Supplementary Material

Appendix A: Data Sources

Table A.1: Data Sources

NIPA Accounts	
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Table 1.3	Current-Cost Depreciation of Fixed Assets & Consumer Durable Goods (1925-2024)
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Table 7.1	Current-Cost Net Stock of Government Fixed Assets (1925-2024)
Table 7.3	Current-Cost Depreciation of Government Fixed Assets (1925-2024)
Table 7.5	Investment in Government Fixed Assets (1925-2024)

Appendix B: Macroeconomic system

In our solutions, we assume the number of firms is the same across the three sectors and equal to the endogenously determined number of blueprints or ideas. Thus,

$$N_{f,t} = N_{i,t} \equiv N_{b,t}.$$

Also, from (5), (12) and (20) in the main text or as seen in (B.4), (B.9) and (B.15) below, $l^r_{f,t} = l^r_{i,t} = l^r_{i,t} \equiv l^r_t$. Finally, since ex-post we impose symmetricity within each type of firm and also $n_{b,t} \equiv 1$, we have the following system:

Final good sector

Production function:

$$y_{f,t} = A_{f,t} \left(N_{b,t} l^w_{f,t}\right)^{\alpha} x^{1-\alpha}_{i,t}, \tag{B.1}$$

Profits:

$$\pi_{f,t} \equiv (1-\tau^f_t)[y_{f,t} - w_t(l^w_{f,t} + l^r_{f,t}) - p_{i,t}x_{i,t}] + \frac{G^p_t}{3N_{b,t}} - r_c w_t(l^w_{f,t} + l^r_{f,t}), \tag{B.2}$$

Demand for labour used for productive work:

$$(1-\tau_t^f+r_c)w_t=(1-\tau_t^f)\frac{\alpha y_{f,t}}{l_{f,t}^w}, \tag{B.3}$$

Demand for the intermediate input:

$$p_{i,t}=\frac{(1-\alpha)\,y_{f,t}}{x_{i,t}}, \tag{B.4}$$

Demand for labour used for rent-seeking activities:

$$l_{f,t}^r=\frac{1}{(1-\tau_t^f+r_c)w_t}\frac{G_t^p}{3N_{b,t}}. \tag{B.5}$$

where $A_{f,t}=A_f\left(\widetilde{k}_t^g\right)^{\phi}$, $\widetilde{k}_t^g\equiv\frac{N_{b,t}k_t^g}{N_t h_{h,t}N_{b,t}}=\frac{K_t^g}{H_t N_{b,t}}$, $L_t^r\equiv N_{f,t}l_{f,t}^r+N_{i,t}l_{i,t}^r+N_{b,t}l_{i,t}^r=3N_{b,t}l_t^r$ and $G_t^p\equiv\kappa(G_t^c+G_t^i)$.

Intermediate goods sector

Production function:

$$x_{i,t}=A_{i,t}\left(N_{b,t}l_{i,t}^w\right)^{\alpha}k_{i,t}^{1-\alpha}, \tag{B.6}$$

Profits:

$$\pi_{i,t}\equiv(1-\tau_t^f)[p_{i,t}x_{i,t}-w_t(l_{i,t}^w+l_{i,t}^r)-q_t]-i_{i,t}+\frac{G_t^p}{3N_{b,t}}-r_c w_t(l_{i,t}^w+l_{i,t}^r), \tag{B.7}$$

Demand for labour used for productive work:

$$(1-\tau_t^f+r_c)w_t=\frac{(1-\tau_t^f)(1-\alpha)^2 y_{f,t}}{x_{i,t}}\frac{\alpha x_{i,t}}{l_{i,t}^w}, \tag{B.8}$$

Demand for labour used for rent-seeking activities:

$$l_{i,t}^r=\frac{1}{(1-\tau_t^f+r_c)w_t}\frac{G_t^p}{3N_{b,t}}, \tag{B.9}$$

Demand for capital:

$$1=\beta_{i,1}\left[1-\delta+\frac{(1-\tau_{t+1}^f)(1-\alpha)^2 y_{f,t+1}}{x_{i,t+1}}\frac{(1-\alpha)x_{i,t+1}}{k_{i,t+1}}\right], \tag{B.10}$$

Price of blueprint:

$$q_t\equiv\frac{(1-\tau_t^f)[p_{i,t}x_{i,t}-w_t(l_{i,t}^w+l_{i,t}^r)]-i_{i,t}+\frac{G_t^p}{3N_{b,t}}-r_c w_t(l_{i,t}^w+l_{i,t}^r)}{(1-\tau_t^f)}, \tag{B.11}$$

where $\beta_{i,1} \equiv \beta \frac{\left(1+\tau^c_{t+1}\right)(c_{h,t}+\lambda g^c_t)^{-\sigma}(1-l^w_{h,t}-l^e_{h,t})^{\psi(1-\sigma)}}{(1+\tau^c_t)(c_{h,t+1}+\lambda g^c_{t+1})^{-\sigma}(1-l^w_{h,t+1}-l^e_{h,t+1})^{\psi(1-\sigma)}}$; $G^p_t \equiv \kappa(G^c_t + G^i_t)$; and $A_{i,t} \equiv A_i \left(\widetilde{k}^g_t\right)^\phi$.

Research sector

Motion of blueprints:

$$N_{b,t+1} = (1-\delta^{n_b})\, N_{b,t} + M_t N_{b,t} l^w_{b,t} N_t h_{h,t} \left(N_{b,t}\right)^\mu , \tag{B.12}$$

Profits:

$$\pi_{b,t} \equiv (1-\tau^f_t)[q_t - w_t(l^w_{b,t} + l^r_{b,t})] + \frac{G^p_t}{3N_{b,t}} - r_c w_t(l^w_{b,t} + l^r_{b,t}), \tag{B.13}$$

Demand for labour used for productive work:

$$(1-\tau^f_t + r_c)w_t = \beta_{b,1}\left(1-\tau^f_{t+1}\right)\frac{N_{b,t}}{N_{b,t+1}} q_{t+1} M_t H_t \left(N_{b,t}\right)^\mu , \tag{B.14}$$

Demand for labour used for rent-seeking activities:

$$l^r_{b,t} = \frac{1}{(1-\tau^f_t + r_c)w_t}\frac{G^p_t}{3N_{b,t}}, \tag{B.15}$$

where $M_t \equiv M\left(\widetilde{k}^g_t\right)^\phi$; $L_{b,t} = N_{b,t} l_{b,t}$; $H_t = N_t h_{h,t}$; $\Pi_{b,t} = N_{b,t}\pi_{b,t}$; $G^p_t \equiv \kappa(G^c_t + G^i_t)$; and $\beta_{b,1} \equiv \beta_{i,t}$.

