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Supplementary Material for The Stable Model Semantics for Higher-Order Logic Programming

BOGAERTS BART[†‡], CHARALAMBIDIS ANGELOS[⊛], CHATZIAGAPIS GIANNOS[⁎], KOSTOPOULOS BABIS[⊛], POLLACI SAMUELE[†‡] and RONDOGIANNIS PANOS[⁎]
†*Vrije Universiteit Brussel, Belgium*
‡*Katholieke Universiteit Leuven, Belgium*
⊛*Harokopio University of Athens, Greece*
⁎*National and Kapodistrian University of Athens, Greece*

A Proofs of Section 4 and Section 5

In the following, we provide the proofs of a few propositions contained in Sections 4 and 5. Notice that most of the results of these two sections are rather straightforward. Propositions 4.1 and 5.1 are algebraic consequences of Definitions 4.1 and 5.1, respectively.

Proposition 4.1
For every predicate type π, $(\llbracket\pi\rrbracket, \leq_\pi)$ is a complete lattice.

Proof
We proceed by induction on the predicate type π.

Let $\pi = o$. Clearly, the set $\llbracket o \rrbracket = \{\mathit{true}, \mathit{false}\}$ with the order $\leq_o$ is a complete lattice, with bottom element *false* and top element *true*.

Now let $\pi = \rho \to \pi'$, and assume $\llbracket\pi'\rrbracket$ is a complete lattice. We have to show that the set of functions $\llbracket\rho\rrbracket \to \llbracket\pi'\rrbracket$ with the order $\leq_{\rho\to\pi}$ is a complete lattice. Since $\llbracket\pi'\rrbracket$ is a complete lattice, for each subset $S \subseteq \llbracket\pi\rrbracket$ we can define $f_{\bigwedge S}, f_{\bigvee S} \in \llbracket\pi\rrbracket$ by $f_{\bigwedge S}(x) := \bigwedge\{g(x) \mid g \in S\}$ and $f_{\bigvee S}(x) := \bigvee\{g(x) \mid g \in S\}$, respectively. By the definition of $\leq_{\rho\to\pi'}$, it is immediate to see that $\bigwedge S = f_{\bigwedge S}$ and $\bigvee S = f_{\bigvee S}$. Hence, $\llbracket\pi\rrbracket$ is a complete lattice, as desired. $\square$

The following proposition draws the correspondence between the two-valued Herbrand models of a program and the pre-fixpoints of its immediate consequence operator.

Proposition 4.3
Let P be a program and $I \in H_{\mathsf{P}}$. Then, I is a model of P iff I is a pre-fixpoint of T_{P}.

Proof
Suppose first that I is a model of P. Suppose, for the sake of contradiction, that I is not a pre-fixpoint of T_{P}. That means that there exists a predicate constant $\mathsf{p} : \rho_1 \to \cdots \to \rho_n \to o$ and a $\overline{d} \in \llbracket\rho_1\rrbracket \times \cdots \times \llbracket\rho_n\rrbracket$ such that $T_{\mathsf{P}}(I)(\mathsf{p})\ \overline{d} \not\leq I(\mathsf{p})\ \overline{d}$. Then, $T_{\mathsf{P}}(I)(\mathsf{p})\ \overline{d}$ must be *true*, so by the definition of T_{P} there is a Herbrand state s and a rule $\mathsf{p}\ \overline{\mathsf{R}} \leftarrow \mathsf{B}$ such that $\llbracket\mathsf{B}\rrbracket_{s[\overline{\mathsf{R}}/\overline{d}]}(I) = \mathit{true}$. Since I is a model, $\llbracket\mathsf{p}\ \overline{\mathsf{R}}\rrbracket_{s[\overline{\mathsf{R}}/\overline{d}]}(I) = \mathit{true}$. Then, $I(\mathsf{p})\ \overline{d} = \mathit{true}$, which is a contradiction.

In the other direction, suppose that I is a pre-fixpoint of T_{P}. Suppose s is a Herbrand state and $\mathsf{p}\ \mathsf{R}_1 \ldots \mathsf{R}_n \leftarrow \mathsf{B}$ is a rule such that $[\![\mathsf{B}]\!]_s(I) = \mathit{true}$. Then, $T_{\mathsf{P}}(I)(\mathsf{p})\ s(\mathsf{R}_1)\ldots s(\mathsf{R}_n) = \mathit{true}$. Since I is a pre-fixpoint of T_{P}, $I(\mathsf{p})\ s(\mathsf{R}_1)\ldots s(\mathsf{R}_n) = \mathit{true}$, which implies that $[\![\mathsf{p}\ \mathsf{R}_1 \ldots \mathsf{R}_n]\!]_s(I) = \mathit{true}$. $\square$

Proposition 5.1
For every predicate type π, $([\![\pi]\!]^*, \leq_\pi)$ is a complete lattice and $([\![\pi]\!]^*, \preceq_\pi)$ is a complete meet-semilattice (i.e., every non-empty subset of $[\![\pi]\!]^*$ has a $\preceq_\pi$-greatest lower bound).

Proof
We proceed by induction on the predicate type π.

If $\pi = o$, then obviously $([\![\pi]\!]^*, \leq_\pi)$ is a complete lattice and $([\![\pi]\!]^*, \preceq_\pi)$ is a complete meet-semilattice.

