TPLP: Page 1–x. © The Author(s), 2024. Published by Cambridge University Press 2024
doi:10.1017/xxxxx

Appendix to On the Foundations of Conflict-Driven Solving for Hybrid MKNF Knowledge Bases

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1 Introduction

The purpose of this document is to provide complete proofs for all non-trivial claims. It is divided into two main sections. The first of which concerns the completion and loop formulas, while the second concerns the related sets of nogoods.

2 Completion and Loop Formulas Proofs

Throughout this section we routinely need to consider MKNF interpretations, and the **K**-interpretations which they induce or extend from. In order to quickly reason about the properties of one, based on the properties of the other, we rely on the following lemma. It provides alternative characterizations of what it means for a **K**-interpretation to be saturated. The reason saturated **K**-interpretations are of interest, is largely due to characterization 3. As it means that all MKNF models induce saturated **K**-interpretations.

Lemma 1
The following are equivalent for a **K**-interpretation $\hat{I}$

1. $\hat{I}$ is saturated.
2. $\hat{I}$ extends to an MKNF interpretation M which induces $\hat{I}$.
3. $\hat{I}$ is induced by some MKNF interpretation M such that $M \models_{\text{MKNF}} \mathbf{K}\pi(\mathcal{O})$.
4. $\hat{I} \models \mathcal{O}_{satr}(\hat{I})$.
5. $OB_{\mathcal{O},\hat{I}} \not\models a$ for every atom $a \in (\mathbf{KA}(\mathcal{K}) \cup \{\bot\}) \setminus \hat{I}$.

Proof
1 to 2: Since $\hat{I}$ is saturated,

$$OB_{\mathcal{O},\hat{I}} \not\models \bot \tag{1}$$

and

$$\bigwedge_{a \in \mathbf{KA}(\mathcal{K})} \Big((OB_{\mathcal{O},\hat{I}} \models a) \supset (a \in \hat{I}) \Big). \tag{2}$$

From (1) there exists a first-order interpretation I, such that $I \models OB_{\mathcal{O},\hat{I}}$ and $I \not\models \bot$, therefore $M = \{J \mid J \models OB_{\mathcal{O},\hat{I}}\}$ is a non-empty set of first-order interpretations and thus an MKNF interpretation. Clearly, $\hat{I}$ extends to M.

Now we show that M induces $\hat{I}$. By construction $\forall I \in M, I \models OB_{\mathcal{O},\hat{I}}$ consequently

$$\forall I \in M, \forall a \in \hat{I}, I \models a \tag{3}$$

and therefore

$$\forall a \in \hat{I}, M \models_{\text{MKNF}} \mathbf{K}a. \tag{4}$$

Assume $M \models_{\text{MKNF}} \mathbf{K}a$ for some $a \in \mathbf{KA}(\mathcal{K}) \setminus \hat{I}$. From the construction of M this means that for any first-order interpretation I which satisifes $OB_{\mathcal{O},\hat{I}}$, $I \models a$. Equivalently,

$$\exists a \in \mathbf{KA}(\mathcal{K}) \setminus \hat{I}, OB_{\mathcal{O},\hat{I}} \models a. \tag{5}$$

This is contradictory with (2) so it must be the case that

$$\forall a \in \mathbf{KA}(\mathcal{K}) \setminus \hat{I}, M \not\models_{\text{MKNF}} \mathbf{K}a. \tag{6}$$

Combining (4) and (6) gives that

$$\hat{I} = \{a \in \mathbf{KA}(\mathcal{K}) \mid M \models_{\text{MKNF}} \mathbf{K}a\} \tag{7}$$

and it follows that M induces $\hat{I}$.

2 to 3: Since $\hat{I}$ extends to M,

$$M = \{I \mid I \models OB_{\mathcal{O},\hat{I}}\} \tag{8}$$

From this,

$$\forall I \in M, I \models \Big(\pi(\mathcal{O}) \land \bigwedge_{a \in \mathbf{KA}(\mathcal{K})} I \models a\Big) \tag{9}$$

consequently $\forall I \in M, I \models \pi(\mathcal{O})$, and therefore

$$M \models_{\text{MKNF}} \mathbf{K}\pi(\mathcal{O}). \tag{10}$$

$\hat{I}$ is induced by M, thereby there is an MKNF interpretation M such that $M \models_{\text{MKNF}} \mathbf{K}\pi(\mathcal{O})$, which induces $\hat{I}$.

3 to 4: Since $M \models_{\text{MKNF}} \mathbf{K}\pi(\mathcal{O})$,

$$\forall I \in M, I \models \pi(\mathcal{O}). \tag{11}$$

Since M induces $\hat{I}$, $\hat{I} = \{a \in \mathbf{KA}(\mathcal{K}) \mid M \models_{\text{MKNF}} \mathbf{K}a\}$, or equivalently

$$\hat{I} = \{a \in \mathbf{KA}(\mathcal{K}) \mid \forall I \in M, I \models a\}. \tag{12}$$

This means that

$$\forall a \notin \hat{I}, \exists I \in M \text{ s.t. } I \not\models a. \tag{13}$$

(12) also means that

$$\forall I \in M, I \models \bigwedge_{a \in \hat{I}} a. \tag{14}$$

By (11) and (14),

$$\forall I \in M, I \models \pi(\mathcal{O}) \land \bigwedge_{a \in \hat{I}} a \tag{15}$$

equivalently,

$$\forall I \in M, I \models OB_{\mathcal{O},\hat{I}}. \tag{16}$$

By (13) and (16)

$$\forall a \in (\mathbf{KA}(\mathcal{K}) \setminus \hat{I}), \exists I \in M \text{ s.t. } I \models OB_{\mathcal{O},\hat{I}} \land I \not\models a \tag{17}$$

consequently

$$\forall a \in (\mathbf{KA}(\mathcal{K}) \setminus \hat{I}), OB_{\mathcal{O},\hat{I}} \not\models a. \tag{18}$$

Since M is an MKNF interpretation, it contains at least one first-order interpretation I, by (16) this means there is at least one first-order interpretation I such that $I \models OB_{\mathcal{O},\hat{I}}$. Therefore

$$OB_{\mathcal{O},\hat{I}} \not\models \bot. \tag{19}$$

From (18) and (19),

$$\{a \in \mathbf{KA}(\mathcal{K}) \cup \{\bot\} \mid OB_{\mathcal{O},\hat{I}} \models a\} \subseteq \hat{I} \tag{20}$$

it directly follows that

$$\hat{I} \models \mathcal{O}_{satr}(\hat{I}). \tag{21}$$

4 to 5: Here we will prove the contra-positive, if $\exists a \in (\mathbf{KA}(\mathcal{K}) \cup \{\bot\}) \setminus \hat{I}$ such that $OB_{\mathcal{O},\hat{I}} \models a$ then $\hat{I} \not\models \mathcal{O}_{satr}(\hat{I})$.

As $a \in (\mathbf{KA}(\mathcal{K}) \cup \{\bot\}) \setminus \hat{I}$, clearly $a \notin \hat{I}$, Therefore

$$\hat{I} \not\models a. \tag{22}$$

Furthermore since $a \in (\mathbf{KA}(\mathcal{K}) \cup \{\bot\})$ and $OB_{\mathcal{O},\hat{I}} \models a$,

$$\hat{I} \not\models \mathcal{O}_{satr}(\hat{I}). \tag{23}$$

5 to 1: $\mathbf{KA}(\mathcal{K})$ contains only atoms a which appear within as $\mathbf{K}a$ or $\mathbf{not}a$ within $\pi(\mathcal{P})$, $\bot$ will never appear within $\pi(\mathcal{P})$, so $\bot \notin \mathbf{KA}(\mathcal{K})$, thereby $\bot \notin \hat{I}$. This combined with the fact that $OB_{\mathcal{O},\hat{I}} \not\models a$ for any $a \in (\mathbf{KA}(\mathcal{K}) \cup \{\bot\}) \setminus \hat{I}$, means that,

$$OB_{\mathcal{O},\hat{I}} \not\models \bot. \tag{24}$$

Since $OB_{\mathcal{O},\hat{I}} \not\models a$ for any atom $a \in (\mathbf{KA}(\mathcal{K}) \cup \{\bot\}) \setminus \hat{I}$, for an atom $a \in \mathbf{KA}(\mathcal{K})$ such that $OB_{\mathcal{O},\hat{I}} \models a$, $a \notin (\mathbf{KA}(\mathcal{K}) \cup \{\bot\}) \setminus \hat{I}$. Equivalently since $\hat{I} \subseteq \mathbf{KA}(\mathcal{K})$,

$$\bigwedge_{a \in \mathbf{KA}(\mathcal{K})} \Big((OB_{\mathcal{O},\hat{I}} \models a) \supset (a \in \hat{I})\Big). \tag{25}$$

From (24) and (25), $\hat{I}$ is saturated. □

Now we bring our attention to Theorem 1, with regard to the connection between **K**-interpretations satisfying the completion and those induced by MKNF models of tight HMKNF-KBs. In order to do so we start with weaker claims, which build upon each other.

Lemma 2
For any **K**-interpretation of an HMKNF-KB $\mathcal{K} = (\mathcal{P}, \mathcal{O})$, if $\hat{I} \models \mathcal{P}_{rule} \wedge \mathcal{O}_{satr}(\hat{I})$, then it extends to an MKNF interpretation M, such that $M \models_{\text{MKNF}} \pi(\mathcal{K})$.

Proof
As M extends from $\hat{I}$, $M = \{I \mid I \models OB_{\mathcal{O},\hat{I}}\}$. Consequently, $\forall I \in M, I \models \pi(\mathcal{O})$ therefore

$$M \models_{\text{MKNF}} \mathbf{K}\pi(\mathcal{O}). \tag{26}$$

It remains to show that $M \models_{\text{MKNF}} \pi(\mathcal{P})$. From $\hat{I} \models \mathcal{P}_{rule}$ we have that

$$\hat{I} \models \bigwedge_{r \in \mathcal{P}} \big(Body(r) \supset \bigvee head(r)\big). \tag{27}$$

Therefore for all rules $r \in \mathcal{P}$ $\hat{I} \models \bigvee head(r)$ or $\hat{I} \models Body(r)$. Equivalently for all rules $r \in \mathcal{P}$

$$\begin{aligned} &(\exists h \in head(r), h \in \hat{I}),\\ &\text{or}\\ &(\exists p \in body^+(r), p \notin \hat{I}, \text{or}\\ &\exists n \in body^-(r), n \in \hat{I}). \end{aligned} \tag{28}$$

From the fact that M induces $\hat{I}$, $\forall a \in \hat{I}, M \models_{\text{MKNF}} \mathbf{K}a$, and $\forall b \in \mathbf{KA}(\mathcal{K}), b \notin \hat{I}, M \not\models_{\text{MKNF}} \mathbf{K}b$. As a consequence, for all rules $r \in \mathcal{P}$,

$$\begin{aligned} &\forall r \in \mathcal{P},\\ &(\exists h \in head(r), M \models_{\text{MKNF}} \mathbf{K}h),\\ &\text{or}\\ &(\exists p \in body^+(r), M \not\models_{\text{MKNF}} \mathbf{K}p, \text{or}\\ &\exists n \in body^-(r), M \models_{\text{MKNF}} \mathbf{K}n) \end{aligned} \tag{29}$$

Equivalently, since $(M \not\models_{\text{MKNF}} \mathbf{K}\phi) \equiv (M \models_{\text{MKNF}} \mathbf{not}\phi)$,

$$\forall r \in \mathcal{P}, M \models_{\text{MKNF}} \pi(r). \tag{30}$$

Consequently, $M \models_{\text{MKNF}} \pi(\mathcal{P})$. In combination with (26), $M \models_{\text{MKNF}} \pi(\mathcal{K})$. □

The following proposition, effectively extends Lemma 2 with another direction.

Proposition 1
For any **K**-interpretation $\hat{I}$ of an HMKNF-KB $\mathcal{K} = (\mathcal{P}, \mathcal{O})$, $\hat{I} \models \mathcal{P}_{rule} \wedge \mathcal{O}_{satr}(\hat{I})$, if and only if, there exists an MKNF interpretation M, such that M induces $\hat{I}$ and $M \models_{\text{MKNF}} \pi(\mathcal{K})$.

