Online Appendix
Auxiliary State Method: Theory and Application*

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*I would like to thank Francois Gourio, Thomas Lubik, and several anonymous referees for useful comments.

1 Appendix A: Model summary

The equilibrium of the model consists of following nonlinear equations.

- Euler equation

$$E_t\left[M_{t,t+1}\left(\frac{R_t}{\Pi_{t+1}}\right)\right]=1, \tag{1}$$

where R_t is the gross short-term interest rate; $\Pi_t = P_t/P_{t-1}$ is the gross inflation rate; $M_{t,t+1}$ is the stochastic discount factor:

$$M_{t,t+1}=\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}, \tag{2}$$

where C denotes composite consumption; β and $1/\gamma$ are the subjective discount factor and the intertemporal elasticity of substitution, respectively.

- Gross wage inflation

$$\Pi_t^w=\frac{w_t}{w_{t-1}}\Pi_t, \tag{3}$$

where $w_t = W_t/P_t$ is the real wage; $\Pi_t^w = W_t/W_{t-1}$ is the gross wage inflation rate.

- New Keynesian Phillips Curve for wages

$$\begin{aligned} 0 \;=\; & (1-\varepsilon_w)\left(1-\Phi\left(\Pi_t^w\right)\right)N_t-\Phi'\left(\Pi_t^w\right)\Pi^w N_t+\varepsilon_w\chi\frac{N_t^{\eta+1}}{w_t C_t^{-\gamma}} \\ & +E_t\left[M_{t,t+1}\frac{\Phi'\left(\Pi_{t+1}^w\right)\left(\Pi^w\right)^2}{\Pi_{t+1}}N_{t+1}\right], \end{aligned} \tag{4}$$

where N denotes composite labor; $\varepsilon_w, 1/\eta, \chi$ are the elasticity of substitution among differentiated types of labor, the elasticity of labor supply, and the weight of working in the per-period flow utility, respectively; $\Phi(\cdot)$ is the wage adjustment cost:

$$\Phi(x)=\phi_w\left(\frac{\exp\left(-\psi_w\left(x-\overline{\Pi}\right)\right)+\psi_w\left(x-\overline{\Pi}\right)-1}{\psi_w^2}\right), \tag{5}$$

where ϕ_w is the level parameter and ψ_w is the asymmetry parameter. If $\psi_w > 0$, the wage adjustment cost is asymmetric. In particular, the cost to lower a wage relative to the reference wage is higher than to increase it by the same amount. When ψ_w approaches 0, this function becomes a symmetric quadratic function

$$\Phi(x) = \frac{\phi_w}{2}\left(x - \overline{\Pi}\right)^2.$$

Note that when $\phi_w = 0$ and $\varepsilon_w \to \infty$, equation (4) becomes a standard marginal rate of substitution between labor and consumption

$$\frac{\chi N_t^{\eta}}{C_t^{-\gamma}} = w_t.$$

- The New Keynesian Phillips curve

$$\left(1 - \varepsilon + \varepsilon \frac{w_t}{Z_t} - \Pi_t \Gamma'(\Pi_t)\right) Y_t + E_t\left(M_{t,t+1}\Pi_{t+1}\Gamma'(\Pi_{t+1}) Y_{t+1}\right) = 0. \tag{6}$$

where Y denotes aggregate output; Z is the technology shock:

$$\begin{aligned} \ln(Z_t) &= \rho_Z \ln(Z_{t-1}) + \varepsilon_{Z,t}, \\ \varepsilon_{Z,t} &\sim i.i.d\ N\left(0, \sigma_Z^2\right), \end{aligned} \tag{7}$$

and $\Gamma(\cdot)$ is the price adjustment cost function:

$$\Gamma(x) = \phi_p \left(\frac{\exp\left(-\psi_p\left(x - \overline{\Pi}\right)\right) + \psi_p\left(x - \overline{\Pi}\right) - 1}{\psi_p^2} \right), \tag{8}$$

where ϕ_p, ψ_p are parameters that determines the level and the asymmetry of price adjustment costs. If $\psi_p > 0$, the price adjustment cost is asymmetric. Particularly, the cost to lower a price relative to the reference price is higher than to increase it by the same amount. The linex function nests the symmetric quadratic cost when ψ_p approaches 0, i.e., it becomes a quadratic function

$$\Gamma(x) = \frac{\phi_p}{2}\left(x - \overline{\Pi}\right)^2.$$

- Taylor rule

$$R_t = R^* \left(\frac{GDP_t}{GDP^*}\right)^{\phi_y} \left(\frac{\Pi_t}{\Pi^*}\right)^{\phi_\pi} \tag{9}$$

where $GDP_t \equiv C_t$ denotes the gross domestic product (GDP); GDP^* and Π^* denote the target GDP and inflation, respectively; R^* denotes the intercept of the Taylor rule; ϕ_π, ϕ_y represents how aggressive the central bank responds to stabilizing inflation and output.