Let now $\pi = \rho \to \pi'$, and assume that $([\![\pi']\!]^*, \leq_\pi)$ is a complete lattice and $([\![\pi']\!]^*, \preceq_\pi)$ is a complete meet-semilattice. We show that $([\![\pi]\!]^*, \leq_\pi)$ is a complete lattice. The proof that $([\![\pi']\!]^*, \preceq_\pi)$ is a complete meet-semilattice is analogous and omitted. Let S be a subset of $[\![\rho]\!]_{U_{\mathsf{P}}} \to [\![\pi']\!]^*_{U_{\mathsf{P}}}$. By induction hypothesis, we can define $f_{\bigwedge S}, f_{\bigvee S} \in [\![\pi]\!]$ by $f_{\bigwedge S}(x) := \bigwedge\{g(x) \mid g \in S\}$ and $f_{\bigvee S}(x) := \bigvee\{g(x) \mid g \in S\}$, respectively. By the definition of $\leq_{\rho\to\pi'}$, it is immediate to see that $\bigwedge S = f_{\bigwedge S}$ and $\bigvee S = f_{\bigvee S}$. Hence, $[\![\pi]\!]$ is a complete lattice, as desired. $\square$

Lemma 5.1
For every argument type ρ and $d^* \in [\![\rho]\!]^*$, there exists $d \in [\![\rho]\!]$ such that $d^* \preceq_\rho d$.

Proof
For the case $\rho = \iota$, we have $[\![\iota]\!] = [\![\iota]\!]^*$, so that $d^* \in [\![\iota]\!]$ and $d^* \preceq_\iota d^*$. On the other hand, if $\rho = \pi$, ρ would be of the form $\rho_1 \to \ldots \to \rho_n \to o$. We define $d \in [\![\rho]\!]$ such that for all $x_1 \in [\![\rho_1]\!], \ldots, x_n \in [\![\rho_n]\!]$:

$$d(\overline{x}) = \begin{cases} d^*(\overline{x}), & \text{if } d^*(\overline{x}) \in \{\mathit{false}, \mathit{true}\} \\ \mathit{false}, & \text{otherwise} \end{cases}$$

It is easy to see that $d \preceq_\rho d^*$. $\square$

In order to establish Lemma 5.2, we need to first show an auxiliary one:

Lemma A.1
Let ρ be an argument type. Then, $(\preceq_\rho)$ restricted to $[\![\rho]\!]$ is the trivial ordering, i.e., for all $d, d' \in [\![\rho]\!]$, $d \preceq_\rho d'$ if and only if $d = d'$.

Proof
By induction on the argument type ρ. If $\rho = \iota$ the result follows from the definition of $\preceq_\iota$ over $[\![\iota]\!]^*$. If $\rho = o$, it can be established by case analysis. Suppose now that $\rho = \rho' \to \pi$ and the statement holds for π. Let $d, d' \in [\![\rho' \to \pi]\!]$ such that $d \preceq_{\rho'\to\pi} d'$. By the definition of $\preceq_{\rho'\to\pi}$, we have $d(x) \preceq_\pi d'(x)$ for all $x \in [\![\rho']\!]$. By the induction hypothesis, we have $d(x) = d'(x)$ for all $x \in [\![\rho']\!]$. Therefore, $d = d'$. $\square$

Lemma 5.2
Let P be a program, $I \in H_{\mathsf{P}}$ and $s \in S_{\mathsf{P}}$. Then, for every expression E, $[\![\mathsf{E}]\!]_s(I) = [\![\mathsf{E}]\!]^*_s(I)$.

Proof
By induction on the structure of E. The only non-trivial case is when E is of the form $(\mathsf{E}_1\ \mathsf{E}_2)$. By Lemma A.1, $\{[\![\mathsf{E}_1]\!]_s^*(I)(d) \mid d \in [\![\rho]\!], [\![\mathsf{E}_2]\!]_s^*(\mathcal{I}) \preceq_\rho d\} = \{[\![\mathsf{E}_1]\!]_s^*(I)([\![\mathsf{E}_2]\!]_s^*(\mathcal{I}))\}$. Therefore,

$$\begin{array}{rcll}[[\![(\mathsf{E}_1\ \mathsf{E}_2)]\!]_s^*(I) & = & [\![\mathsf{E}_1]\!]_s^*(I)([\![\mathsf{E}_2]\!]_s^*(\mathcal{I})) & \\ & = & [\![\mathsf{E}_1]\!]_s(I)([\![\mathsf{E}_2]\!]_s(\mathcal{I})) & \text{(Induction Hypothesis)} \\ & = & [\![(\mathsf{E}_1\ \mathsf{E}_2)]\!]_s(I) & \end{array}$$

This completes the proof of the lemma. □

B Proofs of Section 6

In this appendix, we collect the proofs of the results of Section 6. We start with Proposition 6.1, which concerns the existence, for every predicate type π, of an isomorphism, i.e., an order-preserving bijection, between the set $[\![\pi]\!]^*$ of three-valued meanings and the set $[\![\pi]\!]^c$ of pairs of two-valued ones. We first provide the definition of such functions, and then we prove they are indeed isomorphisms in Proposition 6.1.

Definition B.1
For every predicate type π, we define the functions $\tau_\pi : [\![\pi]\!]^* \to [\![\pi]\!]^c$ and $\tau_\pi^{-1} : [\![\pi]\!]^c \to [\![\pi]\!]^*$, as follows:

- $\tau_o(\mathit{false}) = (\mathit{false}, \mathit{false})$, $\tau_o(\mathit{true}) = (\mathit{true}, \mathit{true})$, $\tau_o(\mathit{undef}) = (\mathit{false}, \mathit{true})$
- $\tau_{\rho\to\pi}(f) = (\lambda d.[\tau_\pi(f(d))]_1, \lambda d.[\tau_\pi(f(d))]_2)$

and

- $\tau_o^{-1}(\mathit{false}, \mathit{false}) = \mathit{false}$, $\tau_o^{-1}(\mathit{true}, \mathit{true}) = \mathit{true}$, $\tau_o^{-1}(\mathit{false}, \mathit{true}) = \mathit{undef}$
- $\tau_{\rho\to\pi}^{-1}(f_1, f_2) = \lambda d.\tau_\pi^{-1}(f_1(d), f_2(d))$

