# Supplementary Material for "Normal Mode Copulas for Nonmonotonic Dependence"

Kentaro Fukumoto*

October 13, 2023

*Professor, Department of Political Science, Gakushuin University. Mailing address: 1-5-1 Mejiro, Toshima, Tokyo 171-8588, Japan. Telephone: +81-3-5904-9258. Fax: +81-3-5992-1006. Email address: `Kentaro.Fukumoto@gakushuin.ac.jp`.

# FORMATION MODEL

CMS model the latent utility of government $j$ as

$$\alpha_{ij} \equiv \boldsymbol{z}'_{1,ij}\boldsymbol{\theta}_1 + \eta_{ij},$$

where $\eta_{ij}$ follows the standard Gumbel distribution. Government $g$ is formed when $\alpha_{ig}$ is the maximum among $\alpha_{ij}$'s of formation opportunity $i$. CMS show that

$$\begin{aligned}\Pr(Y_{1,i} = g) &= \Pr(\max_{j \in \mathcal{J}_i} \alpha_{ij} = \alpha_{ig}) \\ &= G(g \mid \boldsymbol{z}_{1,i}, \boldsymbol{\theta}_1).\end{aligned} \tag{S.1}$$

I define $X_{1,i} \equiv -(\alpha_{ig} - \max_{j \in \mathcal{J}} \alpha_{ij}) + \overline{x}_{1,i}$. It follows that

$$\begin{aligned}\Pr(Y_{1,i} = g) &= \Pr(X_{1,i} \leq \overline{x}_{1,i}) \\ &= \Pr\{U_{1,i} \leq F_1(\overline{x}_{1,i})\} \\ &= F_1(\overline{x}_{1,i}).\end{aligned} \tag{S.2}$$

According to Equations S.1 and S.2,

$$\begin{aligned}G(g \mid \boldsymbol{z}_{1,i}, \boldsymbol{\theta}_1) &= F_1(\overline{x}_{1,i}) \\ \therefore F_1^{-1}\{G(g \mid \boldsymbol{z}_{1,i}, \boldsymbol{\theta}_1)\} &= \overline{x}_{1,i}.\end{aligned}$$

Thus, in the main text, I write, “when a latent utility variable $X_{1,i}$ is smaller than $\overline{x}_{1,i} \equiv F_1^{-1}\{G(g \mid \boldsymbol{z}_{1,i}, \boldsymbol{\theta}_1)\}$, we observe $Y_{1,i} = g$.”

# FULL MODEL

## Model

In the main text, I simplify the original CMS model with respect to two points. In this section, I use the CMS's full model.

First, although CMS employ competing risks approach, my main text pools risks. CMS tell the difference between two types of events: replacement (by another government without an election, 231 governments) and dissolution (of the legislature followed by an election, 112 governments). Below, when we focus on one type of event (e.g., replacement), we regard the other (i.e., dissolution) as censoring.

Second, CMS use 22 covariates (besides 80 party dummy variables) for the formation model and six covariates for the duration model, while my main text pick only two covariates for each model. Below, I employ the same covariates as CMS. The names of the full set of covariates are shown in Tables S2 and S3 (or S4). For the details of the covariates, refer to CMS and Martin and Stevenson (2010).

## Results

### AIC

Like Table 1 of the main text, Table S1 reports the AIC for each copula used in a model. Initially, we focus on the left column, where the event is replacement. Compared with the separate model (first row), the Gaussian copula model CMS used (third row) improves the AIC by 17. The normal mode copula model with $\kappa_1 = 1$ and $\kappa_2 = 3$ (hereafter, the "NM(1, 3) copula model," sixth row) has the best (i.e., smallest) AIC, which is smaller than the AIC of the Gaussian copula model by a larger margin, 61. Note that this is different from the NM(1, 2) copula model, which is the best among the simplified models in the main text; this is unsurprising because the covariates differ across the full and simplified models. No conventional copula model (the last ten rows) outperforms even the Gaussian copula model, much less the NM(1, 3) copula model. When we proceed to the right column, where the event is dissolution, the implications remain the same: the NM(1, 3) copula model is again the best in terms of AIC.

**Table S1:** AICs for models using various copulas.

| Copula | | Replacement | Dissolution |
|---|---|---|---|
| Product | Separate | 5531.4 | 3826.3 |
| | Two-Step | 5521.3 | 3828.2 |
| Gaussian | | 5514.4 | 3826.8 |
| Normal Mode | $\kappa_1 = 1, \kappa_2 = 1$ | 5522.6 | 3828.2 |
| | $\kappa_1 = 1, \kappa_2 = 2$ | 5472.6 | 3815.5 |
| | $\kappa_1 = 1, \kappa_2 = 3$ | 5453.4 | 3782.3 |
| | $\kappa_1 = 1, \kappa_2 = 4$ | 5470.0 | 3782.9 |
| | $\kappa_1 = 2, \kappa_2 = 1$ | 5730.0 | 3828.3 |
| | $\kappa_1 = 2, \kappa_2 = 2$ | 5533.4 | 3822.4 |
| | $\kappa_1 = 2, \kappa_2 = 3$ | 5533.4 | 3812.6 |
| | $\kappa_1 = 2, \kappa_2 = 4$ | 5533.3 | 3805.6 |
| | $\kappa_1 = 3, \kappa_2 = 1$ | 5523.4 | 3783.7 |
| | $\kappa_1 = 3, \kappa_2 = 2$ | 5473.1 | 3821.0 |
| | $\kappa_1 = 3, \kappa_2 = 3$ | 5480.8 | 3826.3 |
| | $\kappa_1 = 3, \kappa_2 = 4$ | 5489.0 | 3828.3 |
| | $\kappa_1 = 4, \kappa_2 = 1$ | 5517.5 | 3800.2 |
| | $\kappa_1 = 4, \kappa_2 = 2$ | 5491.4 | 3817.8 |
| | $\kappa_1 = 4, \kappa_2 = 3$ | 5485.4 | 3822.7 |
| | $\kappa_1 = 4, \kappa_2 = 4$ | 5495.6 | 3827.2 |
| FGM | | 5524.4 | 3830.6 |
| AMH | Original | 5529.5 | 3828.0 |
| | Associate 1 | 5521.6 | 3828.1 |
| | Associate 2 | 5531.4 | 3828.1 |
| | Survival | 5519.5 | 3828.1 |
| Clayton | Original | 5533.4 | 3828.3 |
| | Associate 1 | 5533.4 | 3828.3 |
| | Associate 2 | 5533.4 | 3828.3 |
| | Survival | 5533.4 | 3828.3 |
| Frank | | 5526.6 | 3828.1 |