- Aggregate output

$$Y_t = Z_t N_t, \tag{10}$$

- The resource constraint

$$C_t = (1 - \Gamma(\Pi_t)) Y_t - w_t N_t \Phi\left(\Pi_t^w\right). \tag{11}$$

The equilibrium of the model is governed by the system of seven nonlinear difference equations (1), (3), (4), (6), (9), (10), (11) with respect to seven variables w_t, C_t, R_t, Π_t, Π_t^w, N_t, and Y_t.

2 Appendix B: Deriving the New Keynesian Phillips Curve for wages

We can write household h's problem in the following Bellman equation:

$$V^h\left(B_{t-1}^h, W_{t-1}^h, Z_t\right) = \underbrace{\text{Max}}_{\{C_t^h, N_t^h, W_t^h, B_t^h\}} \left\{ \begin{array}{c} (1-\beta)\left(\frac{\left(C_t^h\right)^{1-\gamma}}{1-\gamma} - \chi\frac{\left(N_t^h\right)^{1+\eta}}{1+\eta}\right) \\ +\beta\left(E_t\left[V^h\left(B_t^h, W_t^h, Z_{t+1}\right)\right]\right) \end{array} \right\} \tag{12}$$

subject to

$$P_t C_t^h + R_t^{-1} B_t^h = W_t^h N_t^h \left(1 - \Phi_t^h\right) + B_{t-1}^h + D_t^h + T_t^h + AC_t^h \tag{13}$$

$$N_t^h = \left(\frac{W_t^h}{W_t}\right)^{-\epsilon_w} N_t \tag{14}$$

where

$$\Phi_t^h = \Phi\left(\frac{W_t^h}{W_{t-1}^h}\right) = \phi_w \left(\frac{\exp\left(-\psi_w\left(\frac{W_t^h}{W_{t-1}^h} - \overline{\Pi}\right)\right) + \psi_w\left(\frac{W_t^h}{W_{t-1}^h} - \overline{\Pi}\right) - 1}{\psi_w^2}\right). \tag{15}$$

Let λ_t^{h1} and λ_t^{h2} be the Lagrangian multipliers for the budget constraint and the labor demand at time t, respectively. The first-order conditions include

$$(C_t) : \left(C_t^h\right)^{-\gamma} - P_t \lambda_t^{h1} = 0, \tag{16}$$

$$(N_t) : -(1-\beta)\chi\left(N_t^h\right)^{\eta} + W_t^h\left(1 - \Phi_t^h\right)\lambda_t^{h1} - \lambda_t^{h2} = 0, \tag{17}$$

$$\begin{aligned}(W_t) \quad : \quad 0 = {} & \beta V_w^h\left(B_t^h, W_t^h, Z_{t+1}\right) \\ & \left(N_t^h\left(1-\Phi_t^h\right) - W_t^h N_t^h \left(\Phi_t^h\right)' \frac{1}{W_{t-1}^h}\right)\lambda_t^{h1} - \varepsilon_w \frac{N_t^h}{W_t^h}\lambda_t^{h2}.\end{aligned} \tag{18}$$

$$(B_t) : \beta V_B^h\left(B_t^h, W_t^h, Z_{t+1}\right) - R_t^{-1}\lambda_t^{h1} = 0. \tag{19}$$

The envelope theorem implies:

$$\frac{\partial V^h\left(B_{t-1}^h, W_{t-1}^h, Z_t\right)}{\partial W_{t-1}^h} \equiv V_w^h\left(B_{t-1}^h, W_{t-1}^h, Z_t\right) \tag{20}$$

$$= \left(W_t^h N_t^h \left(\Phi_t^h\right)' \frac{W_t^h}{\left(W_{t-1}^h\right)^2}\right) \lambda_t^{h1} \tag{21}$$

$$\frac{\partial V^h\left(B_t^h, W_{t-1}^h, Z_t\right)}{\partial B_{t-1}^h} \equiv V_B^h\left(B_{t-1}^h, W_{t-1}^h, Z_t\right) = \lambda_t^{h1} \tag{22}$$