The functions τ_π defined above, can easily be extended to a function between $\mathcal{H}_\mathsf{P}$ and H_P^c: given $\mathcal{I} \in \mathcal{H}_\mathsf{P}$, we define $\tau(\mathcal{I}) = (I, J)$, where for every predicate constant $\mathsf{p} : \pi$ it holds $I(\mathsf{p}) = [\tau_\pi(\mathcal{I}(\mathsf{p}))]_1$ and $J(\mathsf{p}) = [\tau_\pi(\mathcal{I}(\mathsf{p}))]_2$. Conversely, given a pair $(I, J) \in {H_\mathsf{P}}^c$, we define the three-valued Herbrand interpretation $\tau^{-1}(I, J)$, for every predicate constant $\mathsf{p} : \pi$, as follows: $\tau^{-1}(I, J)(\mathsf{p}) = \tau_\pi^{-1}(I(\mathsf{p}), J(\mathsf{p}))$.

Proposition 6.1
For every predicate type π there exists a bijection $\tau_\pi : [\![\pi]\!]^* \to [\![\pi]\!]^c$ with inverse $\tau_\pi^{-1} : [\![\pi]\!]^c \to [\![\pi]\!]^*$, that both preserve the orderings $\leq$ and $\preceq$ of elements between $[\![\pi]\!]^*$ and $[\![\pi]\!]^c$. Moreover, there exists a bijection $\tau : \mathcal{H}_\mathsf{P} \to H_\mathsf{P}^c$ with inverse $\tau^{-1} : H_\mathsf{P}^c \to \mathcal{H}_\mathsf{P}$, that both preserve the orderings $\leq$ and $\preceq$ between $\mathcal{H}_\mathsf{P}$ and H_P^c.

Proof
Consider the functions in Definition B.1. Let π be a predicate type. It follows easily from the definition that τ_π, τ_π^{-1} are well defined functions and a formal proof using induction on the type structure is omitted.

We show that they are also order-preserving, τ_π is a bijection and τ_π^{-1} is the inverse. Specifically, for every $f, g \in [\![\pi]\!]^*$ and for every $(f_1, f_2), (g_1, g_2) \in [\![\pi]\!]^c$, we show that the following statements hold:

1. If $f \preceq_\pi g$ then $\tau_\pi(f) \preceq_\pi \tau_\pi(g)$.

2. If $f \leq_\pi g$ then $\tau_\pi(f) \leq_\pi \tau_\pi(g)$.
3. If $(f_1, f_2) \preceq_\pi (g_1, g_2)$ then $\tau_\pi^{-1}(f_1, f_2) \preceq_\pi \tau_\pi^{-1}(g_1, g_2)$.
4. If $(f_1, f_2) \leq_\pi (g_1, g_2)$ then $\tau_\pi^{-1}(f_1, f_2) \leq_\pi \tau_\pi^{-1}(g_1, g_2)$.
5. $\tau_\pi^{-1}(\tau_\pi(f)) = f$
6. $\tau_\pi(\tau_\pi^{-1}(f_1, f_2)) = (f_1, f_2)$

We will use structural induction on the types. For the base types o and ι the proof is trivial for all the statements. Consider the general case of $\pi : \rho_1 \to \pi_2$.
Statement 1: Assume that $f \preceq g$. Then, for any $d \in [\![\rho_1]\!]$ it is $f(d) \preceq_{\pi_2} g(d)$ and by induction hypothesis $\tau_{\pi_2}(f(d)) \preceq_{\pi_2} \tau_{\pi_2}(g(d))$. Therefore, $[\tau_{\pi_2}(f(d))]_1 \leq_{\pi_2} [\tau_{\pi_2}(g(d))]_1$ and by abstracting $\lambda d.[\tau_{\pi_2}(f(d))]_1 \leq_{\pi_2} \lambda d.[\tau_{\pi_2}(g(d))]_1$. Similarly, it is $[\tau_{\pi_2}(g(d))]_2 \leq_{\pi_2} [\tau_{\pi_2}(f(d))]_2$ and so $\lambda d.[\tau_{\pi_2}(g(d))]_2 \leq_{\pi_2} \lambda d.[\tau_{\pi_2}(f(d))]_2$. We conclude that

$$\tau_\pi(f) = (\lambda d.[\tau_{\pi_2}(f(d))]_1, \lambda d.[\tau_{\pi_2}(f(d))]_2) \preceq_\pi (\lambda d.[\tau_{\pi_2}(g(d))]_1, \lambda d.[\tau_{\pi_2}(g(d))]_2) = \tau_\pi(g)$$

The proof of the second statement is analogous.
Statement 3: Assume that $(f_1, f_2) \preceq_\pi (g_1, g_2)$. Then, for any $d \in [\![\rho_1]\!]$ it is $f_1(d) \leq_{\pi_2} g_1(d)$ and $g_2(d) \leq_{\pi_2} f_2(d)$. therefore, $(f_1(d), f_2(d)) \preceq_\pi (g_1(d), g_2(d))$ and by induction it is $\tau_\pi^{-1}(f_1(d), f_2(d)) \preceq_\pi \tau_\pi^{-1}(g_1(d), g_2(d))$ which by abstracting gives $\lambda d.\tau_\pi^{-1}(f_1(d), f_2(d)) \preceq_\pi \lambda d.\tau_\pi^{-1}(g_1(d), g_2(d))$ therefore $\tau_\pi^{-1}(f_1, f_2) \preceq_\pi \tau_\pi^{-1}(g_1, g_2)$.
The proof of the fourth statement is analogous.
Statement 5: We have:

$$\begin{array}{lll}
 & \tau^{-1}_{\rho_1 \to \pi_2}(\tau_{\rho_1 \to \pi_2}(f)) & \\
= & \tau^{-1}_{\rho_1 \to \pi_2}(\lambda d.[\tau_{\pi_2}(f(d))]_1, \lambda d.[\tau_{\pi_2}(f(d))]_2) & \text{(Definition of } \tau_{\rho_1 \to \pi_2}) \\
= & \lambda d.\tau^{-1}_{\pi_2}([\tau_{\pi_2}(f(d))]_1, [\tau_{\pi_2}(f(d))]_2) & \text{(Definition of } \tau^{-1}_{\rho_1 \to \pi_2}) \\
= & \lambda d.\tau^{-1}_{\pi_2}(\tau_{\pi_2}(f(d))) & \text{(Definition of } [\cdot]_1 \text{ and } [\cdot]_2) \\
= & \lambda d.f(d) & \text{(Induction Hypothesis)} \\
= & f &
\end{array}$$

Statement 6: Similarly to the previous statement:

$$\begin{array}{lll}
 & \tau_{\rho_1 \to \pi_2}(\tau^{-1}_{\rho_1 \to \pi_2}(f_1, f_2)) & \\
= & \tau_{\rho_1 \to \pi_2}(\lambda d.\tau^{-1}_{\pi_2}(f_1(d), f_2(d))) & \text{(Definition of } \tau^{-1}_{\rho_1 \to \pi_2}) \\
= & (\lambda d.[\tau_{\pi_2}(\tau^{-1}_{\pi_2}(f_1(d), f_2(d)))]_1, \lambda d.[\tau_{\pi_2}(\tau^{-1}_{\pi_2}(f_1(d), f_2(d)))]_2) & \text{(Definition of } \tau_{\rho_1 \to \pi_2}) \\
= & (\lambda d.[(f_1(d), f_2(d))]_1, \lambda d.[(f_1(d), f_2(d))]_2) & \text{(Induction Hypothesis)} \\
= & (\lambda d.f_1(d), \lambda d.f_2(d)) & \text{(Definition of } [\cdot]_1 \text{ and } [\cdot]_2) \\
= & (f_1, f_2) &
\end{array}$$

The proof extends for the functions τ, τ^{-1} defined for interpretations. For example, for any interpretation $\mathcal{I} \in \mathcal{H}_{\mathsf{P}}$ and for any predicate constant p it follows easily from the previous result and the definitions of τ, τ^{-1} that $(\tau^{-1}(\tau(\mathcal{I})))(\mathsf{p}) = \mathcal{I}(\mathsf{p})$ and for any $(I, J) \in H^c_{\mathsf{P}}$ it is $\tau(\tau^{-1}(I, J))(\mathsf{p}) = (I(\mathsf{p}), J(\mathsf{p}))$. $\square$

We proceed now to the second result of Section 6, Lemma 6.1, which shows that the mapping A_{P} of Definition 6.2 is a *consistent approximator* of T_{P}, i.e. A_{P} is $\preceq$-monotonic and for every $I \in H_{\mathsf{P}}$, $A_{\mathsf{P}}(I, I) = (T_{\mathsf{P}}(I), T_{\mathsf{P}}(I))$. The term *consistent* comes from the terminology used by Denecker et al. (2004), and it refers to the fact that the mapping is defined over sets of consistent elements, i.e., of the form defined in Definition 6.1. A few intermediate lemmas are needed to ease the proof of Lemma 6.1.

Lemma B.1
Let π be a predicate type and $f \in [\![\pi]\!]$. Then $\tau_\pi^{-1}(f,f) = f$.

Proof
By induction on π. If $\pi = o$ it follows from the definition of τ_o^{-1}. When $\pi = \rho \to \pi'$ assuming that lemma holds for π', then $\tau_{\rho\to\pi'}^{-1}(f,f) = \lambda d.\tau_{\pi'}^{-1}(f(d), f(d)) = \lambda d.f(d) = f$. $\square$

Corollary B.1
Let $I \in H_{\mathsf{P}}$. Then $\tau^{-1}(I,I) = I$.

Lemma B.2
Let π be a predicate type and $f \in [\![\pi]\!]$. Then f is $\preceq_\pi$-maximal over $[\![\pi]\!]^*$.

Proof
By induction on π. If $\pi = o$ it follows from the definition of $\preceq_o$. When $\pi = \rho \to \pi'$ assuming that lemma holds for π', then for every $d \in [\![\rho]\!]$, $f(d) \in [\![\pi']\!]$, so that $f(d)$ is $\preceq_{\pi'}$-maximal over $[\![\pi']\!]^*$. We conclude that f is $\preceq_{\rho\to\pi'}$-maximal over $[\![\rho \to \pi']\!]^*$. $\square$

Lemma B.3
Let P be a program and π be a predicate type. Let I be a non-empty index-set and for any $i \in I$, $d_i, d'_i \in [\![\pi]\!]^*$. If for all $i \in I$, $d_i \preceq_\pi d'_i$, then $\bigvee_{\leq_\pi}\{d_i \mid i \in I\} \preceq_\pi \bigvee_{\leq_\pi}\{d'_i \mid i \in I\}$.