Household

Human capital evolution:

$$h_{h,t+1} = (1-\delta^h) h_{h,t} + D\left(l^e_{h,t} h_{h,t}\right)^\theta \left(h_{h,t}\right)^{1-\theta} , \tag{B.16}$$

Demand for bonds:

$$\frac{\left(1+\tau^c_{t+1}\right)(c_{h,t}+\lambda g^c_t)^{-\sigma}(1-l^w_{h,t}-l^e_{h,t})^{\psi(1-\sigma)}}{(1+\tau^c_t)(c_{h,t+1}+\lambda g^c_{t+1})^{-\sigma}(1-l^w_{h,t+1}-l^e_{h,t+1})^{\psi(1-\sigma)}} = \beta\left(1 + r^b_{t+1}\right), \tag{B.17}$$

Supply of hours at work:

$$\begin{aligned} &\psi(c_{h,t}+\lambda g^c_t)^{1-\sigma}(1-l^w_{h,t}-l^e_{h,t})^{\psi(1-\sigma)-1} = \\ &= \frac{(c_{h,t}+\lambda g^c_t)^{-\sigma}}{(1+\tau^c_t)}(1-l^w_{h,t}-l^e_{h,t})^{\psi(1-\sigma)}(1-\tau^y_t) w_t h_{h,t}, \end{aligned} \tag{B.18}$$

Supply of hours in education:

$$\begin{aligned}&\psi(c_{h,t}+\lambda g_t^c)^{1-\sigma}(1-l_{h,t}^w-l_{h,t}^e)^{\psi(1-\sigma)-1}=\\&=\mu_{h,t}\frac{\theta D_t(l_{h,t}^e h_{h,t})^{\theta}\left(h_{h,t}\right)^{1-\theta}}{l_{h,t}^e},\end{aligned}\tag{B.19}$$

Human capital supply:

$$\begin{aligned}&\mu_{h,t}=\beta\frac{(c_{h,t+1}+\lambda g_{t+1}^c)^{-\sigma}}{\left(1+\tau_{t+1}^c\right)}(1-l_{h,t+1}^w-l_{h,t+1}^e)^{\psi(1-\sigma)}(1-\tau_{t+1}^y)\times\\&\times w_{t+1}l_{h,t+1}^w+\beta\mu_{\times,t+1}\left[1-\delta^h+\frac{\theta D_{t+1}(l_{h,t+1}^e h_{h,t+1})^{\theta}\left(h_{h,t+1}\right)^{1-\theta}}{h_{h,t+1}}\right].\end{aligned}\tag{B.20}$$

where $D_t \equiv D\left(\widetilde{k}_t^g\right)^{\phi}$.

Aggregate constraints

Resource constraint:

$$\begin{aligned}&N_t c_{h,t}+N_{i,t}i_{i,t}+(1-\kappa)(G_t^c+G_t^i)+r_c w_t(l_{f,t}^w+l_{f,t}^r)+\\&+r_c w_t(l_{i,t}^w+l_{i,t}^r)+r_c w_t(l_{b,t}^w+l_{b,t}^r)=N_{f,t}y_{f,t},\end{aligned}\tag{B.21}$$

Government budget constraint:

$$\begin{aligned}&G_t^c+G_t^i+G_t^t+(1+r_t^b)N_t b_{h,t}=\\&=N_t b_{h,t+1}+N_t\tau_t^y(w_t h_{h,t}l_{h,t}^w+\pi_{h,t})+N_t\tau_t^c c_{h,t}+\\&+N_{f,t}\tau_t^f[y_{f,t}-w_t(l_{f,t}^w+l_{f,t}^r)-p_{i,t}x_{i,t}]+\\&+N_{i,t}\tau_t^f[p_{i,t}x_{i,t}-w_t(l_{i,t}^w+l_{i,t}^r)-q_t]+\\&+N_{b,t}\tau_t^f[q_t-w_t(l_{b,t}^w+l_{b,t}^r)].\end{aligned}\tag{B.22}$$

where $G_t^k = N_t g_t^k$; $k \equiv c,i,t$; $g_t^c \equiv \frac{(1-\kappa)G_t^c}{N_t}$; $B_{t+1} \equiv N_t b_{h,t+1}$; and $B_t \equiv N_t b_{h,t}$. Notice that we use the economy's resource constraint instead of the household's budget constraint in the equilibrium system (the latter is satisfied residually).

Public capital and market clearing

Public capital evolution:

$$K_{t+1}^g=(1-\delta^g)K_t^g+(1-\kappa)G_t^i,\tag{B.23}$$

Market-clearing: labour market:

$$N_{b,t}(l_{f,t}^w+l_{f,t}^r+l_{i,t}^w+l_{i,t}^r+l_{b,t}^w+l_{b,t}^r)=N_{b,t}(l_{f,t}+l_{i,t}+l_{b,t}+3l_t^r)=N_t h_{h,t}l_{h,t}^w,\tag{B.24}$$

Market-clearing: dividend market:

$$N_{f,t}\pi_{f,t} + N_{i,t}\pi_{i,t} + N_{b,t}\pi_{b,t} = N_t\pi_{h,t}. \tag{B.25}$$

Therefore, we have 25 equations in the paths of 25 unknowns: $y_{f,t}$, $\pi_{f,t}$, $l^w_{f,t}$, $l^r_{f,t}$. $p_{i,t}$, $x_{i,t}$ $\pi_{i,t}$, $l^w_{i,t}$, $l^r_{i,t}$, $k_{i,t+1}$, q_t, $N_{b,t+1}$, $\pi_{b,t}$, $l^w_{b,t}$, $l^r_{b,t}$, $h_{h,t+1}$, $b_{h,t+1}$, $l^w_{h,t}$, $l^e_{h,t}$, $\mu_{h,t}$, $c_{h,t}$, r^b_t, K^g_{t+1}, w_t, $\pi_{h,t}$. This is given the exogenous variables defined in the main text and Appendix A. Note that here we include the end-of-period public debt, $b_{h,t+1}$, in the list of endogenous variables. If, however, we set the public debt to GDP as in the data, then one of the other fiscal instruments takes its place as an endogenous variable.

