Supplemental Online Appendices to "Optimal Taxes and Basic Income During an Episode of Automation: A Worker's Perspective," by Manoj Atolia, Morgan Holland, and Jonathan Kreamer

Accepted for Publication by *Macroeconomic Dynamics*

August 2024

A Omitted Proofs and Derivations

A.1 Derivation of Optimal Production

We can write the firm optimization problem as

$$\max_{\ell(i),k(i)} \left\{ \left[\int_0^1 \left(a\left(i\right)k\left(i\right) + b\left(i\right)\ell\left(i\right)\right)^{1-\frac{1}{\sigma}} di \right]^{\frac{\sigma}{\sigma-1}} - w\int_0^1 \ell\left(i\right)di - r\int_0^1 k\left(i\right)di \right\}, \tag{A.1}$$

subject to non-negativity constraints for factor inputs, $k\left(i\right)$ and $\ell\left(i\right)$. This yields two optimality conditions,

$$w \geq b\left(i\right)\cdot\left(\frac{Y}{y\left(i\right)}\right)^{\frac{1}{\sigma}} \tag{A.2}$$

and

$$r \geq a\left(i\right)\cdot\left(\frac{Y}{y\left(i\right)}\right)^{\frac{1}{\sigma}}, \tag{A.3}$$

which hold with equality for positive use of labor or capital respectively in the performance of task i.

With linear substitution between capital and labor in the performance of a given task, it will be performed by capital only if $a\left(i\right)/b\left(i\right) > r/w$, and by labor only if $a\left(i\right)/b\left(i\right) < r/w$. If $a\left(i\right)/b\left(i\right) = r/w$, then the task may be performed by both capital or labor. This will often pin down a unique production plan, but if $a(i)/b(i) = r/w$ for all tasks in an interval, then there are many equivalent production plans. To avoid this, we adapt the convention that (1) all tasks are done purely by capital or purely by labor, and (2) if $i < j$, then it is never the case that i is done by labor and j by capital. We can then set α such that tasks $i < \alpha$ are done only by capital, and tasks $i > \alpha$ are done only by labor. Therefore, α is the share of tasks done by capital and it will satisfy:

$$\begin{cases} \frac{a(i)}{b(i)} \geq \frac{r}{w} & \text{for } i < \alpha \\ \frac{a(i)}{b(i)} \leq \frac{r}{w} & \text{for } i > \alpha. \end{cases} \tag{A.4}$$

In the event that $a(i)/b(i)$ is strictly decreasing (at least in the neighborhood of α), α will be uniquely defined by

$$\frac{a(\alpha)}{b(\alpha)} = \frac{r}{w}. \tag{A.5}$$

Otherwise, α will be determined together with factor demands.

For tasks done by capital ($i \leq \alpha$), the optimal production of task i satisfies

$$r = \left(\frac{Y}{a(i) \cdot k(i)}\right)^{\frac{1}{\sigma}} a(i). \tag{A.6}$$

Thus $(a(i))^{1-\sigma} k$ is the same across all $i \leq \alpha$ and aggregate inverse capital demand, given unit measure of entrepreneurs, satisfies:

$$r = (A)^{\frac{\sigma-1}{\sigma}} \left(\frac{\alpha Y}{K}\right)^{\frac{1}{\sigma}}. \tag{A.7}$$

where

$$K \equiv \int_0^{\alpha} k(i)\, di \tag{A.8}$$

is aggregate capital and

$$A(\alpha) \equiv \left[\frac{1}{\alpha} \int_0^{\alpha} (a(i))^{\sigma-1}\, di\right]^{\frac{1}{\sigma-1}} \tag{A.9}$$

is the productivity of capital. Analogously, inverse labor demand, given a unit measure of workers, satisfies

$$w = (B)^{\frac{\sigma-1}{\sigma}} \left(\frac{(1-\alpha) Y}{L}\right)^{\frac{1}{\sigma}}, \tag{A.10}$$

where

$$L \equiv \int_{\alpha}^{1} \ell(i)\, di \tag{A.11}$$

is aggregate labor and

$$B(\alpha) \equiv \left[\frac{1}{1-\alpha}\int_{\alpha}^{1} (b(i))^{\sigma-1}\, di\right]^{\frac{1}{\sigma-1}} \tag{A.12}$$

is labor productivity.

The expressions above allow us to derive the following expression for aggregate production as a function of aggregate capital and labor:

$$Y \equiv F(K,L) = \left[\alpha^{\frac{1}{\sigma}}\left(A(\alpha)K\right)^{1-\frac{1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}}\left(B(\alpha)L\right)^{1-\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{A.13}$$

where $\{A(\alpha), B(\alpha), \alpha\}$ are defined as above and

$$w = F_L(K,L) = B^{1-1/\sigma}\left((1-\alpha)Y/L\right)^{1/\sigma} \tag{A.14}$$

$$r = F_K(K,L) = A^{1-1/\sigma}\left(\alpha Y/K\right)^{1/\sigma} \tag{A.15}$$

The expression for q is just equal to r/w.

To see that the aggregate production function can also be written as:

$$Y = F(K,L) = \max_{\alpha}\left\{\left[\alpha^{\frac{1}{\sigma}}\left(A(\alpha)K\right)^{1-\frac{1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}}\left(B(\alpha)L\right)^{1-\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}\right\}$$

We observe that the expression being maximized is equal to production with

$$k(i) = \frac{(a(i))^{\sigma-1}}{\int_0^{\alpha}(a(i))^{\sigma-1}\, di}K, \quad \ell(i) = \frac{(b(i))^{\sigma-1}}{\int_{\alpha}^{1}(b(i))^{\sigma-1}\, di}L$$

Then we simply observe that this is the expression that holds under optimal production, and if a different choice of α yielded a higher level of production for given $\{K, L\}$, then this would not be the solution to the firm profit maximization problem.

A.2 Proof of Proposition 1

In steady state, C_e and K are constant, and thus from equation (9):

$$r = \frac{\rho+\delta}{1-\tau^k} \equiv r^* > 0 \tag{A.16}$$

Then we may rewrite (9) as

$$\dot{C}_e/C_e = \dot{K}/K = (1-\tau^k)(r-r^*)$$

where $r = F_K$ is the marginal product of capital, and we derive the growth rate of capital from the fact that $C_e = \rho K$.

Next we prove two statements which together suffice to prove the claim:

1. When $L = 0$, the production function is $Y = A(1) \cdot K$, and $r = A(1)$.

2. The marginal product of capital is increasing in L.

From these it follows that $r \geq A(1)$, and therefore when $A(1) > r^*$, it follows that the growth rate of the capital stock is bounded below by the value $(1-\tau^k)(A(1)-r^*) > 0$. Thus the economy does not reach a steady state, but experiences continuous growth. By contrast, when $A(1) < r^*$, K and therefore K/L will grow until eventually r falls to $r = r^*$, and the economy reaches a steady state. (Since $r = F_K$ is decreasing in K/L, as we prove below.)

