpha$ and $x_R^k = \bar{\mu} + (1+\beta)\alpha$ for $k = N$.

In a district $k < N$, we need to consider two cases.

Case 1. When $x_L^k, x_L^A \leq x_N^{*k} \leq x_R^k, x_R^A$, the indifference point of election B^k is $x^{*k} = \frac{x_L^k + x_R^k + \beta(x_L^A + x_R^A)}{2(1+\beta)}$. Then the candidate from Party L need to solve the question:

$$\max_{x_L^k} \frac{1}{2\alpha}(x^{*k} - \mu^k + \alpha)(x_R^k - x_L^k).$$

So we have $x_L^k = (1+\beta)(\mu^k - \alpha) - \beta\bar{\mu}$. Similarly, we can solve that $x_R^k = (1+\beta)(\mu^k + \alpha) - \beta\bar{\mu}$. These solutions hold if and only if $x_L^k, x_L^A \leq x^{*k} \leq x_R^k, x_R^A$. Since $x^{*k} = \mu^k$ in equilibrium, therefore, we need $\max\{\bar{\mu} - (1 + \frac{1}{\beta})\alpha, \bar{\mu} - T\} \leq \mu^k$. Because $T \leq (1+\gamma)\alpha$ and $\gamma < \beta$, we may conclude when $\bar{\mu} - T \leq \mu^k$, $x_L^k = (1+\beta)(\mu^k - \alpha) - \beta\bar{\mu}$ and $x_R^k = (1+\beta)(\mu^k + \alpha) - \beta\bar{\mu}$.

Case 2. When $x_L^k \leq x_N^{*k} \leq x_R^k, x_L^A, x_R^A$, the indifference point of election B^k is $x^{*k} = \frac{x_L^k + x_R^k + \beta(x_R^A - x_L^A)}{2}$. Using the similar way, we can find that $x_L^k = \mu^k - \alpha - \beta T$ and $x_R^k = \mu^k + \alpha - \beta T$. Condition $x_L^k \leq x^{*k} \leq x_R^k, x_L^A, x_R^A$ requires $x_L^k \leq x_N^{*k} \leq x_R^k$, which is satisfied by $T \leq (1+\gamma)\alpha$ and $\beta(1+\gamma) < 1$ (Assumption 1), and $x^{*k} \leq x_L^A, x_R^A$, which requires $\mu^k < \bar{\mu} - T$.

Similarly, in a district $k > N$, when $\mu^k \leq \bar{\mu} + T$, $x_L^k = (1+\beta)(\mu^k - \alpha) - \beta\bar{\mu}$ and $x_R^k = (1+\beta)(\mu^k + \alpha) - \beta\bar{\mu}$. When $\mu^k > \bar{\mu} + T$, $x_L^k = \mu^k - \alpha + \beta T$ and $x_R^k = \mu^k + \alpha + \beta T$ □

Proof of Proposition 2. Lemma A.1 has proven the equilibrium positions in election B^k. In election A, assume x_L^A and x_R^A have the following forms on the equilibrium path: $x_L^A = \bar{\mu} - T$ and $x_R^A = \bar{\mu} + T$ with $T \leq (1+\gamma)\alpha$. Then the policy positions can be calculated from Lemma A.1. Since $(2+\gamma)\alpha < d$ by Assumption 2, we know $\mu^k \notin [\bar{\mu} - (1+\gamma)\alpha, \bar{\mu} + (1+\gamma)\alpha]$ for $k \neq N$. Therefore, when $\mu^k < \bar{\mu}$, $x_L^k = \mu^k - \alpha - \beta T$ and $x_R^k = \mu^k + \alpha - \beta T$.

Consider election A in the $N-1$th district with the expected median voter position $\bar{\mu} - d$, the indifference position of election A in this district is $x_{N-1}^{*A} = \frac{x_L^A + x_B^A - \gamma(x_R^{N-1} - x_L^{N-1})}{2} = \bar{\mu} - \gamma\alpha.$, which is greater than $\mu^{N-1} + \alpha = \mu - d + \alpha$. It implies Party L will win this district. When we consider one party deviate from the equilibrium position in election such that $\frac{x_L^A + x_R^B}{2} = \bar{\mu} - \alpha$, then $x_{N-1}^{*A} = \mu - (1+\gamma)\alpha$, which is still greater than $\mu^{N-1} + \alpha = \mu - d + \alpha$ by Assumption 2. This result indicates that for any policy positions of x_L^A and x_R^A, as long as one candidate has a positive but less than 1 probability to win the Nth district in election A, party L always wins the district with $\mu^k < \bar{\mu}$. Similarly, under the same condition, party R always wins the district with $\mu^k > \bar{\mu}$. Therefore, we can conclude that the Nth district with $\bar{\mu}$ is the only pivotal district for election, i.e, win the election as long as win this district.