Appendix C: Stationary macroeconomic system

Since population, N_t, individual human capital, $h_{h,t}$, and ideas, $N_{b,t}$, can generate long-term growth, we need to transform the system presented in Appendix B to make it stationary. In particular, we define all quantities in efficiency units, namely, $\widetilde{y}_{f,t} \equiv \frac{N_{f,t}y_{f,t}}{N_t h_{h,t} N_{b,t}} = \frac{Y_{,t}}{H_t N_{b,t}}$, $\widetilde{x}_{i,t} \equiv \frac{N_{i,t}x_{i,t}}{N_t h_{h,t} N_{b,t}} = \frac{X_t}{H_t N_{b,t}}$, $\widetilde{k}_{i,t} \equiv \frac{N_{i,t}k_{i,t}}{N_t h_{h,t} N_{b,t}} = \frac{K_t}{H_t N_{b,t}}$, $\widetilde{i}_{i,t} \equiv \frac{N_{i,t}i_{i,t}}{N_t h_{h,t} N_{b,t}} = \frac{I_t}{H_t N_{b,t}}$, $\widetilde{c}_{h,t} \equiv \frac{N_t c_{h,t}}{N_t h_{h,t} N_{b,t}} = \frac{C_t}{H_t N_{b,t}}$, $\widetilde{b}_{h,t} \equiv \frac{N_t b_{h,t}}{N_t h_{h,t} N_{b,t}} = \frac{B_t}{H_t N_{b,t}}$, $\widetilde{k}^g_t \equiv \frac{N_{b,t}k^g_t}{N_t h_{h,t} N_{b,t}} = \frac{K^g_t}{H_t N_{b,t}}$, where $H_t = N_t h_{h,t}$ is total human capital. We also redefine the prices $\widetilde{w}_t \equiv \frac{w_t}{H_t}$ and $\widetilde{q}_t \equiv \frac{q_t}{H_t}$, the human capital multiplier $\widetilde{\mu}_{h,t} = \frac{\mu_{h,t}(h_{h,t})^\sigma}{(N_{b,t})^{1-\sigma}}$, and we add the auxiliary variable $\widetilde{\psi}_{b,t} \equiv \frac{H_t}{N_{b,t}}$ (see also Jones (2022b)). Further, note that $l^w_{f,t}$, $l^r_{f,t}$, $l^w_{i,t}$, $l^r_{i,t}$, $l^w_{b,t}$, $l^r_{b,t}$, $l^w_{h,t}$, $l^e_{h,t}$, $p_{i,t}$, r^b_t are ratios so they are not transformed. Finally, recall that $N_{f,t} = N_{i,t} \equiv N_{b,t}$, $\mu = -1$, and $l^r_{f,t} = l^r_{i,t} = l^r_{b,t} \equiv l^r_t$. Thus, the stationary equilibrium can be written as follows:

Final good sector

Production function:

$$\widetilde{y}_{f,t} = A_f \left(\frac{l^w_{f,t}}{\widetilde{\psi}_{b,t}}\right)^\alpha (\widetilde{x}_{i,t})^{1-\alpha} \left(\widetilde{k}^g_t\right)^\phi, \tag{C.1}$$

Demand for labour used for productive work:

$$(1 - \tau^f_t + r_c)\widetilde{w}_t = (1 - \tau^f_t)\frac{\alpha \widetilde{y}_{f,t}}{l^w_{f,t}}, \tag{C.2}$$

Demand for the intermediate input:

$$p_{i,t} = \frac{(1-\alpha)\widetilde{y}_{f,t}}{\widetilde{x}_{i,t}}, \tag{C.3}$$

Demand for labour used for rent-seeking activities:

$$l_t^r = \frac{\kappa\left(s_t^c + s_t^i\right)\widetilde{y}_{f,t}}{3(1-\tau_t^f + r_c)\widetilde{w}_t}. \tag{C.4}$$

Intermediate goods sector

Production function:

$$\widetilde{x}_{i,t} = A_i \left(\frac{l_{i,t}^w}{\widetilde{\psi}_{b,t}}\right)^{\alpha} (\widetilde{k}_{i,t})^{1-\alpha} \left(\widetilde{k}_t^g\right)^{\phi}, \tag{C.5}$$

Motion of private capital:

$$\widetilde{k}_{i,t+1}\left(1+\gamma^n\right)\left(1+\gamma_t^h\right) = (1-\delta)\widetilde{k}_{i,t} + \widetilde{i}_{i,t}, \tag{C.6}$$

Demand for labour used for productive work:

$$(1-\tau_t^f + r_c)\widetilde{w}_t = \frac{(1-\tau_t^f)(1-a)^2\widetilde{y}_{f,t}}{\widetilde{x}_{i,t}}\frac{a\widetilde{x}_{i,t}}{l_{i,t}^w}, \tag{C.7}$$

Demand for capital:

$$1 = \beta_{i,1}\left[1-\delta + \frac{(1-\tau_{t+1}^f)(1-\alpha)^3\widetilde{y}_{f,t+1}}{\widetilde{k}_{i,t+1}}\right], \tag{C.8}$$