Proof
We proceed by induction on the predicate type π. If $\pi = o$ the lemma follows by case analysis of $\bigvee_{\leq_\pi}\{d_i \mid i \in I\}$. Suppose now that $\pi = \rho \to \pi'$ and the lemma holds for π'. By the proof of Proposition 5.1, we have that $\bigvee_{\leq_\pi}\{d_i \mid i \in I\} = \lambda x.\bigvee_{\leq_{\pi'}}\{d_i(x) \mid i \in I\}$ and $\bigvee_{\leq_\pi}\{d'_i \mid i \in I\} = \lambda x.\bigvee_{\leq_{\pi'}}\{d'_i(x) \mid i \in I\}$. For any $x \in [\![\rho]\!]$, by induction hypothesis, we have that $\bigvee_{\leq_{\pi'}}\{d_i(x) \mid i \in I\} \preceq_{\pi'} \bigvee_{\leq_{\pi'}}\{d'_i(x) \mid i \in I\}$. We conclude that $\bigvee_{\leq_\pi}\{d_i \mid i \in I\} \preceq_\pi \bigvee_{\leq_\pi}\{d'_i \mid i \in I\}$. $\square$

Lemma B.4
Let P be a program, let $\mathcal{I}, \mathcal{J} \in \mathcal{H}_{\mathsf{P}}$, and let s be a Herbrand state of P. For every expression $\mathsf{E} : \pi$, if $\mathcal{I} \preceq \mathcal{J}$ then $[\![\mathsf{E}]\!]_s^*(\mathcal{I}) \preceq_\pi [\![\mathsf{E}]\!]_s^*(\mathcal{J})$.

Proof
Using induction on E. The only interesting case is when $\mathsf{E} = (\mathsf{E}_1\ \mathsf{E}_2)$ where $\mathsf{E}_1 : \rho \to \pi$ and $\mathsf{E}_2 : \rho$. Suppose that when $\mathcal{I} \preceq \mathcal{J}$ then $[\![\mathsf{E}_1]\!]_s^*(\mathcal{I}) \preceq_{\rho\to\pi} [\![\mathsf{E}_1]\!]_s^*(\mathcal{J})$ and $[\![\mathsf{E}_2]\!]_s^*(\mathcal{I}) \preceq_\rho [\![\mathsf{E}_2]\!]_s^*(\mathcal{J})$. Suppose $x \in \{[\![\mathsf{E}_1]\!]_s^*(\mathcal{J})(d) \mid d \in [\![\rho]\!], [\![\mathsf{E}_2]\!]_s^*(\mathcal{J}) \preceq_\rho d\}$. Then, there exists some d such that $[\![\mathsf{E}_2]\!]_s^*(\mathcal{J}) \preceq_\rho d$ and $x = [\![\mathsf{E}_1]\!]_s^*(\mathcal{J})(d)$. Since $[\![\mathsf{E}_2]\!]_s^*(\mathcal{I}) \preceq_\rho [\![\mathsf{E}_2]\!]_s^*(\mathcal{J}) \preceq_\rho d$, we have $[\![\mathsf{E}_1]\!]_s^*(\mathcal{I})(d) \in \{[\![\mathsf{E}_1]\!]_s^*(\mathcal{I})(d) \mid d \in [\![\rho]\!], [\![\mathsf{E}_2]\!]_s^*(\mathcal{I}) \preceq_\rho d\}$. Also, by inductive hypothesis, $[\![\mathsf{E}_1]\!]_s^*(\mathcal{I})(d) \preceq_\pi x$, so that $\bigwedge_{\preceq_\pi}\{[\![\mathsf{E}_1]\!]_s^*(\mathcal{I})(d) \mid d \in [\![\rho]\!], [\![\mathsf{E}_2]\!]_s^*(\mathcal{I}) \preceq_\rho d\} \preceq_\pi x$. Since that holds for any x, we have $[\![\mathsf{E}]\!]_s^*(\mathcal{I}) = \bigwedge_{\preceq_\pi}\{[\![\mathsf{E}_1]\!]_s^*(\mathcal{I})(d) \mid d \in [\![\rho]\!], [\![\mathsf{E}_2]\!]_s^*(\mathcal{I}) \preceq_\rho d\} \preceq_\pi \bigwedge_{\preceq_\pi}\{[\![\mathsf{E}_1]\!]_s^*(\mathcal{J})(d) \mid d \in [\![\rho]\!], [\![\mathsf{E}_2]\!]_s^*(\mathcal{J}) \preceq_\rho d\} = [\![\mathsf{E}]\!]_s^*(\mathcal{J})$. $\square$

We are finally ready to prove Lemma 6.1.

Lemma 6.1
Let P be a program. In the terminology of Denecker et al. (2004), $A_{\mathsf{P}} : H^c_{\mathsf{P}} \to H^c_{\mathsf{P}}$ is a consistent approximator of T_{P}.

Proof
We have to show that A_{P} is $\preceq$-monotone and extends T_{P}. For the monotonicity, it follows from the definition of $\mathcal{T}_{\mathsf{P}}$ together with Lemma B.4 and Lemma B.3 that $\mathcal{T}_{\mathsf{P}}$ is $\preceq$-monotone. Also, by Proposition 6.1, τ and τ^{-1} preserve $\preceq$, so that A_{P} is $\preceq$-monotone.

Now, we have to show that A_{P} extends T_{P}, i.e., for every $I \in H_{\mathsf{P}}$, $A_{\mathsf{P}}(I, I) = (T_{\mathsf{P}}(I), T_{\mathsf{P}}(I))$. By Corollary B.1, $\tau^{-1}(I, I) = I$. Since $I \in H_{\mathsf{P}}$, by Lemma 5.2, we have that for every expression E, $[\![E]\!]_s(\tau^{-1}(I, I)) = [\![E]\!]^*_s(I)$. Now we have

$$\begin{aligned} A_{\mathsf{P}}(I,I) = {} & \tau(\mathcal{T}_{\mathsf{P}}(\tau^{-1}(I,I))) \\ = {} & \tau\big(\textstyle\bigvee_{\leq_o}\{[\![\mathsf{B}]\!]^*_{s[\overline{\mathsf{R}}/\overline{d}]}(\tau^{-1}(I,I)) \mid s \in S_{\mathsf{P}} \text{ and } (\mathsf{p}\ \overline{\mathsf{R}} \leftarrow \mathsf{B}) \text{ in } \mathsf{P}\}\big) \\ = {} & \tau\big(\textstyle\bigvee_{\leq_o}\{[\![\mathsf{B}]\!]_{s[\overline{\mathsf{R}}/\overline{d}]}(I) \mid s \in S_{\mathsf{P}} \text{ and } (\mathsf{p}\ \overline{\mathsf{R}} \leftarrow \mathsf{B}) \text{ in } \mathsf{P}\}\big) \\ = {} & \tau(T_{\mathsf{P}}(I)) \end{aligned}$$

Since $T_{\mathsf{P}}(I) \in H_{\mathsf{P}}$, by Corollary B.1, we have $\tau(T_{\mathsf{P}}(I)) = (T_{\mathsf{P}}(I), T_{\mathsf{P}}(I))$. $\square$

We conclude this appendix by showing that Definition 6.3 and Definition 5.4 for three-valued models agree.

