# Online Appendix for

# Optimal Inflation Target with

# Expectations-Driven Liquidity Traps

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This technical appendix is organized as follows:

- Section A describes the details of the stylized model.
- Section B describes the conditions under which the equilibrium exists for the model with a demand shock and/or sunspot shock.
- Section C presents some analytical results on the effect of a higher inflation target in a semi-loglinear New Keynesian model.
- Section D discusses the effect of a higher inflation target in the nonlinear model with alternative values of $\alpha$.

## A Details of the Stylized Model

This section describes a stylized DSGE model with a representative household, a final good producer, a continuum of intermediate goods producers with unit measure, and government policies.

## A.1 Household

The representative household chooses its consumption level, amount of labor, and bond holdings so as to maximize the expected discounted sum of utility in future periods. As is common in the literature, the household enjoys consumption and dislikes labor. Assuming that period utility is separable, the household problem can be defined by

$$\max_{C_t,N_t,B_t} \mathrm{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \Big[\prod_{s=0}^{t-1} \delta_s\Big] \Big[\frac{C_t^{1-\chi_c}}{1-\chi_c} - \frac{N_t^{1+\chi_n}}{1+\chi_n}\Big], \tag{A1}$$

subject to the budget constraint

$$P_t C_t + R_t^{-1} B_t \leq W_t N_t + B_{t-1} + P_t \Phi_t, \tag{A2}$$

or equivalently

$$C_t + \frac{B_t}{R_t P_t} \leq w_t N_t + \frac{B_{t-1}}{P_t} + \Phi_t, \tag{A3}$$

where $C_t$ is consumption, $N_t$ is the labor supply, $P_t$ is the price of the consumption good, $W_t$ ($w_t$) is the nominal (real) wage, $\Phi_t$ is the profit share (dividends) of the household from the intermediate goods producers, $B_t$ is a one-period risk free bond that pays one unit of money at period $t+1$, and $R_t^{-1}$ is the price of the bond.

The discount rate at time $t$ is given by $\beta\delta_t$ where $\delta_t$ is the discount factor shock altering the weight of future utility at time $t+1$ relative to the period utility at time $t$. This increase in $\delta_t$ is a preference imposed by the household to increase the relative valuation of future utility flows, resulting in decreased consumption today (when considered in the absence of changes in the nominal interest rate).

## A.2 Firms

There is a final good producer and a continuum of intermediate goods producers indexed by $i \in [0,1]$. The final good producer purchases the intermediate goods $Y_{i,t}$ at the intermediate price $P_{i,t}$ and aggregates them using CES technology to produce and sell the final good $Y_t$ to the household and government at price $P_t$. Its problem is then summarized as

$$\max_{Y_{i,t}, i\in[0,1]} P_t Y_t - \int_0^1 P_{i,t} Y_{i,t} di, \tag{A4}$$

subject to the CES production function

$$Y_t = \Big[\int_0^1 Y_{i,t}^{\frac{\theta-1}{\theta}} di\Big]^{\frac{\theta}{\theta-1}}. \tag{A5}$$

Intermediate goods producers use labor to produce the imperfectly substitutable intermediate goods according to a linear production function ($Y_{i,t} = N_{i,t}$) and then sell the product to the final good producer. Each firm maximizes its expected discounted sum of future profits[1] by setting the price of its own good. We can assume that each firm receives a production subsidy $\tau$ so that the economy is fully efficient in the steady state.[2] In our baseline, we set $\tau = 1/\theta$. Price changes are subject to quadratic adjustment costs.

$$\max_{P_{i,t}} \mathrm{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \Big[\prod_{s=0}^{t-1} \delta_s\Big] \lambda_t \Big[P_{i,t} Y_{i,t} - (1-\tau) W_t N_{i,t} - P_t \frac{\varphi}{2} [\frac{P_{i,t}}{P_{i,t-1}(\Pi^{targ})^{\alpha}} - 1]^2 Y_t\Big], \tag{A6}$$

such that

$$Y_{i,t} = \Big[\frac{P_{i,t}}{P_t}\Big]^{-\theta} Y_t. \tag{A7}$$

[1]Each period, as it is written below, is in *nominal* terms. However, we want each period's profits in *real* terms so the profits in each period must be divided by that period's price level $P_t$ which we take care of further along in the document.

[2]$(\theta - 1) = (1-\tau)\theta$ which implies zero profits in the zero inflation steady state. In a welfare analysis, this would extract any inflation bias from the second-order approximated objective welfare function. $\tau$ therefore represents the size of a steady state distortion (see Chapter 5 Appendix, Galí (2008)).

[3] $\lambda_t$ is the Lagrange multiplier on the household's budget constraint at time $t$ and $\beta^{t-1}\left[\prod_{s=0}^{t-1}\delta_s\right]\lambda_t$ is the marginal value of an additional profit to the household. The positive time zero price is the same across firms (i.e. $P_{i,0} = P_0 > 0$).

### A.3 Monetary Policy

It is assumed that the central bank determines the short term nominal interest rate according to a Taylor rule

$$R_t = \max\left[1, \quad \frac{\Pi^{targ}}{\beta\delta_t}\left(\frac{\Pi_t}{\Pi^{targ}}\right)^{\phi_\pi}\right], \tag{A8}$$

where $\Pi_t = \frac{P_t}{P_{t-1}}$.

### A.4 Market clearing conditions

The market clearing conditions for the final good, labor, and government bond are given by

$$Y_t = C_t + \int_0^1 \frac{\varphi}{2}\left[\frac{P_{i,t}}{P_{i,t-1}(\Pi^{targ})^\alpha} - 1\right]^2 Y_t di, \tag{A9}$$

$$N_t = \int_0^1 N_{i,t} di, \tag{A10}$$

and

$$B_t = 0. \tag{A11}$$

### A.5 Recursive equilibrium

Given $P_0$ and a two-state Markov shock process establishing $\delta_t$, an equilibrium consists of allocations $\{C_t, N_t, N_{i,t}, Y_t, Y_{i,t}\}_{t=1}^{\infty}$, prices $\{W_t, P_t, P_{i,t}\}_{t=1}^{\infty}$, and a policy instrument $\{R_t\}_{t=1}^{\infty}$

[3] This expression is derived from the profit maximizing input demand schedule when solving for the final good producer's problem above. Plugging this expression back into the CES production function implies that the final good producer will set the price of the final good $P_t = \left[\int_0^1 P_{i,t}^{1-\theta} di\right]^{\frac{1}{1-\theta}}$.

such that (i) given the determined prices and policies, allocations solve the problem of the household, (ii) $P_{i,t}$ solves the problem of firm $i$, (iii) $R_t$ follows a specified rule, and (iv) all markets clear.