Price of blueprint:

$$\widetilde{q}_t \equiv \frac{(1-\tau_t^f)[p_{i,t}\widetilde{x}_{i,t} - \widetilde{w}_t(l_{i,t}^w + l_t^r)] - \widetilde{i}_{i,t} + \frac{\kappa\left(s_t^c+s_t^i\right)\widetilde{y}_{f,t}}{3} - r_c\widetilde{w}_t(l_{i,t}^w + l_t^r)}{(1-\tau_t^f)}\ . \tag{C.9}$$

Research sector

Demand for labour used for productive work:

$$\begin{aligned}&(1+\gamma_t^{n_b})(1-\tau_t^f + r_c)\widetilde{w}_t = \beta_{b,1}(1-\tau_{t+1}^f)\left(1+\gamma^n\right)\left(1+\gamma_t^h\right)\times \\ &\times\widetilde{q}_{t+1}\widetilde{\psi}_{b,t}M\left(\widetilde{k}_t^g\right)^{\phi},\end{aligned}\quad , \tag{C.10}$$

Evolution of the auxillary variable, $\widetilde{\psi}_b$:

$$\frac{\widetilde{\psi}_{b,t+1}}{\widetilde{\psi}_{b,t}} = \frac{(1+\gamma^n)\left(1+\gamma_t^h\right)}{1+\gamma_t^{n_b}}, \tag{C.11}$$

Household

Demand for bonds:

$$\begin{aligned} &\tfrac{(1+\tau_{t+1}^c)(\widetilde{c}_{h,t}+\lambda\widetilde{g}_t^c)^{-\sigma}(1-l_{h,t}^w-l_{h,t}^e)^{\psi(1-\sigma)}}{(1+\tau_t^c)(\widetilde{c}_{h,t+1}+\lambda\widetilde{g}_{t+1}^c)^{-\sigma}(1-l_{h,t+1}^w-l_{h,t+1}^e)^{\psi(1-\sigma)}} = \\ &= \beta\left[\left(1+\gamma_t^h\right)\left(1+\gamma_t^{n_b}\right)\right]^{-\sigma}\left(1+r_{t+1}^b\right), \end{aligned} \tag{C.12}$$

Supply of hours at work:

$$\tfrac{\psi}{(1-l_{h,t}^w-l_{h,t}^e)} = \tfrac{(1-\tau_t^y)\widetilde{\psi}_{b,t}\widetilde{w}_t}{(1+\tau_t^c)(\widetilde{c}_{h,t}+\lambda\widetilde{g}_t^c)}, \tag{C.13}$$

Supply of hours in education:

$$\begin{aligned} &\psi(\widetilde{c}_{h,t}+\lambda\widetilde{g}_t^c)^{1-\sigma}(1-l_{h,t}^w-l_{h,t}^e)^{\psi(1-\sigma)-1} = \\ &= \widetilde{\mu}_{h,t}\theta(l_{h,t}^e)^{\theta-1}D\left(\widetilde{k}_t^g\right)^{\phi}, \end{aligned} \tag{C.14}$$

Human capital supply:

$$\begin{aligned} &\widetilde{\mu}_{h,t} = \beta\left(1+\gamma_t^h\right)^{-\sigma}\left(1+\gamma_t^{n_b}\right)^{1-\sigma}\widetilde{\psi}_{b,t+1}\tfrac{(\widetilde{c}_{h,t+1}+\lambda\widetilde{g}_{t+1}^c)^{-\sigma}}{(1+\tau_{t+1}^c)}\times \\ &\times(1-l_{h,t+1}^w-l_{h,t+1}^e)^{\psi(1-\sigma)}(1-\tau_{t+1}^y)\widetilde{w}_{t+1}l_{h,t+1}^w + \\ &+\beta\widetilde{\mu}_{h,t+1}\left(1+\gamma_t^h\right)^{-\sigma}\left(1+\gamma_t^{n_b}\right)^{1-\sigma}[1-\delta^h+\theta(l_{h,t+1}^e)^{\theta}D(\widetilde{k}_{t+1}^g)^{\phi}]. \end{aligned} \tag{C.15}$$

Aggregate constraints

Resource constraint:

$$\begin{aligned} &\widetilde{c}_{h,t}+\widetilde{i}_{i,t}+(1-\kappa)(s_t^c+s_t^i)\widetilde{y}_{f,t}+r_c\widetilde{w}_t(l_{f,t}^w+l_t^r)+ \\ &+r_c\widetilde{w}_t(l_{i,t}^w+l_t^r)+r_c\widetilde{w}_t(l_{b,t}^w+l_t^r)=\widetilde{y}_{f,t}, \end{aligned} \tag{C.16}$$

Government budget constraint:

$$\begin{aligned} &\left(s_t^c+s_t^i+s_t^t\right)\widetilde{y}_{f,t}+(1+r_t^b)\widetilde{b}_{h,t} = \left(1+\gamma_t^h\right)\left(1+\gamma_t^{n_b}\right)\widetilde{b}_{h,t+1}+ \\ &+\tau_t^y\widetilde{w}_t\widetilde{\psi}_{b,t}l_{h,t}^w+\tau_t^y[(1-\tau_t^f)\{\widetilde{y}_{f,t}-\widetilde{w}_t(l_{f,t}^w+l_{i,t}^w+l_{b,t}^w+3l_t^r)\}+ \\ &+\kappa\left(s_t^c+s_t^i\right)\widetilde{y}_{f,t}-\widetilde{i}_{i,t}-r_c\widetilde{w}_t(l_{f,t}^w+l_{i,t}^w+l_{b,t}^w+3l_t^r)]+ \\ &+\tau_t^f\left(\widetilde{y}_{f,t}-\widetilde{w}_t(l_{f,t}^w+l_{i,t}^w+l_{b,t}^w+3l_t^r)\right)+\tau_t^c\widetilde{c}_{h,t}. \end{aligned} \tag{C.17}$$

Public capital and labour market clearing

Public capital evolution:

$$\widetilde{k}^g_{t+1}\left(1+\gamma^n\right)\left(1+\gamma^h_t\right)\left(1+\gamma^{n_b}_t\right) = (1-\delta^g)\widetilde{k}^g_t + (1-\kappa)s^i_t\widetilde{y}_{f,t}, \tag{C.18}$$