Lemma 6.2
Let P be a program and $(I, J) \in H^c_{\mathsf{P}}$. Then, (I, J) is a pre-fixpoint of A_{P} if and only if $\tau^{-1}(I, J)$ is a three-valued model of P.

Proof
First notice that by Definition 5.5 and Proposition 6.1, (I, J) is a pre-fixpoint of A_{P} if and only if $A_{\mathsf{P}}(I, J) = \tau(\mathcal{T}_{\mathsf{P}}(\tau^{-1}(I, J))) \leq (I, J)$ if and only if $\mathcal{T}_{\mathsf{P}}(\tau^{-1}(I, J)) \leq \tau^{-1}(I, J)$, i.e. (I, J) is a pre-fixpoint of A_{P} if and only if $\tau^{-1}(I, J)$ is a fixpoint of $\mathcal{T}_{\mathsf{P}}$. We conclude by Proposition 5.2. $\square$

C Proofs of Section 7

The following theorem states that our stable model semantics coincides with the classical stable model semantics for the class of propositional programs.

Theorem 7.1
Let P be a propositional logic program. Then, $\mathcal{M}$ is a (three-valued) stable model of P iff $\mathcal{M}$ is a classical (three-valued) stable model of P.

Proof
In (Denecker et al. 2004, Section 6, pages 107–108), the well-founded semantics of propositional logic programs is derived. The language used there allows arbitrary nesting of conjunction, disjunction and negation in bodies of the rules which fully encompasses our syntax when we restrict our programs to be propositional. In addition, the immediate consequence operator T_{P} is the same and so is the approximation space which we have denoted as H^c_{P} in this work.

It is easy to see that the approximator A_{P} we give in our approach and the one given in Denecker et al. (2004) fully coincide for propositional programs therefore produce equivalent semantics. Notice how our three-valued operator $\mathcal{T}_{\mathsf{P}}$ fully coincides with the three-valued immediate consequence operator in Denecker et al. (2004) since the fourth rule in Definition 5.3 is never used. $\square$

In order to establish Theorem 7.2, we use the following proposition which is a restatement of Proposition 3.14 found in Denecker et al. (2004) that refers to pre-fixpoints of the approximator A_{P} instead of fixpoints. The proof is almost identical but is presented here, nonetheless, for reasons of completeness.

Proposition C.1
A stable fixpoint (x, y) of A_{P} is a $\leq$-minimal pre-fixpoint of A_{P}. Furthermore, if (x, x) is a stable fixpoint of A_{P} then x is a minimal pre-fixpoint of T_{P}.

Proof
Let (x, y) be a stable fixpoint of A_{P} and let (x', y') such that $(x', y') \leq (x, y)$ and (x', y') is a pre-fixpoint of A_{P}, so $A_{\mathsf{P}}(x', y') \leq (x', y')$. We have that $x' \leq y' \leq y$ which gives us that $A_{\mathsf{P}}(x', y)_1 \leq A_{\mathsf{P}}(x', y')_1 \leq (x', y')_1 = x'$. Therefore, x' is a pre-fixpoint of the operator $A_{\mathsf{P}}(\cdot, y)_1$ and since x is its least fixpoint we get that $x \leq x'$. By the assumption that $x' \leq x$ we conclude that $x = x'$.

Since we have shown that $x = x'$ we have that $x = x' \leq y'$ and $A_{\mathsf{P}}(x, y')_2 \leq A_{\mathsf{P}}(x', y')_2 \leq (x', y')_2 = y'$ which makes y' a pre-fixpoint of $A_{\mathsf{P}}(x, \cdot)_2$. Since y is its least fixpoint we have that $y \leq y'$ and by assumption $y' \leq y$. We conclude that $y = y'$ and finally $(x, y) = (x', y')$.

Assume that $x' \leq x$ and x' is a pre-fixpoint of T_{P} therefore $T_{\mathsf{P}}(x') \leq x'$. Since A_{P} is an approximator of T_{P} we have that $A_{\mathsf{P}}(x', x') = (T_{\mathsf{P}}(x'), T_{\mathsf{P}}(x')) \leq (x', x')$. But then (x', x') is a pre-fixpoint of A_{P} and $(x', x') \leq (x, x)$. By the result of the previous paragraph we conclude that $(x', x') = (x, x)$ and $x' = x$. $\square$

Theorem 7.2
All (three-valued) stable models of a $\mathcal{HOL}$ program P are $\leq$-minimal models of P.

Proof
Let $\mathcal{M}$ be a three-valued stable model of P and $\mathcal{M}'$ a three-valued model of P, such that $\mathcal{M}' \leq \mathcal{M}$ which also implies $\tau(\mathcal{M}') \leq \tau(\mathcal{M})$. By Lemma 6.2, $\tau(\mathcal{M}')$ is a pre-fixpoint of A_{P}. But $\tau(\mathcal{M})$ is a stable fixpoint of A_{P} so by Proposition C.1 it is a minimal pre-fixpoint of A_{P}. We conclude that $\tau(\mathcal{M}') = \tau(\mathcal{M})$ and so $\mathcal{M}' = \mathcal{M}$. Since every two-valued model is a three-valued model it follows that that every stable model is also a $\leq$-minimal two-valued model. $\square$

Theorem 7.3
Let P be a $\mathcal{HOL}$ program. If the well-founded model of P is two-valued, then this is also its unique stable model.