Therefore, two candidates in election A only need to consider how to win the Nth district. Since $x_L^k = \bar{\mu} - (1+\beta)\alpha$ and $x_R^k = \bar{\mu} + (1+\beta)\alpha$ from Lemma A.1, so using the same way in Proposition 1, we can get $x_L^A = \bar{\mu} - (1+\gamma)\alpha$ and $x_L^B = \bar{\mu} + (1+\gamma)\alpha$, i.e., $T = (1+\gamma)\alpha$ in Lemma A.1. This proof process also indicates the uniqueness of the equilibrium. □

Proof of Proposition 3. In election B^k, when $\mu^k = \bar{\mu}$, a candidate with ideal point $\mu^k + q\bar{v}$ from Party R has the expected payoff $-q\bar{v}$, and the candidate with an ideal point $\mu^k - q\bar{v}$ from Party L has the same payoff $-q\bar{v}$. If a candidate participate in the election, her expected payoff is $\frac{1}{M}(-q\bar{v}) + (1 - \frac{1}{M}(-w))$, and if her does not, the payoff is $-w$. Therefore, a candidate joins the election if and only if $q\bar{v} > w$. This means that in Party L, only candidates with ideal points greater than $\mu^k - w$ will join the election. In Party R, only candidates with ideal points less than $\mu^k + w$ will join the election.

Similarly, when $\mu^k < \bar{\mu}$, a candidate with ideal point $\mu^k + q\bar{v}$ from Party R has a payoff $-[q\bar{v} + \beta(1+\gamma)\alpha]$. The candidate from Party L has the payoff $-[q\bar{v} - \beta(1+\gamma)\alpha]$. Then the rest of the procedure is the same as the case when $\mu^k = \bar{\mu}$. Similarly, the result can be easily found for the case when $\mu^k > \bar{\mu}$. $\square$

B Extensions

B.1 When candidates Care about Another Election

In the main context, from the candidates' perspective, competition in other elections only affects their probability of winning, without influencing their payoff from policy outcomes. In this section, we explore an extension where the candidates' policy payoff is also impacted by the outcome of another election. This extension accounts for scenarios where a candidate in one district might consider how the outcome of a national election could affect the implementation of their local policies in the future.

As in the benchmark model, we consider two elections, A and B. The setup of the model is exactly the same as the benchmark model, except for how the candidate's utility is defined. In election $k \in \{A, B\}$, candidate j^k chooses $\widetilde{x}_j^k$ to solve the following problem:

$$\max_{\widetilde{x}_j^k} \rho[\pi_j^k \cdot (-|I_j - \widetilde{x}_j^k|) + (1-\pi_j^k) \cdot (-|I_j - \widetilde{x}_{-j}^k|)] + (1-\rho)[\pi_j^{-k} \cdot (-|I_j - \widetilde{x}_j^{-k}|) + (1-\pi_j^{-k}) \cdot (-|I_j - \widetilde{x}_{-j}^{-k}|)], \tag{8}$$

Here, $\rho \in [0, 1]$ denotes the weight of the payoff that candidate j^k places on her own election. π_j^k denotes the probability that j^k wins the election. The symbols $\widetilde{x}_j^k$ and $\widetilde{x}_{-j}^k$ represent the policy choices of j^k and her opponent, respectively. π_j^{-k} signifies the probability that j^k's co-partisan candidate j^{-k} secures a win in another election. The terms $\widetilde{x}_j^{-k}$ and $\widetilde{x}_{-j}^{-k}$ refer to the policy choices of j^{-k} and her opponent in that other election, respectively.