Proof
($\Rightarrow$) From Lemma 2, $\hat{I}$ extends to an MKNF model M, such that $M \models_{\text{MKNF}} \pi(\mathcal{K})$. The fact that $\hat{I} \models \mathcal{O}_{satr}(\hat{I})$ along with Lemma 1, means that M induces $\hat{I}$ as well.
($\Leftarrow$) For a ground MKNF knowledge base all rules in $\pi(\mathcal{P})$ are of the form

$$\pi(r) = (\mathbf{K}h_0 \vee ... \vee \mathbf{K}h_z) \subset (\mathbf{K}p_0 \wedge \cdots \wedge \mathbf{K}p_j \wedge \mathbf{not}n_0 \wedge ... \wedge \mathbf{not}n_k).$$

$M \models_{\text{MKNF}} \pi(\mathcal{K})$ by assumption and thereby $M \models_{\text{MKNF}} \pi(\mathcal{P})$, so

$$\forall r \in \mathcal{P}, M \models_{\text{MKNF}} \pi(r). \tag{31}$$

Equivalently, since $M \not\models_{\text{MKNF}} \mathbf{K}\phi \equiv M \models_{\text{MKNF}} \mathbf{not}\phi$, for all rules $r \in \mathcal{P}$,

$$\begin{aligned}
&(\exists h \in \mathit{head}(r), M \models_{\text{MKNF}} \mathbf{K}h),\\
&\text{or}\\
&(\exists p \in \mathit{body}^+(r), M \not\models_{\text{MKNF}} \mathbf{K}p, \text{or}\\
&\exists n \in \mathit{body}^-(r), M \models_{\text{MKNF}} \mathbf{K}n^1).
\end{aligned} \tag{32}$$

$\hat{I}$ is induced by M so by defintion,

$$\hat{I} = \{a \in \mathbf{KA}(\mathcal{K}) \mid M \models_{\text{MKNF}} \mathbf{K}a\}. \tag{33}$$

As $\forall r \in \mathcal{P}, (\mathit{head}(r) \cup \mathit{body}^+(r) \cup \mathit{body}^-(r)) \subseteq \mathbf{KA}(\mathcal{K})$, this means that $\forall r \in \mathcal{P}$,

$$\begin{aligned}
&(\exists h \in \mathit{head}(r), h \in \hat{I}),\\
&\text{or}\\
&(\exists p \in \mathit{body}^+(r), p \notin \hat{I}, \text{or}\\
&\exists n \in \mathit{body}^-(r), n \in \hat{I}).
\end{aligned} \tag{34}$$

As a consequence,

$$\forall r \in \mathcal{P}, \hat{I} \models \bigvee \mathit{head}(r) \vee \Big(\hat{I} \not\models \bigwedge \mathit{body}^+(r) \vee \hat{I} \models \bigvee \mathit{body}^-(r)\Big). \tag{35}$$

This can be simplified further using the shorthand '$\mathit{Body}(r) = \bigwedge \mathit{body}^+(r) \wedge \neg \bigvee \mathit{body}^-(r)$' to

$$\hat{I} \models \bigwedge_{r \in \mathcal{P}} \big(\mathit{Body}(r) \supset \bigvee \mathit{head}(r)\big), \tag{36}$$

therefore $\hat{I} \models \mathcal{P}_{rule}$. $\hat{I}$ is induced by M and $M \models_{\text{MKNF}} \pi(\mathcal{K})$, so by Lemma 1, $\hat{I} \models \mathcal{O}_{satr}(\hat{I})$. It is clear then, that $\hat{I} \models \mathcal{P}_{rule} \wedge \mathcal{O}_{satr}(\hat{I})$. □

The gap between this claim and Theorem 1, is that Proposition 1 only requires that $M \models_{\text{MKNF}} \pi(\mathcal{K})$, but not that M is a model. For M to be a model, it must be that there is no MKNF interpretation $M' \supset M$, for which $\forall I' \in M', (I', M', M) \models \pi(\mathcal{K})$. We reason about the existence of M', by considering the MKNF interpretation $\hat{I}'$ it induces, in particular how it differs from $\hat{I}$. In the following lemma we show $\hat{I}'$ is a strict subset of $\hat{I}$.

Lemma 3
Let $\hat{I}$ be a saturated **K**-interpretation of an HMKNF-KB $\mathcal{K} = (\mathcal{P}, \mathcal{O})$, which extends to an MKNF interpretation M. If $M \models_{\text{MKNF}} \pi(\mathcal{K})$ and there is an MKNF interpretation $M' \supset M$ such that $\forall I' \in M', (I', M', M) \models \pi(\mathcal{K})$, then M' induces a **K**-interpretation $\hat{I}'$ such that $\hat{I}' \subset \hat{I}$.

Proof
Proof by contradiction. If M' induces $\hat{I}$ for any first-order interpretation $I' \in M'$, $\{a \mid I' \models a\} \supseteq \hat{I}$. Also it follows from the fact that $\forall I' \in M', (I', M', M) \models \pi(\mathcal{K})$

that for any $I' \in M'$, $I' \models \pi(\mathcal{O})$. Consequently for any $I' \in M'$, $I' \models OB_{\mathcal{O},\hat{I}}$. This implies it must be a subset of the MKNF interpretation which $\hat{I}$ extends to therefore $M' \subseteq M$, This is a contradiction of the fact that $M' \supset M$, therefore M' induces $\hat{I}'$ such that $\hat{I}' \neq \hat{I}$. Furthermore since $M' \supset M$, $\hat{I}' \subset \hat{I}$. □

As aforementioned we have interest in showing that given two MKNF interpretations M and M' of an HMKNF-KB $\mathcal{K}$, which induce the **K**-interpretations $\hat{I}$ and $\hat{I}'$, that $\forall I' \in M', (I', M', M) \not\models \pi(\mathcal{K})$. To do so we can consider the set of atoms $(\hat{I} \setminus \hat{I}')$. The general strategy is to show that $\hat{I}'$ is sufficient to imply some atom $p \in (\hat{I} \setminus \hat{I}')$, in order to reason that $\forall I' \in M', (I', M', M) \not\models \pi(\mathcal{K})$. Under the assumptions that $\hat{I} \models \mathcal{K}_{comp}(\hat{I})$ and $M \models_{\text{MKNF}} \pi(\mathcal{K})$, we do so by analyzing the section of the dependency graph $G(\mathcal{K})$ which $(\hat{I} \setminus \hat{I}')$ exists within. The approach is to take any individual atom $g \in (\hat{I} \setminus \hat{I}')$ and restrict the graph to only those atoms which g can reach within $(\hat{I} \setminus \hat{I}')$. We denote this subgraph as G. If G is acyclic then the following lemma applies. It relies on the fact that there is some atom $p \in (\hat{I} \setminus \hat{I}')$ which has no outgoing edges within G, and the restrictions imposed on the full graph $G(\mathcal{K})$ by the fact that $\hat{I} \models \mathcal{K}_{comp}(\hat{I})$. In doing so it is able to show p is implied by the knowledge contained within $\hat{I}'$, and thereby that $\forall I' \in M', (I', M', M) \not\models \pi(\mathcal{K})$.

Lemma 4
For two MKNF interpretations M and M' of the knowledge base $\mathcal{K} = (\mathcal{P}, \mathcal{O})$, which induce the **K**-interpretations $\hat{I}$ and $\hat{I}'$ respectively. Let G be the subgraph of the dependency graph $G(\mathcal{K})$, containing only the atoms within $(\hat{I} \setminus \hat{I}')$ reachable in $G(\mathcal{K})$ from some atom $g \in (\hat{I} \setminus \hat{I}')$. If $M' \supset M$, $(\hat{I} \setminus \hat{I}') \neq \emptyset$, $\hat{I} \models \mathcal{K}_{comp}(\hat{I})$, $M \models_{\text{MKNF}} \pi(\mathcal{K})$ and G is acyclic, then $\forall I' \in M', (I', M', M) \not\models \pi(\mathcal{K})$.

Proof
As $\hat{I} \models \mathcal{K}_{comp}(\hat{I})$, $\hat{I} \models \mathcal{K}_{sup}(\hat{I})$ and

$$\hat{I} \models \bigvee_{\substack{r \in \mathcal{P},\\ a \in head(r)}} \big(Body(r) \wedge \bigwedge_{p \in head(r) \setminus \{a\}} \neg p\big). \tag{37}$$

for all $a \in \mathbf{KA}(\mathcal{K})$ such such that $OB_{\mathcal{O},\hat{I} \setminus \{a\}} \not\models a$. As G is acyclic, there is at least one atom $a \in G$ which has no outgoing edges. By the fact that all atoms in G are from $(\hat{I} \setminus \hat{I}')$, $\hat{I} \models a$. As $a \in \mathbf{KA}(\mathcal{K})$ either

$$OB_{\mathcal{O},\hat{I} \setminus \{a\}} \models a$$

$$\text{or} \tag{38}$$

$$\hat{I} \models \bigvee_{\substack{r \in \mathcal{P},\\ a \in head(r)}} \big(Body(r) \wedge \bigwedge_{p \in head(r) \setminus \{a\}} \neg p\big). \tag{39}$$

Case 1:

$$OB_{\mathcal{O},\hat{I} \setminus \{a\}} \models a. \tag{40}$$

Since $M \models_{\text{MKNF}} \pi(\mathcal{K})$, Lemma 1 tells us that $\hat{I}$ is saturated, meaning that $OB_{\mathcal{O},\hat{I}} \not\models \bot$. Consequently,

$$\forall S \subseteq \hat{I}, OB_{\mathcal{O},S} \not\models \bot. \tag{41}$$

From (40) and (41)

$$\exists S \subseteq (\hat{I} \setminus \{a\}) \text{ such that } OB_{\mathcal{O},S} \models a, \tag{42}$$

$$a \notin S, OB_{\mathcal{O},S} \not\models \bot, \text{ and} \tag{43}$$

$$\forall S' \subseteq S, OB_{\mathcal{O},S'} \not\models a. \tag{44}$$

Therefore there is an edge in $G(\mathcal{K})$ from a to each atom in S. As a has no outgoing edges within $(\hat{I} \setminus \hat{I}')$, this means that $S \subseteq \hat{I}'$. Therefore

$$OB_{\mathcal{O},\hat{I}'} \models a. \tag{45}$$

By defintion this means that for any first-order interpretation I such that $I \models OB_{\mathcal{O},\hat{I}'}, I \models a$. Conversely, M' must contain at least one first-order interpretation I' such that $I' \not\models a$, since it induces $\hat{I}'$ and $a \notin \hat{I}'$. Therefore

$$\exists I' \in M', I' \not\models OB_{\mathcal{O},\hat{I}'} \tag{46}$$

As M' induces $\hat{I}'$, $\forall I' \in M', I' \models \bigwedge_{p \in \hat{I}'} p$, therefore (46) implies that $\exists I' \in M', I' \not\models \pi(\mathcal{O})$. Consequently,

$$M' \not\models_{\text{MKNF}} \mathbf{K}\pi(\mathcal{O}). \tag{47}$$

This implies that $\forall I' \in M', (I', M', M) \not\models \mathbf{K}\pi(\mathcal{O})$, and thereby

$$\forall I' \in M', (I', M', M) \not\models \pi(\mathcal{K}). \tag{48}$$

Case 2:

$$\hat{I} \models \bigvee_{\substack{r \in \mathcal{P}, \\ a \in head(r)}} \big(Body(r) \land \bigwedge_{p \in head(r) \setminus \{a\}} \neg p\big). \tag{49}$$

For the above to be true, there must be a rule $r \in \mathcal{P}$ such that,

$$a \in head(r), \tag{50}$$

$$head(r) \setminus \{a\} \cap \hat{I} = \emptyset, \tag{51}$$

$$body^+(r) \subseteq \hat{I}, \text{ and} \tag{52}$$

$$body^-(r) \cap \hat{I} = \emptyset. \tag{53}$$

From the fact that a has no outgoing edges in G, along with (50) proves $body^+(r) \cap (\hat{I} \setminus \hat{I}') = \emptyset$. This and (52) mean that

$$body^+(r) \subseteq \hat{I}'. \tag{54}$$

As $a \in (\hat{I} \setminus \hat{I}')$, $a \notin \hat{I}'$ and since M' induces $\hat{I}'$ this means that $M' \not\models_{\text{MKNF}} \mathbf{K}a$, consequently

$$\forall I' \in M', (I', M', M) \not\models \mathbf{K}a. \tag{55}$$

From (51) we can also derive that

$$\forall h \in head(r),\ \text{s.t. } h \neq a, h \notin \hat{I} \tag{56}$$

therefore $\forall h \in head(r),\ \text{s.t. } h \neq a, M \not\models_{\text{MKNF}} \mathbf{K}h$ which together with (55) implies

$$\forall I' \in M', (I', M', M) \not\models (\mathbf{K}h_0 \wedge \cdots \wedge \mathbf{K}h_i). \tag{57}$$

