Optimal monetary policy in a two-country new Keynesian model with deep consumption habits: Technical Appendix

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Appendix A Aggregate Profits

Following Leith, Moldovan and Rossi (2009), we express the aggregate profits of intermediate goods firms as follows:

$$
\begin{aligned}
&\int_0^1 \int_0^1 \Phi_t^m(i,j)djdi \\
&= \int_0^1 \int_0^1 (P_t^m(i,j) - MC_t^n)Y_t(i,j)djdi \\
&= \int_0^1 \int_0^1 \left(P_{D,t}(i,j) - \frac{W_t}{A_t}\right)\left(\frac{P_t^m(i,j)}{P_t^m(i)}\right)^{-\xi}(Y_{D,t}(i) + Y_{EX,t}(i))djdi \\
&= \int_0^1 \int_0^1 \left[\frac{P_t^m(i,j)^{1-\xi}}{P_t^m(i)^{-\xi}}(Y_{D,t}(i) + Y_{EX,t}(i)) - \frac{W_t}{A_t}A_tN_t(i,j)\right]djdi \\
&= \int_0^1 (P_t^m(i))^{\xi}(Y_{D,t}(i) + Y_{EX,t}(i))\left(\int_0^1 (P_t^m)^{1-\xi}(i,j)dj\right)di - W_t\int_0^1\int_0^1 N_t(i,j)djdi \\
&= \int_0^1 (P_t^m(i))^{\xi}(Y_{D,t}(i) + Y_{EX,t}(i))(P_t(i)^m)^{1-\xi}di - W_tN_t \\
&= \int_0^1 (P_t^m(i))(Y_{D,t}(i) + Y_{EX,t}(i))di - W_tN_t
\end{aligned}
$$

Aggregate profits of final goods firms are:

$$
\begin{aligned}
&\int_0^1 \Phi_{D,t}(i)di + \int_0^1 \Phi_{M,t}^*(i)di \\
&= \int_0^1 (P_{D,t}(i) - P_t^m(i))Y_{D,t}(i)di + \int_0^1 (\mathcal{E}_tP_{M,t}^*(i) - P_t^m(i))Y_{EX,t}(i)di \\
&= \int_0^1 (P_{D,t}(i)Y_{D,t}(i) + \mathcal{E}_tP_{M,t}^*(i)Y_{EX,t}(i))di - \int_0^1 P_t^m(i)(Y_{D,t}(i) + Y_{EX,t}(i))di.
\end{aligned}
$$

Thus, aggregate profits in a home country are as follows:

$$\begin{aligned}
\Phi_t &\equiv \int_0^1 \int_0^1 \Phi_t^m(i,j)dj di + \int_0^1 \Phi_{D,t}(i)di + \int_0^1 \Phi_{M,t}^*(i)di \\
&= \int_0^1 (P_t^m(i))(Y_{D,t}(i) + Y_{EX,t}(i))di - W_t N_t \\
&+ \int_0^1 (P_{D,t}(i)Y_{D,t}(i) + \mathcal{E}_t P_{M,t}^*(i) Y_{EX,t}(i))di \\
&- \int_0^1 P_t^m(i)(Y_{D,t}(i) + Y_{EX,t}(i))di \\
&= \int_0^1 (P_{D,t}(i)Y_{D,t}(i) + \mathcal{E}_t P_{M,t}^*(i) Y_{EX,t}(i))di - W_t N_t \\
&= P_{D,t}Y_{D,t} + \mathcal{E}_t P_{M,t}^* Y_{EX,t} - W_t N_t
\end{aligned}$$

The first and second terms on the right-hand side of the last equation can be derived by combining the CES aggregation properties of the production function and its prices with the cost minimization problem from the final goods firm.

Appendix B System of Equilibrium Conditions

Complete-market condition

$$\left(\frac{1-\omega}{\omega}\right)^{\frac{1}{\eta}} \left(\frac{X_t^*}{X_t}\right)^{-\sigma+1/\eta} \left(\frac{X_{M,t}^*}{X_{D,t}}\right)^{-\eta} = \psi_t \tag{B.1}$$

Habit-adjusted aggregate consumption

$$X_t = \left(\omega^{\frac{1}{\eta}} X_{D,t}^{\frac{\eta-1}{\eta}} + (1-\omega)^{\frac{1}{\eta}} X_{M,t}^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}} \tag{B.2}$$

$$X_t^* = \left(\omega^{\frac{1}{\eta}} X_{D,t}^{*\frac{\eta-1}{\eta}} + (1-\omega)^{\frac{1}{\eta}} X_{M,t}^{*\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}} \tag{B.3}$$