We have the following result:

Proposition B.1. *There exists a unique pure strategy Nash equilibrium:*

In Election A:

$$\widetilde{x}_L^A = \mu + (1+\gamma)\alpha(\eta_1 - 1); \qquad \widetilde{x}_R^A = \mu + (1+\gamma)\alpha(1 - \eta_1).$$

Here $\eta_1 = \frac{\frac{1-\rho}{\rho}\beta(1-\frac{1-\rho}{\rho}\gamma)}{1-(\frac{1-\rho}{\rho})^2\beta\gamma}$.

In Election B:

$$\widetilde{x}_L^B = \mu + (1+\beta)\alpha(\eta_2 - 1); \qquad \widetilde{x}_R^B = \mu + (1+\beta)\alpha(1-\eta_2).$$

Here $\eta_2 = \frac{\frac{1-\rho}{\rho}\gamma(1-\frac{1-\rho}{\rho}\beta)}{1-(\frac{1-\rho}{\rho})^2\beta\gamma}$.

Compared to the benchmark model, we find that $x_L^A = \mu - (1+\gamma)\alpha < \widetilde{x}_L^A < \mu - \alpha$ and $\mu + \alpha < \widetilde{x}_R^A < x_R^A = \mu + (1+\gamma)\alpha$. This result indicates that the interaction effect still exists, causing candidates' policy positions to become more divergent than in the Calvert-Wittman model. However, since candidates take the outcome of another election into consideration, the magnitude of divergence is less than that in the benchmark model. A similar result can also be found for Election B.

B.2 Coordination and Manipulation

In multidistrict elections, candidates running in districts with a disadvantage may receive a low expected payoff (e.g., a conservative candidate running in a liberal district). This scenario introduces the possibility of coordination among candidates from the same party. In this section, we extend the multidistrict model to consider situations where a party might manipulate a lower-level election to enhance its chances in an upper-level election.

For simplicity, we keep the model setup consistent with that described in Section 4.1. We consider only three districts: a central district, a left-leaning district, and a right-leaning district. The expected median voter's positions in these districts are still denoted as $\{\mu^k\}$ with $k = 1, 2$ or 3. Notably, $\mu^2 \equiv \bar{\mu}$ for the central district.

Parties prioritize winning the upper-level election over winning any single district in the lower-level election. Parties may offer an individual benefit M to their candidates in lower-level elections to increase their chances of winning the upper-level election. Candidates in Election B^k must prioritize maximizing their party's probability of winning in their district in Election A over their own expected utility in Election B^k to receive the benefit. M compensates candidates for sacrifices they make, such as committing to future promotions. We assume that

Assumption B.1. $M < \bar{\mu} + 1$.

This assumption eliminates situations where the benefit is so significant that most local election candidates prioritize the upper-level election over their own.

For Party R, without considering M, a local election candidate's expected payoff is $\bar{\mu} - 1$ in the central district and $\mu^k - \beta(1+\gamma)\alpha - 1$ in the left district. Assumption B.1 implies that right party candidates in the right-leaning and central districts prefer to focus on their own elections rather than accept the offer from the party. Now we focus on the case where $\mu^k < \hat{\mu} \equiv \beta(1+\gamma)\alpha + 1 + M$. The expected payoff for a right party candidate from the left-leaning district is so low that they might opt to accept the party's offer. In this situation, the candidate's problem becomes:

$$\max_{x_R^1} P(x_m^1 > x^{*1}), \tag{9}$$

where x_m^1 is the local median voter's position in the left district and x^{*1} is the indifference point of Election A in the left district. Meanwhile, their opponent from Party L still will try to maximize their own expected payoff in this local election. That is,

$$\max_{x_L^1} \pi_L^1 \cdot (-|1 - x_L^1|) + (1 - \pi_L^1) \cdot (-|1 - x_R^1|), \tag{10}$$

where π_L^1 is the probability that the candidate from Party L will win the local election in the left district.