From (52), $M \models_{\text{MKNF}} (\mathbf{K}p_0 \wedge \cdots \wedge \mathbf{K}p_j)$ which implies that

$$\forall I' \in M', (I', M', M) \models (\mathbf{K}p_0 \wedge \cdots \wedge \mathbf{K}p_j). \tag{58}$$

Similarly from (53), $M \models_{\text{MKNF}} (\mathbf{not} n_0 \wedge ... \wedge \mathbf{not} n_k)$ implying that

$$\forall I' \in M', (I', M', M) \models (\mathbf{not} n_0 \wedge ... \wedge \mathbf{not} n_k). \tag{59}$$

Clearly from (57) (58) and (59),

$$\forall I' \in M, (I', M', M) \not\models \Big((\mathbf{K}h_0 \wedge \cdots \wedge \mathbf{K}h_i) \supset \tag{60}$$

$$(\mathbf{K}p_0 \wedge \cdots \wedge \mathbf{K}p_j \wedge \mathbf{not} n_0 \wedge ... \wedge \mathbf{not} n_k)\Big) \tag{61}$$

more simply $\forall I' \in M', (I', M', M) \not\models \pi(r)$. Therefore $\forall I' \in M', (I', M', M) \not\models \pi(\mathcal{P})$ and thereby

$$\forall I' \in M', (I', M', M) \not\models \pi(\mathcal{K}). \tag{62}$$

In either case the conclusion holds, so the lemma is proven. □

We are now ready to prove Theorem 1. The first direction is massively aided by the previous lemmas, while the other is relatively straightforward.

Theorem 1
For any **K**-interpretation $\hat{I}$, of a tight HMKNF-KB $\mathcal{K} = (\mathcal{P}, \mathcal{O})$, $\hat{I} \models \mathcal{K}_{comp}(\hat{I})$, if and only if, $\mathcal{K}$ has an MKNF model M, such that M induces $\hat{I}$.

Proof
($\Rightarrow$) Let M be the interpretation which $\hat{I}$ extends to. As $\hat{I} \models \mathcal{P}_{rule} \wedge \mathcal{O}_{satr}(\hat{I})$, Lemma 2, implies that $M \models_{\text{MKNF}} \pi(\mathcal{K})$. Also, since $\hat{I} \models \mathcal{O}_{satr}(\hat{I})$, by Lemma 1, M induces $\hat{I}$.

It remains to show that there is no MKNF interpretation $M' \supset M$ such that $\forall I' \in M', (I', M', M) \models \pi(\mathcal{K})$, therefore we let M' be any MKNF interpretation such that $M' \supset M$. Take $\hat{I}'$ to be the **K**-interpretation that M' induces, and assume $\forall I' \in M', (I', M', M) \models \pi(\mathcal{K})$. By Lemma 3 $\hat{I}' \subset \hat{I}$.

Let G be the subgraph of the dependency graph $G(\mathcal{K})$ containing only atoms in $(\hat{I} \setminus \hat{I}')$ which are reachable from any atom $g \in (\hat{I} \setminus \hat{I}')$. Clearly by the assumption that the knowledge base is tight, $G(\mathcal{K})$ is acyclic, and so is G. Therefore by Lemma 4, $\forall I' \in M', (I', M', M) \not\models \pi(\mathcal{K})$. Consequently M is a model of $\pi(\mathcal{K})$, which induces $\hat{I}$.
($\Leftarrow$) $M \models_{\text{MKNF}} \pi(\mathcal{K})$ so by Proposition 1, $\hat{I} \models \mathcal{P}_{rule} \wedge \mathcal{O}_{satr}(\hat{I})$.

Let $a \in \mathbf{KA}(\mathcal{K})$, be any atom such that $OB_{\mathcal{O},\hat{I}\setminus\{a\}} \not\models a$ clearly

$$\{a \in \mathbf{KA}(\mathcal{K}) \mid OB_{\mathcal{O},\hat{I}\setminus\{a\}} \models a\} \subseteq \{a \in \mathbf{KA}(\mathcal{K}) \mid OB_{\mathcal{O},\hat{I}} \models a\} \tag{63}$$

therefore since $\hat{I}$ is saturated, $\hat{I}\setminus\{a\} \subseteq \{a \in \mathbf{KA}(\mathcal{K}) \mid OB_{\mathcal{O},\hat{I}\setminus\{a\}} \models a\} \subseteq \hat{I}$. Consequently,

$$\{a \in \mathbf{KA}(\mathcal{K}) \mid OB_{\mathcal{O},\hat{I}\setminus\{a\}} \models a\} = \hat{I} \setminus \{a\}. \tag{64}$$

therefore $\hat{I} \setminus \{a\}$ is saturated. Let M' be the MKNF model which induces $\hat{I} \setminus \{a\}$. As $\hat{I} \setminus \{a\}$ is saturated, by Lemma 1 $M' \models_{\text{MKNF}} \mathbf{K}\pi(\mathcal{O})$, and $\forall I' \in M', I' \models \pi(\mathcal{O})$. M is a model so it is also the case that $\forall I \in M, I \models \pi(\mathcal{O})$. Let M'' be $M \cup M'$, clearly $M'' \models_{\text{MKNF}} \mathbf{K}\pi(\mathcal{O})$. As $\pi(\mathcal{O})$ is a standard first order formula with no MKNF modal operators

$$\forall I'' \in M'', (I'', M'', M) \models \mathbf{K}\pi(\mathcal{O}). \tag{65}$$

Conversely, since M is a model by assumption, and $M'' \supset M$,

$$\forall I'' \in M'', (I'', M'', M) \not\models \pi(\mathcal{K}). \tag{66}$$

Consequently,

$$\forall I'' \in M'', (I'', M'', M) \not\models \pi(\mathcal{P}) \tag{67}$$

equivalently there is some rule r in $\mathcal{P}$ such that $\forall I'' \in M'', (I'', M'', M) \not\models \pi(r)$. This means that

$$\forall I'' \in M'', (I'', M'', M) \models (\mathbf{K}p_0 \land \cdots \land \mathbf{K}p_j \land \mathbf{not} n_0 \land ... \land \mathbf{not} n_k) \tag{68}$$

but

$$\forall I'' \in M'', (I'', M'', M) \not\models (\mathbf{K}h_0 \lor ... \lor \mathbf{K}h_z). \tag{69}$$

As $M \subset M''$,

$$\forall I \in M, (I, M, M) \models (\mathbf{K}p_0 \land \cdots \land \mathbf{K}p_j \land \mathbf{not} n_0 \land ... \land \mathbf{not} n_k) \tag{70}$$

therefore as M induces $\hat{I}$,

$$\hat{I} \models Body(r). \tag{71}$$

Also since $M \models_{\text{MKNF}} \pi(r)$,

$$M \models_{\text{MKNF}} (\mathbf{K}h_0 \lor ... \lor \mathbf{K}h_z). \tag{72}$$

Consequently, $\exists h \in head(r)$ such that $h \in \hat{I}$. Conversely, from (69),

$$M'' \not\models_{\text{MKNF}} (\mathbf{K}h_0 \lor ... \lor \mathbf{K}h_z), \tag{73}$$

and M'' induces $\hat{I} \cap (\hat{I} \setminus \{a\}) = \hat{I} \setminus \{a\}$, therefore $\nexists h \in head(r)$ such that $h \in (\hat{I} \setminus \{a\})$. This means that a is the only atom in $head(r)$ and $\hat{I}$. As a result

$$\hat{I} \models \bigwedge_{p \in head(r) \setminus \{a\}} \neg p. \tag{74}$$

It follows from (71) and (74) that

$$\hat{I} \models \bigvee_{\substack{r \in \mathcal{P}, \\ a \in head(r)}} \big(Body(r) \land \bigwedge_{p \in head(r) \setminus \{a\}} \neg p\big). \tag{75}$$

As this is the case for any arbitrary atom $a \in \mathbf{KA}(\mathcal{K})$ such that $OB_{\mathcal{O}, \hat{I} \setminus \{a\}} \not\models a$, $\hat{I} \models \mathcal{K}_{sup}(\hat{I})$. $\square$

We now bring our attention to proving Theorem 2. The previous theorem relied on an assumption that there exists a subgraph G of $G(\mathcal{K})$, containing all atoms within $(\hat{I} \setminus \hat{I}')$

reachable from some atom $g \in (\hat{I} \setminus \hat{I}')$, which is acyclic. This assumption is sure to hold in the case where $\mathcal{K}$ is tight, as $G(\mathcal{K})$ itself is acyclic, however it cannot be relied on in general. A better approach when $G(\mathcal{K})$ is not acyclic, is to find a loop L, whose atoms reach no other loop which is not its subset. The following lemma states that such a loop must always exist.

Lemma 5
Within a (finite) dependency graph with at least one loop, there is a loop L, from which no loop $L' \nsubseteq L$ can be reached via any atom $l \in L$.

The proof for this lemma is ommitted as it is both intuitive and well established.

Given a loop L which reaches no other loops which are not its subset, we can take a similar approach to the case of an acyclic graph. In the simple case, there exists some atom $h \notin L$ which L reaches, in which case we can take H to be the subgraph of $G(\mathcal{K})$ containing only atoms from $(\hat{I} \setminus \hat{I}')$ reachable from h. H will be acyclic so Lemma 4 directly applies. Otherwise the set of atoms reachable from any atom within L is simply L. Under the additional assumption that $\hat{I} \models \mathcal{K}_{loop}(\hat{I})$, we can show that some atom within L is implied by the knowledge of $\hat{I}'$. Once again showing that $\forall I' \in M', (I', M', M) \not\models \pi(\mathcal{K})$. This result is characterized by the following lemma.

Lemma 6
For two MKNF interpretations M and M' of the knowledge base $\mathcal{K} = (\mathcal{P}, \mathcal{O})$, which induce the **K**-interpretations $\hat{I}$ and $\hat{I}'$ respectively. Let G be any subgraph of the dependency graph $G(\mathcal{K})$, containing only the atoms within $(\hat{I} \setminus \hat{I}')$ reachable in $G(\mathcal{K})$ from some atom $g \in (\hat{I} \setminus \hat{I}')$. If $M' \supset M$, $M \models_{\text{MKNF}} \pi(\mathcal{K})$, $(\hat{I} \setminus \hat{I}') \neq \emptyset$, $\hat{I} \models \mathcal{K}_{loop}(\hat{I})$, and G contains only atoms from a single loop L, then $\forall I' \in M', (I', M', M) \not\models \pi(\mathcal{K})$.