From habit-adjusted consumption (**??**), assuming that goods i are symmetric for households, we can write the following:

$$X_{D,t} = C_{D,t} - \theta_D \bar{S}_{D,t-1} \tag{B.4}$$

$$X_{M,t} = C_{M,t} - \theta_M \bar{S}_{M,t-1} \tag{B.5}$$

$$X_{D,t}^* = C_{D,t}^* - \theta_D^* \bar{S}_{D,t-1}^* \tag{B.6}$$

$$X_{M,t}^* = C_{M,t}^* - \theta_M^* \bar{S}_{M,t-1}^* \tag{B.7}$$

The stocks of habit consumption

$$\bar{S}_{H,t} = \varrho_H \bar{S}_{H,t-1} + (1 - \varrho_H)\, C_{H,t} \tag{B.8}$$

$$\bar{S}_{F,t} = \varrho_F \bar{S}_{F,t-1} + (1 - \varrho_F)\, C_{F,t} \tag{B.9}$$

$$\bar{S}^*_{H,t} = \varrho^*_H \bar{S}^*_{H,t-1} + (1 - \varrho^*_H)\, C^*_{H,t} \tag{B.10}$$

$$\bar{S}^*_{F,t} = \varrho^*_F \bar{S}^*_{F,t-1} + (1 - \varrho^*_F)\, C^*_{F,t} \tag{B.11}$$

First order conditions

$$\left(\frac{\omega}{1-\omega}\right)^{1/\eta} \left(\frac{X_{D,t}}{X_{M,t}}\right)^{-1/\eta} = \frac{P_{D,t}}{P_{M,t}} = TOT_t^{-1} \tag{B.12}$$

$$\omega^{-1/\eta} \chi (X_t)^{\sigma - 1/\eta} X_{D,t}^{1/\eta} N_t^{v} = \frac{W_t}{P_{D,t}} = w_t \tag{B.13}$$

where $w_t \equiv W_t / P_{D,t}$.

Euler equation of consumption

$$1 = \beta E_t \left[\left(\frac{X_{t+1}}{X_t}\right)^{-\sigma + 1/\eta} \left(\frac{X_{D,t+1}}{X_{D,t}}\right)^{-1/\eta} \left(\frac{P_{D,t}}{P_{D,t+1}}\right) \right] R_t \tag{B.14}$$

$$1 = \beta E_t \left[\left(\frac{X^*_{t+1}}{X^*_t}\right)^{-\sigma + 1/\eta} \left(\frac{X^*_{D,t+1}}{X^*_{D,t}}\right)^{-1/\eta} \left(\frac{P^*_{D,t}}{P^*_{D,t+1}}\right) \right] R^*_t \tag{B.15}$$

the distribution of intermediate goods prices

$$(P_t^m)^{1-\xi} = \alpha (P_{t-1}^m)^{1-\xi} + (1-\alpha)(P_t^{mo}(j))^{1-\xi}$$

Therefore,

$$1 = \alpha \left(\frac{1}{\Pi_t^m}\right)^{1-\xi} + (1-\alpha) \left(\frac{P_t^{mo}}{P_t^m}\right)^{1-\xi} \tag{B.16}$$

The evolution of price dispersion

$$\Delta_t^m = \alpha (\Pi_t^m)^{\xi} \Delta_{t-1}^m + (1-\alpha) \left(\frac{P_t^{mo}}{P_t^m}\right)^{-\xi} \tag{B.17}$$

Optimal intermediate price

$$\frac{P_t^{mo}}{P_{D,t}} = \left(\frac{\zeta_t}{\zeta_t - 1}\right) \frac{K_{1,t}}{K_{2,t}} \tag{B.18}$$

$$\frac{P_t^{mo*}}{P^*_{D,t}} = \left(\frac{\zeta^*_t}{\zeta^*_t - 1}\right) \frac{K^*_{1,t}}{K^*_{2,t}} \tag{B.19}$$

where

$$K_{1,t} = X_t^{-\sigma+1/\eta} X_{D,t}^{-1/\eta} mc_t \left(\frac{P_{D,t}}{P_t^m}\right)^{-\xi} Y_t + \alpha\beta E_t \left[K_{1,t+1}(\Pi_{D,t+1})^{\xi}\right] \tag{B.20}$$