Combining all of the results from (i)-(iv), a symmetric equilibrium can be characterized recursively by $\{C_t, N_t, Y_t, w_t, \Pi_t, R_t\}_{t=1}^{\infty}$ satisfying the following equilibrium conditions:

$$C_t^{-\chi_c} = \beta\delta_t R_t \mathrm{E_t} C_{t+1}^{-\chi_c} \Pi_{t+1}^{-1}, \tag{A12}$$

$$w_t = N_t^{\chi_n} C_t^{\chi_c}, \tag{A13}$$

$$\begin{aligned} &\frac{Y_t}{C_t^{\chi_c}}\left[\varphi\left(\frac{\Pi_t}{(\Pi^{targ})^{\alpha}}-1\right)\frac{\Pi_t}{(\Pi^{targ})^{\alpha}}-(1-\theta)-(1-\tau)\theta w_t\right] \\ &\qquad = \beta\delta_t \mathrm{E_t}\frac{Y_{t+1}}{C_{t+1}^{\chi_c}}\varphi\left(\frac{\Pi_{t+1}}{(\Pi^{targ})^{\alpha}}-1\right)\frac{\Pi_{t+1}}{(\Pi^{targ})^{\alpha}}, \end{aligned} \tag{A14}$$

$$Y_t = C_t + \frac{\varphi}{2}\left[\frac{\Pi_t}{(\Pi^{targ})^{\alpha}}-1\right]^2 Y_t, \tag{A15}$$

$$Y_t = N_t, \tag{A16}$$

$$R_t = \max\left[1,\quad \frac{\Pi^{targ}}{\beta\delta_t}\left(\frac{\Pi_t}{\Pi^{targ}}\right)^{\phi_\pi}\right]. \tag{A17}$$

Equation (A12) is the consumption Euler equation, Equation (A13) is the intratemporal optimality condition of the household, Equation (A14) is the optimal condition of the intermediate good producing firms (forward-looking Phillips curve) relating today's inflation to real marginal cost today and expected inflation tomorrow, Equation (A15) is the aggregate resource constraint capturing the resource cost of price adjustment, and Equation (A16) is the aggregate production function. Equation (A17) is the interest-rate feedback rule.

# B On the existence and multiplicity of a sunspot equilibrium

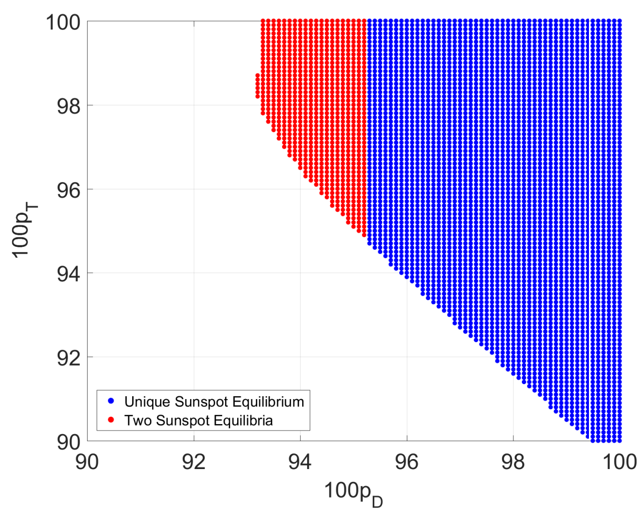
The left panel of Figure A1 shows pairs of $p_N$ and $p_C$ that are consistent with the equilibrium existence in the model with a demand shock only, whereas the right panel shows pairs of $p_T$ and $p_D$ that are consistent with the equilibrium existence in the model with a sunspot shock only.

Figure A1: Transition probabilities and the equilibrium existence


Consistent with the analytical result based on a semi-loglinear New Keynesian model in Nakata (2019), in the model with a demand shock only, the equilibrium exists if and only if $p_N$ is sufficiently high and $p_C$ is sufficiently low. When $p_N$ is low, there is a high probability of moving to the crisis state in the next period when the economy is currently in the normal state. The anticipation effect of moving to the crisis state in the next period causes the policy rate to decline. When $p_N$ is sufficiently low, the policy rate in the normal state becomes negative, violating the conditions for the equilibrium existence. When $p_C$ is high, there is a high probability of staying in the crisis state in the next period when the economy is currently in the crisis state. When $p_C$ is too high, inflation and output consistent with the private sector equilibrium conditions are positive, which implies a positive policy rate and thus inconsistent with the equilibrium existence.[4]

In the model with a sunspot shock only, the equilibrium exists if and only if $p_T$ and $p_D$ are both sufficiently high (see Nakata and Schmidt (2022) for a proof of this statement in a semi-loglinear model). The reason for why $p_T$ needs to be high for the equilibrium to exist in this model is the same as the reason for why $p_N$ needs to be high for the equilibrium to exist in the model with a demand shock only. Why does $p_D$ have to be sufficiently high for the equilibrium to exist? When $p_D$ is low, there is a low probability of staying in the deflationary state in the next period when the economy is currently in the target state. When $p_D$ is too low, inflation and output consistent with the private sector equilibrium conditions are positive, which implies a positive policy rate and thus inconsistent with the equilibrium existence.[5]

Figure A2: Transition probabilities and the sunspot equilibria multiplicity



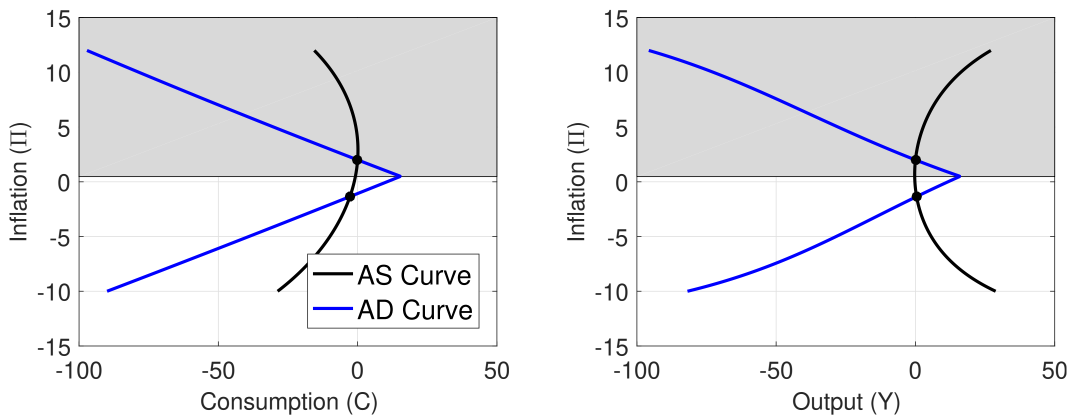

Consistent with Boneva, Braun, and Waki (2016) and as shown by the red dots in Figure A2, there are two pairs of inflation and the output gap in the deflationary regime satisfying the equilibrium conditions when $p_D$ is sufficiently small, but large enough so that an

[4]A version of this equilibrium existence result with $p_N = 1$ is well known in the literature. See, for example, Eggertsson (2011) and Boneva, Braun, and Waki (2016).