Now we prove the first claim. When $L = 0$, the production function for given K may be written as:

$$F(K,0) = \max_{\alpha}\left\{\left[\alpha^{\frac{1}{\sigma}}\left(A(\alpha)\cdot K\right)^{1-\frac{1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}}\,0^{1-\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}\right\}$$

Suppose first that $\sigma > 1$. Then output satisfies:

$$F(\alpha, K, 0) = \max_{\alpha}\left\{\alpha^{1/(\sigma-1)}A(\alpha)\cdot K\right\}$$

Is this maximized at $\alpha = 1$? Note that we have:

$$\alpha^{1/(\sigma-1)}A(\alpha) = \left[\int_0^{\alpha}(a(i))^{\sigma-1}\,di\right]^{1/(\sigma-1)}$$

Taking the derivative with respect to α, we obtain:

$$\frac{d}{d\alpha}\left[\alpha^{1/(\sigma-1)}A(\alpha)\right] = \frac{1}{\sigma-1}\left[\int_0^{\alpha}(a(i))^{\sigma-1}\,di\right]^{1/(\sigma-1)-1}(a(\alpha))^{\sigma-1} \geq 0$$

which is positive because $\sigma > 1$ and $a(i) \geq 0$. Thus it is always weakly better to increase α than not, and thus output is maximized at $\alpha = 1$, and therefore $Y = A(1)\cdot K$, and $r = A(1)$.

Suppose instead that $\sigma < 1$. Then clearly if $\alpha < 1$, $Y = 0$ (since $L^{1-1/\sigma} \to \infty$ as $L \to 0$, and $x^{(\sigma-1)/\sigma} \to 0$ as $x \to \infty$). Thus we must set $\alpha = 1$ to have any hope of positive output. In this case, the production function immediately becomes $Y = A(1)\cdot K$, and therefore $r = A(1)$. And the same argument applies when $\sigma = 1$, as the Cobb-Douglas form $K^{\alpha}L^{1-\alpha}$ is likewise 0 for $L = 0$ unless $\alpha = 1$.

Next we prove that the marginal product of capital (r) is increasing in L. This amounts to proving that the derivative $F_K(K,L)$ is increasing in L, where:

$$F(K,L) = \max_{\alpha} \left\{ \left[\alpha^{\frac{1}{\sigma}} (A(\alpha) \cdot K)^{1-\frac{1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}} (B(\alpha) \cdot L)^{1-\frac{1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \right\}$$

We begin by proving the following lemmas.

Lemma 1. *Optimal capital task share α is decreasing in labor/capital ratio L/K.*

Proof. The equilibrium condition that holds at α can be written as:

$$\begin{cases} Z(i) \geq X(\alpha) \cdot \frac{L}{K} & \text{for } i < \alpha \\ Z(i) \leq X(\alpha) \cdot \frac{L}{K} & \text{for } i > \alpha \end{cases}$$

where:

$$X(\alpha) = \frac{\int_0^\alpha (a(i))^{\sigma-1} \, di}{\int_\alpha^1 (b(i))^{\sigma-1} \, di}$$

$$Z(i) = \left[\frac{a(i)}{b(i)} \right]^\sigma$$

(This condition is found by substituting the expressions for r and w into the condition for α, and substituting in the expressions for A and B). By assumption, $Z(i)$ is decreasing in i. Furthermore, since $a(i), b(i) \geq 0$, it is clear that $X(\alpha)$ is increasing in α, since:

$$X'(\alpha) = X(\alpha) \cdot \left(\frac{(a(\alpha))^{\sigma-1}}{\int_0^\alpha (a(i))^{\sigma-1} \, di} + \frac{(b(\alpha))^{\sigma-1}}{\int_\alpha^1 (b(i))^{\sigma-1} \, di} \right) > 0$$

Now suppose that $L_2/K_2 > L_1/K_1$. We show that $\alpha_2 < \alpha_1$ by contradiction. Suppose not, so that $\alpha_2 > \alpha_1$. Take some $i \in (\alpha_1, \alpha_2)$. Since $i > \alpha_1$, it follows that:

$$Z(i) \leq X(\alpha_1) \cdot \frac{L_1}{K_1}$$

and since $i < \alpha_2$, it follows that:

$$Z(i) \geq X(\alpha_2) \cdot \frac{L_2}{K_2}$$

Therefore:

$$X(\alpha_1) \cdot \frac{L_1}{K_1} \geq X(\alpha_2) \cdot \frac{L_2}{K_2}$$

Since by assumption $L_2/K_2 > L_1/K_2$, it follows that:

$$X(\alpha_1) > X(\alpha_2)$$

Since X is increasing in α, this implies that

$$\alpha_1 > \alpha_2$$

which contradicts our assumption. Therefore $\alpha_2 \leq \alpha_1$, which proves the first part of the claim. □

Lemma 2. *The ratio $w/r = F_L(K,L)/F_K(K,L)$ is increasing in $\kappa = K/L$.*

Proof. We know that:

$$\begin{cases} \frac{a(i)}{b(i)} \geq \frac{r}{w} & \text{for } i < \alpha \\ \frac{a(i)}{b(i)} \leq \frac{r}{w} & \text{for } i > \alpha \end{cases}$$

and that α is decreasing in L/K, and therefore increasing in κ. We further know that a/b is decreasing in i.

Now consider what happens if there is an increase in L/K. We can consider two cases. One case is that a/b is continuously decreasing at α. In this case, $r/w = a/b$ holds with equality in the neighborhood of α. Therefore an increase in L/K will cause α to decline, and therefore $a/b = r/w$ will increase.

Conversely, suppose that a/b is discontinuous at i. Then a marginal change in L/K will not change α. But then from:

$$\frac{r}{w} = \left(\frac{\int_0^\alpha (a(i))^{\sigma-1}\,di}{\int_\alpha^1 (b(i))^{\sigma-1}\,di}\right)^{\frac{1}{\sigma}} \left(\frac{L}{K}\right)^{\frac{1}{\sigma}}$$

we immediately get that r/w must increase.

Thus in either case a marginal increase in L/K causes a marginal increase in r/w. □

Now we are in position to prove our final lemma:

Lemma 3. *The marginal product of capital $F_K(K,L)$ is increasing in L, since $F_{KL}(K,L) > 0$.*

Proof. We can rewrite the production function as:

$$F(K,L) = \max_{\alpha}\left\{\left(\alpha^{\frac{1}{\sigma}}(A)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}}\left(B\frac{L}{K}\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} K\right\}$$

This implies that the choice of α that maximizes production for a given (K, L) depends only on the ratio $\kappa = K/L$.