$$K_{1,t}^* = (X_t^*)^{-\sigma+1/\eta} (X_{D,t}^*)^{-1/\eta} mc_t^* \left(\frac{P_{D,t}^*}{P_t^{m*}}\right)^{-\xi} Y_t^* + \alpha\beta E_t \left[K_{1,t+1}^*(\Pi_{D,t+1}^*)^{\xi}\right] \tag{B.21}$$

and

$$K_{2,t} = X_t^{-\sigma+1/\eta} X_{D,t}^{-1/\eta} \left(\frac{P_{D,t}}{P_t^m}\right)^{-\xi} Y_t + \alpha\beta E_t \left[K_{2,t+1}(\Pi_{D,t+1})^{\xi-1}\right] \tag{B.22}$$

$$K_{2,t}^* = (X_t^*)^{-\sigma+1/\eta} (X_{D,t}^*)^{-1/\eta} \left(\frac{P_{D,t}^*}{P_t^{m*}}\right)^{-\xi} Y_t^* + \alpha\beta E_t \left[K_{2,t+1}^*(\Pi_{D,t+1}^*)^{\xi-1}\right] \tag{B.23}$$

Where $\Pi_{D,t} = P_{D,t}/P_{D,t-1}$.

When intermediate goods firms can decide their price, subscript j can be removed from its optimal price since all firms behave in the same manner, $P_t^{mo}(j) = P_t^{mo}$ for all j.

Aggregate production function

$$Y_t = Y_{D,t} + Y_{EX,t} = \frac{A_t}{\Delta_t} N_t \tag{B.24}$$

$$Y_t^* = Y_{D,t}^* + Y_{EX,t}^* = \frac{A_t^*}{\Delta_t^*} N_t^* \tag{B.25}$$

Aggregate resource constraints are derived from home and foreign households' budget constraints,

$$\begin{aligned} P_{D,t}Y_{D,t} + \mathcal{E}_t P_{M,t}^* Y_{EX,t} &= P_{D,t}C_{D,t} + \mathcal{E}_t P_{M,t}^* C_{M,t}^* \\ \Rightarrow Y_{D,t} + \psi_t Y_{EX,t} &= C_{D,t} + \psi_t C_{M,t}^* \end{aligned} \tag{B.26}$$

$$\begin{aligned} P_{D,t}^*Y_{D,t}^* + P_{M,t} Y_{EX,t}/\mathcal{E}_t &= P_{D,t}^*C_{D,t}^* + P_{M,t} C_{M,t}/\mathcal{E}_t \\ \Rightarrow Y_{D,t}^* + \psi_t^* Y_{EX,t}^* &= C_{D,t}^* + \psi_t^* C_{M,t} \end{aligned} \tag{B.27}$$

Behavior of deep habit formation

$$\Lambda_{D,t}(i) = (P_{D,t}(i) - P_t^m(i)) + \theta_D E_t \left[Q_{t,t+1}\Lambda_{D,t+1}(i)\right]$$

$$Q_{t,t+1} = \beta \left(\frac{X_{t+1}}{X_t}\right)^{-\sigma+1/\eta} \left(\frac{X_{D,t}}{X_{D,t+1}}\right)^{1/\eta} \left(\frac{P_{D,t}}{P_{t+1}}\right)$$

Dropping i and eliminate $Q_{t,t+1}$ from the first equation,

$$\Lambda_{D,t} = (P_{D,t} - P_t^m) + \beta\theta_D E_t \left[\left(\frac{X_{t+1}}{X_t}\right)^{-\sigma+1/\eta} \left(\frac{X_{D,t}}{X_{D,t+1}}\right)^{1/\eta} \left(\frac{P_{D,t}}{P_{D,t+1}}\right) \Lambda_{D,t+1}\right]$$

$$\Rightarrow \lambda_{D,t} = \left(1 - \frac{P_t^m}{P_{D,t}}\right) + \beta\theta_D E_t \left[\left(\frac{X_{t+1}}{X_t}\right)^{-\sigma+1/\eta} \left(\frac{X_{D,t}}{X_{D,t+1}}\right)^{1/\eta} \lambda_{D,t+1}\right] \tag{B.28}$$