From equations (16) and (17)

$$\begin{aligned}
\lambda_t^{h1} &= \frac{\left(C_t^h\right)^{-\gamma}}{P_t}; \lambda_{t+1}^{h1} = \frac{\left(C_{t+1}^h\right)^{-\gamma}}{P_{t+1}}; \\
\lambda_t^{h2} &= -\chi\left(N_t^h\right)^{\eta} + W_t^h\left(1-\Phi_t^h\right)\frac{\left(C_t^h\right)^{-\gamma}}{P_t}
\end{aligned}$$

Equation (18) can be simplified to

$$\begin{aligned}
0 = & \left(N_t^h\left(1-\Phi_t^h\right) - W_t^h N_t^h \left(\Phi_t^h\right)' \frac{1}{W_{t-1}^h}\right)\frac{\left(C_t^h\right)^{-\gamma}}{P_t} \\
& -\varepsilon_w \frac{N_t^h}{W_t^h}\left(-\chi\left(N_t^h\right)^{\eta} + W_t^h\left(1-\Phi_t^h\right)\frac{\left(C_t^h\right)^{-\gamma}}{P_t}\right) \\
& +\beta\left(N_{t+1}^h\left(\Phi_{t+1}^h\right)'\left(\frac{W_{t+1}^h}{W_t^h}\right)^2\right)\frac{\left(C_{t+1}^h\right)^{-\gamma}}{P_{t+1}},
\end{aligned}$$

$$
\begin{aligned}
0 &= \left(N_t^h\left(1-\Phi_t^h\right) - N_t^h\left(\Phi_t^h\right)'\frac{W_t^h}{W_{t-1}^h}\right) \\
&+\varepsilon_w\chi\frac{\left(N_t^h\right)^{\eta+1}}{W_t^h}\frac{P_t}{\left(C_t^h\right)^{-\gamma}} - \varepsilon_w N_t^h\left(1-\Phi_t^h\right) \\
&+\beta\left(N_{t+1}^h\left(\Phi_{t+1}^h\right)'\left(\frac{W_{t+1}^h}{W_t^h}\right)^2\right)\frac{\left(C_{t+1}^h\right)^{-\gamma}}{\left(C_t^h\right)^{-\gamma}}\frac{P_t}{P_{t+1}},
\end{aligned}
$$

$$
\begin{aligned}
0 &= \left(1-\varepsilon_w\right)N_t^h\left(1-\Phi_t^h\right) - N_t^h\left(\Phi_t^h\right)'\frac{W_t^h}{W_{t-1}^h} + \varepsilon_w\chi\frac{\left(N_t^h\right)^{\eta+1}}{w_t^h\left(C_t^h\right)^{-\gamma}} \\
&+\beta\frac{\left(C_{t+1}^h\right)^{-\gamma}}{\left(C_t^h\right)^{-\gamma}}\left(N_{t+1}^h\left(\Phi_{t+1}^h\right)'\left(\frac{W_{t+1}^h}{W_t^h}\right)^2\right)\frac{P_t}{P_{t+1}}, \qquad (23)
\end{aligned}
$$

Equation (19) becomes

$$
0 = \beta\lambda_{t+1}^{h1} - R_t^{-1}\lambda_t^{h1} \qquad (24)
$$

$$
0 = \beta\frac{\left(C_{t+1}^h\right)^{-\gamma}}{P_{t+1}} - R_t^{-1}\frac{\left(C_t^h\right)^{-\gamma}}{P_t} \qquad (25)
$$

In a symmetric equilibrium the optimal wage setting becomes the wage Phillips curve:

$$
\begin{aligned}
0 &= \left(1-\varepsilon_w\right)\left(1-\Phi_t\right)N_t - \Phi_t'\Pi_t^w N_t + \varepsilon_w\chi\frac{N_t^{\eta+1}}{w_t C_t^{-\gamma}} \qquad (26) \\
&+E_t\left[M_{t,t+1}\left(\frac{\left(\Phi_{t+1}\right)'\left(\Pi_{t+1}^w\right)^2}{\Pi_t}N_{t+1}\right)\right],
\end{aligned}
$$

and the optimality condition for bonds satisfies:

$$
E_t\left[M_{t,t+1}\left(\frac{R_t}{\Pi_{t+1}}\right)\right] = 1, \qquad (27)
$$

where $w_t = W_t/P_t$ is the real wage, $\Pi_t^w = W_t/W_{t-1}$ is the gross wage inflation, $\Pi_t =$

P_t/P_{t-1} is the gross (price) inflation rate, and the stochastic discount factor is given by

$$M_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}. \tag{28}$$