Proof
Since L is a loop within G, which contains only atoms in $(\hat{I} \setminus \hat{I}')$, $L \subseteq \hat{I}$, thereby

$$\hat{I} \models \bigvee L. \tag{76}$$

Since $\hat{I} \models \mathcal{K}_{loop}(\hat{I})$ and $L \in \mathit{Loops}(\mathcal{K})$ this means that either

$$\begin{aligned} & OB_{\mathcal{O}, \hat{I} \setminus L} \models \bigvee L \\ & \text{or} \\ & \hat{I} \models \bigvee_{\substack{r \in \mathcal{P} \\ head(r) \cap L \neq \emptyset \\ body^+(r) \cap L = \emptyset}} \Big(Body(r) \land \bigwedge_{a \in head(r) \setminus L} \neg a \Big). \end{aligned} \tag{77}$$

Case 1:

$$OB_{\mathcal{O}, \hat{I} \setminus L} \models \bigvee L \tag{78}$$

Since $M \models_{\text{MKNF}} \pi(\mathcal{K})$, Lemma 1 tells us that $\hat{I}$ is saturated, meaning that $OB_{\mathcal{O}, \hat{I}} \not\models \bot$. Consequently,

$$\forall S \subseteq \hat{I}, OB_{\mathcal{O}, S} \not\models \bot. \tag{79}$$

From (78) and (79)

$$\exists l \in L, \exists S \subseteq (\hat{I} \setminus L) \text{ such that } OB_{\mathcal{O},S} \models l, \tag{80}$$
$$l \notin S, OB_{\mathcal{O},S} \not\models \bot, \text{ and} \tag{81}$$
$$\forall S' \subseteq S, OB_{\mathcal{O},S'} \not\models l. \tag{82}$$

For this l and S there is an edge in $G(\mathcal{K})$ from l to each atom in S. As l has no outgoing edges in $(\hat{I} \setminus \hat{I}') \setminus L$ meaning that $S \subseteq (\hat{I}' \cup L)$. Of course since $S \subseteq (\hat{I} \setminus L)$, it is actually the case that $S \subseteq \hat{I}'$. Therefore

$$OB_{\mathcal{O},\hat{I}'} \models l. \tag{83}$$

By defintion this means that for any first-order interpretation I such that $I \models OB_{\mathcal{O},\hat{I}'}, I \models l$. Conversely, M' must contain at least one first-order interpretation I' such that $I' \not\models l$, since it induces $\hat{I}'$ and $a \notin \hat{I}'$. Therefore

$$\exists I' \in M', I' \not\models OB_{\mathcal{O},\hat{I}'} \tag{84}$$

As M' induces $\hat{I}'$, $\forall I' \in M', I' \models \bigwedge_{p \in \hat{I}'} p$, therefore (84) implies that $\exists I' \in M', I' \not\models \pi(\mathcal{O})$. Consequently,

$$M' \not\models_{\text{MKNF}} \mathbf{K}\pi(\mathcal{O}). \tag{85}$$

This implies that $\forall I' \in M', (I', M', M) \not\models \mathbf{K}\pi(\mathcal{O})$, and thereby

$$\forall I' \in M', (I', M', M) \not\models \pi(\mathcal{K}). \tag{86}$$

Case 2:

$$\hat{I} \models \bigvee_{\substack{r \in \mathcal{P} \\ head(r) \cap L \neq \emptyset \\ body^+(r) \cap L = \emptyset}} \Big(Body(r) \wedge \bigwedge_{a \in head(r) \setminus L} \neg a \Big). \tag{87}$$

For the above to be true, there must be a rule $r \in \mathcal{P}$ such that,

$$head(r) \cap L \neq \emptyset, \tag{88}$$
$$body^+(r) \cap L = \emptyset \tag{89}$$
$$(head(r) \setminus L) \cap \hat{I} = \emptyset, \tag{90}$$
$$body^+(r) \subseteq \hat{I}, \text{ and} \tag{91}$$
$$body^-(r) \cap \hat{I} = \emptyset. \tag{92}$$

From the fact that no atom outside L in G is reachable from any atom in L, along with (88), $body^+(r) \cap ((\hat{I} \setminus \hat{I}') \setminus L) = \emptyset$. Combined with (91) this means $body^+(r) \subseteq \hat{I}' \cup L$, of course due to (89)

$$body^+(r) \subseteq \hat{I}'. \tag{93}$$

As $L \subseteq (\hat{I} \setminus \hat{I}')$, $L \cap \hat{I}' = \emptyset$, and since M' induces $\hat{I}'$ this means that for each $l \in L$, $M' \not\models_{\text{MKNF}} \mathbf{K}l$. Consequently,

$$\forall l \in L, \forall I' \in M', (I', M', M) \not\models \mathbf{K}l. \tag{94}$$

From (90) we can also derive that

$$\forall h \in head(r), \text{ s.t. } h \neq a, h \notin \hat{I} \tag{95}$$

therefore $\forall h \in head(r), \text{ s.t. } h \notin L, M \not\models_{\text{MKNF}} \mathbf{K}h$ which together with (94) implies

$$\forall I' \in M', (I', M', M) \not\models (\mathbf{K}h_0 \wedge \cdots \wedge \mathbf{K}h_i). \tag{96}$$

From (91), $M \models_{\text{MKNF}} (\mathbf{K}p_0 \wedge \cdots \wedge \mathbf{K}p_j)$ which implies that

$$\forall I' \in M', (I', M', M) \models (\mathbf{K}p_0 \wedge \cdots \wedge \mathbf{K}p_j). \tag{97}$$

Similarly from (92), $M \models_{\text{MKNF}} (\mathbf{not} n_0 \wedge ... \wedge \mathbf{not} n_k)$ implying that

$$\forall I' \in M', (I', M', M) \models (\mathbf{not} n_0 \wedge ... \wedge \mathbf{not} n_k). \tag{98}$$

Clearly from (96) (97) and (98),

$$\forall I' \in M, (I', M', M) \not\models \Big((\mathbf{K}h_0 \wedge \cdots \wedge \mathbf{K}h_i) \supset \tag{99}$$

$$(\mathbf{K}p_0 \wedge \cdots \wedge \mathbf{K}p_j \wedge \mathbf{not} n_0 \wedge ... \wedge \mathbf{not} n_k)\Big) \tag{100}$$

more simply $\forall I' \in M', (I', M', M) \not\models \pi(r)$. Therefore $\forall I' \in M', (I', M', M) \not\models \pi(\mathcal{P})$ and thereby

$$\forall I' \in M', (I', M', M) \not\models \pi(\mathcal{K}). \tag{101}$$

In both cases the conclusion holds, so the lemma is proven. □

We are now able to prove Theorem 2. As with the Theorem 1, the first direction relies heavily on the previous lemmas. The other direction is relatively straightforward.

Theorem 2
For any **K**-interpretation $\hat{I}$, of an HMKNF-KB $\mathcal{K} = (\mathcal{P}, \mathcal{O})$, $\hat{I} \models \mathcal{K}_{comp}(\hat{I}) \wedge \mathcal{K}_{loop}(\hat{I})$, if and only if, $\mathcal{K}$ has an MKNF model M such that M induces $\hat{I}$.

Proof
($\Rightarrow$) Let M be the interpretation which $\hat{I}$ extends to. As $\hat{I} \models \mathcal{P}_{rule} \wedge \mathcal{O}_{satr}(\hat{I})$, Lemma 2, implies that $M \models_{\text{MKNF}} \pi(\mathcal{K})$. Also, since $\hat{I} \models \mathcal{O}_{satr}(\hat{I})$, by Lemma 1, M induces $\hat{I}$.

It remains to show that there is no MKNF interpretation $M' \supset M$ such that $\forall I' \in M', (I', M', M) \models \pi(\mathcal{K})$, therefore we let M' be any MKNF interpretation such that $M' \supset M$. Take $\hat{I}'$ to be the **K**-interpretation that M' induces, and assume $\forall I' \in M', (I', M', M) \models \pi(\mathcal{K})$. By Lemma 3 $\hat{I}' \subset \hat{I}$.

Let G be the subgraph of the dependency graph $G(\mathcal{K})$ containing only atoms in $(\hat{I} \setminus \hat{I}')$ which are reachable from any atom $g \in (\hat{I} \setminus \hat{I}')$. Either G is acyclic or it is not.

Case 1: If G is acyclic, by Lemma 4, $\forall I' \in M', (I', M', M) \not\models \pi(\mathcal{K})$.

Case 2: If G is not acyclic, then $(\hat{I} \setminus \hat{I}')$ must contain at least one loop. Take L to be a loop in G such that no other loop $L' \nsubseteq L$ in G is reachable from any atom in $l \in L$. Such a loop exists by Lemma 5.

Case 2.1: If there is an atom $h \in (\hat{I} \setminus \hat{I}')$ such that $h \notin L$ and which is reachable from L. Then take H to be the subgraph of $G(\mathcal{K})$ reachable from h, and containing only atoms in $(\hat{I} \setminus \hat{I}')$. As h is reachable from L and is not in L it cannot be part of a loop, thereby H is acyclic. Consequently by Lemma 4, $\forall I' \in M', (I', M', M) \not\models \pi(\mathcal{K})$.

Case 2.2: Otherwise the set of atoms reachable by L within $(\hat{I} \setminus \hat{I}')$ is L. By Lemma 6 this means that $\forall I' \in M', (I', M', M) \not\models \pi(\mathcal{K})$.

As in all cases $\forall I' \in M', (I', M', M) \not\models \pi(\mathcal{K})$. M is an MKNF model which induces $\hat{I}$. ($\Leftarrow$) Assume that there is an MKNF model M of $\pi(\mathcal{K})$, such that M induces $\hat{I}$. Clearly by Theorem 1,

$$\hat{I} \models \mathcal{K}_{comp}(\hat{I}). \tag{102}$$

Assume that there exists $L \in \mathit{Loops}(\mathcal{K})$ such that $\psi(L) \in \mathcal{K}_{loop}(\hat{I})$ and $\hat{I} \not\models \psi(L)$.

Clearly $\hat{I} \models \psi(L)$ for any loop L such that $\hat{I} \cap L = \emptyset$. Therefore

$$\hat{I} \cap L \neq \emptyset. \tag{103}$$

It is also clear that $OB_{\mathcal{O}, I \setminus L} \not\models \bigvee L$, since $\psi(L) \in \mathcal{K}_{loop}(\hat{I})$. Due to the fact that $\hat{I}$ is saturated, $OB_{\mathcal{O}, I \setminus L} \not\models p$ for any atom $p \in (\mathbf{KA}(\mathcal{K}) \cup \{\bot\}) \setminus \hat{I}$, it follows that $OB_{\mathcal{O}, I \setminus L} \not\models p$ for any atom $p \in (\mathbf{KA}(\mathcal{K}) \cup \{\bot\} \setminus (\hat{I} \setminus L))$. By Lemma 1 $(\hat{I} \setminus L)$ is saturated. As a result for any clause $\psi(L) \in \mathcal{K}_{loop}(\hat{I})$,

$$(\hat{I} \setminus L) \text{ is a saturated } \mathbf{K}\text{-interpretation.} \tag{104}$$

Consequently by Lemma 1, $(\hat{I} \setminus L)$ is induced by some MKNF interpretation M' such that $M' \models_{\text{MKNF}} \mathbf{K}\pi(\mathcal{O})$. Let $M'' = M \cup M'$, since all interpretations in M and M' satisfy $\pi(\mathcal{O})$, $M'' \models_{\text{MKNF}} \mathbf{K}\pi(\mathcal{O})$. As $\pi(\mathcal{O})$ has no atoms under the '**not**' quantifer,

$$\forall I'' \in M'', (I, M'', M) \models \mathbf{K}\pi(\mathcal{O}). \tag{105}$$

In contrast, since M is an MKNF model of $\pi(\mathcal{K})$ and $M \subset M''$,

$$\forall I'' \in M'', (I, M'', M) \not\models \pi(\mathcal{K}). \tag{106}$$

It follows that $\forall I'' \in M'', (I, M'', M) \not\models \pi(\mathcal{P})$. Of course this means that there is some rule $r \in \mathcal{P}$ such that $\forall I'' \in M'', (I, M'', M) \not\models \pi(r)$. Therefore

$$\forall I'' \in M'', (I, M'', M) \models \mathbf{K}p_0 \wedge \cdots \wedge \mathbf{K}p_j \wedge \mathbf{not} n_0 \wedge ... \wedge \mathbf{not} n_k \tag{107}$$

but

$$\forall I'' \in M'', (I, M'', M) \not\models \mathbf{K}h_0 \wedge \cdots \wedge \mathbf{K}h_i \tag{108}$$

where $\mathit{head}(r) = \{h_0, \ldots, h_i\}$, $\mathit{body}^+(r) = \{p_0, \ldots, p_j\}$, and $\mathit{body}^-(r) = \{n_0, \ldots, n_k\}$. It follows that

$$M'' \models_{\text{MKNF}} \mathbf{K}p_0 \wedge \cdots \wedge \mathbf{K}p_j \tag{109}$$

and that

$$M'' \not\models_{\text{MKNF}} \mathbf{K}h_0 \wedge \cdots \wedge \mathbf{K}h_i. \tag{110}$$

M induces $\hat{I}$ and M' induces $\hat{I} \setminus L$, therefore M'' induces $\hat{I} \cap (\hat{I} \setminus L)$, aka $\hat{I} \setminus L$. This along with (109) means that,

$$(\hat{I} \setminus L) \models \bigwedge \mathit{body}^+(r) \tag{111}$$

and similarly from (110),

$$(\hat{I} \setminus L) \not\models \bigvee \mathit{head}(r). \tag{112}$$

Of course since $\hat{I} \supseteq (\hat{I} \setminus L)$, it is also true that
$M'' \supseteq M$, so by (107),

$$M \models_{\text{MKNF}} \mathbf{K}p_0 \wedge \cdots \wedge \mathbf{K}p_j \wedge \mathbf{not}n_0 \wedge ... \wedge \mathbf{not}n_k. \tag{113}$$