where $\lambda_{D,t} \equiv \Lambda_{D,t}/P_{D,t}$ is Lagrange multiplier in real terms. And,

$$C_{D,t} = \epsilon \lambda_{D,t} X_{D,t} \tag{B.29}$$

Similarly,

$$\Lambda^*_{M,t} = (\mathcal{E}_t P^*_{M,t} - P^m_t) + \beta\theta^*_M E_t \left[\left(\frac{X_{t+1}}{X_t} \right)^{-\sigma+1/\eta} \left(\frac{X_{D,t}}{X_{D,t+1}} \right)^{1/\eta} \left(\frac{P_{D,t}}{P_{D,t+1}} \right) \Lambda^*_{M,t+1} \right]$$

Dividing both sides by $\mathcal{E}_t P^*_{M,t}$ and using $\psi_t = \mathcal{E}_t P^*_{M,t}/P_{D,t}$,

$$\lambda^*_{M,t} = \left(1 - \frac{P^m_t}{\psi_t P_{D,t}} \right) + \beta\theta^*_M E_t \left[\left(\frac{X_{t+1}}{X_t} \right)^{-\sigma+1/\eta} \left(\frac{X_{D,t}}{X_{D,t+1}} \right)^{1/\eta} \left(\frac{\psi_{t+1}}{\psi_t} \right) \lambda^*_{M,t+1} \right] \tag{B.30}$$

where $\lambda^*_{M,t} = \Lambda^*_{M,t}/(\mathcal{E}_t P^*_{M,t})$. Also,

$$C^*_{M,t} = \epsilon \lambda^*_{M,t} X^*_{M,t} \tag{B.31}$$

The demand for goods produced in a foreign country, considering the deep habit, can be derived as follows:

$$\Lambda^*_{D,t}(i) = (P^*_{D,t}(i) - P^{m*}_t(i)) + \theta^*_D E_t \left[Q^*_{t,t+1} \Lambda^*_{D,t+1}(i) \right]$$
$$Q^*_{t,t+1} = \beta \left(\frac{X^*_{t+1}}{X^*_t} \right)^{-\sigma+1/\eta} \left(\frac{X^*_{D,t}}{X^*_{D,t+1}} \right)^{1/\eta} \left(\frac{P^*_{D,t}}{P^*_{t+1}} \right)$$

Dropping i and eliminate $Q^*_{t,t+1}$ from the first equation,

$$\Lambda^*_{D,t} = (P^*_{D,t} - P^{m*}_t) + \beta\theta^*_D E_t \left[\left(\frac{X^*_{t+1}}{X^*_t} \right)^{-\sigma+1/\eta} \left(\frac{X^*_{D,t}}{X^*_{D,t+1}} \right)^{1/\eta} \left(\frac{P^*_{D,t}}{P^*_{D,t+1}} \right) \Lambda^*_{D,t+1} \right]$$

$$\Rightarrow \lambda^*_{D,t} = \left(1 - \frac{P^{m*}_t}{P^*_{D,t}} \right) + \beta\theta^*_D E_t \left[\left(\frac{X^*_{t+1}}{X^*_t} \right)^{-\sigma+1/\eta} \left(\frac{X^*_{D,t}}{X^*_{D,t+1}} \right)^{1/\eta} \lambda^*_{D,t+1} \right] \tag{B.32}$$

where $\lambda^*_{D,t} \equiv \Lambda^*_{D,t}/P^*_{D,t}$ is Lagrange multiplier in real terms. And,

$$C^*_{D,t} = \epsilon \Lambda^*_{D,t} (P^*_{D,t})^{-1} X^*_{D,t} \Rightarrow C^*_{D,t} = \epsilon \lambda^*_{D,t} X^*_{D,t} \tag{B.33}$$

Similarly,

$$\Lambda_{M,t} = (P_{M,t}/\mathcal{E}_t - P^{m*}_t) + \beta\theta_M E_t \left[\left(\frac{X^*_{t+1}}{X^*_t} \right)^{-\sigma+1/\eta} \left(\frac{X^*_{D,t}}{X^*_{D,t+1}} \right)^{1/\eta} \left(\frac{P^*_{D,t}}{P^*_{D,t+1}} \right) \Lambda_{M,t+1} \right]$$