Market-clearing condition in the labour market

$$\left(l^w_{f,t} + l^w_{i,t} + l^w_{b,t} + 3l^r_t\right) = \widetilde{\psi}_{b,t} l^w_{h,t}, \tag{C.19}$$

Drivers of long-run endogenous growth

Motion of household's human capital growth:

$$1+\gamma^h_t = 1-\delta^h + (l^e_{h,t})^\theta D(\widetilde{k}^g_t)^\phi, \tag{C.20}$$

Motion of the ideas growth:

$$1+\gamma^{n_b}_t = 1-\delta^{n_b} + l^w_{b,t}\widetilde{\psi}_{b,t} M(\widetilde{k}^g_t)^\phi. \tag{C.21}$$

In the above we use:

$$\begin{array}{llll}
\frac{N_{t+1}}{N_t} \equiv 1+\gamma^n; & \frac{h_{h,t+1}}{h_{h,t}} \equiv 1+\gamma^h_t; & \frac{N_{b,t+1}}{N_{b,t}} \equiv 1+\gamma^{n_b}_t; & \widetilde{g}^c_t \equiv (1-\kappa)s^c_t\widetilde{y}_{f,t}; \\
\widetilde{b}_{h,t+1} = \frac{B_h}{Y_f}\widetilde{y}_{f,t}; & \beta_{i,1} = \beta_{b,1}; & s^k_t = \overline{s}^k + \varepsilon^k_t; & \text{and} \\
\multicolumn{4}{l}{\beta_{i,1} \equiv \beta \frac{(1+\tau^c_t)(\widetilde{c}_{h,t+1}+\lambda\widetilde{g}^c_{t+1})^{-\sigma}(1-l^w_{h,t+1}-l^e_{h,t+1})^{\psi(1-\sigma)}}{(1+\tau^c_{t+1})(\widetilde{c}_{h,t}+\lambda\widetilde{g}^c_t)^{-\sigma}(1-l^w_{h,t}-l^e_{h,t})^{\psi(1-\sigma)}}\left[\left(1+\gamma^h_t\right)\left(1+\gamma^{n_b}_t\right)\right]^{-\sigma},}
\end{array}$$

where for the spending shares $k \equiv c, i, t$.

Therefore, we now have 21 equations in the paths of 21 unknowns: $\widetilde{y}_{f,t}$, $l^w_{f,t}$, $p_{i,t}$, $\widetilde{x}_{i,t}$, $\widetilde{k}_{i,t+1}$, $l^w_{i,t}$, $\widetilde{i}_t$, $\widetilde{q}_t$, $1+\gamma^{n_b}_t$, $l^w_{b,t}$, $\widetilde{\psi}_{b,t}$, l^r_t, $\widetilde{c}_{h,t}$, $l^w_{h,t}$, $l^e_{h,t}$, $1+\gamma^h_t$, $\widetilde{w}_t$, r^b_t, $\widetilde{b}_{h,t+1}$, $\widetilde{\mu}_{h,t}$, $\widetilde{k}^g_{t+1}$. Note that here, relative to Appendix B, we have substituted out $\pi_{f,t}$, $\pi_{i,t}$, $\pi_{b,t}$ and $\pi_{h,t}$. Also note that, as in Appendix B, we include $\widetilde{b}_{h,t+1}$ in the list of endogenous variables. Finally, note that from those 21 unknowns, $\widetilde{k}_i$, ψ_b, r^b, $\widetilde{b}_h$, $\widetilde{k}^g$ are state-like variables.

Appendix D: Per capita levels and growth rates

To calculate any per capita quantity, denoted as x_t, we start with the definition $\widetilde{x}_t \equiv \frac{N_{f,t} x_t}{N_t h_{h,t} N_{b,t}} \equiv \frac{X_t}{N_t h_{h,t} N_{b,t}}$ which can be rewritten in per capita terms as $x_t \equiv \frac{X_t}{N_t} \equiv \widetilde{x}_t h_{h,t} N_{b,t}$. Thus, along the transition path for $t \geq 1$:

$$x_t = \left(\frac{\widetilde{x}_t}{\widetilde{x}_{t-1}} \right) \left(\frac{h_{h,t}}{h_{h,t-1}} \right) \left(\frac{N_{b,t}}{N_{b,t-1}} \right) x_{t-1}, \tag{D.1}$$

where the initial value, x_0, is given and $\frac{h_{h,t}}{h_{h,t-1}} \equiv 1 + \gamma_{t-1}^h$ and $\frac{N_{b,t}}{N_{b,t-1}} \equiv 1 + \gamma_{t-1}^{n_b}$ have been defined above. We use U.S. data from 2022 as starting values for the analysis reported in the main text. Per capita growth rates in turn are simply $\gamma_t^x = \frac{x_t}{x_{t-1}} - 1$.

We can also find per capita final output growth along the transition and on the BGP analytically. We start by repeating equation (C.1) here for convenience:

$$\widetilde{y}_{f,t} = A_f \left(\frac{l_{f,t}^w}{\widetilde{\psi}_{b,t}} \right)^{\alpha} (\widetilde{x}_{i,t})^{1-\alpha} \left(\widetilde{k}_t^g \right)^{\phi},$$

so that, since $\widetilde{y}_{f,t} \equiv \frac{N_{f,t} y_{f,t}}{N_t h_{h,t} N_{b,t}} \equiv \frac{Y_{f,t}}{N_t h_{h,t} N_{b,t}}$, per capita GDP is:

$$\frac{Y_{f,t}}{N_t} = A_f \left(\frac{l_{f,t}^w}{\widetilde{\psi}_{b,t}} \right)^{\alpha} (\widetilde{x}_{i,t})^{1-\alpha} \left(\widetilde{k}_t^g \right)^{\phi} N_{b,t} h_{h,t}. \tag{D.2}$$

Note that we can compare this to the Jones' baseline model. For example, equation (D.2) is like equation (18) in Jones (2019) if we ignore intermediate goods, public capital and human capital and if we notice that $\frac{l_{f,t}^w}{\widetilde{\psi}_{b,t}} = \frac{l_{f,t}^w N_{b,t}}{N_t h_{h,t}}$ here is like $(1-s)$ in Jones (2019). This follows in the sense that $\frac{l_{f,t}^w}{\widetilde{\psi}_{b,t}}$ is the fraction of the population that works in the production of the final good as is $(1-s)$.