[5]A version of this equilibrium existence result with $p_T = 1$ is well known in the literature. See, for example, Mertens and Ravn (2014) and Boneva, Braun, and Waki (2016).

equilibrium exists. To illustrate the multiplicity of the sunspot equilibrium, Figure A3 and A4 show the AD and AS curves in the deflationary regime when $p_D = 0.975$ and $p_D = 0.95$, respectively. Focusing on the part of the AD curve consistent with the binding ELB constraint, there is only one intersection of the AS and AD curves when $p_D = 0.975$, as shown by Figure A3. When $p_D = 0.95$, there are two intersections of the AS and AD curves, as shown by Figure A4. One intersection features moderate declines in inflation and consumption, as well as a moderate increase in output (output increases due to price-adjustment costs associated with the mild decline in inflation). The other intersection features very large declines in inflation and consumption, as well as a very large increase in output.

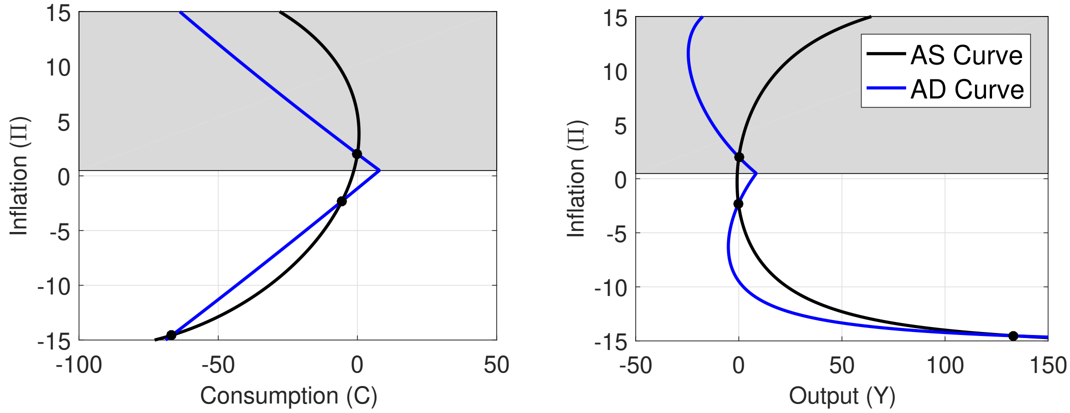
Figure A3: AD and AS Curves in the Deflationary Regime
—High $p_D$—




Note: Grey shades indicate the region in which the ELB is not binding.

When there are more than one sunspot equilibrium, we focus on the equilibrium with moderate declines in inflation and consumption in the deflationary regime for two reasons. First, the allocation in this equilibrium changes continuously with a change in $p_D$ even at the threshold value of $p_D$ above which there is one sunspot equilibrium and below which there are two sunspot equilibrium. In contrast, the other sunspot equilibrium with very large declines in inflation and consumption "suddenly" shows up when $p_D$ declines below the threshold. Second, we would like our deflationary regime to look like the Japanese economy over the past two decades, which features mild deflation.

Figure A4: AD and AS Curves in the Deflationary Regime
—Low $p_D$—

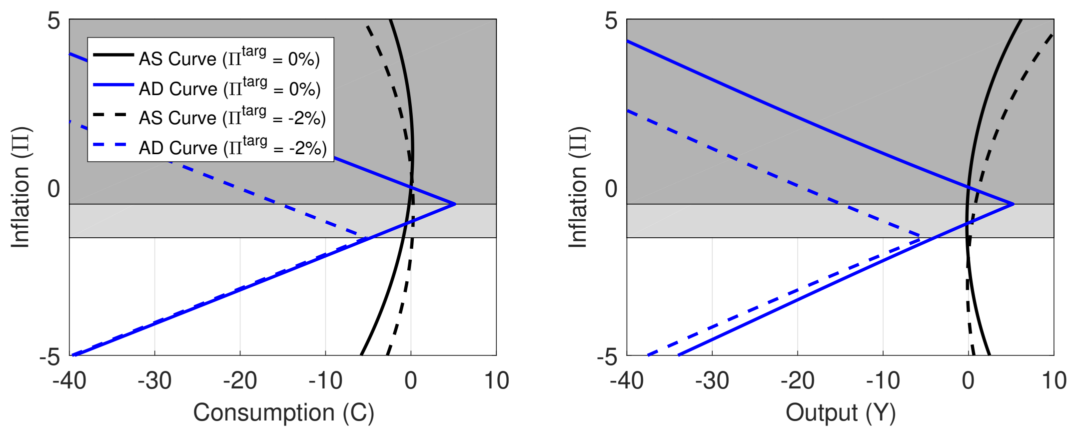



Note: Grey shades indicate the region in which the ELB is not binding.

## B.1 Equilibrium existence and the inflation target

As noted in Section 3 of the main text, the sunspot equilibrium ceases to exist when the inflation target is sufficiently low. To see this result, Figure A5 shows the AS and AD curves when the inflation target is 0 percent and -2 percent. According to the figure, the AD curve shifts to the left and the AS curve shifts to the right when the inflation target declines. With the inflation target of -2 percent, there is no intersection for the region of inflation rate consistent with the binding ELB constraint.

Figure A5: AD and AS Curves in the Deflationary Regime
—Low $\Pi^{targ}$—




Note: Grey shades indicate the region in which the ELB is not binding.

# C Some analytical results from a semi-loglinear model

In this section, we will investigate the effect of raising the inflation target in a semi-loglinear model that allows for analytical results.[6]

## C.1 Model

$$y_t = \mathbb{E}_t\{y_{t+1}\} - \sigma\left[i_t - \mathbb{E}_t\{\hat{\pi}_{t+1}\} - (r^* + \pi^*) + \delta_t\right], \tag{A18}$$

$$\hat{\pi}_t = \kappa y_t + \beta \mathbb{E}_t\{\hat{\pi}_{t+1}\}, \tag{A19}$$

$$\hat{\pi}_t = \pi_t - \pi^*, \tag{A20}$$

$$i_t = \max\left[0, (r^* + \pi^*) + \phi_\pi \hat{\pi}_t\right]. \tag{A21}$$

Here, equation (A18) is the log-linearized consumption Euler Equation, equation (A19) is the forward looking Phillips curve, and equation (A21) is the interest rate feedback rule with an effective lower bound (ELB) constraint. Notice that $\hat{\pi}_t$ is the inflation gap—deviation of the inflation rate ($\pi_t$) from the inflation target ($\pi^*$). $y_t$ is the deviation of output from its deterministic steady state level. Note that, in deriving these semi-loglinear equilibrium conditions, we assume that the price "indexation" parameter in the original fully nonlinear model is unity so that the level of the inflation target does not affect the deterministic steady state level of output.

As in the main body of the paper, we will first discuss the effect of an increase in the inflation target parameter in the version of the model with a crisis shock only, and then move on to the analysis in the version of the model with a sunspot shock only.