Dividing both sides by $P_{M,t}/\mathcal{E}_t$ and using $\psi_t^* = \frac{P_{M,t}/\mathcal{E}_t}{P_{D,t}^*}$,

$$\frac{\Lambda_{M,t}}{P_{M,t}/\mathcal{E}_t} = \left(1 - \frac{P_t^{m*}}{\psi_t^* P_{D,t}^*}\right) + \beta\theta_M E_t \left[\left(\frac{X_{t+1}^*}{X_t^*}\right)^{-\sigma+1/\eta} \left(\frac{X_{D,t}^*}{X_{D,t+1}^*}\right)^{1/\eta} \left(\frac{\psi_{t+1}^*}{\psi_t^*}\right) \left(\frac{\Lambda_{M,t+1}^*}{\mathcal{E}_{t+1} P_{M,t+1}^*}\right)\right]$$

$$\Rightarrow \lambda_{M,t} = \left(1 - \frac{P_t^{m*}}{\psi_t^* P_{D,t}^*}\right) + \beta\theta_M E_t \left[\left(\frac{X_{t+1}^*}{X_t^*}\right)^{-\sigma+1/\eta} \left(\frac{X_{D,t}^*}{X_{D,t+1}}\right)^{1/\eta} \left(\frac{\psi_{t+1}^*}{\psi_t^*}\right) \lambda_{M,t+1}\right] \quad \text{(B.34)}$$

where $\lambda_{M,t} = \Lambda_{M,t}/(P_{M,t}/\mathcal{E}_t)$. Also,

$$C_{M,t} = \epsilon \Lambda_{M,t} \frac{1}{P_{M,t}/\mathcal{E}_t} X_{M,t}$$

$$\Rightarrow C_{M,t} = \epsilon \lambda_{M,t} X_{M,t} \quad \text{(B.35)}$$

Inflation, relative prices and real variables

Terms of trade

$$TOT_t = \frac{P_{M,t}}{P_{D,t}} \quad \text{(B.36)}$$

$$TOT_t^* = \frac{P_{M,t}^*}{P_{D,t}^*} \quad \text{(B.37)}$$

Good-specific real exchange rate

$$\psi_t = \frac{\mathcal{E}_t P_{M,t}^*}{P_{D,t}} \quad \text{(B.38)}$$

$$\psi_t^* = \frac{P_{M,t}/\mathcal{E}_t}{P_{D,t}^*} \quad \text{(B.39)}$$

Terms of trade and good-specific real exchange rates are combined as follows:

$$\psi_t \psi_t^* = \frac{\mathcal{E}_t P_{M,t}^* P_{M,t}/\mathcal{E}_t}{P_{D,t} P_{D,t}^*} = TOT_t \cdot TOT_t^* \quad \text{(B.40)}$$

Optimal relative intermediate goods price

$$p_t^{mo} \equiv \frac{P_t^{mo}}{P_{D,t}} \quad \text{(B.41)}$$

$$p_t^{mo*} \equiv \frac{P_t^{mo*}}{P_{D,t}^{m*}} \quad \text{(B.42)}$$

Price markup of final goods firms

$$\mu_t \equiv \frac{P_{D,t}}{P_t^m} \quad \text{(B.43)}$$

$$\mu_t^* \equiv \frac{P_{D,t}^*}{P_t^{m*}} \quad \text{(B.44)}$$

We can express the relative optimal price to the average price of intermediate goods as follows:

$$\frac{P_t^{mo}}{P_t^m} = \frac{P_t^{mo}}{P_{D,t}}\frac{P_{D,t}}{P_t^m} = p_t^{mo}\mu_t \tag{B.45}$$

$$\frac{P_t^{mo*}}{P_t^{m*}} = p_t^{mo*}\mu_t^* \tag{B.46}$$

Intermediate goods price inflation

$$\Pi_t^m = \frac{P_t^m}{P_{t-1}^m} \tag{B.47}$$

$$\Pi_t^{m*} = \frac{P_t^{m*}}{P_{t-1}^{m*}} \tag{B.48}$$

Domestic price inflation

$$\Pi_{D,t} \equiv \frac{P_{D,t}}{P_{D,t-1}} = \frac{P_{D,t}}{P_t^m}\frac{P_{t-1}^m}{P_{D,t-1}}\frac{P_t^m}{P_{t-1}^m} = \frac{\mu_t}{\mu_{t-1}}\Pi_t^m \tag{B.49}$$