As M is a model this also means that

$$M \models_{\text{MKNF}} \mathbf{K}h_0 \wedge \cdots \wedge \mathbf{K}h_i. \tag{114}$$

Consequently,

$$\hat{I} \models \bigwedge body^+(r), \tag{115}$$

$$\hat{I} \models \bigwedge body^-(r), \text{ and} \tag{116}$$

$$\hat{I} \models \bigvee head(r). \tag{117}$$

It is clear from (115) and (116) that

$$\hat{I} \models Body(r). \tag{118}$$

From (112) and (117)

$$\hat{I} \models \bigwedge_{p \in head(r) \setminus L} \neg p, \text{ and} \tag{119}$$

$$head(r) \cap L \neq \emptyset. \tag{120}$$

Furthermore from (111),

$$body^+(r) \cap L = \emptyset. \tag{121}$$

The combination of (118), (119), (120), and (121), directly imply that $\hat{I} \models \psi(L)$, therefore there can be no loop $L \in Loops(\mathcal{K})$ for which $\psi(L) \in \mathcal{K}_{comp}(\hat{I})$ and $\hat{I} \not\models \psi(L)$ as it leads to a contradiction. Clearly this implies that $\hat{I} \models \mathcal{K}_{comp}(\hat{I})$. $\square$

3 Nogood Proofs

The focus of this section is on showing that solutions to our nogoods correspond with **K**-interpretations which satisfy our formulas. As is described in Theorem 3 and Theorem 4. To help us with these goals we first introduce two lemmas, which aid in characterizing the relationship between a **K**-interpretation and the assignment it induces.

In order for an assignment to be a solution with respect to a set of nogoods it must be total. The following lemma shows this to be the case for the assignment induced by any **K**-interpretation of an HMKNF-KB, with respect to either relevant sets of nogoods.

Lemma 7
Let $\hat{I}$ be a **K**-interpretation of an HMKNF-KB $\mathcal{K} = (\mathcal{P}, \mathcal{O})$, then the assignment $\hat{I}$ induces, $A^{\hat{I}}_{\mathcal{K}}$, is a total assignment for $\Delta_{\mathcal{K}} \cup \{\mathbf{T}\bot\}$ and $\Delta_{\mathcal{K}} \cup \Lambda_{\mathcal{K}} \cup \{\mathbf{T}\bot\}$.

Proof
$\Delta_{\mathcal{K}}$ contains the sets $\Phi_{\mathcal{P}}$, $\Phi_{\mathcal{O}}$, $\Psi_{\mathcal{O}}$, $\Gamma_{\mathcal{P}}$, and $\Gamma_{\mathcal{O}}$.

$\Phi_{\mathcal{P}} = \{\phi(r) \mid r \in \mathcal{P}\}$ where $\phi_{\mathcal{P}}(r) = \{\mathbf{F}p_1, \ldots, \mathbf{F}p_t, \mathbf{T}\beta(r) \mid head(r) = \{p_1, \ldots, p_t\}\}$. Clearly $\phi_{\mathcal{P}}(r)^T = \{\beta(r)\}$ and $\phi_{\mathcal{P}}(r)^F = \{p \mid p \in head(r)\}$. Consequently $\Phi^T_{\mathcal{P}} =$

$\{\beta(r) \mid r \in \mathcal{P}\}$ and $\Phi^F_{\mathcal{P}} = \{p \in head(r) \mid r \in \mathcal{P}\}$. It follows that

$$var(\Phi_{\mathcal{P}}) = \{\beta(r) \mid r \in \mathcal{P}\} \cup \{p \in head(r) \mid r \in \mathcal{P}\}. \tag{122}$$

$\Phi_{\mathcal{O}} = \{\phi_{\mathcal{O}}(p) \mid p \in \mathbf{KA}(\mathcal{O}) \cup \{\bot\}\}$ where $\phi_{\mathcal{O}}(p) = \{\mathbf{F}p, \mathbf{T}\beta_{\mathcal{O}}(p)\}$. Clearly $\phi_{\mathcal{O}}(p)^T = \{\beta_{\mathcal{O}}(p)\}$ and $\phi_{\mathcal{O}}(p)^F = \{p\}$. Consequently $\Phi^T_{\mathcal{O}} = \{\beta_{\mathcal{O}}(p) \mid p \in \mathbf{KA}(\mathcal{O}) \cup \{\bot\}\}$, and $\Phi^F_{\mathcal{O}} = \{p \mid p \in \mathbf{KA}(\mathcal{O}) \cup \{\bot\}\}$. It follows that

$$var(\Phi_{\mathcal{O}}) = \{\beta_{\mathcal{O}}(p) \mid p \in \mathbf{KA}(\mathcal{O}) \cup \{\bot\}\} \cup \mathbf{KA}(\mathcal{O}) \cup \{\bot\}. \tag{123}$$

$\Psi_{\mathcal{K}} = \{\psi_{\mathcal{K}}(p) \mid p \in \mathbf{KA}(\mathcal{K})\}$ where $\psi_{\mathcal{K}}(p) = \{\mathbf{T}p\} \cup \{\mathbf{F}\beta_{\mathcal{P}}(r,p) \mid r \in \mathcal{P}, p \in head(r)\} \cup \{\mathbf{F}\beta_{\mathcal{O}}(p)\}$ if $p \in \mathbf{KA}(\mathcal{O})$ and $\psi_{\mathcal{K}} = \psi_{\mathcal{K}}(p) = \{\mathbf{T}p\} \cup \{\mathbf{F}\beta_{\mathcal{P}}(r,p) \mid r \in \mathcal{P}, p \in head(r)\}$ otherwise. Clearly $\psi^T_{\mathcal{K}} = \{p\}$, $\psi^F_{\mathcal{K}} = \{\beta_{\mathcal{P}}(r,p) \mid r \in \mathcal{P}, p \in head(r)\} \cup \{\beta_{\mathcal{O}}(p)\}$ for all $p \in \mathbf{KA}(\mathcal{O})$, and $\psi^F_{\mathcal{K}} = \{\beta_{\mathcal{P}}(r,p) \mid r \in \mathcal{P}, p \in head(r)\}$ for all $p \in \mathbf{KA}(\mathcal{K}) \setminus \mathbf{KA}(\mathcal{O})$. Consequently $\Psi^T_{\mathcal{K}} = \mathbf{KA}(\mathcal{K})$, and $\Psi^F_{\mathcal{K}} = \{\beta_{\mathcal{P}}(r,p) \mid r \in \mathcal{P}, p \in head(r)\} \cup \{\beta_{\mathcal{O}}(p) \mid p \in \mathbf{KA}(\mathcal{O})\}$. It follows that

$$\begin{aligned} var(\Psi_{\mathcal{K}}) = \mathbf{KA}(\mathcal{K}) &\cup \{\beta_{\mathcal{P}}(r,p) \mid r \in \mathcal{P}, p \in head(r)\} \\ &\cup \{\beta_{\mathcal{O}}(p) \mid p \in \mathbf{KA}(\mathcal{O})\}. \end{aligned} \tag{124}$$

$\Gamma_{\mathcal{P}} = \bigcup_{\beta \in \{\beta(r) | r \in \mathcal{P}\} \cup \{\beta_{\mathcal{P}}(r,p) | r \in \mathcal{P}, p \in head(r)\}} \gamma(\beta)$ where $\gamma_{\mathcal{P}}(\beta) = \{\{F\beta\} \cup \beta\} \cup \{\{\mathbf{T}\beta, \overline{\sigma}\} \mid \sigma \in \beta\}$. It should be clear that for any $\beta \in \{\beta(r) | r \in \mathcal{P}\} \cup \{\beta_{\mathcal{P}}(r,p) | r \in \mathcal{P}, p \in head(r)\}$, $var(\gamma_{\mathcal{P}}(\beta)) \subseteq \{\mathbf{KA}(\mathcal{K}) \cup \{\beta(r) \mid r \in \mathcal{P}\} \cup \{\beta(r,p) \mid r \in \mathcal{P}, p \in head(r)\}\}$. It follows that

$$\begin{aligned} var(\Gamma_{\mathcal{P}}) \subseteq &\{\mathbf{KA}(\mathcal{K}) \cup \{\beta(r) \mid r \in \mathcal{P}\} \\ &\cup \{\beta(r,p) \mid r \in \mathcal{P}, p \in head(r)\}\}. \end{aligned} \tag{125}$$

The entailment nogoods of $\pi(\mathcal{K})$ are

$$\Gamma_{\mathcal{O}} = \bigcup_{\substack{p \in \mathbf{KA}(\mathcal{O}) \cup \{\bot\}, S \subseteq \mathbf{KA}(\mathcal{O}), \\ OB_{\mathcal{O}, S \setminus \{p\}} \models p}} \gamma^+_{\mathcal{O}}(p,S) \cup \bigcup_{\substack{p \in \mathbf{KA}(\mathcal{O}), S \subseteq \mathbf{KA}(\mathcal{O}), \\ OB_{\mathcal{O}, \mathbf{KA}(\mathcal{O}) \setminus (S \cup \{p\})} \not\models p}} \gamma^-_{\mathcal{O}}(p,S). \tag{126}$$

where $\gamma^+_{\mathcal{O}}(p,S) = \{\mathbf{F}\beta_{\mathcal{O}}(p)\} \cup \{\mathbf{T}s \mid s \in S\}$ and $\gamma^-_{\mathcal{O}}(p,S) = \{\mathbf{T}\beta_{\mathcal{O}}(p)\} \cup \{\mathbf{F}s \mid s \in S\}$. Clearly $var(\gamma^+_{\mathcal{O}}(p,S)) = var(\gamma^-_{\mathcal{O}}(p,S)) = \{s \mid s \in S\} \cup \{\beta_{\mathcal{O}}(p)\}$. It follows that

$$var(\Gamma_{\mathcal{O}}) \subseteq \mathbf{KA}(\mathcal{O}) \cup \{\bot\} \cup \{\beta_{\mathcal{O}}(p) \mid p \in \mathbf{KA}(\mathcal{O}) \cup \{\bot\}\}\} \tag{127}$$

From (122), (123), (124), (125), (127), and since $\{p \in head(r) \mid r \in \mathcal{P}\} \subseteq \mathbf{KA}(\mathcal{K})$ and $\mathbf{KA}(\mathcal{O}) \subseteq \mathbf{KA}(\mathcal{K})$ it directly follows that

$$\begin{aligned} var(\Delta_{\mathcal{K}}) = \mathbf{KA}(\mathcal{K}) &\cup \{\bot\} \cup \{\beta(r) \mid r \in \mathcal{P}\} \\ &\cup \{\beta_{\mathcal{P}}(r,p) \mid r \in \mathcal{P}, p \in head(r)\} \\ &\cup \{\beta_{\mathcal{O}}(p) \mid p \in \mathbf{KA}(\mathcal{O}) \cup \{\bot\}\} \end{aligned} \tag{128}$$

$\Lambda_{\mathcal{K}} = \bigcup_{\substack{L \subseteq \mathbf{KA}(\mathcal{K}), S \subseteq \mathbf{KA}(\mathcal{O}), \\ L \cap S = \emptyset, OB_{\mathcal{O}, \mathbf{KA}(\mathcal{K}) \setminus (S \cup L)} \not\models \bigvee L}} \lambda_{\mathcal{K}}(L,S)$ where

$$\begin{aligned} \lambda_{\mathcal{K}}(L,S) = \{\{&\mathbf{T}p, \sigma_1, \dots, \sigma_k, \mathbf{F}s_1, \dots, \mathbf{F}s_n\} \mid p \in L, \mathcal{E}_{\mathcal{P}}(L) = \{r_1, \dots, r_k\}, \\ &\sigma_1 \in \rho(r_1, L), \dots, \sigma_k \in \rho(r_k, L), S = \{s_1, \dots, s_n\}\}, \end{aligned} \tag{129}$$