Next we calculate the derivatives of the aggregate production function. These are:

$$F_K(K,L) = \left(\frac{\alpha A^{\sigma-1} Y}{K}\right)^{\frac{1}{\sigma}} = \left(\alpha A^{\sigma-1}\right)^{\frac{1}{\sigma}} \left(\alpha^{\frac{1}{\sigma}} (A)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}} \left(B\frac{L}{K}\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}$$
$$F_L(K,L) = \left(\frac{(1-\alpha) B^{\sigma-1} Y}{L}\right)^{\frac{1}{\sigma}} = \left((1-\alpha) B^{\sigma-1}\right)^{\frac{1}{\sigma}} \left(\alpha^{\frac{1}{\sigma}} \left(A\frac{K}{L}\right)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}} (B)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}$$

The key thing to observe is that both equations depend only on $\kappa = K/L$, and not on K or L independently. Therefore we can write each as:

$$F_K(K,L) = r(\kappa)$$
$$F_L(K,L) = w(\kappa)$$

Moreover, from Lemma 2 we know that:

$$\frac{w(\kappa)}{r(\kappa)} = \frac{F_L(K,L)}{F_K(K,L)} = \left(\frac{(1-\alpha) B^{\sigma-1}}{\alpha A^{\sigma-1}}\kappa\right)^{\frac{1}{\sigma}}$$

is increasing in κ.

Now we can prove the claim. Note that changes in L and K affect F_K and F_L only by altering κ. Therefore we have:

$$\begin{aligned} F_{KL} &= \frac{d}{dL}(F_K) = -r'(\kappa) \cdot \frac{K}{L^2} \\ &= \frac{d}{dK}(F_L) = w'(\kappa) \cdot \frac{1}{L} \end{aligned}$$

Therefore:

$$-r'(\kappa) \cdot \kappa = w'(\kappa)$$

This implies that $w'(\kappa)$ and $r'(\kappa)$ have opposite signs.

In Lemma 2 we showed that w/r is increasing in κ. Therefore:

$$\frac{d}{d\kappa}\left(\frac{w(\kappa)}{r(\kappa)}\right) = \frac{w}{r}\left(\frac{w_\kappa}{w} - \frac{r_\kappa}{r}\right) \geq 0$$

Now, since r_κ and w_κ have opposite signs, the only way that:

$$\frac{w_\kappa}{w} \geq \frac{r_\kappa}{r}$$

is if $w_\kappa \geq 0$ and $r_\kappa \leq 0$. From this we conclude that $r_\kappa \leq 0$, and therefore:

$$F_{KL} = -r'(\kappa) \cdot \frac{K}{L^2} \geq 0$$

That is, the return on capital is increasing in L/K, and therefore in L. □

A.3 Proofs From Section 4

Proof of Proposition 2. We would like to reduce the model to a minimal set of restrictions on the choices of main variables $\{C_w, C_e, L, K\}$. For given (K, L), output and factor prices are defined by equations (13) – (19), and thus we have $\{Y, r, w\}$ as functions of (K, L). Next observe that choosing τ^k will imply a choice of r^k according to (7). Since entrepreneur consumption satisfies (10), the choice of r^k allows the planner to choose the path of K.

Next, we observe that the planner can effectively determine the level of labor supply by altering the tax on labor income, so as to satisfy workers' labor supply condition, (3). Finally, we observe that the remaining conditions only enter as part of the resource constraint. In particular, combining the worker, entrepreneur, and government budget constraints (2), (8), and (20), with expressions for equilibrium factor prices, we obtain the aggregate resource constraint:

$$C_w + C_e + \dot{K} = (1 - \omega)F(K, L) - \delta K. \quad \text{(A.17)}$$

We can therefore state the Ramsey Planning Problem for the social planner which maximizes a worker's life-time utility under majority voting as

$$\max_{\{C_w, C_e, K, L\}} \left\{ \int e^{-\gamma t} U(C_w, L) \right\}, \quad \text{(A.18)}$$

subject to (10), (A.17), and

$$L \geq 0 \quad \text{(A.19)}$$

We can simplify the statement of the Ramsey problem by substituting the entrepreneur's behavioral rule (10) directly into the resource constraint (A.17). The Hamiltonian for the Ramsey Planning problem is then:

$$\mathcal{H} = U(C_w, L) + \lambda\left[(1 - \omega)F(K, L) - \delta K - \rho K - C_w\right] + \mu_L L$$

The choice variables are (C_w, L) and the state variable is K. This yields the optimality conditions:

$$U_C(C_w, L) = \lambda \tag{A.20}$$

$$-U_L(C_w, L) \geq \lambda(1-\omega)F_L(K, L) \tag{A.21}$$

$$-\dot{\lambda}/\lambda = (1-\omega)F_K(K, L) - \delta - \rho - \gamma \tag{A.22}$$

In addition, the entrepreneur consumption rule (10) and aggregate resource constraint (A.17) must hold. These five equations, along with with complementary slackness conditions on L, determine the four choice variables of the Ramsey problem and the Lagrange multiplier λ.

□

Proof of Corollary 2. Combining equations (26) and (27) yields:

$$-U_L(C_w, L) \geq (1-\omega)U_C(C_w, L) \cdot F_L(K, L) \tag{A.23}$$

which holds with equality when $L > 0$. Comparing with (3), and noting that $w = F_L$, we observe that this condition can be implemented by setting the labor tax equal to the government expenditure share, $\tau^\ell = \omega$.

To derive the expression for the capital tax rate, we substitute the entrepreneur consumption rule (10) into the entrepreneur Euler equation (9). Combining this with the optimality condition (28), and noting that $r = F_K$, yields the given expression.