$$\Pi_{D,t}^* = \frac{\mu_t^*}{\mu_{t-1}^*}\Pi_t^{m*} \tag{B.50}$$

Real wage

$$w_t = \frac{W_t}{P_{D,t}} \tag{B.51}$$

$$w_t^* = \frac{W_t^*}{P_{D,t}^*} \tag{B.52}$$

Real marginal cost

$$mc_t = \frac{MC_t}{P_{D,t}} = \frac{w_t}{A_t} \tag{B.53}$$

$$mc_t^* = \frac{w_t^*}{A_t^*} \tag{B.54}$$

Appendix C Steady state

$$\left(\frac{1-\omega}{\omega}\right)^{\frac{1}{\eta}}\left(\frac{X^*}{X}\right)^{-\sigma+1/\eta}\left(\frac{X_M^*}{X_D}\right)^{-\eta} = \psi \tag{C.1}$$

$$X = \left(\omega^{\frac{1}{\eta}}X_D^{\frac{\eta-1}{\eta}} + (1-\omega)^{\frac{1}{\eta}}X_M^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}} \tag{C.2}$$

$$X^* = \left(\omega^{\frac{1}{\eta}}X_D^{*\frac{\eta-1}{\eta}} + (1-\omega)^{\frac{1}{\eta}}X_M^{*\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}} \tag{C.3}$$

$$X_D = C_D - \theta_D S_D \tag{C.4}$$

$$X_M = C_M - \theta_M S_M \tag{C.5}$$

$$X_D^* = C_D^* - \theta_D^* S_D^* \tag{C.6}$$

$$X_M^* = C_M^* - \theta_M^* S_M^* \tag{C.7}$$

$$S_D = C_D \tag{C.8}$$

$$S_M = C_M \tag{C.9}$$

$$S_D^* = C_D^* \tag{C.10}$$

$$S_M^* = C_M^* \tag{C.11}$$

$$\left(\frac{\omega}{1-\omega}\right)^{1/\eta}\left(\frac{X_D}{X_M}\right)^{-1/\eta} = TOT^{-1} \tag{C.12}$$

$$\left(\frac{\omega}{1-\omega}\right)^{1/\eta}\left(\frac{X_D^*}{X_M^*}\right)^{-1/\eta} = TOT(\psi\psi^*)^{-1} \tag{C.13}$$

$$\omega^{-1/\eta}\chi(X)^{\sigma-1/\eta}(X_D)^{1/\eta}(N)^{\upsilon} = w \tag{C.14}$$

$$\omega^{-1/\eta}\chi(X^*)^{\sigma-1/\eta}(X_D^*)^{1/\eta}(N^*)^{\upsilon} = w^* \tag{C.15}$$

$$1 = \beta\Pi_D^{-1}R \tag{C.16}$$

$$1 = \beta\Pi_D^{*-1}R^* \tag{C.17}$$

$$1 = \alpha(\Pi^m)^{-1+\xi} + (1-\alpha)(p^{mo}\mu)^{1-\xi} \tag{C.18}$$

$$1 = \alpha(\Pi^{m*})^{-1+\xi} + (1-\alpha)(p^{mo*}\mu^*)^{1-\xi} \tag{C.19}$$

$$\Delta^m = \frac{(1-\alpha)(p^{mo}\mu)^{-\xi}}{1-\alpha(\Pi^m)^{\xi}} \tag{C.20}$$

$$\Delta^{m*} = \frac{(1-\alpha)(p^{mo*}\mu^*)^{-\xi}}{1-\alpha(\Pi^{m*})^{\xi}} \tag{C.21}$$

$$p^{mo} = \left(\frac{\zeta}{\zeta-1}\right)\frac{K_1}{K_2}$$

$$p^{mo*} = \left(\frac{\zeta^*}{\zeta^*-1}\right)\frac{K_1^*}{K_2^*}$$

$$K_1 = \frac{(X)^{-\sigma+1/\eta}(X_D)^{-1/\eta}mc(\mu)^{-\xi}Y}{1-\alpha\beta(\Pi_D)^{\xi}} \tag{C.22}$$

$$K_1^* = \frac{(X^*)^{-\sigma+1/\eta}(X_D^*)^{-1/\eta}mc^*(\mu^*)^{-\xi}Y^*}{1-\alpha\beta(\Pi_D^*)^{\xi}} \tag{C.23}$$