$\mathcal{E}_{\mathcal{P}}(L) = \{r \in \mathcal{P} \mid head(r) \cap L = \emptyset, body^+(r) \cap L = \emptyset\}$, and $\rho(r,L) = \{\mathbf{F}\beta(r)\} \cup \{\mathbf{T}p \mid p \in head(r) \setminus L\}$. Clearly for any $r \in \mathcal{P}$ and $L \subseteq \mathbf{KA}(\mathcal{K})$, $var(\rho(r,L)) \subseteq \{\{\beta(r) \mid r \in$

$\mathcal{P}\} \cup \{p \mid p \in \mathbf{KA}(\mathcal{K})\}\}$. As a result for any $L \subseteq \mathbf{KA}(\mathcal{K})$, $S \subseteq \mathbf{KA}(\mathcal{O})$, $var(\lambda_{\mathcal{K}}(L,S)) \subseteq \{\{p \mid p \in \mathbf{KA}(\mathcal{K})\} \cup \{\beta(r) \mid r \in \mathcal{P}\}\}$. It follows that

$$var(\Lambda_{\mathcal{K}}) \subseteq \{\mathbf{KA}(\mathcal{K}) \cup \{\beta(r) \mid r \in \mathcal{P}\}\}. \tag{130}$$

Combined with (128) this means

$$\begin{aligned} var(\Delta_{\mathcal{K}}) = var(\Delta_{\mathcal{K}} \cup \Lambda_{\mathcal{K}}) =& \{p \mid p \in \mathbf{KA}(\mathcal{K})\} \cup \{\bot\} \\ & \cup \{\beta(r) \mid r \in \mathcal{P}\} \\ & \cup \{\beta_{\mathcal{P}}(r,p) \mid r \in \mathcal{P}, p \in head(r)\} \\ & \cup \{\beta_{\mathcal{O}}(p) \mid p \in \mathbf{KA}(\mathcal{O}) \cup \{\bot\}\}. \end{aligned} \tag{131}$$

For all $p \in \mathbf{KA}(\mathcal{K})$ either $p \in \hat{I}$ or $p \in \mathbf{KA}(\mathcal{K}) \setminus \hat{I}$ therefore

$$\mathbf{KA}(\mathcal{K}) \subseteq ({A_{\mathcal{K}}^{\hat{I}}}^{T} \cup {A_{\mathcal{K}}^{\hat{I}}}^{F}) \tag{132}$$

Clearly since $\mathbf{F}\bot \in A_{\mathcal{K}}^{\hat{I}}$

$$\bot \in ({A_{\mathcal{K}}^{\hat{I}}}^{T} \cup {A_{\mathcal{K}}^{\hat{I}}}^{F}). \tag{133}$$

For all $r \in \mathcal{P}$ either $\hat{I} \models Body(r)$ or $\hat{I} \not\models Body(r)$ therefore

$$\{\beta(r) \mid r \in \mathcal{P}\} \subseteq ({A_{\mathcal{K}}^{\hat{I}}}^{T} \cup {A_{\mathcal{K}}^{\hat{I}}}^{F}). \tag{134}$$

For all (r,p) such that $r \in \mathcal{P}$ and $p \in head(r)$, $\hat{I} \models Body(r), head(r) \cap (\hat{I} \cup \{p\}) = \{p\}$, or either $\hat{I} \not\models Body(r)$ or $head(r) \cap \hat{I} \not\subseteq \{p\}$, therefore

$$\{\beta(r,p) \mid r \in \mathcal{P}, p \in head(r)\} \subseteq ({A_{\mathcal{K}}^{\hat{I}}}^{T} \cup {A_{\mathcal{K}}^{\hat{I}}}^{F}). \tag{135}$$

Finally for all $p \in \mathbf{KA}(\mathcal{O}) \cup \{\bot\}$ either $OB_{\mathcal{O},\hat{I}\setminus\{p\}} \models p$ or $OB_{\mathcal{O},\hat{I}\setminus\{p\}} \not\models p$, therefore

$$\{\beta_{\mathcal{O}}(p) \mid p \in \mathbf{KA}(\mathcal{O}) \cup \{\bot\}\} \subseteq ({A_{\mathcal{K}}^{\hat{I}}}^{T} \cup {A_{\mathcal{K}}^{\hat{I}}}^{F}). \tag{136}$$

From (132), (133), (134), (135), and (136), it follows that

$$\begin{aligned} ({A_{\mathcal{K}}^{\hat{I}}}^{T} \cup {A_{\mathcal{K}}^{\hat{I}}}^{F}) \supseteq & \{\{p \mid p \in \mathbf{KA}(\mathcal{K})\} \cup \{\bot\} \cup \{\beta(r) \mid r \in \mathcal{P}\} \\ & \cup \{\beta_{\mathcal{P}}(r,p) \mid r \in \mathcal{P}, p \in head(r)\} \\ & \cup \{\beta_{\mathcal{O}}(p) \mid p \in \mathbf{KA}(\mathcal{O}) \cup \{\bot\}\}\}. \end{aligned} \tag{137}$$

The construction of $A_{\mathcal{K}}^{\hat{I}}$ makes it impossible for any kind of literal to appear within it, therefore

$$\begin{aligned} ({A_{\mathcal{K}}^{\hat{I}}}^{T} \cup {A_{\mathcal{K}}^{\hat{I}}}^{F}) =& \{p \mid p \in \mathbf{KA}(\mathcal{K})\} \cup \{\bot\} \cup \{\beta(r) \mid r \in \mathcal{P}\} \\ & \cup \{\beta_{\mathcal{P}}(r,p) \mid r \in \mathcal{P}, p \in head(r)\} \\ & \cup \{\beta_{\mathcal{O}}(p) \mid p \in \mathbf{KA}(\mathcal{O}) \cup \{\bot\}\}. \end{aligned} \tag{138}$$

It is clear from (131) and (138) that

$$({A_{\mathcal{K}}^{\hat{I}}}^{T} \cup {A_{\mathcal{K}}^{\hat{I}}}^{F}) = var(\Delta_{\mathcal{K}}) = var(\Delta_{\mathcal{K}} \cup \Lambda_{\mathcal{K}}). \tag{139}$$

As $\bot$ is already within each of these sets its addition makes no difference

$$({A_{\mathcal{K}}^{\hat{I}}}^{T} \cup {A_{\mathcal{K}}^{\hat{I}}}^{F}) = var(\Delta_{\mathcal{K}} \cup \{\bot\}) = var(\Delta_{\mathcal{K}} \cup \Lambda_{\mathcal{K}} \cup \{\bot\}). \tag{140}$$

Thus $A^{\hat{I}}_{\mathcal{K}}$ is a total assignment for both $var(\Delta_{\mathcal{K}} \cup \{\bot\})$ and $var(\Delta_{\mathcal{K}} \cup \Lambda_{\mathcal{K}} \cup \{\bot\})$. □

The final lemma characterizes the relationship between a **K**-interpretation satisfying the body of a rule, and the literals which appear within its induced assignment.

Lemma 8

For any **K**-interpretation $\hat{I}$ of an HMKNF-KB $\mathcal{K} = (\mathcal{P}, \mathcal{O})$, and any rule $r \in \mathcal{P}$, $\hat{I} \models Body(r)$, if and only if, $\forall p \in body^{+}(r), \mathbf{T}p \in A^{\hat{I}}_{\mathcal{K}}$ and $\forall n \in body^{-}(r), \mathbf{F}p \in A^{\hat{I}}_{\mathcal{K}}$.

Proof

($\Rightarrow$) Assume $\hat{I} \models Body(r)$. By definition $\hat{I} \models \bigwedge body^{+}(r) \land \neg \bigvee body^{-}(r)$. Consequently, $\forall p \in body^{+}(r), p \in \hat{I} \land \forall n \in body^{-}(r), n \notin \hat{I}$. Note that $\forall n \in body^{-}(r), n \in \mathbf{KA}(\mathcal{K})$ therefore $\forall n \in body^{-}(r), n \in \mathbf{KA}(\mathcal{K}) \setminus \hat{I}$. Clearly then $\forall p \in body^{+}(r), \mathbf{T}p \in A^{\hat{I}}_{\mathcal{K}}$ and $\forall n \in body^{-}(r), \mathbf{F}p \in A^{\hat{I}}_{\mathcal{K}}$.

($\Leftarrow$) Assume $\forall p \in body^{+}(r), \mathbf{T}p \in A^{\hat{I}}_{\mathcal{K}}$ and $\forall n \in body^{-}(r), \mathbf{F}p \in A^{\hat{I}}_{\mathcal{K}}$. As a result $\forall p \in body^{+}(r), p \in \hat{I} \land \forall n \in body^{-}(r), n \notin \hat{I}$. Clearly this means that $\hat{I} \models Body(r)$. □

We now are able to prove Theorem 3. In order to do this, we show that there is no nogood which is a subset of $A^{\hat{I}}_{\mathcal{K}}$, whenever $\hat{I} \models \mathcal{K}_{comp}(\hat{I})$, and that there is such a nogood when $\hat{I} \not\models \mathcal{K}_{comp}(\hat{I})$.

Theorem 3

Let $\hat{I}$ be a **K**-interpretation for an HMKNF-KB $\mathcal{K} = (\mathcal{P}, \mathcal{O})$, then we have that $\hat{I} \models \mathcal{K}_{comp}(\hat{I})$ if and only if $A^{\hat{I}}_{\mathcal{K}}$ is a solution to $\Delta_{\mathcal{K}} \cup \{\mathbf{T}\bot\}$.

Proof

($\Rightarrow$) Let $\hat{I}$ be a **K**-interpretation of the HMKNF-KB $\mathcal{K} = (\mathcal{P}, \mathcal{O})$ such that $\hat{I} \models \mathcal{K}_{comp}(\hat{I})$, then $A^{\hat{I}}_{\mathcal{K}}$ is well defined.

Assume there exists $\delta \in \Gamma_{\mathcal{O}}$ such that $\delta \subseteq A^{\hat{I}}_{\mathcal{K}}$. Clearly since $\delta \in \Gamma_{\mathcal{O}}$ either $\delta = \gamma^{+}_{\mathcal{O}}(p, S)$ for $p \in \mathbf{KA}(\mathcal{O}) \cup \{\bot\}, S \subseteq \mathbf{KA}(\mathcal{O})$, such that $OB_{\mathcal{O}, S \setminus \{p\}} \models p$ or, $\delta = \gamma^{-}_{\mathcal{O}}(p, S)$ for $p \in \mathbf{KA}(\mathcal{O}), S \subseteq \mathbf{KA}(\mathcal{O})$ such that $OB_{\mathcal{O}, \mathbf{KA}(\mathcal{O}) \setminus (S \cup \{p\})} \not\models p$.

Case 1: $\delta = \gamma^{+}_{\mathcal{O}}(p, S)$ for $p \in \mathbf{KA}(\mathcal{O}) \cup \{\bot\}, S \subseteq \mathbf{KA}(\mathcal{O})$, such that $OB_{\mathcal{O}, S \setminus \{p\}} \models p$. $\gamma^{+}_{\mathcal{O}}(p, S) \subseteq A^{\hat{I}}_{\mathcal{K}}$ implies that $\forall s \in S, s \in \hat{I}$, or more simply $S \subseteq \hat{I}$. This also means that $OB_{\mathcal{O}, \hat{I} \setminus \{p\}} \models p$. Consequently, $\mathbf{T}\beta_{\mathcal{O}}(p) \in A^{\hat{I}}_{\mathcal{K}}$. Conversely, by the fact that $\gamma^{+}_{\mathcal{O}}(p, S) \subseteq A^{\hat{I}}_{\mathcal{K}}$, $\mathbf{F}\beta_{\mathcal{O}}(p) \in A^{\hat{I}}_{\mathcal{K}}$. Thereby this case results in a contradiction.