In this left district, since both policy choices, x_L^A and x_R^A, from Election A are still on the right side of the local median voter's position, the indifference point should be $x^{*1} = \frac{x_L^A + x_R^A}{2} - \frac{\gamma}{2}(x_R^1 - x_L^1)$. For R^1, in order to bolster the vote share in Election A within her district, she must select a policy position that makes the indifference point x^{*1} as low as possible. Therefore, the optimal choice is to choose $x_R^1 = x^{*1}$ such that $\gamma(x_R^1 - x_L^1) = x_R^A - x_L^A$. Following a similar procedure, we can calculate the optimal choice for the candidate from Party L in the right district. The equilibrium strategies can be summarized as follows:

Proposition B.2. *There exists a unique Nash equilibrium such that in Election A,*

$$x_L^k = \bar{\mu} - (1+\gamma)\alpha, \qquad x_R^k = \bar{\mu} + (1+\gamma)\alpha;$$

in Election B^k,

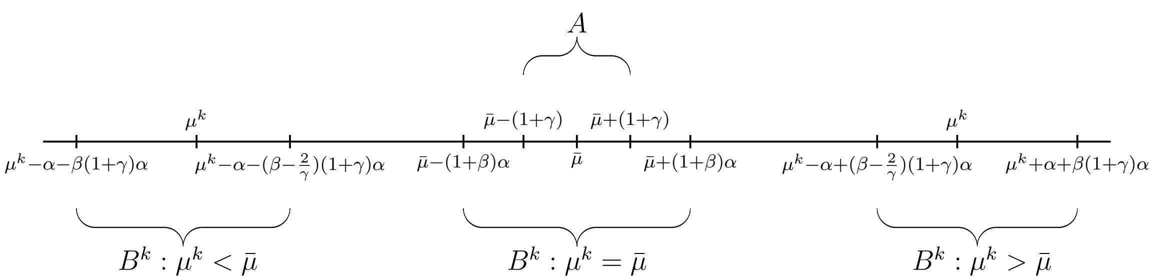
- *when $\mu^k = \bar{\mu}$,*
$$x_L^k = \bar{\mu} - (1+\beta)\alpha, \qquad x_R^k = \bar{\mu} + (1+\beta)\alpha;$$

- *when $\mu^k < \bar{\mu}$,*
$$x_L^k = \mu^k - \alpha - \beta(1+\gamma)\alpha, \qquad x_R^k = \mu^k - \alpha - \beta(1+\gamma)\alpha + \frac{2(1+\gamma)}{\gamma}\alpha;$$

- *when $\mu^k > \bar{\mu}$,*
$$x_L^k = \mu^k + \alpha + \beta(1+\gamma)\alpha - \frac{2(1+\gamma)}{\gamma}\alpha; \qquad x_R^k = \mu^k + \alpha + \beta(1+\gamma)\alpha.$$

Figure B.1 shows that upper-level candidates prioritize winning the central district, and their policy choices remain the same as without coordination (Part 1 of Proposition B.2). However, as illustrated in Parts 2 and 3 of the proposition, in districts where a party is at a significant disadvantage, its candidate might forgo personal success to bolster the prospects in the upper-level election. Proposition B.2 states that accepting the party's offer would lead them to choose a less polarized policy closer to the national median voter's position instead of campaigning for the local median voter.

Figure B.1: Equilibrium Positions in Multidistrict Elections with Coordination



B.3 Math Proof of Extensions

Proof of Proposition B.1. Candidate j^k's question in Eq.(8) can be rewritten as follows:

$$\max_{\widetilde{x}_j^k} \pi_j^k \cdot (\widetilde{x}_{-j}^k - \widetilde{x}_j^k) + \frac{1-\rho}{\rho}[\pi_j^{-k} \cdot (\widetilde{x}_{-j}^{-k} - \widetilde{x}_j^{-k}) - 1 - \widetilde{x}_{-j}^{-k}]$$

Similar to the proof of Proposition 1, we take the F.O.C. for four maximization problems, resulting in four equations with four variables. Following the same approach in th proof of Proposition 1, the rest of the calculation is straightforward,

□

Proof of Proposition B.2. We first assume

$$x_L^A = \bar{\mu} - (1+\gamma)\alpha; \qquad x_R^A = \bar{\mu} + (1+\gamma)\alpha.$$

Then use the same way in Proposition 1, we know, when $\mu^k = \bar{\mu}$

$$x_L^k = \bar{\mu} - (1+\beta)\alpha; \qquad x_R^k = \bar{\mu} + (1+\beta)\alpha.$$

In the left district, for L^1, she still try to maximize her payoff in election B^1. From the proof of Proposition 1, we know L^1's decision is dependent with her opponent's decision, so we can calculate her optimal choice is the same as that in the previous section without manipulation:

$$x_L^k = \mu^k - \alpha - \beta(1+\gamma)\alpha$$

.