$$K_2 = \frac{(X)^{-\sigma+1/\eta}(X_D)^{-1/\eta}\mu^{-\xi}Y}{1-\alpha\beta(\Pi_D)^{\xi-1}} \tag{C.24}$$

$$K_2^* = \frac{(X^*)^{-\sigma+1/\eta}(X_D^*)^{-1/\eta}(\mu^*)^{-\xi}Y^*}{1-\alpha\beta(\Pi_D^*)^{\xi-1}} \tag{C.25}$$

$$\Rightarrow (p^{mo})^{-1} = \left(\frac{\zeta}{\zeta-1}\right)\frac{1-\alpha\beta(\Pi_D)^{\xi-1}}{1-\alpha\beta(\Pi_D)^{\xi}}mc \tag{C.26}$$

$$(p^{mo*})^{-1} = \left(\frac{\zeta^*}{\zeta^*-1}\right)\frac{1-\alpha\beta(\Pi_D^*)^{\xi-1}}{1-\alpha\beta(\Pi_D^*)^{\xi}}mc^* \tag{C.27}$$

$$Y = \frac{A}{\Delta^m}N \tag{C.28}$$

$$Y^* = \frac{A^*}{\Delta^{m*}}N^* \tag{C.29}$$

$$mc = \frac{w}{A} \tag{C.30}$$

$$mc^* = \frac{w^*}{A^*} \tag{C.31}$$

$$Y_D + \psi Y_{EX} = C_D + \psi_t C_M^* \tag{C.32}$$

$$Y_D^* + \psi_t^* Y_{EX}^* = C_D^* + \psi^* C_M \tag{C.33}$$

$$\mu = [1-(1-\beta\theta_D)\lambda_D]^{-1} \tag{C.34}$$

$$\psi\mu = [1-(1-\beta\theta_M^*)\lambda_M^*]^{-1} \tag{C.35}$$

$$\mu^* = [1-(1-\beta\theta_D^*)\lambda_D^*]^{-1} \tag{C.36}$$

$$\psi^*\mu^* = [1-(1-\beta\theta_M)\lambda_M]^{-1} \tag{C.37}$$

$$C_D = \epsilon \lambda_D X_D \tag{C.38}$$

$$C_M^* = \epsilon \lambda_M^* X_M^* \tag{C.39}$$

$$C_D^* = \epsilon \lambda_D^* X_D^* \tag{C.40}$$

$$C_M = \epsilon \lambda_M X_M \tag{C.41}$$

$$A = 1 \tag{C.42}$$

$$A^* = 1 \tag{C.43}$$

$$\zeta = \xi \tag{C.44}$$

$$\zeta^* = \xi^* \tag{C.45}$$

$$\Pi^m = \Pi_D \tag{C.46}$$

$$\Pi^{*m} = \Pi_D^* \tag{C.47}$$

For tractability, we make some simplifying and symmetry assumptions.

Eliminating λ, C in both countries

$$\begin{aligned} \Rightarrow \mu &= \left[1 - \frac{1-\beta\theta_D}{(1-\theta_D)\epsilon}\right]^{-1} \\ \psi\mu &= \left[1 - \frac{1-\beta\theta_M^*}{(1-\theta_M^*)\epsilon}\right]^{-1} \\ \mu^* &= \left[1 - \frac{(1-\beta\theta_D^*)}{(1-\theta_D^*)\epsilon}\right]^{-1} \\ \psi^*\mu^* &= \left[1 - \frac{(1-\beta\theta_M)}{(1-\theta_M)\epsilon}\right]^{-1} \end{aligned}$$

Symmetry assumptions, $\theta_D = \theta_D^*, \theta_M = \theta_M^*$.

$$\begin{aligned} \Rightarrow \mu &= \mu^* \\ \psi &= \psi^* \end{aligned}$$

Further assumptions, $\theta_D = \theta_M$. $\psi = \psi^* = 1$

$$\left(\frac{1-\omega}{\omega}\right)^{\frac{1}{\eta}} \left(\frac{X_M^*}{X_D}\right)^{-\eta} = 1$$

Finally, assuming $\Pi^m = \Pi_D = 1$, steady state prices are determined, and the deterministic steady state system equations are uniquely determined analytically.