To derive the expression for the transfer to workers, we note that the government budget constraint implies

$$T^w = \tau^l wL + \tau^k rK - \omega Y$$

Substituting in the optimal tax rates, plus $w = F_L$ and $r = F_K = r^*$, we obtain:

$$\begin{aligned} T^w &= \omega\left(F_L L + \omega F_K K - Y\right) + \left(\gamma - \dot{\lambda}/\lambda - \dot{K}/K\right) K \\ &= \left(\gamma - \dot{\lambda}/\lambda - \dot{K}/K\right) K \end{aligned}$$

□

Proof of Corollary 1. Suppose that a steady state is reached. Then equation (28) implies:

$$F_K(K, L) = \frac{\delta + \rho + \gamma}{1-\omega} = r^*$$

That is, the marginal product of capital reaches a specific level, r^*. From (36), with $\dot{K} = \dot{\lambda} = 0$, the steady state capital tax rate is:

$$\tau_{ss}^{k} = 1 - (1-\omega)\left(\frac{\delta+\rho}{\rho+\gamma+\delta}\right) = \omega + \frac{\gamma}{r^*} \tag{A.24}$$

and likewise from (37) with $\dot{\lambda} = \dot{K} = 0$, we obtain

$$T^w = \gamma K$$

When is the steady state reached? From Proposition 1, we know that a steady state will be reached iff $A(1) < r^*$. Here this requires the condition:

$$A(1) < \frac{\delta+\rho+\gamma}{1-\omega} \tag{A.25}$$

Note that in a steady state, we will have $L > 0$ (since otherwise the model would be AK and would achieve continuous growth), and therefore the worker labor supply condition holds with equality. □

A.4 Proofs from Section 5

We begin with the following lemma:

Lemma 4. *Given that (38) holds and $b(i) = 1$, and given a value of $L > 0$, there exists a $\bar{K}(a, \bar{\alpha}) = \frac{\bar{\alpha}}{1-\bar{\alpha}}\frac{L}{a}$ such that for $0 \leq K \leq \bar{K}(a, \bar{\alpha})$, the production function has the following linear representation*

$$Y(K, L) = aK + L. \tag{A.26}$$

Proof. Using the expressions from equations (13) – (19), note that $a(i)/b(i) = a$ for $\alpha \leq \bar{\alpha}$. Thus the $a(i)/b(i)$ curve is constant over this region, and therefore at any interior point $\alpha < \bar{\alpha}$, condition (16) implies that $q = a$. Further, for $\alpha < \bar{\alpha}$, we have:

$$q = \left(\frac{\alpha}{1-\alpha}a^{\sigma-1}\frac{L}{K}\right)^{\frac{1}{\sigma}}$$

and therefore $q = a$ implies:

$$\alpha = \frac{aK}{aK+L}$$

Plugging this into the production function (13), with $A = a$ and $B = 1$, yields:

$$Y = \left(\left(\frac{aK}{aK + L} \right)^{\frac{1}{\sigma}} (aK)^{1-\frac{1}{\sigma}} + \left(\frac{L}{aK + L} \right)^{\frac{1}{\sigma}} L^{1-\frac{1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} = aK + L$$

This holds as long as $\alpha < \bar{\alpha}$, which using the expression for α above, is equivalent to:

$$K < \frac{\bar{\alpha}}{1 - \bar{\alpha}} \frac{L}{a} \equiv \bar{K}$$

Meanwhile, when $K = \bar{K}$ exactly, we have $\alpha = \bar{\alpha}$, and therefore $Y = aK + L$ still holds. □

Intuitively, when capital is below a threshold value, there is insufficient capital to perform all tasks that capital is capable of performing, and thus labor and capital are perfectly substitutable at the margin.[1]

We now turn to characterizing what happens in steady state. The following lemma summarize the possibilities, depending on initial parameters:

Lemma 5. *In steady state, (i) if $a < r^*$, then $\alpha = 0$ and equilibrium capital stock $K = 0$ implying $Y(K, L) = Y(L) = L$, (ii) if $a = r^*$, then there is a continuum of steady states with $\alpha \in [0, \bar{\alpha}]$ and equilibrium capital stock $K = \bar{K}(r^*, \alpha)$ so that $Y(K, L) = Y(\bar{K}, L) = a\bar{K}(r^*, \alpha) + L$, and (iii) if $a > r^*$, then $\alpha = \bar{\alpha}$ and equilibrium capital stock $K = \hat{K}(a, \bar{\alpha}, r^*) = \frac{\bar{\alpha}}{(1-\bar{\alpha})^{-\frac{1}{\sigma-1}}((\frac{r^*}{a})^{\sigma-1}-\bar{\alpha})^{\frac{\sigma}{\sigma-1}}} \frac{L}{a} > \bar{K}(r^*, \bar{\alpha})$ with $Y(K, L) = \left((\bar{\alpha})^{\frac{1}{\sigma}} (aK)^{\frac{\sigma-1}{\sigma}} + (1 - \bar{\alpha})^{\frac{1}{\sigma}} L^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$ so the production function exhibits the usual diminishing returns in both factors.*

Proof. The results follow easily from the linear representation in Lemma 4. (i) follows from the fact that the return on capital (a) is below the return that the entrepreneurs want (r^*); (ii) arises because in the knife edge case of $a = r^*$ any value of $0 \leq K \leq \bar{K}(r^*, \bar{\alpha})$ is consistent with the entrepreneurs' Euler equation in the steady state because with a linear production function there are no diminishing returns to capital accumulation in that range of K; and (iii) is the case where Lemma 4 is not applicable so the argument in (ii) about no diminishing returns does not apply.

To derive the expression for $\hat{K}$, first note that when $a > r^*$, we know that $\alpha = \bar{\alpha}$ binds

[1] The linear representation of production in Lemma 4 has a strong resemblance to the "cone of diversification" for production in a small-open economy (see Dixit and Norman (1980), Bhagwati et al. (1998), and Grossman and Helpman (1991)). For example, in a two-good, two-factor economy with goods prices fixed in international markets, zero profit conditions for the production of the two goods imply fixed values of the two factor prices. Thus, within the cone of diversification, the aggregate economy displays no diminishing returns to changes in (relative) quantity (or endowment) of factors just as our linear representation in (A.26).

in steady state. Therefore in steady state we have:

$$F_K = a\left(\frac{\alpha Y}{aK}\right)^{1/\sigma} = r^*$$

where production is:

$$Y = \left(\bar{\alpha}^{1/\sigma}(aK)^{1-1/\sigma} + (1-\bar{\alpha})^{1/\sigma} L^{1-1/\sigma}\right)^{\frac{\sigma}{\sigma-1}}$$

Combining these and solving for K yields:

$$K = \frac{\bar{\alpha}(1-\bar{\alpha})^{\frac{1}{\sigma-1}}}{\left[(r^*/a)^{\sigma-1} - \bar{\alpha}\right]^{\frac{\sigma}{\sigma-1}}}\frac{L}{a}$$

which is the expression for $\hat{K}$ given above. Note that $\hat{K}(r^*, \bar{\alpha}, r^*) = \bar{K}(r^*, \bar{\alpha})$. Finally, to see that $K > 0$, recall that we assumed $a > r^*$ and $r^* > A(1)$. When $\sigma < 1$, the first implies the denominator is positive. When $\sigma > 1$, the second, together with $A(1) = a(\bar{\alpha})^{\frac{1}{\sigma-1}}$, implies the denominator is positive. □

Now it is fairly straightforward to prove Proposition 3:

Proof of Proposition 3. Given assumption (39), we are in case (iii) of Lemma 5. The expression for K/L therefore follows immediately from the expression for $\hat{K}$. The expression from output follows from the expressions in equations (13) – (19), with $\alpha = \bar{\alpha}$, $A(\alpha) = a$, and $B = 1$. Output is positive since $K > 0$. The wage satisfies $w = F_L = ((1-\alpha)Y/L)^{1/\sigma}$, which given the expression for $\hat{K}$, has the given expression. Finally, the labor share is

$$s_L = \frac{wL}{Y} = (1-\alpha)^{1/\sigma}(Y/L)^{1/\sigma-1}$$

which, given the expressions for Y and K/L, yields the given expression for the labor share. Both are positive because $K > 0$ and $Y > 0$. □

The results of corollaries 3 and 4 follow immediately from the expressions for the wage and labor share:

Proof of Corollary 3. Differentiating the expression for steady state wage (43) and labor share (51) yields equations (44) and (45). Note that the denominator of (44) is positive by the same argument that $K > 0$. □

Proof of Corollary 4. Follows directly from differentiating equations (43) and (51). The expression in (46) is positive because $(r^*/a)^{\sigma-1} > \bar{\alpha}$, by the same argument as for $K > 0$, and since $a > r^*$, $1 - (r^*/a)^{\sigma-1}$ has the same sign as $\sigma - 1$. □

A.5 Proof of Proposition 4

Proof. When $\alpha = \bar{\alpha}$, output is now:

$$Y = \left(\bar{\alpha}^{1/\sigma}(aK)^{1-1/\sigma} + (1-\bar{\alpha})^{1/\sigma}(BL)^{1-1/\sigma}\right)^{\frac{\sigma}{\sigma-1}}$$

where now $B \neq 1$ is possible. Since $r = r^*$ and $\alpha = \bar{\alpha}$ hold in steady state, the steady state effective capital-labor ratio is still:

$$\frac{aK}{BL} = \frac{\bar{\alpha}(1-\bar{\alpha})^{\frac{1}{\sigma-1}}}{\left[(r^*/a)^{\sigma-1} - \bar{\alpha}\right]^{\frac{\sigma}{\sigma-1}}}$$

Then the steady state wage can be written as:

$$w^* = B\left(\frac{1-\bar{\alpha}(r^*/a)^{1-\sigma}}{1-\bar{\alpha}}\right)^{\frac{1}{1-\sigma}}$$

Let w_0 be the steady state wage initially, and let w_1 be the steady state wage after the episode of automation. Before automation, we have:

$$B_0 = \left(\frac{\bar{\alpha}_1 - \bar{\alpha}_0}{1-\bar{\alpha}_0} b_m^{\sigma-1} + \frac{1-\bar{\alpha}_1}{1-\bar{\alpha}_0} b_1^{\sigma-1}\right)^{\frac{1}{\sigma-1}}$$

and after automation we have $B_1 = b_1$. Using these expressions, the ratio of w_1 to w_0 satisfies:

$$\left(\frac{w_1}{w_0}\right)^{1-\sigma} = \frac{1-\bar{\alpha}_1\left(\frac{r^*}{a}\right)^{1-\sigma}}{1-\bar{\alpha}_0\left(\frac{r^*}{a}\right)^{1-\sigma}}\left\{1 + \left(\frac{\bar{\alpha}_1-\bar{\alpha}_0}{1-\bar{\alpha}_1}\right)\left(\frac{b_1}{b_m}\right)^{1-\sigma}\right\} \tag{A.27}$$

We are interested in knowing when the steady state wage declines following automation, i.e., when $w_0 < w_1$. From (A.27), we can work out that this will happen when condition (49) holds.

□

A.6 Proofs from Section 7

Proof of Proposition 5. The Hamiltonian of the stated problem is:

$$H = U(C_w, L) + \lambda \{(1-\omega) F(K_1, K_2, L) - \delta K - \rho K - C_w\} \\ + \kappa (K - K_1 - K_2) + \phi_1 K_1 + \phi_2 K_2 + \mu_L L$$

This yields optimality conditions:

$$U_C(C_w, L) = \lambda \tag{A.28}$$

$$-U_L(C_w, L) \geq \lambda (1-\omega) \tag{A.29}$$

$$-\dot{\lambda}/\lambda = \kappa/\lambda - \delta - \rho \tag{A.30}$$

$$\kappa \geq \lambda (1-\omega) F_{K_1}(K_1, K_2, L) \tag{A.31}$$

$$\kappa \geq \lambda (1-\omega) F_{K_2}(K_1, K_2, L) \tag{A.32}$$

Then when $K_1, K_2 > 0$ at the solution, this implies $F_{K_1} = F_{K_2}$, where $F_{K_1} > F_{K_2}$ occurs only when $K_2 = 0$, and $F_{K_1} < F_{K_2}$ occurs only when $K_1 = 0$. Comparing this with the allocation of capital across tasks by entrepreneurs (i.e., condition (53)), we see that this allocation is implemented by $\tau_1^k = \tau_2^k$. The conditions are then identical to those given in Proposition 2, with $F_L(K, L) = F_L(K_1, K_2, L)$ and $F_K(K, L) = \max(F_{K_1}(K_1, K_2, L), F_{K_2}(K_1, K_2, L))$, implying identical tax rates. □

B Quantitative analysis with high σ

In section 5 of the paper, we analyzed an episode of automation under majority voting and compared it to the case with fixed taxes at optimal value. That analysis assumed a value of $\sigma = 0.8$. However, given the variation in estimates in the empirical literature and the importance of the value of σ to various other results in the paper, and especially the qualitative difference observed for $\sigma < 1$ vs. $\sigma > 1$, this section repeats the quantitative experiment with $\sigma = 1.2$.

B.1 Calibration

In order to formulate a comparable case for the higher value of σ, we use the concept of a "normalized" CES function for our production function as in Klump et al. (2012). This requires changing some other parameters of the model when changing σ to target the same levels of production, capital, labor, and the labor share at a particular point, which is the point of normalization. We choose the initial steady state under optimal policy as our point of normalization, since this is the initial point for analyzing the episode of automation that is our focus. One advantage of normalizing around the steady state is that the same point will likewise be the steady state of the normalized model, making comparisons between the two easier.

Applied to our case, the normalized CES approach requires that $\{\alpha, a, b\}$ satisfy:

$$\alpha^{1/\sigma} a^{1-1/\sigma} = (1 - s_{L0})\,(Y_0/K_0)^{1-1/\sigma} \tag{A.33}$$

$$(1-\alpha)^{1/\sigma}\, b^{1-1/\sigma} = (s_{L0})\,(Y_0/L_0)^{1-1/\sigma} \tag{A.34}$$

where $\{K_0, L_0, Y_0, s_{L0}\}$ are the levels of capital, labor, output, and the labor share respectively at the point of normalization. Assuming that $\alpha = \bar{\alpha}$ holds at the new point, as is indeed the case, this gives us two conditions in three technological parameters $\{\bar{\alpha}, a, b\}$. However, there is an additional condition we require these parameters to satisfy, which is the inequality in equation (39), which requires that tasks with $i \leq \bar{\alpha}$ are cheaper to perform using capital (rather than labor) in the steady state, and therefore implies positive capital in steady state.