Since R^1 tries to maximize the vote share for R^A in election A in the left district, so she needs to make the indifference point, x_B^1, of A in this district as small as possible. Consider this situation first, given other players' positions, if $x_R^k = x_L^k$, then $x_B^1 = \frac{x_L^A + x_R^A}{2}$. Now let x_R^k increases to a point $\bar{x}$ such that $x_B^1 = x_L^A$, this step is valid because if it is not, it means when $x_R^k = x_L^A$, we would have $x_B^1 > x_L^A$. If so, then we must have $x_B^1 = \frac{x_L^A + x_R^A}{2} - \frac{\gamma}{2}(x_L^k - x_L^A)$, which is greater than $\frac{x_L^A + x_R^A}{2}$ and we have a contradiction.

Now when we have $x_B^1 = x_L^A$, voters between $[x_L^A, +\infty)$ will choose R^A in election A, voters between $(-\infty, \bar{x}]$ will choose L^A, voters between $(\bar{x}, x_L^A)$ is indifferent between two candidates. If x_R^k increases from $\bar{x}$ a little bit (a small ε), then all voters in $(\bar{x}, x_L^A)$ will vote for R^A. Therefore, this $\bar{x}$ is the optimal choice for x_R^k, i.e.,

$$x_R^k = \bar{x} = x_L^k + \frac{x_R^A - x_L^A}{\gamma} = \mu^k - \alpha - \beta(1+\gamma)\alpha + \frac{2(1+\gamma)}{\gamma}\alpha.$$

Similarly, by the symmetry, in the right district with $\bar{\mu} < \mu^k$, we have

$$x_R^k = \mu^k + \alpha + \beta(1+\gamma)\alpha,$$

$$x_L^k = \bar{x} = x_L^k + \frac{x_R^A - x_L^A}{\gamma} = \mu^k + \alpha + \beta(1+\gamma)\alpha - \frac{2(1+\gamma)}{\gamma}\alpha.$$

Next, we begin to prove $x_L^A = \bar{\mu} - (1+\gamma)\alpha$ and $x_R^A = \bar{\mu} + (1+\gamma)\alpha$. Given this equilibrium position, we may find $x_B^1 = \mu^k - \alpha - \beta(1+\gamma)\alpha + \frac{2(1+\gamma)}{\gamma}\alpha$ in the left district, which is greater than $\mu^1 + \alpha$; similarly, x_B^2 is less than $\mu^3 - \alpha$. It implies R loses the left district and wins the right district. When R^A increases x_R^A, the indifference position in the left district will increase, while its counterpart in the right district will decrease. This means the probability of winning the election is still fully determined by the election result in the middle district. Therefore, this deviation is not profitable for R^A, because $x_R^A = \bar{\mu} + (1+\gamma)\alpha$ is the optimal choice in this single pivotal district case.

If R^A decreases x_R^A, we use p_1 to denote the probability that R^A wins the left district, p_3 to denote the probability that R^A wins the right district, and $\bar{p}$ to denote the probability that R^A wins the middle district. Then the probability for R^A to win election A is:

$$p_R^A \equiv p_1 p_3 + \bar{p} p_3 + \bar{p} p_1 - 2\bar{p} p_1 p_3,$$

and we also know $p_3 = 1 - p_1$ by the symmetry. Therefore $p_R^A = (2\bar{p} - 1)p_1^2 + (1 - 2\bar{p})p_1 + \bar{p}$. Since $\bar{p} \geq 1/2$ when $x_R^A \leq +(1+\gamma)\alpha$, it follows that $P_R^A \leq \bar{p}$ for $p_1 \in [0,1]$, and $P_R^A = \bar{p}$ when $p_1 = 0$ or 1. Therefore, R^A's expected payoff is $P_R^A(x_R^A - x_L^A) \leq \bar{p}(x_R^A - x_L^A)$, and when $x_R^A = \bar{\mu} + (1+\gamma)\alpha$, we have the equality and R^A gets the maximum payoff.

Finally, by the symmetry, we know L^A also has no incentive to deviate from the equilibrium position.

□