[6]See Bilbiie (2022) for a related analysis. He analytically shows that a *temporary* increase in inflation in the target regime lowers inflation and output in a deflationary regime in a similar setup. We analytically show that a permanent increase in inflation in the target regime lowers inflation and output in a deflationary regime.

## C.2 Model with a crisis shock only

As in the main text, we assume that the demand shock, $\delta_t$, follows a two-state Markov shock process. $Prob(N|N) := p_N$ is the probability of staying in the normal state in the next period when the economy is in the normal state today. $Prob(C|C) := p_C$ is the probability of staying in the crisis state in the next period when the economy is in the crisis state today.

The equilibrium conditions of the economy with a crisis shock only are given by

$$y_N = p_N y_N + (1-p_N)y_C + \sigma\left[p_N\pi_N + (1-p_N)\pi_C - \pi^*\right] - \sigma\left[i_N - (r^* + \pi^*) + \delta_N\right], \tag{A22}$$

$$\pi_N - \pi^* = \kappa y_N + \beta\left(p_N\pi_N + (1-p_N)\pi_C - \pi^*\right), \tag{A23}$$

$$i_N = r^* + \pi^* + \phi_\pi(\pi_N - \pi^*), \tag{A24}$$

$$y_C = p_C y_C + (1-p_C)y_N + \sigma\left[p_C\pi_C + (1-p_C)\pi_N - \pi^*\right] - \sigma\left[i_C - (r^* + \pi^*) + \delta_C\right], \tag{A25}$$

$$\pi_C - \pi^* = \kappa y_C + \beta\left(p_C\pi_C + (1-p_C)\pi_N - \pi^*\right), \tag{A26}$$

$$i_C = \max\left[0, r^* + \pi^* + \phi_\pi\pi_C - \pi^*\right]. \tag{A27}$$

To analyze how an increase in the inflation target affects the crisis shock state in a transparent way, we assume that the normal state is an absorbing state (i.e., $p_N = 1$). Then,

$$y_N = 0, \tag{A28}$$

$$\pi_N = \pi^*, \tag{A29}$$

$$i_N = r^* + \pi^*, \tag{A30}$$

$$y_C = \frac{\sigma(1-\beta p_C)}{(1-\beta p_C)(1-p_C) - \kappa\sigma p_C}\pi^* + \frac{\sigma(r^* - \delta_C)(1-\beta p_C)}{(1-\beta p_C)(1-p_C) - \kappa\sigma p_C}, \tag{A31}$$

$$\pi_C = \left[\frac{\kappa\sigma}{(1-\beta p_C)(1-p_C) - \kappa\sigma p_C} + 1\right]\pi^* + \frac{\kappa\sigma(r^* - \delta_C)}{(1-\beta p_C)(1-p_C) - \kappa\sigma p_C}, \tag{A32}$$

$$i_C = 0. \tag{A33}$$

As shown by Nakata (2019) and Nakata and Schmidt (2022), for the equilibrium to exist, $(1-\beta p_C)(1-p_C)-\kappa\sigma p_C>0$. Thus, the coefficients in front of $\pi^*$ in equations (A31) and (A32) are both positive, meaning that $y_C$ and $\pi_C$ increase with the inflation target.

To understand the mechanism behind this result, it is useful to investigate the aggregate demand curve—a set of pairs of inflation and output in the crisis state consistent with the Euler equation—and the aggregate supply curve—a set of pairs of inflation and output in the crisis state consistent with the Phillips curve. They are given by

$$\pi_C = \frac{1-p_C}{\sigma p_C}y_C - \frac{1-p_C}{p_C}\pi^* - \frac{r^*-\delta_C}{p_C}, \tag{A34}$$

$$\pi_C = \frac{\kappa}{1-\beta p_C}y_C + \pi^*. \tag{A35}$$

According to equation (A34), an increase in the inflation target shifts down the AD curve. An increase in the inflation target parameter translates into an increase in the normal state inflation. According to the Euler equation, to support the same level of output in the crisis state, the expected real interest rate has to remain unchanged. Thus, if the normal state inflation is higher, the crisis state inflation has to decline.

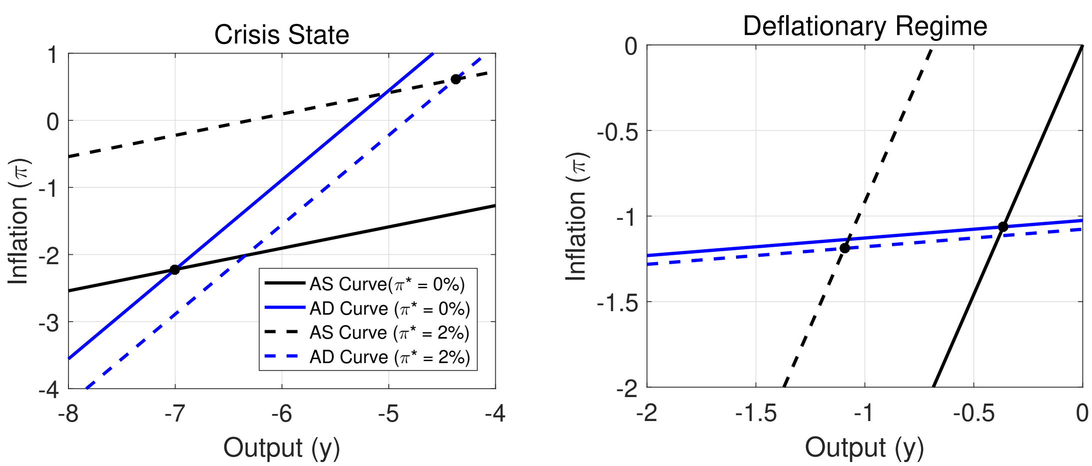
According to equation (A35), an increase in the inflation target shifts up the AS curve. An increase in the inflation target parameter translates into an increase in the normal state inflation. According to the Phillips curve, the crisis state inflation positively depends on the expected inflation in the next period, which also positively depends on the normal state inflation. As a result, a higher normal state inflation leads to a higher crisis state inflation for any given level of the crisis state output.

As shown by the left panel of Figure A6 which plots the effect of an increase in $\pi^*$ from 0 percent to 2 percent on the AS and AD curves, these shifts in the AS and AD curves mean a higher inflation and output in the crisis state. Table A1 shows the calibration of the semi-loglinear model used to generate the left panel of Figure A6.