Taking logs and differentiating (D.2) with respect to time, the growth rate of per capita final output between t and $t-1$ is:

$$\gamma_t^{y_f} \simeq \alpha \gamma_t^{l_f^w} - a \gamma_t^{\widetilde{\psi}_b} + (1-\alpha) \gamma_t^{\widetilde{x}_i} + \phi_f \gamma_t^{\widetilde{k}^g} + \gamma_t^h + \gamma_t^{n_b}, \tag{D.3}$$

where

$$
\begin{aligned}
&\gamma_t^{y_f} \equiv \frac{\frac{Y_{f,t}}{N_t} - \frac{Y_{f,t-1}}{N_{t-1}}}{\frac{Y_{f,t-1}}{N_{t-1}}}, \quad \gamma_t^{l_f^w} \equiv \frac{l_{f,t}^w - l_{f,t-1}^w}{l_{f,t-1}^w}, \quad \gamma_t^{\widetilde{\psi}_b} \equiv \frac{\widetilde{\psi}_{b,t} - \widetilde{\psi}_{b,t-1}}{\widetilde{\psi}_{b,t-1}},\\
&\gamma_t^{\widetilde{x}_i} \equiv \frac{\widetilde{x}_{i,t} - \widetilde{x}_{i,t-1}}{\widetilde{x}_{i,t-1}}, \quad \gamma_t^{\widetilde{k}^g} \equiv \frac{\widetilde{k}_t^g - \widetilde{k}_{t-1}^g}{\widetilde{k}_{t-1}^g},\\
&\gamma_t^h \equiv \frac{h_{h,t} - h_{h,t-1}}{h_{h,t-1}} = -\delta^h + (l_{h,t-1}^e)^\theta D(\widetilde{k}_{t-1}^g)^\phi,\\
&\gamma_t^{n_b} \equiv \frac{N_{b,t} - N_{b,t-1}}{N_{b,t-1}} = -\delta^{n_b} + l_{b,t-1}\widetilde{\psi}_{b,t-1} M(\widetilde{k}_{t-1}^g)^\phi.
\end{aligned}
$$

On the BGP, stationary variables do not change so that the long-run endogenous growth rate reduces to (we now omit time subscripts for rates and other stationary variables):

$$\gamma^{y_f} = \gamma^h + \gamma^{n_b}, \tag{D.4}$$

where:

$$\gamma^h = -\delta^h + (l_h^e)^\theta D(\widetilde{k}^g)^\phi, \tag{D.5}$$

$$\gamma^{n_b} = -\delta^{n_b} + l_b \widetilde{\psi}_b M \left(\widetilde{k}^g\right)^\phi. \tag{D.6}$$

Note that we can again compare with Jones (2019). For example, if first, we drop the growth rate of human capital, γ_h, given by (D.5), and next drop public capital in efficiency units, $M\left(\widetilde{k}^g\right)^\phi$, the level of human capital h_h and set $\delta^{n_b} = 0$ in (D.6), we have since $\widetilde{\psi}_b \equiv \frac{N_t h_{h,t}}{N_{b,t}}$:

$$\gamma^{n_b} = l_b \frac{N_t}{N_{b,t}}. \tag{D.7}$$

Thus, taking logs and totally differentiating (D.7) with respect to time, since γ^{n_b} and l_b remain constant on the BGP, gives $\gamma^{n_b} = \gamma^n$, which is like equation (21) in Jones (2019). In this case, the long-run per capita GDP growth rate is driven by the creation of new ideas and the latter by population growth only. This special case of our model is the so-called semi-endogenous growth model.

Appendix E: Welfare

Recall that households' period utility function is:

$$u_t = \frac{(c_{h,t} + \lambda g_t^c)^{1-\sigma}(1 - l_{h,t}^w - l_{h,t}^e)^{\psi(1-\sigma)}}{1-\sigma}, \tag{E.1}$$

which we rewrite as:

$$\begin{aligned} u_t &= \frac{\left(\frac{N_t c_{h,t}}{N_t h_{h,t} N_{b,t}} + \lambda \frac{G_t^c}{N_t h_{h,t} N_{b,t}}\right)^{1-\sigma}\left(h_{h,t} N_{b,t}\right)^{1-\sigma}(1-l_{h,t}^w - l_{h,t}^e)^{\psi(1-\sigma)}}{1-\sigma} = \\ &= \frac{\left(\widetilde{c}_{h,t} + \lambda \widetilde{g}_t^c\right)^{1-\sigma}\left(h_{h,t} N_{b,t}\right)^{1-\sigma}(1-l_{h,t}^w - l_{h,t}^e)^{\psi(1-\sigma)}}{1-\sigma}, \end{aligned} \tag{E.2}$$

where notice that $\widetilde{g}_t^c = (1-\kappa)s_t^c \widetilde{y}_{f,t}$.

Moreover, we have for $h_{h,t}$:

$$\begin{aligned} h_{h,t} &\equiv (1+\gamma_{t-1}^h)h_{h,t-1} = \\ &= (1+\gamma_0^h)(1+\gamma_1^h)...(1+\gamma_{t-1}^h)h_{h,0} = \\ &= \prod_{j=0}^{t-1}(1+\gamma_j^h)h_{h,0}, \end{aligned} \tag{E3}$$

where $h_{h,0}$ is the initial value of the individual human capital stock.

Similarly we have for $N_{b,t}$:

$$\begin{aligned} N_{b,t} &\equiv (1+\gamma_{t-1}^{n_b})N_{b,t-1} = \\ &= (1+\gamma_0^{n_b})(1+\gamma_1^{n_b})...(1+\gamma_{t-1}^{n_b})N_{b,0} = \\ &= \prod_{j=0}^{t-1}(1+\gamma_j^{n_b})N_{b,0}, \end{aligned} \tag{E.4}$$

where $N_{b,0}$ is the initial value of the stock of ideas.