Appendix D Welfare gains from international cooperation

We derive the gains from international cooperation in terms of steady-state consumption.[1]

$$E_t \sum_{j=0}^{\infty} \beta^j \left[(1-\gamma) U((1+\Omega) X_{t+j}^b, N_{t+j}^b) + \gamma U^*(C_{t+j}^{*b}, N_{t+j}^{*b}) \right] = (1-\gamma) W_t^a + \gamma W_t^{*a}$$

where superscript b and a indicate Nash equilibrium and cooperative equilibrium, respectively. Ω represents the unit of steady-state consumption that must be given up to achieve the same level of welfare as the Nash equilibrium in international cooperation.

Define the global welfare as follows:

$$W_{w,t}^i \equiv (1-\gamma) W_t^i + \gamma W_t^{*i}, \text{ for } i \in [b, a].$$

By using this, the welfare gain of cooperation from non-cooperation can be calculated as follows:

$$\begin{aligned}
& W_{w,t}^a - W_{w,t}^b \\
= & E_t \sum_{j=0}^{\infty} \beta^j \left[(1-\gamma) U((1+\Omega) X_{t+j}^b, N_{t+j}^b) + \gamma U(C_{t+j}^{*b}, N_{t+j}^{*b}) \right] \\
& - E_t \sum_{j=0}^{\infty} \beta^j \left[(1-\gamma) U(X_{t+j}^b, N_{t+j}^b) + \gamma U(C_{t+j}^{*b}, N_{t+j}^{*b}) \right] \\
= & E_t \sum_{j=0}^{\infty} \beta^j \left[(1-\gamma) \left\{ U((1+\Omega) X_{t+j}^b, N_{t+j}^b) - U(X_{t+j}^b, N_{t+j}^b) \right\} \right]
\end{aligned}$$

We specify the period utility function as $U(X_t, N_t) = \dfrac{(X_t)^{1-\sigma}}{1-\sigma} - \chi \dfrac{(N_t)^{1+\upsilon}}{1+\upsilon}$,

$$\begin{aligned}
& W_{w,t}^a - W_{w,t}^b \\
= & E_t \sum_{j=0}^{\infty} \beta^j \left[(1-\gamma) \left\{ \frac{\left((1+\Omega) X_{t+j}^b\right)^{1-\sigma}}{1-\sigma} - \chi \frac{\left(N_{t+j}^b\right)^{1+\upsilon}}{1+\upsilon} - \frac{\left(X_{t+j}^b\right)^{1-\sigma}}{1-\sigma} + \chi \frac{\left(N_{t+j}^b\right)^{1+\upsilon}}{1+\upsilon} \right\} \right] \\
= & E_t \sum_{j=0}^{\infty} \beta^j \left[(1-\gamma) \left\{ \frac{\left((1+\Omega) X_{t+j}^b\right)^{1-\sigma}}{1-\sigma} - \frac{\left(X_{t+j}^b\right)^{1-\sigma}}{1-\sigma} \right\} \right] \\
= & E_t \sum_{j=0}^{\infty} \beta^j \left[(1-\gamma) \left\{ \frac{\left((1+\Omega)^{1-\sigma} - 1\right) (X_{t+j}^b)^{1-\sigma}}{1-\sigma} \right\} \right]
\end{aligned}$$

Evaluating this at the steady state, $X^b_{t+j} = X$,

$$\begin{aligned}
W^a_{w,t} - W^b_{w,t} &= E_t \sum_{j=0}^{\infty} \beta^j \left[(1-\gamma) \left\{ \frac{\left((1+\Omega)^{1-\sigma} - 1\right) (X)^{1-\sigma}}{1-\sigma} \right\} \right] \\
&= \left[(1-\gamma) \left\{ \frac{\left((1+\Omega)^{1-\sigma} - 1\right) (X)^{1-\sigma}}{1-\sigma} \right\} \right] E_t \sum_{j=0}^{\infty} \beta^j \\
&= \frac{(1-\gamma)\left((1+\Omega)^{1-\sigma} - 1\right) (X)^{1-\sigma}}{(1-\sigma)(1-\beta)}
\end{aligned}$$

Therefore,

$$\Omega = \left[(W^a_{w,t} - W^b_{w,t})(1-\sigma)(1-\beta)(1-\gamma)^{-1} X^{\sigma-1} + 1 \right]^{\frac{1}{1-\sigma}} - 1$$

References

Leith, Campbell, Ioana Moldovan, and Raffaele Rossi (2009) “Monetary and Fiscal Policy under Deep Habits,” SIRE Discussion Papers 2009-47, Scottish Institute for Research in Economics (SIRE).