Case2: $\delta = \gamma^{-}_{\mathcal{O}}(p, S)$ for $p \in \mathbf{KA}(\mathcal{O}), S \subseteq \mathbf{KA}(\mathcal{O})$ such that $OB_{\mathcal{O}, \mathbf{KA}(\mathcal{O}) \setminus (S \cup \{p\})} \not\models p$. $\gamma^{-}_{\mathcal{O}}(p, S) \subseteq A^{\hat{I}}_{\mathcal{K}}$ implies that $\forall s \in S, s \notin \hat{I}$, or more simply $S \subseteq (\mathbf{KA}(\mathcal{O}) \setminus \hat{I})$. This also means that $OB_{\mathcal{O}, \mathbf{KA}(\mathcal{O}) \setminus ((\mathbf{KA}(\mathcal{O}) \setminus \hat{I}) \cup \{p\})} \not\models p$, or equivalently $OB_{\mathcal{O}, \hat{I} \setminus \{p\}} \not\models p$. Consequently, $\mathbf{F}\beta_{\mathcal{O}}(p) \in A^{\hat{I}}_{\mathcal{K}}$. Conversely, by the fact that $\gamma^{-}_{\mathcal{O}}(p, S) \subseteq A^{\hat{I}}_{\mathcal{K}}$, $\mathbf{T}\beta_{\mathcal{O}}(p) \in A^{\hat{I}}_{\mathcal{K}}$. Thereby this case also results in a contradiction.

As both cases where there exists $\delta \in \Gamma_{\mathcal{O}}$ such that $\delta \subseteq A^{\hat{I}}_{\mathcal{K}}$ result in a contradiction,

$$\nexists \delta \in \Gamma_{\mathcal{O}}, \text{ s.t. } \delta \subseteq A^{\hat{I}}_{\mathcal{K}}. \tag{141}$$

Assume there exists some nogood $\delta \in \Gamma_{\mathcal{P}}$ such that $\delta \subseteq A^{\hat{I}}_{\mathcal{K}}$. Clearly for some $\beta \in \{\beta(r) \mid r \in \mathcal{P}\} \cup \{\beta_{\mathcal{P}}(r, p) \mid r \in \mathcal{P}, p \in head(r)\}$, $\delta \in \gamma_{\mathcal{P}}(\beta)$. As a result either $\mathbf{F}\beta \in A^{\hat{I}}_{\mathcal{K}}$ and $\beta \subseteq A^{\hat{I}}_{\mathcal{K}}$, or $\mathbf{T}\beta \in A^{\hat{I}}_{\mathcal{K}}$ and $\overline{\sigma} \in A^{\hat{I}}_{\mathcal{K}}$ for some $\sigma \in \beta$.

Case 1: $\beta \in \{\beta(r) \mid r \in \mathcal{P}\}$.

Case 1.1: $\mathbf{F}\beta \in A_{\mathcal{K}}^{\hat{I}}$ and $\beta \subseteq A_{\mathcal{K}}^{\hat{I}}$. In this case $\hat{I} \not\models \mathit{Body}(r)$, so by Lemma 7 and Lemma 8 $\mathbf{T}p \notin A_{\mathcal{K}}^{\hat{I}}$ or $\mathbf{F}n \notin A_{\mathcal{K}}^{\hat{I}}$. Conversely, since $p \in \mathit{body}^+(r)$, $\mathbf{T}p \in \beta(r) \subseteq A_{\mathcal{K}}^{\hat{I}}$ and since $n \in \mathit{body}^-(r)$, $\mathbf{F}n \in \beta(r) \subseteq A_{\mathcal{K}}^{\hat{I}}$. This leads to a contradiction.

Case 1.2: $\mathbf{T}\beta \in A_{\mathcal{K}}^{\hat{I}}$ and $\overline{\sigma} \in A_{\mathcal{K}}^{\hat{I}}$ for some $\sigma \in \beta$. In this case $\hat{I} \models \mathit{Body}(r)$, so by Lemma 7 and Lemma 8, $\forall p \in \mathit{body}^+(r), \mathbf{F}p \notin A_{\mathcal{K}}^{\hat{I}}$ and $\forall n \in \mathit{body}^-(r), \mathbf{T}n \notin A_{\mathcal{K}}^{\hat{I}}$. Conversely, either for some $p \in \mathit{body}^+(r)$, $\overline{\sigma} = \mathbf{F}p$ or for some $n \in \mathit{body}^-(r)$ $\overline{\sigma} = \mathbf{T}p$. Therefore either $\exists p \in \mathit{body}^+(r), \mathbf{F}p \in A_{\mathcal{K}}^{\hat{I}}$ or $\exists n \in \mathit{body}^-(r), \mathbf{T}p \in A_{\mathcal{K}}^{\hat{I}}$. This leads to a contradiction.

Case 2: $\beta \in \{\beta(r,p) \mid r \in \mathcal{P}, p \in \mathit{head}(r)\}$. In this case either $\overline{\sigma} = \mathbf{F}\beta(r)$ or $\overline{\sigma} = \mathbf{T}q$ for some $q \in \mathit{head}(r) \setminus \{p\}$.

Case 2.1: $\mathbf{F}\beta \in A_{\mathcal{K}}^{\hat{I}}$ and $\beta \subseteq A_{\mathcal{K}}^{\hat{I}}$. In this case $\mathbf{T}\beta(r) \in A_{\mathcal{K}}^{\hat{I}}$ and $\forall q \in \mathit{head}(r) \setminus \{p\}, \mathbf{F}q \in A_{\mathcal{K}}^{\hat{I}}$. Also either $I \not\models \mathit{Body}(r)$ or $\mathit{head}(r) \cap \hat{I} \not\subseteq \{p\}$. This means that $\mathbf{F}\beta(r) \in A_{\mathcal{K}}^{\hat{I}}$ or $\exists q \in \mathit{head}(r) \setminus \{p\}, \mathbf{T}q \in A_{\mathcal{K}}^{\hat{I}}$. Following Lemma 7, $\mathbf{T}\beta(r) \notin A_{\mathcal{K}}^{\hat{I}}$ or $\exists q \in \mathit{head}(r) \setminus \{p\}, \mathbf{F}q \notin A_{\mathcal{K}}^{\hat{I}}$, so we have a contradiction.

Cases 2.2: $\mathbf{T}\beta \in A_{\mathcal{K}}^{\hat{I}}$ and $\overline{\sigma} \in A_{\mathcal{K}}^{\hat{I}}$ for some $\sigma \in \beta$. In this case either $\overline{\sigma} = \mathbf{F}\beta(r)$ or $\overline{\sigma} = \mathbf{T}q$ for some $q \in \mathit{head}(r) \setminus \{p\}$. Also $\hat{I} \models \mathit{Body}(r)$, and $\mathit{head}(r) \cap (\hat{I} \cup \{p\}) = \{p\}$. This means that $\mathbf{T}\beta(r) \in A_{\mathcal{K}}^{\hat{I}}$, and $\forall q \in \mathit{head}(r) \setminus \{p\}, \mathbf{F}q \in A_{\mathcal{K}}^{\hat{I}}$. Following Lemma 7, $\mathbf{F}\beta(r) \notin A_{\mathcal{K}}^{\hat{I}}$ and $\forall q \in \mathit{head}(r) \setminus \{p\}, \mathbf{T}q \notin A_{\mathcal{K}}^{\hat{I}}$, so we have a contradiction.

As all possible cases where there exists a nogood $\delta \in \Gamma_{\mathcal{P}}$ such that $\delta \subseteq A_{\mathcal{K}}^{\hat{I}}$ result in a contradiction.

$$\nexists \delta \in \Gamma_{\mathcal{P}}, \text{ s.t. } \delta \subseteq A_{\mathcal{K}}^{\hat{I}}. \tag{142}$$

Assume that there is some nogood $\delta \in \Phi_{\mathcal{P}}$ such that $\delta \subseteq A_{\mathcal{K}}^{\hat{I}}$. Clearly $\delta = \phi_{\mathcal{P}}(r)$ for some rule $r \in \mathcal{P}$. As a consequence $\forall p \in \mathit{head}(r), \mathbf{F}p \in A_{\mathcal{K}}^{\hat{I}}$ and $\mathbf{T}\beta(r) \in A_{\mathcal{K}}^{\hat{I}}$. This means that $\forall p \in \mathit{head}(r), \hat{I} \not\models p$ and $\hat{I} \models \mathit{Body}(r)$. More simply $\hat{I} \not\models \bigvee \mathit{head}(r)$ and $\hat{I} \models \mathit{Body}(r)$. This contradicts the assumption that $\hat{I} \models \mathcal{P}_{rule}$, therefore

$$\nexists \delta \in \Phi_{\mathcal{P}}, \text{ s.t. } \delta \subseteq A_{\mathcal{K}}^{\hat{I}}. \tag{143}$$

Assume that there is some nogood $\delta \in \Phi_{\mathcal{O}}$ such that $\delta \subseteq A_{\mathcal{K}}^{\hat{I}}$. Clearly $\delta = \phi_{\mathcal{O}}(p)$ for some $p \in \mathbf{KA}(\mathcal{O}) \cup \{\bot\}$. As a consequence $\mathbf{F}p \in A_{\mathcal{K}}^{\hat{I}}$ and $\mathbf{T}\beta_{\mathcal{O}}(p) \in A_{\mathcal{K}}^{\hat{I}}$. This means that $p \in \mathbf{KA}(\mathcal{K}) \setminus \hat{I}$, and that $OB_{\mathcal{O},\hat{I}\setminus\{p\}} \models p$. More simply $p \notin \hat{I}$ and $OB_{\mathcal{O},\hat{I}} \models p$. This contradicts the assumption that $\hat{I} \models \mathcal{O}_{satr}(\hat{I})$, therefore

$$\nexists \delta \in \Phi_{\mathcal{O}} \text{ s.t. } \delta \subseteq A_{\mathcal{K}}^{\hat{I}}. \tag{144}$$

Assume that there is some nogood $\delta \in \Psi_{\mathcal{K}}$ such that $\delta \subseteq A_{\mathcal{K}}^{\hat{I}}$. Clearly $\delta = \psi_{\mathcal{K}}(p)$ for some $p \in \mathbf{KA}(\mathcal{K})$. As a consequence $\mathbf{T}p \in A_{\mathcal{K}}^{\hat{I}}$, $\mathbf{F}\beta_{\mathcal{P}}(r,p) \in A_{\mathcal{K}}^{\hat{I}}$ for each $r \in \mathcal{P}$ such that $p \in \mathit{head}(r)$, and $\mathbf{F}\beta_{\mathcal{O}}(p) \in A_{\mathcal{K}}^{\hat{I}}$ if $p \in \mathbf{KA}(\mathcal{O})$. This means that $p \in \hat{I}$, either $\hat{I} \not\models \mathit{Body}(r)$ or $\mathit{head}(r) \cap \hat{I} \not\subseteq \{p\}$ for each $r \in \mathcal{P}$ such that $p \in \mathit{head}(r)$, and $OB_{\mathcal{O},\hat{I}\setminus\{p\}} \not\models p$ if $p \in \mathbf{KA}(\mathcal{O})$. Equivalently, $\hat{I} \models p$, $\hat{I} \not\models \mathit{Body}(r) \wedge \bigwedge_{q \in \mathit{head}(r)\setminus\{p\}} \neg q$ for each $r \in \mathcal{P}$ such that $p \in \mathit{head}(r)$. More simply,

$$\hat{I} \not\models p \supset \bigvee_{\substack{r \in \mathcal{P}, \\ p \in \mathit{head}(r)}} \big(\mathit{Body}(r) \wedge \bigwedge_{q \in \mathit{head}(r)\setminus\{p\}} \neg q\big). \tag{145}$$

As $OB_{\mathcal{O},\hat{I}\setminus\{p\}} \not\models p$, $\mathcal{K}_{sup}(\hat{I})$ includes this formula so $\hat{I} \not\models \mathcal{K}_{sup}(\hat{I})$. This is a contradiction, therefore

$$\nexists \delta \in \Psi_{\mathcal{K}} \text{ s.t. } \delta \subseteq A^{\hat{I}}_{\mathcal{K}}. \tag{146}$$