Hence, we have for discounted lifetime utility or welfare:

$$\begin{aligned} U \equiv \sum_{t=0}^{\infty} \beta^t \Bigg[&\frac{\left(h_{h,0} N_{b,0}\right)^{1-\sigma}(\widetilde{c}_{h,t} + \lambda \widetilde{g}_t^c)^{1-\sigma}(1-l_{h,t}^w - l_{h,t}^e)^{\psi(1-\sigma)}}{1-\sigma} \times \\ &\times \frac{\left(\prod_{j=0}^{t-1}\left(1+\gamma_j^h\right)\right)^{1-\sigma}\left(\prod_{j=0}^{t-1}\left(1+\gamma_j^{n_b}\right)\right)^{1-\sigma}}{1-\sigma} \Bigg], \end{aligned} \tag{E.5}$$

where $h_{h,0}$ and $N_{b,0}$ are given by initial conditions. This sum will be bounded to the extent that $\beta(1+\gamma_t^h)^{1-\sigma}(1+\gamma_t^{n_b})^{1-\sigma} < 1$ at least after a point in time.

By definition, U is also the household's value function at the beginning of the time horizon. Thus, to compare two regimes, we denote discounted lifetime utilities as U^S and U^R, and then calculate the constant consumption subsidy, χ, that would make the household indifferent between them. Thus, χ solves $U^R = (1+\chi)^{1-\sigma} U^S$, i.e. $\chi = \left(\frac{U^R}{U^S}\right)^{\frac{1}{1-\sigma}} - 1$. If $\chi > 0$, regime R is superior, and vice versa.

We next consider welfare along the BGP, on which $\widetilde{c}_{h,t}$, $l^w_{h,t}$, $l^e_{h,t}$ remain constant. At the same time, individual human capital, ideas and population grow at constant rates so that, in this case, welfare simplifies to (we now omit time subscripts since all variables included here are constant over time):

$$U^{BGP} = \frac{\left(h_{h,0} N_{b,0}\right)^{1-\sigma} \left(\widetilde{c}_h + \lambda \widetilde{g}^c\right)^{1-\sigma} (1 - l^w_h - l^e_h)^{\psi(1-\sigma)}}{1-\sigma} \times \\ \times \sum_{t=0}^{\infty} \left[\beta \left(1+\gamma^h\right)^{1-\sigma} \left(1+\gamma^{n_b}\right)^{1-\sigma}\right]^t, \tag{E.6}$$

or

$$U^{BGP} = \frac{\left(h_{h,0} N_{b,0}\right)^{1-\sigma} \left(\widetilde{c}_h + \lambda \widetilde{g}^c\right)^{1-\sigma} \left(1 - l^w_h - l^e_h\right)^{\psi(1-\sigma)}}{(1-\sigma)\left[1 - \beta \left(1+\gamma^h\right)^{1-\sigma} \left(1+\gamma^{n_b}\right)^{1-\sigma}\right]}. \tag{E.6$'$}$$

Appendix F: Higher population growth

This Appendix studies what happens when population growth increases from 1.1% in the data to 1.2%. Jones (2019, 2022a, 2022b) argued that a larger population means more researchers, more ideas, and higher growth. On the other hand, a larger population size may reduce per capita output and welfare. Also, in our model, an increase in the supply of researchers will translate to the production of more ideas and, hence, higher growth only if firms find it profitable to increase their output.

Balanced growth path

Table F.1 shows that raising the exogenous population growth rate by 0.1 percentage points (from 1.1% to 1.2%) accelerates the ideas growth rate, γ^{n_b}, from 1.59% to 1.67%, a 5.1% increase. This is the familiar Romer–Jones channel: more people $\Rightarrow$ more researchers $\Rightarrow$ more ideas $\Rightarrow$ faster growth. In our decentralised setting, however, this operates through optimal labour demand in each sector rather than a fixed labour share, so the growth pickup is achieved with firms endogenously expanding research and production employment when it is profitable. Consequently, the long run per capita GDP growth, $\gamma^h + \gamma^{n_b}$, moves from 2.09% to 2.15%.

Table F.1: Higher population growth

Output & welfare drivers	Base	Shock	% diff.
$\widetilde{y}_f$: detrended final output	0.0407	0.0405	-0.5184
$\widetilde{c}_h$: detrended private consumption	0.0326	0.0325	-0.5212
l_h^w: hours at work	0.3100	0.3104	0.1305
l_h^e: hours in education	0.1051	0.1038	-1.2105
$1 - l_h^w - l_h^e$: leisure time	0.5849	0.5858	0.1483
γ^h: human capital growth	0.00500	0.00481	-3.8384
γ^{n_b}: ideas growth	0.01585	0.01667	5.1143
$\gamma^h + \gamma^{n_b}$: total growth	0.02085	0.02147	2.9677
Compensating consumption supplement		% gain	
χ_{bgp} : CCS on the BGP		2.4237	
χ_{tran} : CCS on the transition (long-run)		2.4115	