Proof
Let $\mathcal{M}$ be the well founded model of P. Then $\tau(\mathcal{M})$ is the $\preceq$-least three-valued stable model. It immediately follows that since $\mathcal{M}$ is two-valued, by Corollary B.1 it is $\tau(\mathcal{M}) = (\mathcal{M}, \mathcal{M})$. Then for any (x, y) three-valued stable fixpoint of A_{P} it is $(\mathcal{M}, \mathcal{M}) \preceq (x, y)$. Since it also must hold $x \leq y$ we conclude that $x = y = \mathcal{M}$ and $\tau^{-1}(x, y) = \tau^{-1}(\mathcal{M}, \mathcal{M}) = \mathcal{M}$. $\square$

In order to establish Theorem 7.4, we first prove some auxiliary results.

Lemma C.1
Let P be a stratified $\mathcal{HOL}$ program and E be an expression. Let $I, J \in H_{\mathsf{P}}$ be two interpretations such that $I(\mathsf{p}) = J(\mathsf{p})$ for every predicate constant p occurring in E. Then, for every state $s \in S_{\mathsf{P}}$, $[\![\mathsf{E}]\!]_s(I) = [\![\mathsf{E}]\!]_s(J)$.

Proof
Trivial using induction on the structure of E. $\Box$

Corollary C.1
Let S be a stratification function of the $\mathcal{HOL}$ program P and $I, J \in H_{\mathsf{P}}$ be two interpretations. If for some $n \in \omega$, $I(\mathsf{p}) = J(\mathsf{p})$ for every predicate constant p with $S(\mathsf{p}) < n$, then $T_{\mathsf{P}}(I)(\mathsf{p}) = T_{\mathsf{P}}(J)(\mathsf{p})$ for every predicate constant p with $S(\mathsf{p}) < n$.

Lemma C.2
Let S be a stratification function of the $\mathcal{HOL}$ program P and $(I, J) \in H^c_{\mathsf{P}}$. If for some $n \in \omega$, $I(\mathsf{p}) = J(\mathsf{p})$ for every predicate constant p with $S(\mathsf{p}) < n$, then $A_{\mathsf{P}}(I, J)(\mathsf{p}) = (T_{\mathsf{P}}(I)(\mathsf{p}), T_{\mathsf{P}}(J)(\mathsf{p}))$ for every predicate constant p with $S(\mathsf{p}) \leq n$.

Proof
We will show the following auxiliary statement that suffices to show the lemma. For any expression $\mathsf{E} : \pi$ such that the following three statements hold:

1. $S(\mathsf{q}) \leq n$ for every predicate constant q occurring in E,
2. if E is of the form $(\mathsf{E}_1\ \mathsf{E}_2)$, then $S(\mathsf{q}) < n$ for every predicate constant q occurring in E_2,
3. if E is of the form $(\sim \mathsf{E}_1)$, then $S(\mathsf{q}) < n$ for every predicate constant q occurring in E_1,

and for any Herbrand state $s \in S_{\mathsf{P}}$ it follows that $[\![\mathsf{E}]\!]^*_s(\tau^{-1}(I, J)) = \tau^{-1}_{\pi}([\![\mathsf{E}]\!]_s(I), [\![\mathsf{E}]\!]_s(J))$. This can be established using induction on the structure of E. The interesting cases are when E is of the form $(\mathsf{E}_1\ \mathsf{E}_2)$ or of the form $(\sim \mathsf{E}_3)$. Suppose that $\mathsf{E} : \pi$ is of the form $(\mathsf{E}_1\ \mathsf{E}_2)$ where $\mathsf{E}_1 : \rho \to \pi$ and $\mathsf{E}_2 : \rho$ and suppose the statement holds for E_1 and E_2. By Lemma C.1, $[\![\mathsf{E}_2]\!]_s(I) = [\![\mathsf{E}_2]\!]_s(J)$. So, by the induction hypothesis and Lemma B.1, $[\![\mathsf{E}_2]\!]^*_s(\tau^{-1}(I, J)) = [\![\mathsf{E}_2]\!]_s(I)$ and therefore $[\![\mathsf{E}_2]\!]^*_s(\tau^{-1}(I, J)) \in [\![\rho]\!]$. By Lemma B.2, $\bigwedge_{\preceq_\pi}\{[\![\mathsf{E}_1]\!]^*_s(\tau^{-1}(I, J))(d) \mid d \in [\![\rho]\!], [\![\mathsf{E}_2]\!]^*_s(\tau^{-1}(I, J)) \preceq_\rho d\} = [\![\mathsf{E}_1]\!]^*_s(\tau^{-1}(I, J))([\![\mathsf{E}_2]\!]^*_s(\tau^{-1}(I, J))) = [\![\mathsf{E}_1]\!]^*_s(\tau^{-1}(I, J))([\![\mathsf{E}_2]\!]_s(I))$. Therefore,