We considered two possible normalizations: setting $b = 1$, or setting $\bar{\alpha} = 0.5$, both as in our baseline calibration. However, fixing $\bar{\alpha}$ results in a violation of the constraint (39), and thus we opted for setting $b = 1$. This gives us $a = 1.3984$ and $\bar{\alpha}_o = 0.23905$. Finally, the targeted 10 percentage point decrease in labor share due to automation yields $\bar{\alpha}_1 = 0.30429$. In the calibrated steady state for $\sigma = 1.2$, as expected *all* the levels and ratios are identical to that in Table 1 for $\sigma = 0.8$, except, of course, for the value of $\bar{\alpha}$ as seen in Table 3.

B.2 Initial Policy and Steady State Under Majority Voting

Once again, we first compare the initial calibrated steady state with the initial steady state under majority voting, both of which are outlined in Table 1. The broad picture painted by Table 3 is similar to that by Table 1. Tax policy under majority voting remains unaffected by the change in σ. As before with $\sigma = 0.8$, given the higher capital tax rate, steady state capital is lower, which likewise reduces steady state output as well as consumption of both workers and entrepreneurs. Quantitatively, the decline in capital due to the higher capital taxes is larger with $\sigma = 1.2$ than it was with $\sigma = 0.8$. This is natural, because a higher capital tax rate raises the equilibrium rental rate of capital r, and therefore the relative factor price of capital r/w. σ is precisely the elasticity of K/L to r/w, and thus a higher σ prompts a greater decline in K/L, and therefore of K.

The only qualitative difference in macroeconomic outcomes in the high σ case is that the

	Calibrated	Majority		Calibrated	Majority
Y	1.5899	1.4498	K/Y	2.8993	2.2212
C_w	1.0308	1.0283	C_w/Y	0.6483	0.7093
C_e	0.1844	0.1289	C_e/Y	0.1160	0.0889
w	2.8574	2.5017	G/Y	0.0907	0.0907
L	0.3432	0.3671	$\bar{\alpha}$	0.23905	0.23905
K	4.6098	3.2214	τ^k	0.3190	0.4544
labor share	0.6168	0.6334	τ^ℓ	0.2310	0.0907
capital share	0.3832	0.3666	T^w/Y	0.1740	0.1333
			T^w/Y adj.	0.0875	0.1333

Table 1: Initial Steady States with Calibrated Policy vs. Majority Voting: $\sigma = 1.2$

labor share rises under majority voting, which also supports a slightly higher consumption of workers as a percentage of GDP (70.93 percent in Table 3 vs. 69.08 percent in Table 1). The reason this occurs can be seen by referring to the expression for labor share (51), which shows that labor share is increasing in K/L when $\sigma < 1$ (as in the baseline $\sigma = 0.8$ case), and decreasing in K/L when $\sigma > 1$ (as in this case). The intuition for this is given in footnote 41.

B.3 Quantitative Analysis of an Episode of Automation

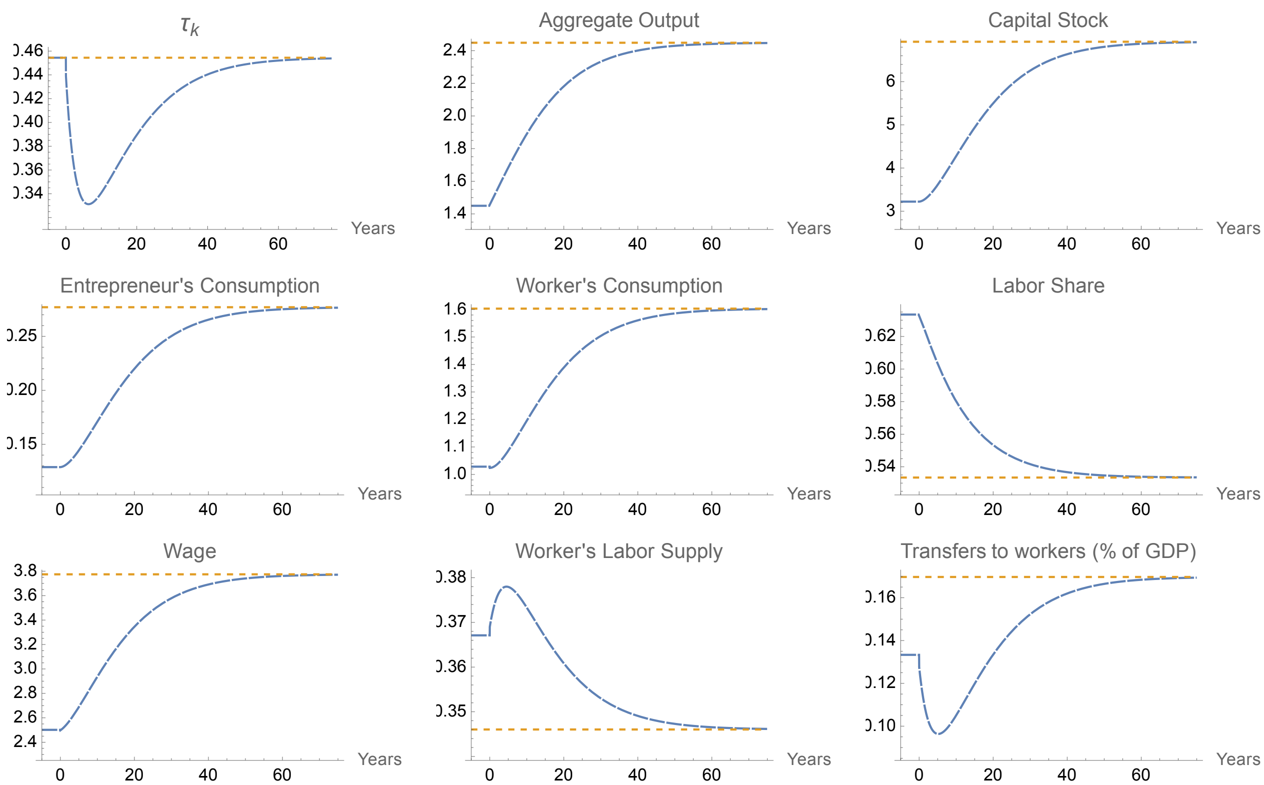
We now consider the effect of the episode of automation. The effect across steady states is qualitatively as well as quantitatively very similar to that for $\sigma = 0.8$ as seen from comparison of Table 2 and Table 2. We, therefore, abstract from a detailed discussion.