From (141), (142), (143), (144), and (146) it clearly follows that there is no nogood $\delta \in \Delta_{\mathcal{K}}$ such that $\delta \subseteq A^{\hat{I}}_{\mathcal{K}}$. Combined with Lemma 7, and the fact that $\mathbf{T}\bot \notin A^{\hat{I}}_{\mathcal{K}}$ this means that $A^{\hat{I}}_{\mathcal{K}}$ is a solution for $\Delta_{\mathcal{K}} \cup \{\mathbf{T}\bot\}$.
($\Leftarrow$) Assume that $\hat{I} \not\models \mathcal{P}_{rule}$. Clearly there is some $r \in \mathcal{P}$ such that $\hat{I} \models Body(r)$ and $\hat{I} \not\models \bigvee head(r)$. As a consequence $\mathbf{T}\beta(r) \in A^{\hat{I}}_{\mathcal{K}}$ but $\forall p \in head(r), \mathbf{F}p \in A^{\hat{I}}_{\mathcal{K}}$. This means that $\phi_{\mathcal{P}}(r) \subseteq A^{\hat{I}}_{\mathcal{K}}$, since $\phi_{\mathcal{P}}(r) \in \Phi_{\mathcal{P}}$ this violates the assumption that $A^{\hat{I}}_{\mathcal{K}}$ is a solution for $\Delta_{\mathcal{K}}$. Therefore

$$\hat{I} \models \mathcal{P}_{rule}. \tag{147}$$

Assume that $\hat{I} \not\models \mathcal{O}_{satr}(\hat{I})$. Clearly there is some atom $p \in \mathbf{KA}(\mathcal{O}) \cup \{\bot\}$ such that $OB_{\mathcal{O},\hat{I}} \not\models p$ but $p \notin \hat{I}$. As a result $\mathbf{T}\beta_{\mathcal{O}}(p) \in A^{\hat{I}}_{\mathcal{K}}$, however $\mathbf{F}p \in A^{\hat{I}}_{\mathcal{K}}$. This means that $\phi_{\mathcal{O}}(p) \subseteq A^{\hat{I}}_{\mathcal{K}}$, since $\phi_{\mathcal{O}}(p) \in \Phi_{\mathcal{O}}$ this violates the assumption that $A^{\hat{I}}_{\mathcal{K}}$ is a solution for $\Delta_{\mathcal{K}}$. Therefore

$$\hat{I} \models \mathcal{O}_{satr}(\hat{I}). \tag{148}$$

Assume that $\hat{I} \not\models \mathcal{K}_{sup}(\hat{I})$. Clearly there is some atom $p \in \hat{I}$ such that $OB_{\mathcal{O},\hat{I}\setminus\{p\}} \not\models p$ for which for all $r \in \mathcal{P}$ where $p \in head(r)$ $\hat{I} \not\models Body(r)$ or $\exists a \in head(r) \setminus \{p\}$ such that $a \in \hat{I}$. From $p \in \hat{I}$, $\mathbf{T}p \in A^{\hat{I}}_{\mathcal{K}}$. From $OB_{\mathcal{O},\hat{I}\setminus\{p\}} \not\models p$, either $p \in (\mathbf{KA}(\mathcal{K}) \setminus \mathbf{KA}(\mathcal{O}))$ or $\mathbf{F}\beta_{\mathcal{O}} \in A^{\hat{I}}_{\mathcal{K}}$. Finally, since for all $r \in \mathcal{P}$ where $p \in head(r)$, $\hat{I} \not\models Body(r)$ or $\exists a \in head(r) \setminus \{p\}$ such that $a \in \hat{I}$, $\mathbf{F}\beta_{\mathcal{P}}(r, p) \in A^{\hat{I}}_{\mathcal{K}}$. It follows that $\psi_{\mathcal{K}}(p) \subseteq A^{\hat{I}}_{\mathcal{K}}$, since $\psi_{\mathcal{K}}(p) \in \Psi_{\mathcal{K}}$ this violates the assumption that $A^{\hat{I}}_{\mathcal{K}}$ is a solution for $\Delta_{\mathcal{K}}$. Therefore

$$\hat{I} \models \mathcal{K}_{sup}(\hat{I}). \tag{149}$$

From (147), (148), and (149), $\hat{I} \models \mathcal{K}_{comp}(\hat{I})$. $\square$

We now are able to prove Theorem 4, by expanding upon Theorem 3. In particular, we show that when $\hat{I} \models \mathcal{K}_{comp}(\hat{I}) \wedge \mathcal{K}_{loop}(\hat{I})$, there is no nogood which is a subset of $A^{\hat{I}}_{\mathcal{K}}$, but there is one whenever $\hat{I} \models \mathcal{K}_{comp}(\hat{I})$ but $\hat{I} \not\models \mathcal{K}_{loop}(\hat{I})$.

Theorem 4
Let $\hat{I}$ be a **K**-interpretation for an HMKNF-KB $\mathcal{K} = (\mathcal{P}, \mathcal{O})$, then we have that $\hat{I} \models \mathcal{K}_{comp}(\hat{I}) \wedge \mathcal{K}_{loop}(\hat{I})$ if and only if $A^{\hat{I}}_{\mathcal{K}}$ is a solution to $\Delta_{\mathcal{K}} \cup \Lambda_{\mathcal{K}} \cup \{\mathbf{T}\bot\}$.

Proof
($\Rightarrow$) Assume that $\hat{I} \models \mathcal{K}_{comp}(\hat{I}) \wedge \mathcal{K}_{loop}(\hat{I})$. By Theorem 3 $A^{\hat{I}}_{\mathcal{K}}$ is a solution for $\Delta_{\mathcal{K}} \cup \{\mathbf{T}\bot\}$. Meaning that no nogood $\delta \in \Delta_{\mathcal{K}} \cup \{T\bot\}$ such that $\delta \subseteq A^{\hat{I}}_{\mathcal{K}}$. We also know that $A^{\hat{I}}_{\mathcal{K}}$ is a total assignment for $\Delta_{\mathcal{K}} \cup \Lambda_{\mathcal{K}} \cup \{\mathbf{T}\bot\}$ by Lemma 7. Therefore we must simply show that there is no nogood $\delta \in \Lambda_{\mathcal{K}}$ such that $\delta \subseteq A^{\hat{I}}_{\mathcal{K}}$. Suppose that there is such a nogood δ. Clearly, $\delta \in \lambda_{\mathcal{K}}(L, S)$ for some $L \in Loops(\mathcal{K})$, $S \subseteq \mathbf{KA}(\mathcal{O})$ for which $L \cap S = \emptyset$ and $OB_{\mathcal{O},\mathbf{KA}(\mathcal{O})\setminus(S \cup L)} \not\models \bigvee L$.

Consequently there is an atom $p \in L$ such that $\mathbf{T}p \in \delta$, so $\mathbf{T}p \in A_{\mathcal{K}}^{\hat{I}}$, therefore

$$\hat{I} \models \bigvee L. \tag{150}$$

Since $\hat{I} \subseteq \mathbf{KA}(\mathcal{K})$, $OB_{\mathcal{O},\mathbf{KA}(\mathcal{O})\setminus(S\cup L)} \not\models \bigvee L$, and only atoms in $\mathbf{KA}(\mathcal{O})$ are relevant for entailing other atoms through the ontology, $OB_{\mathcal{O},\hat{I}\setminus(S\cup L)} \not\models \bigvee L$. Due to the fact that $\forall s \in S, \mathbf{F}s \in \delta$, $\mathbf{F}s \in A_{\mathcal{K}}^{\hat{I}}$ thereby $\hat{I} \cap S = \emptyset$. As a consequence

$$OB_{\mathcal{O},\hat{I}\setminus L} \not\models \bigvee L. \tag{151}$$

For each rule $r \in \mathcal{P}$ for which $L \cap head(r) \neq \emptyset$ and $body^+(r) \cap L = \emptyset$, either $\mathbf{F}\beta(r) \in \delta$, or $\mathbf{T}a \in \delta$ such that $a \in head(r) \setminus L$. This means that for all such rules $\mathbf{F}\beta \in A_{\mathcal{K}}^{\hat{I}}$ or $\mathbf{T}a \in A_{\mathcal{K}}^{\hat{I}}$ for such a. Consequently for each rule either $\hat{I} \not\models Body(r)$ or $\hat{I} \models \bigvee_{a \in head(r)\setminus L} a$. More simply

$$\hat{I} \not\models \bigvee_{\substack{r\in\mathcal{P} \\ head(r)\cap L\neq\emptyset \\ body^+(r)\cap L=\emptyset}} \Big(Body(r) \wedge \bigwedge_{a\in head(r)\setminus L} \neg a\Big). \tag{152}$$

From (150), (151), (152), and the fact that $L \in Loops(\mathcal{K})$, it is clear that $\hat{I} \not\models \mathcal{K}_{loop}(\hat{I})$. Therefore there can be no nogood $\delta \in \Lambda_{\mathcal{K}}$ such that $\delta \subseteq A_{\mathcal{K}}^{\hat{I}}$, or else a contradiction would occur. We have already shown all other necessary conditions, so $A_{\mathcal{K}}^{\hat{I}}$ is a solution to $\Delta_{\mathcal{K}} \cup \Lambda_{\mathcal{K}} \cup \{\mathbf{T}\bot\}$.
($\Leftarrow$) Assume that $A_{\mathcal{K}}^{\hat{I}}$ is a solution to $\Delta_{\mathcal{K}} \cup \Lambda_{\mathcal{K}} \cup \{\mathbf{T}\bot\}$, clearly by Lemma 7 it is also a solution for $\Delta_{\mathcal{K}} \cup \{\mathbf{T}\bot\}$, therefore by Theorem 3 $\hat{I} \models \mathcal{K}_{comp}(\hat{I})$.

Suppose $\hat{I} \not\models \mathcal{K}_{loop}(\hat{I})$. Then there would be some loop $L \in Loops(\mathcal{K})$ for which $OB_{\mathcal{O},\hat{I}\setminus L} \not\models \bigvee L$, $\hat{I} \models \bigvee L$, and

$$\hat{I} \not\models \bigvee_{\substack{r\in\mathcal{P} \\ head(r)\cap L\neq\emptyset \\ body^+(r)\cap L=\emptyset}} \Big(Body(r) \wedge \bigwedge_{a\in head(r)\setminus L} \neg a\Big). \tag{153}$$

Clearly for $S = \mathbf{KA}(\mathcal{O}) \setminus (\hat{I} \cup L)$, $L \cap S = \emptyset$, and since $\mathbf{KA}(\mathcal{O}) \setminus (S \cup L) = \hat{I} \setminus L$, $OB_{\mathcal{O},\mathbf{KA}(\mathcal{O})\setminus(S\cup L)} \not\models \bigvee L$. Therefore

$$\lambda_{\mathcal{K}}(L, S) \subseteq \Lambda_{\mathcal{K}} \tag{154}$$

As $\hat{I} \models \bigvee L$, $\exists p \in L, \hat{I} \models p$, therefore

$$\exists p \in L, \mathbf{T}p \in A_{\mathcal{K}}^{\hat{I}}. \tag{155}$$

Also since $S = \mathbf{KA}(\mathcal{O}) \setminus (\hat{I} \cup L)$, $\forall s \in S, \hat{I} \not\models s$, and consequently

$$\forall s \in S, \mathbf{F}s \in A_{\mathcal{K}}^{\hat{I}}. \tag{156}$$

From (153), for all rules $r \in \mathcal{P}$ such that $head(r) \cap L \neq \emptyset$ and $body^+(r) = \emptyset$, either $\hat{I} \not\models Body(r)$ or there is some atom $a \in head(r) \setminus L$ for which $\hat{I} \models a$. Therefore either $\mathbf{F}\beta(r) \in A_{\mathcal{K}}^{\hat{I}}$ or $a \in head(r) \setminus L$ such that $\mathbf{T}a \in A_{\mathcal{K}}^{\hat{I}}$ for each such rule. Note that these rules make up the set $\mathcal{E}_{\mathcal{P}}(L)$, thereby

$$\forall r \in \mathcal{E}_{\mathcal{P}}(L), \exists \sigma \in \rho(r, L), \text{ such that } \sigma \in A_{\mathcal{K}}^{\hat{I}}. \tag{157}$$

From (155), (156), and (157) there is some nogood $\delta \in \lambda_{\mathcal{K}}(L, S)$ such that $\delta \subseteq A^{\hat{I}}_{\mathcal{K}}$. Due to (154) this means that

$$\exists \delta \in \Lambda_{\mathcal{K}}, \delta \subseteq A^{\hat{I}}_{\mathcal{K}}. \tag{158}$$

This directly contradicts the assumption that $A^{\hat{I}}_{\mathcal{K}}$ is a solution for $\Delta_{\mathcal{K}} \cup \Lambda_{\mathcal{K}} \cup \{\mathbf{T}\bot\}$. As the supposition that $\hat{I} \not\models \mathcal{K}_{loop}(\hat{I})$ leads to a contradiction, it must actually be the case that $\hat{I} \models \mathcal{K}_{loop}(\hat{I})$. $\square$