Table A1: Calibrations for the Semi-Loglinear Model

| Parameter | Calibration | Description |
|---|---|---|
| $\beta$ | 1/1.0025 | Discount factor |
| $\sigma$ | 1 | Elasticity of Intertemporal Substitution |
| $\kappa$ | 0.02 | Slope of the Phillips curve |
| $\phi_\pi$ | 2 | Coefficient on inflation in Taylor Rule |
| $r^*$ | 1% | Annualized steady state nominal interest rate |
| $\delta_N$ | 0 | Demand shock in normal state |
| $\delta_C$ | 1.6/100 | Demand shock in crisis shock state |
| $p_T$ | 1 | Persistence of Target Regime |
| $p_D$ | 0.975 | Persistence of Deflationary Regime |
| $p_N$ | 1 | Persistence of Normal State |
| $p_C$ | 0.75 | Persistence of Crisis Shock State |
| $\pi^*$ | $\{0\%, 2\%\}$ | Annualized inflation target |

Figure A6: AD and AS Curves in the Crisis State and in the Deflationary Regime
—Semi-Loglinear Model—




## C.3 Model with a sunspot shock only

We now examine the effect of an increase in the inflation target on the deflationary regime output and inflation using the version of the semi-loglinear model with a sunspot shock only. Let $Prob(T|T) = p_T$ be the probability of staying in the target regime in the next period when the economy is in the target regime today and let $Prob(D|D) = p_D$ be the probability of staying in the deflationary regime in the next period when the economy is in the deflationary regime today.

The equilibrium conditions of the model with a sunspot shock only are given by

$$
\begin{aligned}
y_T =& p_T y_T + (1-p_T)y_D + \sigma\left[p_T\pi_T + (1-p_T)\pi_D - \pi^*\right] - \sigma\left[i_T - (r^* + \pi^*)\right], && \text{(A36)}\\
\pi_T - \pi^* =& \kappa y_T + \beta\left(p_T\pi_T + (1-p_T)\pi_D - \pi^*\right), && \text{(A37)}\\
i_T =& r^* + \pi^* + \phi_\pi(\pi_T - \pi^*), && \text{(A38)}\\
y_D =& p_D y_D + (1-p_D)y_T + \sigma\left[p_D\pi_D + (1-p_D)\pi_T - \pi^*\right] - \sigma\left[i_D - (r^* + \pi^*)\right], && \text{(A39)}\\
\pi_D - \pi^* =& \kappa y_D + \beta\left(p_D\pi_D + (1-p_D)\pi_T - \pi^*\right), && \text{(A40)}\\
i_D =& 0. && \text{(A41)}
\end{aligned}
$$

To analyze how an increase in the inflation target affects the deflationary regime in a transparent way, we assume that the target regime is an absorbing state (i.e., $p_T = 1$). Then,

$$
\begin{aligned}
y_T =& 0, && \text{(A42)}\\
\pi_T =& \pi^*, && \text{(A43)}\\
i_T =& r^* + \pi^*, && \text{(A44)}\\
y_D =& \frac{\sigma(1-\beta p_D)}{(1-\beta p_D)(1-p_D) - \kappa\sigma p_D}\pi^* + \frac{\sigma r^*(1-\beta p_D)}{(1-\beta p_D)(1-p_D) - \kappa\sigma p_D}, && \text{(A45)}\\
\pi_D =& \left[\frac{\kappa\sigma}{(1-\beta p_D)(1-p_D) - \kappa\sigma p_D} + 1\right]\pi^* + \frac{\kappa\sigma r^*}{(1-\beta p_D)(1-p_D) - \kappa\sigma p_D}, && \text{(A46)}\\
i_D =& 0. && \text{(A47)}
\end{aligned}
$$

As shown by Nakata and Schmidt (2022), for the equilibrium to exist, $(1-\beta p_D)(1-p_D) - \kappa\sigma p_D < 0$. Thus, the coefficient in front of $\pi^*$ in equation (A45) is negative, meaning that $y_D$ increases with the inflation target. Provided that $p_D$ is sufficiently large, the coefficient in front of $\pi^*$ in equation (A46) is negative, and thus $\pi_D$ increases with the inflation target.

To understand the mechanism behind this result, it is useful to investigate the aggregate demand curve and the aggregate supply curve in the deflationary regime:

$$\pi_D = \frac{1 - p_D}{\sigma p_D} y_D - \frac{1 - p_D}{p_D} \pi^* - \frac{r^*}{p_D}, \tag{A48}$$

$$\pi_D = \frac{\kappa}{1 - \beta p_D} y_D + \pi^*. \tag{A49}$$

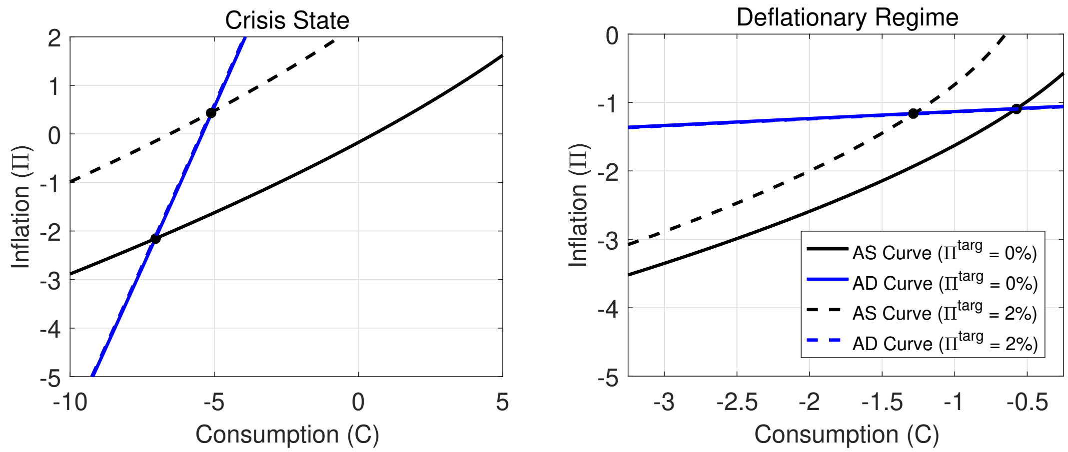
As in the model with a crisis shock only, an increase in the inflation target shifts down the AD curve and shifts up AS curves. Because the AS curve is steeper than the AD curve in the model with a sunspot shock only, these shifts in the AS and AD curves mean lower inflation and output in the deflationary regime, as shown by the right panel of Figure A6.

# D Effects of a Higher Inflation Target with $\alpha = 0$ and $\alpha = 1$

While we set $\alpha$ to 0.893 in our baseline calibration, the key property of the model—a higher inflation target leads to lower inflation and consumption in the deflationary regime—does not hinge on the specific value of $\alpha$. In this section, we examine the effect of a higher inflation target under $\alpha = 0$ (no price indexation) and $\alpha = 1$ (full price indexation). The assumption of no price indexation is common in papers analyzing the effect of a non-zero trend inflation, including those papers on the optimal inflation target (see, for example, Ascari (2004) and Ascari, Castelnuovo, and Rossi (2011)). The assumption of full price indexation is widespread in the literature estimating DSGE models and in papers using policy-oriented, medium-scale DSGE models, as the full price indexation make the dynamics of the loglinear version of those models invariant to the level of the inflation target (see, for example, Smets and Wouters (2007)).