	Initial	Automation		Initial	Automation
Y	1.4498	2.4485	K/Y	2.2212	2.8284
C_w	1.0283	1.6031	C_w/Y	0.7093	0.6547
C_e	0.1289	0.2770	C_e/Y	0.0889	0.1131
w	2.5017	3.7743	G/Y	0.0907	0.0907
L	0.3671	0.3460	$\bar{\alpha}$	0.23905	0.30429
K	3.2214	6.9254	τ^k	0.4544	0.4544
labor share	0.6334	0.5334	τ^ℓ	0.0907	0.0907
capital share	0.3666	0.4666	T^w/Y	0.1333	0.1697
			T^w/Y adj.	0.1333	0.1697

Table 2: Steady States Before and After an Episode of Automation under Optimal Policy: $\sigma = 1.2$

Transitional Dynamics. In a similar vein, we find that the transition dynamics of the episode of automation, shown in Figure 1 over a period of 75 years, are qualitatively very

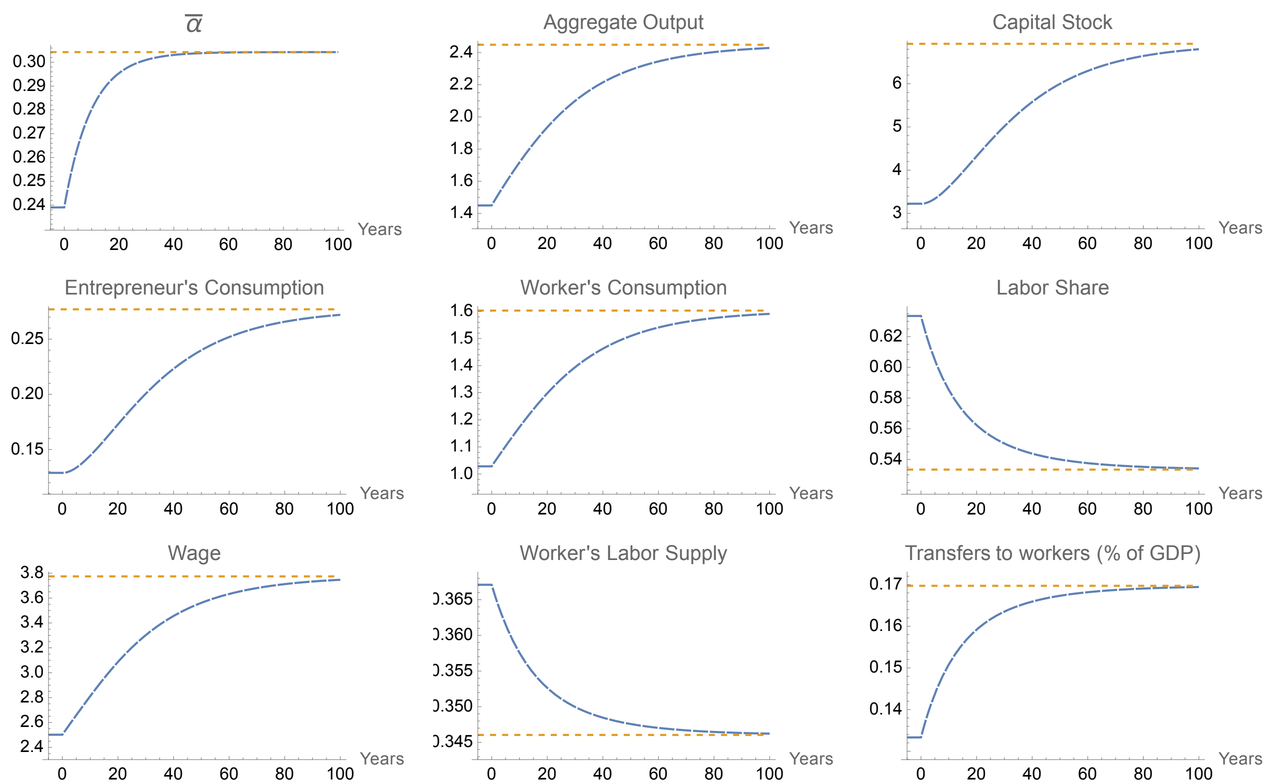
Figure 1: Dynamics for Gradual Increase with $\sigma = 1.2$ under Optimal Policy



similar to those in Figure 1 for $\sigma = 0.8$. Thus, the intuition remains the same as in that case, and we refrain from repeating it here. Instead, we directly move to examining the welfare gains for workers and entrepreneurs, which amount respectively to consumption equivalents of 21.57 percent and 55.02 percent.

Fixed Tax Counterfactual. As before, for a true assessment of the impact of these policies we consider a counterfactual in which tax rates remain fixed over the course of the transition. The transition paths under the fixed tax counterfactual are shown in Figure 2 and comparison with Figure 2 reveals that the transition is now monotonic for all variables, without any overshooting for labor supply related driven accompanied by overshooting of transfers in case of $\sigma = 0.8$. In terms of welfare implications, which is our focus, we find that worker's welfare gain is 21.10 percent whereas entrepreneur's gain is 34.09 percent (both in terms of consumption equivalents.) Thus we see that optimally varying tax rates over the transition increases workers' welfare by only 0.47 percentage points, whereas entrepreneurs'

Figure 2: Dynamics for Gradual Increase with $\sigma = 1.2$ with fixed taxes



gain by a whopping 20.93 percentage points–a result that mirrors the outcome for $\sigma = 0.8$ in the main body of the paper.

C Automation under Continuous Growth

In our analysis so far, we have focused on the case that a steady state is reached. However, as Proposition 1 in the main text makes clear, it is quite possible that no steady state exists, and the economy experiences continuous growth through capital accumulation. In this section, we consider this case.

Fixed Tax Rates. First suppose that tax rates are fixed at some level $\{\tau^k, \tau^\ell\}$. Then we can say the following about the conditions under which continuous growth will occur:

Corollary 1. *Suppose that $a(i)$, $b(i)$, τ^ℓ, and τ^k are all constant, and let $r^* = \frac{\rho+\delta}{1-\tau^k}$. Then, (i) If $a(i) > r^*$ for all i, then $A(1) > r^*$ and there is sustained growth. (ii) If $\sigma < 1$ and $a(i) = 0$ for a positive measure of tasks, then $A(1) = 0 < r^*$ and no long-run growth is possible. (iii) If $\sigma > 1$, a sufficient condition for sustained growth is that there exists m such that for all $i \in [0, m]$ we have $a(i) \geq m^{\frac{1}{1-\sigma}} r^*$.*

Proof. All results follow immediately from Proposition 1 in the main text. □

Result (i) gives a simple sufficient condition for sustained growth: sufficient capital productivity at all tasks. Result (ii) demonstrates the importance of the elasticity of substitution to whether continuous growth is possible. When $\sigma < 1$, each task is necessary, and thus sustained growth requires positive capital productivity at all tasks, excepting a measure 0 subset. In the absence of this, labor commands a positive share of income that is bounded away from zero and constrains the benefit of continued capital accumulation, and hence, sustained economic growth. This is related to Baumol's cost disease (Baumol and Bowen, 1966). Finally, result (iii) gives a sufficient condition for continuous growth under incomplete automation when $\sigma > 1$.