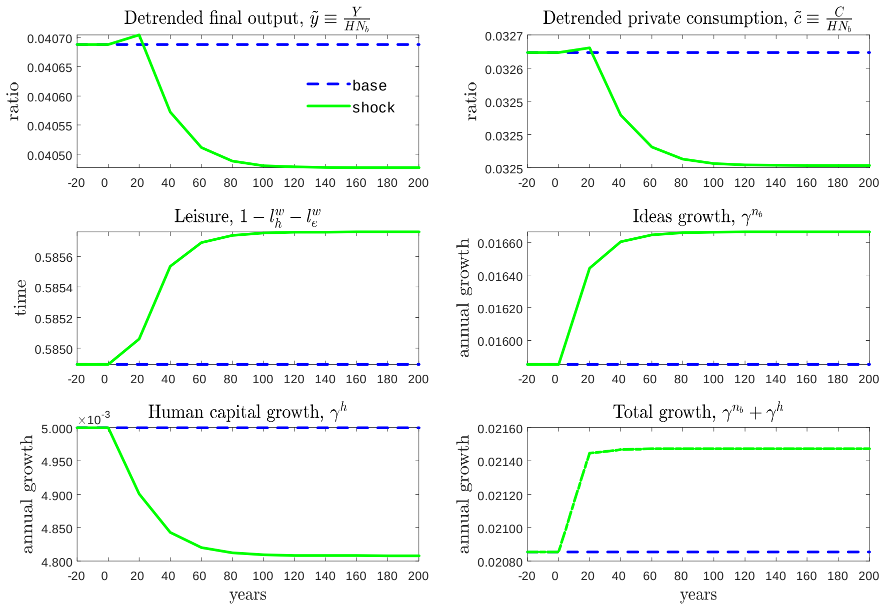
Despite stronger growth, detrended per capita private consumption falls slightly (a 0.52% decline). This is a denominator effect: with a larger population, per capita quantities are lower even as aggregate activity rises. Households also reallocate time: work hours, l_h^w, edge up, education time, l_h^e, declines, and leisure, $1 - l_h^w - l_h^e$, rises marginally, reflecting an optimal response to the new returns. Importantly, BGP welfare increases by 2.42% (the compensating consumption supplement, CCS), because the model's utility weights growth and leisure alongside consumption; the higher growth and slightly more leisure outweigh the small drop in detrended consumption. As in the regulatory cost experiment, discounted lifetime utility during the transition (2.4115%) is almost the same as welfare on the new BGP (2.4237%), reflecting the absence of significant intertemporal trade-offs.

Mechanically, this echoes the semi-endogenous growth logic noted in Jones (2019, 2022): population growth shifts the long-run path via ideas, although our model embeds human capital and public capital as additional channels, and firms choose labour inputs optimally rather than via exogenous shares.

Transition

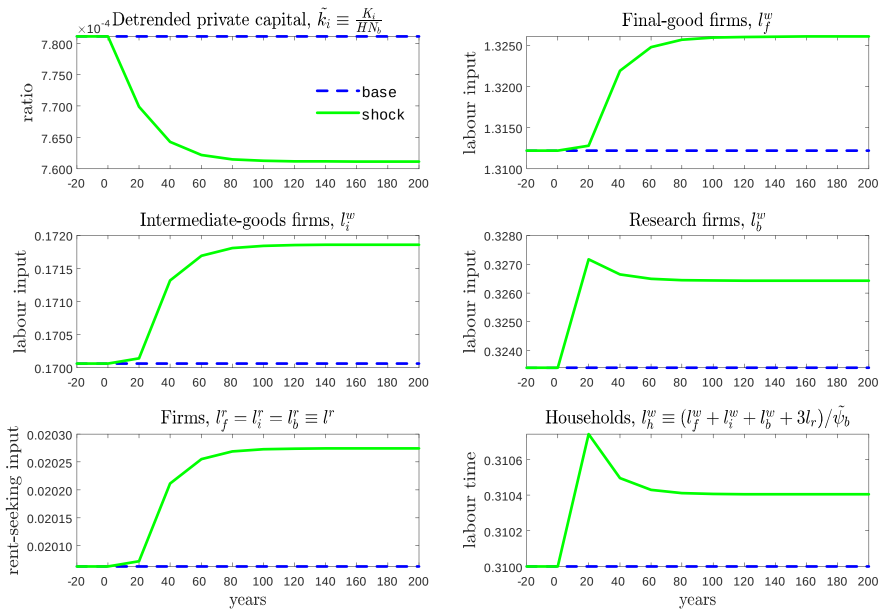
Along the transition to the new BGP, detrended output and total growth rise monotonically toward their higher long run levels, ideas growth, γ^{n_b}, steadily climbs, detrended per capita consumption dips slightly, and leisure ticks up. The paths are smooth and intuitive: the population shock expands productive capacity, and the economy reallocates optimally across work, education, and leisure, gradually converging to the higher-growth BGP.

Figure F.2a: Population shock (growth & welfare)



On the input side, Figure F.2b shows that productive labour in each sector, final goods, l_f^w, intermediate goods, l_i^w, and research, l_b^w, increases vs.

Figure F.2b: Population shock (inputs)



baseline, and rent seeking labour, l_r, rises modestly since the contestable pie expands with the economy (a larger public spending base implies a larger "prize" to contest). Private capital ultimately trends and higher profitability encourage investment. The core message is a bigger population allows firms to hire more across productive and research activities, which is what sustains the ideas channel and the higher γ^{n_b} over the transition.

Reform ranges

Finally, Table F.2 reports the values of variables shaping welfare on the BGP when we assume bigger increases in pupulation growth, say by 2 ppt and 3 ppt, instead of the 1 ppt assumed above. Following the same logic as discussed in Table F.1, the results are monotonic.

Table F.2: Higher population growth

	% gain		% dev. from base		
	χ_{bgp}	χ_{trn}	$\widetilde{c}_h$	$1 - l_h^w - l_h^e$	$\gamma^h + \gamma^{n_b}$
$\uparrow$ n by 0.1 ppt	2.4237	2.4115	-0.5212	0.1483	2.9678
$\uparrow$ n by 0.2 ppt	4.8671	4.8563	-1.0374	0.2955	5.9438
$\uparrow$ n by 0.3 ppt	7.3300	7.3347	-1.5489	0.4414	8.9287

Increasing the population growth shock from +0.1 ppt to +0.2 and +0.3 ppt produces monotonic changes in the same directions: larger welfare gains (CCS on BGP: 2.42%, 4.87%, 7.33%), bigger increases in total growth, $\gamma^h + \gamma^{n_b}$, (roughly +3.0, +5.94, +8.93 per cent deviations from the base), larger declines in detrended per capita consumption (from -0.52% down to -1.55%), and more leisure. The transition CCS mirrors the BGP pattern. The monotonicity across rows underscores that the ideas channel expands with population in this framework, while the per capita arithmetic continues to nudge $\widetilde{c}_h$ downward as the population denominator grows.