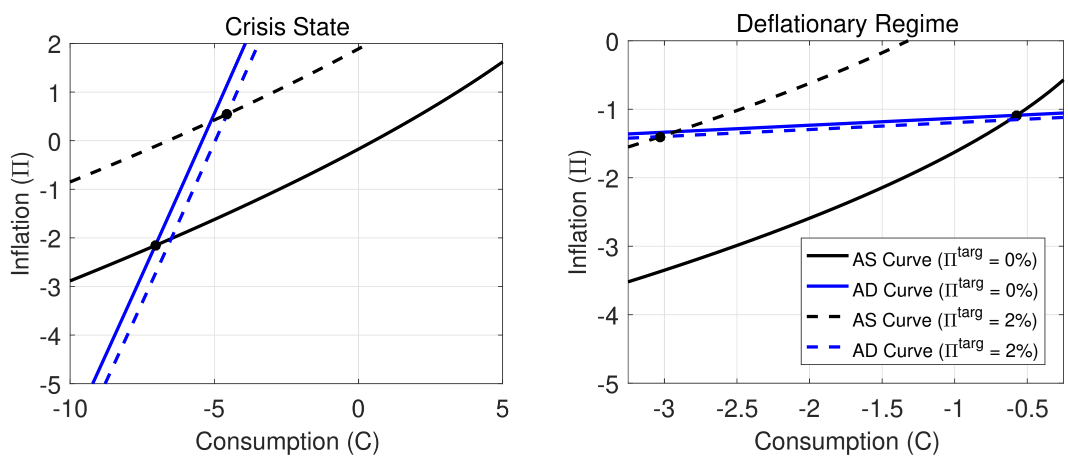
Figures A7 and A8 shows the effect of a higher inflation target with $\alpha = 0$ and $\alpha = 1$, respectively. When $\alpha = 0$, the effect of a higher inflation target is negligible for the AD curve. When $\alpha = 1$, the AD curves in both the crisis state and the deflationary regime shift

Figure A7: AD and AS Curves: $\alpha = 0$




down by a small amount, as in the case discussed in Section 3 of the main text. For both $\alpha = 0$ and $\alpha = 1$, the AS curves shift up in both the crisis state and the deflationary regime, as in the case discussed in Section 3 of the main text. Thus, a higher inflation target leads to lower inflation and consumption regardless of the value of $\alpha$.

Figure A8: AD and AS Curves: $\alpha = 1$




# E Analysis from model with an AR(1) demand shock

In this section, we examine the implications of introducing the possibility of falling into expectations-driven LTs on the optimal inflation target using a model with an AR(1) demand shock and a two-state sunspot shock.

The structure of the model is identical to the model we have examined thus far. The only difference is that the discount factor shock follows an AR(1) process:

$$\delta_t - 1 = \rho(\delta_{t-1} - 1) + \epsilon_t \tag{A50}$$

where $\epsilon_t$ is normally distributed with mean zero and standard deviation of $\sigma_\epsilon$.

We apply the same calibration principle used for the model with a two-state demand shock to this model with an AR(1) demand shock. We first set $\chi_c$, $\chi_n$, $\theta$, and $\varphi$ to the same values described in Table 1 in the main text. Conditional on these parameter values, we choose $\alpha = 0.943$, and $\sigma_\epsilon = 0.285/100$ so that the optimal inflation target is 2 percent in the absence of the sunspot shock. Appendix F describes the numerical solution method used to solve this model.

Figure A9: Optimal Inflation Target with Alternative Probabilities of Moving to the Deflationary Regime

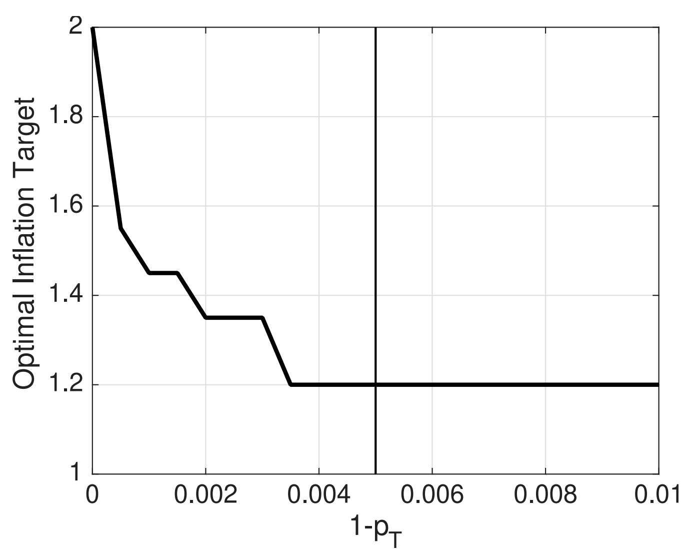



Figure A9 shows how the optimal inflation target varies with the probability of falling into an expectations-driven LT in this model. Consistent with what we saw in Figure 4 of the main text for the model with a two-state demand shock, a very small probability of falling into an expectations-driven LT lowers the optimal inflation target nontrivially. The thin black vertical line denotes our baseline value of the transition probability to the deflationary regime ($1-p_T = 0.005$); under this calibration the optimal inflation target is 1.2 percent, the lowest value of the inflation target consistent with the equilibrium existence in this model.[7] With the target regime persistence of 0.999, denoted by the thin dashed black vertical line, the optimal inflation target is 1.45 percent. Although these optimal inflation targets are higher than those in the model with a two-state demand shock and a two-state sunspot shock, they are still nontrivially lower than the 2 percent inflation target that is optimal in the absence of the sunspot shock.

# F Solution method for model with an AR(1) demand shock and a two-state sunspot shock

To find the regime-specific policy functions for the models endogenous state variables, we approximate true policy functions by degree 4 Chebyshev polynomials. We solve for a set of basis coefficients ($\Theta$) that satisfy the equilibrium conditions at 101 grid points over $\delta$ and the sunspot shock $s$. The discount rate shock is on the interval of $[1-4\sigma_\delta,\ 1+4\sigma_\delta]$. We use a global solution method similar to Judd (1998) and Aruoba, Cuba-Borda, and Schorfheide (2018), where policy functions are approximated by a projection method (as opposed to a time-iteration method). The minimum set of state variables is given by $\mathcal{S}_t = \{\delta_t, s_t\}$.

The problem is to find a set of policy functions $\{C(\mathcal{S};\Theta), N(\mathcal{S};\Theta), Y(\mathcal{S};\Theta), w(\mathcal{S};\Theta), \Pi(\mathcal{S};\Theta), R(\mathcal{S};\Theta)\}$ by minimizing the sum of squared residuals that satisfy the following system of

[7] To search for the optimal inflation target, we solve the model with various inflation targets with 5 basis points interval.