$$
\begin{array}{lll}
 & [\![\mathsf{E}]\!]^*_s(\tau^{-1}(I, J)) & \\
= & [\![\mathsf{E}_1]\!]^*_s(\tau^{-1}(I, J))([\![\mathsf{E}_2]\!]_s(I)) & \\
= & \tau^{-1}_{\rho\to\pi}([\![\mathsf{E}_1]\!]_s(I), [\![\mathsf{E}_1]\!]_s(J))([\![\mathsf{E}_2]\!]_s(I)) & \text{(Induction Hypothesis)} \\
= & (\lambda d.\tau^{-1}_{\pi}([\![\mathsf{E}_1]\!]_s(I)(d), [\![\mathsf{E}_1]\!]_s(J)(d)))([\![\mathsf{E}_2]\!]_s(I)) & \text{(Definition of } \tau^{-1}_{\rho\to\pi}) \\
= & \tau^{-1}_{\pi}([\![\mathsf{E}_1]\!]_s(I)([\![\mathsf{E}_2]\!]_s(I)), [\![\mathsf{E}_1]\!]_s(J)([\![\mathsf{E}_2]\!]_s(I))) & \\
= & \tau^{-1}_{\pi}([\![\mathsf{E}_1]\!]_s(I)([\![\mathsf{E}_2]\!]_s(I)), [\![\mathsf{E}_1]\!]_s(J)([\![\mathsf{E}_2]\!]_s(J))) & \text{(Since } [\![\mathsf{E}_2]\!]_s(I) = [\![\mathsf{E}_2]\!]_s(J)) \\
= & \tau^{-1}_{\pi}([\![\mathsf{E}]\!]_s(I), [\![\mathsf{E}]\!]_s(J)) &
\end{array}
$$

Now, suppose that $\mathsf{E} : o$ is of the form $(\sim \mathsf{E}_3)$ where $\mathsf{E}_3 : o$. By Lemma C.1, $[\![\mathsf{E}_3]\!]_s(I) = [\![\mathsf{E}_3]\!]_s(J)$. So, by Lemma B.1, $[\![\mathsf{E}_3]\!]^*_s(\tau^{-1}(I, J)) = [\![\mathsf{E}_3]\!]_s(I)$. Since E_3 is of type o, $[\![\mathsf{E}_3]\!]^*_s(\tau^{-1}(I,J))$ can be either *true* or *false*. In any of the two cases, it is easy to show that $[\![\mathsf{E}]\!]^*_s(\tau^{-1}(I,J)) = \tau_o^{-1}([\![\mathsf{E}]\!]_s(I), [\![\mathsf{E}]\!]_s(J))$. $\square$

Theorem 7.4
Let P be a stratified $\mathcal{HOL}$ program. Then, the well-founded model of P is two-valued.

Proof
Let S be a stratification function of P and $(I_w, J_w) \in H^c_{\mathsf{P}}$ be the well-founded model of P. Suppose, for the sake of contradiction, that $I_w \neq J_w$. Let n be the least number such that there exists some predicate constant $\mathsf{p} : \pi_1$ with $S(\mathsf{p}) = n$ and $I_w(\mathsf{p}) \neq J_w(\mathsf{p})$. We define an interpretation J such that for any predicate constant q:

$$J(\mathsf{q}) = \begin{cases} I_w(\mathsf{q}), & \text{if } S(\mathsf{q}) \leq n \\ J_w(\mathsf{q}), & \text{if } S(\mathsf{q}) > n \end{cases}$$

It is obvious, by definition, that $I_w \leq J \leq J_w$ and therefore $(I_w, J) \in H^c_{\mathsf{P}}$. For any predicate constant q with $S(\mathsf{q}) \leq n$, we have

$$\begin{array}{rcll} [A_{\mathsf{P}}(I_w, J)]_2(\mathsf{q}) & = & T_{\mathsf{P}}(J)(\mathsf{q}) & \text{(Lemma C.2)} \\ & = & T_{\mathsf{P}}(I_w)(\mathsf{q}) & \text{(Corollary C.1)} \\ & = & [A_{\mathsf{P}}(I_w, J_w)]_1(\mathsf{q}) & \text{(Lemma C.2)} \\ & = & I_w(\mathsf{q}) & ((I_w, J_w) \text{ is a fixpoint of } A_{\mathsf{P}}) \\ & = & J(\mathsf{q}) & \text{(Definition of } J) \end{array}$$

Since $[A_{\mathsf{P}}(I_w, \cdot)]_2$ is monotone and $J \leq J_w$, it follows $[A_{\mathsf{P}}(I_w, J)]_2 \leq [A_{\mathsf{P}}(I_w, J_w)]_2 = J_w$. Thus, for any predicate constant $\mathsf{q} : \pi_2$ with $S(\mathsf{q}) > n$, we have $[A_{\mathsf{P}}(I_w, J)]_2(\mathsf{q}) \leq_{\pi_2} J_w(\mathsf{q}) = J(\mathsf{q})$. Since $[A_{\mathsf{P}}(I_w, J)]_2(\mathsf{q}) \leq_{\pi_2} J(\mathsf{q})$ for any predicate constant $\mathsf{q} : \pi_2$, we have $[A_{\mathsf{P}}(I_w, J)]_2 \leq J$, or J is a pre-fixpoint of $[A_{\mathsf{P}}(I_w, \cdot)]_2$. Since (I_w, J_w) is a stable fixpoint of A_{P}, J_w is the least pre-fixpoint of $[A_{\mathsf{P}}(I_w, \cdot)]_2$. Therefore, we have $J_w \leq J$. So, $J_w(\mathsf{p}) \leq_{\pi_1} J(\mathsf{p}) = I_w(\mathsf{p})$. But, we have $I_w \leq J_w$, so that $I_w(\mathsf{p}) = J_w(\mathsf{p})$, which is a contradiction. We conclude that $I_w = J_w$. Using Corollary B.1, $\tau^{-1}(I_w, J_w) = I_w$, so that $\tau^{-1}(I_w, J_w) \in H_{\mathsf{P}}$. $\square$

References

Denecker, M., Marek, V. W., and Truszczynski, M. 2004. Ultimate approximation and its application in nonmonotonic knowledge representation systems. *Inf. Comput.*, *192*, 1, 84–121.