Majority Voting. Next we consider what happens in our baseline case of majority voting when no steady state exists. Suppose that condition (23) is not satisfied, so that:

$$A(1) > \frac{\delta + \rho + \gamma}{1 - \omega} \tag{A.35}$$

Then continuous growth through the accumulation of capital occurs. Here we make a further simplifying assumption: suppose that worker utility is separable in consumption and labor, and is CRRA in consumption, i.e.,

$$U(C_w, L) = \frac{C_w^{1-\psi}}{1 - \psi} - h(L) \tag{A.36}$$

This assumption allows us to compute a balanced growth rate for the economy.

The planning problem and solution is just as in Proposition 2 in the main text, but now no steady state is obtained. Instead, the long-run result is as follows:

Proposition 1. *Suppose that technology and preferences satisfy (A.35), and that worker preferences satisfy (A.36), with*

$$\psi > 1 - \frac{\gamma}{(1-\omega)A(1) - \delta - \rho} \tag{A.37}$$

Then in the equilibrium under majority voting, no steady state is reached, and the economy converges to a balanced growth path, in which the variables $\{K, Y, C_w, C_e\}$ *all grow at rate:*

$$g = \frac{(1-\omega)A(1) - \delta - \rho - \gamma}{\psi}$$

Along the growth path, output is $Y = A(1) \cdot K$*, entrepreneur consumption is* $C_e = \rho K$*, and worker consumption is:*

$$C_w = \left[(\psi - 1)\, g + \gamma\right] K \tag{A.38}$$

The capital tax rate on this balanced growth path is:

$$\tau_{bg}^{k} = \omega + \frac{(\psi - 1)\, g + \gamma}{A(1)}$$

Proof. The statement of the planning problem and the optimality conditions are the same as in Proposition 2, with $U_C(C_w, L) = C_w^{-\psi}$ and $U_L = -h'(L)$. As we proved in section A.2, the marginal product of capital F_K is increasing in L. Since as $L/K \to 0$, $F_K \to A(1)$, it follows that $F_K \geq A(1)$, and therefore condition (A.35), together with condition (28), implies C_w is continuously growing at some rate bounded below by a value above 0. As the economy grows, eventually $L/K \to 0$, the production function converges to $Y \to A(1) \cdot K$, and $F_K \to A(1)$.

Now we check whether a balanced growth path exists that satisfies the optimality conditions. From (28), we find that worker consumption is growing at the rate:

$$\frac{\dot{C}_w}{C_w} = \frac{(1-\omega)A(1) - \delta - \rho - \gamma}{\psi} = g \tag{A.39}$$

From the constraint (25), we find that capital grows at rate:

$$\frac{\dot{K}}{K} = (1-\omega)A(1) - \delta - \rho - \frac{C_w}{K} \tag{A.40}$$

Since a balanced growth path requires that all variables $\{C_w, C_e, K\}$ grow at the same rate, this rate must be g. If this is true for K, it will also be true for C_e, since $C_e = \rho K$. Then we can simply ask what level of C_w/K is consistent with $\dot{K}/K = g$. Using the expression above yields:

$$\frac{C_w}{K} = [(1-\omega)A(1) - \delta - \rho]\left(\frac{\psi - 1}{\psi}\right) + \frac{\gamma}{\psi} \tag{A.41}$$

Since a solution requires $C_w > 0$ (since this implies finite λ), this expression implies that a solution requires:

$$\psi > 1 - \frac{\gamma}{(1-\omega)A(1) - \delta - \rho}$$

which is equivalent to the infinite integral in the objective function in the optimization problem converging. Note that the growth rate will be positive when

$$A(1) > \frac{\delta + \rho + \gamma}{1 - \omega} = r^* \tag{A.42}$$

which is just the condition (A.35). □

Note that the growth rate is decreasing in ψ. Intuitively, for lower ψ workers have less desire to smooth consumption, and thus choose to grow the capital stock more quickly. Condition (A.37) is necessary because, for sufficiently low ψ, the implied growth rate is fast enough that, on the balanced growth path, the integral of worker lifetime utility diverges and the planning problem (24) is no longer well-defined.[2]

The capital tax rate is increasing in the growth rate when $\psi > 1$, and decreasing in the growth rate when $\psi < 1$. This is for the usual reason — taxes are a means that workers can consume more now at the cost of less consumption in the future. When $\psi > 1$, the desire to smooth consumption by consuming more today is dominant, and thus a higher growth rate causes higher taxes. When $\psi < 1$, the desire to grow future consumption faster dominates.

We can compare the solution under majority voting to the optimal growth problem with a representative agent (i.e., that owns capital and supplies labor, rather than separate workers and entrepreneurs). In that case, if a balanced growth path is achieved, it would feature growth rate:

$$g = \frac{(1-\omega)A(1) - \delta - \rho}{\psi}$$

which corresponds to a capital tax of $\tau^k = \omega$. By contrast, here the tax rate is greater than ω. Thus, compared to the representative agent case, our model features slower growth and

[2]Note that, if condition (A.37) did not hold, (A.38) implies that worker consumption would be non-positive. To see that lifetime utility diverges, note that on a balanced growth path, worker flow utility grows at rate $(1-\psi)g$, and condition (A.37) can be written as $(1-\psi)g < \gamma$.

higher capital taxes.

References

William J Baumol and William G Bowen. *Performing arts, the economic dilemma : a study of problems common to theater, opera, music and dance.* MIT Press, 1966.

Jagdish N Bhagwati, Arvind Panagariya, and Thirukodikaval Nilakanta Srinivasan. *Lectures on international trade.* MIT press, 1998.

Avinash Dixit and Victor Norman. *Theory of international trade: A dual, general equilibrium approach.* Cambridge University Press, 1980.

Gene M Grossman and Elhanan Helpman. *Innovation and growth in the global economy.* MIT press, 1991.

Rainer Klump, Peter McAdam, and Alpo Willman. The normalized ces production function: theory and empirics. *Journal of Economic surveys*, 26(5):769–799, 2012.