*functional* equations:

$$C(\mathcal{S}_t;\Theta)^{-\chi_c} = \beta\delta_t R(\mathcal{S}_t;\Theta)\mathrm{E_t} C(\mathcal{S}_{t+1};\Theta)^{-\chi_c}\Pi(\mathcal{S}_{t+1};\Theta)^{-1}, \tag{A51}$$

$$w(\mathcal{S}_t;\Theta) = N(\mathcal{S}_t;\Theta)^{\chi_n} C(\mathcal{S}_t;\Theta)^{\chi_c}, \tag{A52}$$

$$\begin{aligned}\frac{Y(\mathcal{S}_t;\Theta)}{C(\mathcal{S}_t;\Theta)^{\chi_c}} &\left[\varphi\left(\frac{\Pi(\mathcal{S}_t;\Theta)}{(\Pi^{targ})^{\alpha}} - 1\right)\frac{\Pi(\mathcal{S}_t;\Theta)}{(\Pi^{targ})^{\alpha}} - (1-\theta) - (1-\tau)\theta w(\mathcal{S}_t;\Theta)\right] \\ &= \beta\delta_t \mathrm{E_t}\frac{Y(\mathcal{S}_{t+1};\Theta)}{C(\mathcal{S}_{t+1};\Theta)^{\chi_c}}\varphi\left(\frac{\Pi(\mathcal{S}_{t+1};\Theta)}{(\Pi^{targ})^{\alpha}} - 1\right)\frac{\Pi(\mathcal{S}_{t+1};\Theta)}{(\Pi^{targ})^{\alpha}},\end{aligned} \tag{A53}$$

$$Y(\mathcal{S}_t;\Theta) = C(\mathcal{S}_t;\Theta) + \frac{\varphi}{2}\left[\frac{\Pi(\mathcal{S}_t;\Theta)}{(\Pi^{targ})^{\alpha}} - 1\right]^2 Y(\mathcal{S}_t;\Theta), \tag{A54}$$

$$Y(\mathcal{S}_t;\Theta) = N(\mathcal{S}_t;\Theta), \tag{A55}$$

$$R(\mathcal{S}_t;\Theta) = \max\left[R_{ELB}, \quad \frac{\Pi^{targ}}{\beta\delta_t}\left(\frac{\Pi(\mathcal{S}_t;\Theta)}{\Pi^{targ}}\right)^{\phi_\pi}\right]. \tag{A56}$$

Substituting out $N(\mathcal{S};\Theta)$ and $w(\mathcal{S};\Theta)$, this system can be reduced to a system of four functional equations. The problem then becomes finding a set of policy functions $\{C(\mathcal{S};\Theta), Y(\mathcal{S};\Theta), \Pi(\mathcal{S};\Theta), R(\mathcal{S};\Theta)\}$ by minimizing the sum of squared residuals that satisfy the following system of functional equations:

$$C(\mathcal{S}_t;\Theta)^{-\chi_c} = \beta\delta_t R(\mathcal{S}_t;\Theta)\mathrm{E_t} C(\mathcal{S}_{t+1};\Theta)^{-\chi_c}\Pi(\mathcal{S}_{t+1};\Theta)^{-1}, \tag{A57}$$

$$\begin{aligned}\frac{Y(\mathcal{S}_t;\Theta)}{C(\mathcal{S}_t;\Theta)^{\chi_c}} &\left[\varphi\left(\frac{\Pi(\mathcal{S}_t;\Theta)}{(\Pi^{targ})^{\alpha}} - 1\right)\frac{\Pi(\mathcal{S}_t;\Theta)}{(\Pi^{targ})^{\alpha}} - (1-\theta) - (1-\tau)\theta Y(\mathcal{S}_t;\Theta)^{\chi_n} C(\mathcal{S}_t;\Theta)^{\chi_c}\right] \\ &= \beta\delta_t \mathrm{E_t}\frac{Y(\mathcal{S}_{t+1};\Theta)}{C(\mathcal{S}_{t+1};\Theta)^{\chi_c}}\varphi\left(\frac{\Pi(\mathcal{S}_{t+1};\Theta)}{(\Pi^{targ})^{\alpha}} - 1\right)\frac{\Pi(\mathcal{S}_{t+1};\Theta)}{(\Pi^{targ})^{\alpha}},\end{aligned} \tag{A58}$$

$$Y(\mathcal{S}_t;\Theta) = C(\mathcal{S}_t;\Theta) + \frac{\varphi}{2}\left[\frac{\Pi(\mathcal{S}_t;\Theta)}{(\Pi^{targ})^{\alpha}} - 1\right]^2 Y(\mathcal{S}_t;\Theta), \tag{A59}$$

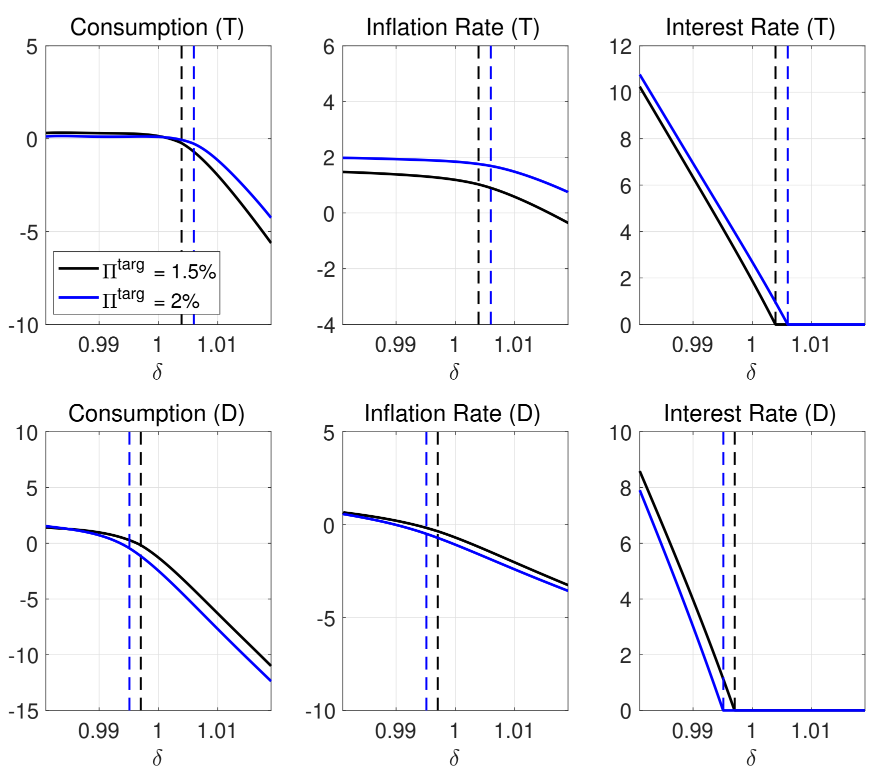
$$R(\mathcal{S}_t;\Theta) = \max\left[R_{ELB}, \quad \frac{\Pi^{targ}}{\beta\delta_t}\left(\frac{\Pi(\mathcal{S}_t;\Theta)}{\Pi^{targ}}\right)^{\phi_\pi}\right]. \tag{A60}$$

The max operator present in equation (A60) introduces a kink in the policy functions for sufficiently large realizations of $\delta$. While Chebyshev polynomials can approximate smooth functions well, there are potential problems for functions that do not have a continuous derivative with low order approximations. To account for these nonlinearities, we follow the idea of Christiano and Fisher (2000), in which we decompose policy functions in the target and deflationary regime into two parts using an indicator: one where the policy rate is allowed to be less than zero and the other part in which the policy rate is assumed to be zero. Thus, for any policy function of a given regime $X^s$:

$$X^s(\mathcal{S};\Theta) = \mathbb{I}_{\{R^s(\mathcal{S};\Theta)\geq 1\}} X^s_{unc}(\mathcal{S};\Theta) + (1 - \mathbb{I}_{\{R^s(\mathcal{S};\Theta)\geq 1\}}) X^s_{ELB}(\mathcal{S};\Theta) \tag{A61}$$

The problem then becomes finding a pair of policy functions for a given regime $\{[C^s_{unc}(\mathcal{S};\Theta), C^s_{ELB}(\mathcal{S};\Theta)], [Y^s_{unc}(\mathcal{S};\Theta), Y^s_{ELB}(\mathcal{S};\Theta)], [\Pi^s_{unc}(\mathcal{S};\Theta), \Pi^s_{ELB}(\mathcal{S};\Theta)], [R^s_{unc}(\mathcal{S};\Theta), R^s_{ELB}(\mathcal{S};\Theta)]\}$ that solves the system of functional equations above. This approach will yield more accurate policy functions. Figure A10 displays the policy functions for consumption, inflation, and the interest rate for two different inflation targets considered by the central bank.

Figure A10: Sample Policy Functions for Target and Deflationary Regimes



The projection method begins by specifying a guess for $\Theta$: $\Theta_0$. The initial parameterization of $\Theta_0$ corresponds to a set of values the policy functions take on given the discretized state space. For the target regime, this set of parameters corresponds to the set of policy functions that solves the model when the ELB constraint is not present; for the deflationary regime, this set of parameters corresponds to flat line functions at the deflationary deterministic steady state. Then for the current value of $\Theta_k$ and a given grid point $\mathcal{S}_i$, compute $C(\mathcal{S}_i;\Theta_k) = C^s_{unc}(\mathcal{S}_i;\Theta_k)$ and $\Pi(\mathcal{S}_i;\Theta_k) = \Pi^s_{unc}(\mathcal{S}_i;\Theta_k)$, assuming that the ELB does not bind, and obtain $Y(\mathcal{S}_i;\Theta_k) = Y^s_{unc}(\mathcal{S}_i;\Theta_k)$ and $R(\mathcal{S}_i;\Theta_k) = R^s_{unc}(\mathcal{S}_i;\Theta_k)$ from their functional forms in equations (A59) and (A60), respectively. If $R(\mathcal{S}_i;\Theta_k) \leq 1$, then $R(\mathcal{S}_i;\Theta_k) = R^s_{ELB}(\mathcal{S}_i;\Theta_k)$ and recompute $C(\mathcal{S}_i;\Theta_k) = C^s_{ELB}(\mathcal{S}_i;\Theta_k)$, $\Pi(\mathcal{S}_i;\Theta_k) = \Pi^s_{ELB}(\mathcal{S}_i;\Theta_k)$ and obtain $Y(\mathcal{S}_i;\Theta_k) = Y^s_{ELB}(\mathcal{S}_i;\Theta_k)$ from equation (A59). In calculating expectation terms in the Euler equation and Phillips curve, we use Gaussian quadrature and the values of these future variables not on grid points are evaluated using Chebyshev interpolation. Finally, the residuals for each regime and each set of policy functions are calculated via the functional forms above. $\Theta$ is updated to minimize the sum of squared residuals of the objective function. We use MATLAB's least-squares minimization algorithm `lsqnonlin` to do this. Following Aruoba, Cuba-Borda, and Schorfheide (2018), we include analytically derived derivatives of our objective function to decrease the run-time of the solution method.

# G Solution Accuracy for model with an AR(1) demand shock and a two-state sunspot shock

In this section, we report the accuracy of our numerical solution for the model with an AR(1) demand shock and a two-state sunspot shock. Following Maliar and Maliar (2015), we evaluate the residuals for the consumption Euler equation and Phillips curve along a simulated equilibrium path. The length of the path is 101,000, where the first 1,000 simulations are discarded.

Table A2: Solution Accuracy

| | Mean[$\log_1 0(R_{k,t})$] | 95th-percentile of [$\log_1 0(R_{k,t})$] |
|---|---|---|
| $k = 1$ Euler Equation error | $-3.90$ | $-3.31$ |
| $k = 2$ Phillips curve error | $-4.43$ | $-3.52$ |

The residuals for the consumption Euler equation and Phillips curve are given as follows:

$$
\begin{aligned}
R_{1,t} &= \left|1 - C_t^{\chi_c} \beta \delta_t R_t \mathrm{E_t} C_{t+1}^{-\chi_c} \Pi_{t+1}^{-1}\right|, && \text{(A62)}\\
R_{2,t} &= \left| \left( \frac{\Pi_t}{(\Pi^{targ})^{\alpha}} - 1 \right) \frac{\Pi_t}{(\Pi^{targ})^{\alpha}} - \frac{(1-\theta) - (1-\tau)\theta w_t}{\varphi} \right. \\
&\qquad \left. - \frac{C_t^{\chi_c}}{Y_t} \beta \delta_t \mathrm{E_t} \frac{Y_{t+1}}{C_{t+1}^{\chi_c}} \left( \frac{\Pi_{t+1}}{(\Pi^{targ})^{\alpha}} - 1 \right) \frac{\Pi_{t+1}}{(\Pi^{targ})^{\alpha}} \right|. && \text{(A63)}
\end{aligned}
$$

With our log-utility specification, $R_{1,t}$ measures the difference between the chosen consumption today and today's consumption consistent with the optimization behavior of the household, as a percentage of the chosen consumption. $R_{2,t}$ is given by the difference between $\left( \frac{\Pi_t}{(\Pi^{targ})^{\alpha}} - 1 \right) \frac{\Pi_t}{(\Pi^{targ})^{\alpha}}$ and the sum of the term involving today's real wage and the term involving expectations. $\left( \frac{\Pi_t}{(\Pi^{targ})^{\alpha}} - 1 \right) \frac{\Pi_t}{(\Pi^{targ})^{\alpha}}$ represents the deviation of inflation from its target rate of inflation. Thus, the difference between this term and the sum of the terms involving today's real wages and expectations measures how much the chosen inflation rate differs from the inflation rate consistent with optimization behavior. This degree of accuracy is similar to what is reported in Aruoba, Cuba-Borda, and Schorfheide (2018).

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