The number of parameters in the models is 111, except for the model with the separate model, where the number is 110; note that the product copula has no parameter. Readers can calculate the log-likelihood of the models; in particular, those with the separate model and the Gaussian copula model correspond to those of the "[w]ithout selection" and "[w]ith selection" models in Table 3 of CMS, respectively.

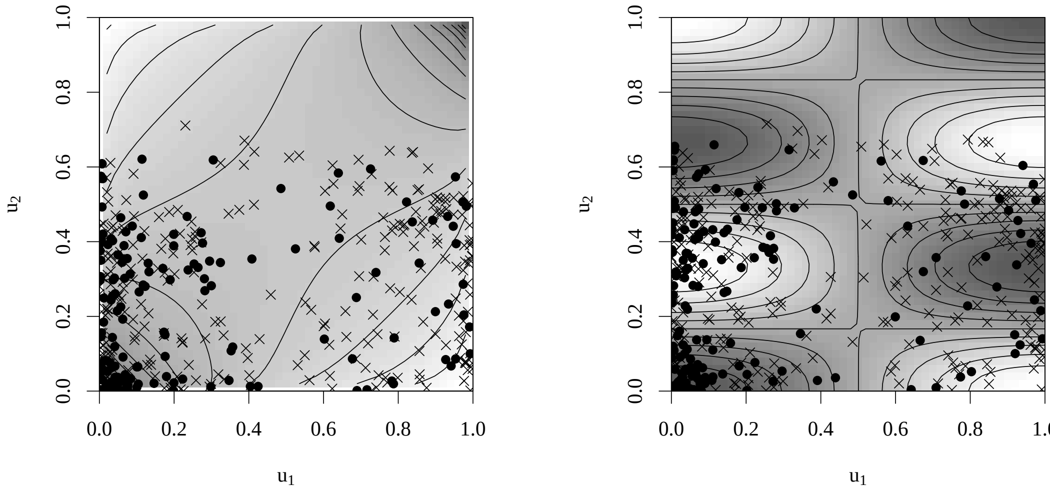
### Scatter Plots

Like Figure 6 of the main text, Figures S1 and S2 illustrate the scatter plots of estimated $\overline{u}_{1,i}$'s and $\underline{u}_{2,i}$'s, where the event is replacement and dissolution, respectively. The left and right panels correspond to the Gaussian copula model and the NM(1, 3) copula model, respectively.[S1] It does not seem that the Gaussian copula fits the data well. In particular, units near the right vertical axis are situated where the density is low. CMS argue that "the governments that parties choose to form are those that have a *lower probability of ending in early replacement*" (p. 55, emphasis original). Nevertheless, apparently, some likely governments end unexpectedly earlier (units in the bottom-right corner), while other unlikely governments survive (units near and in the middle of the left vertical axis). By contrast, the NM(1, 3) copula well captures units in the high-density areas. This is why it achieves the best performance among the studied copulas in Table S1.

### Coefficient Estimates

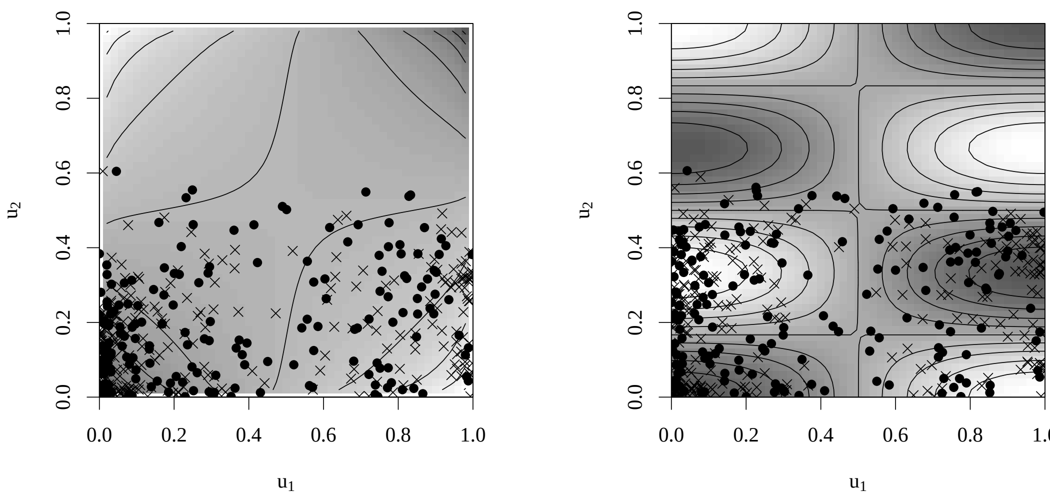
Table S2 focuses on the estimation results of the duration parameters ($\boldsymbol{\theta}_2$). The first and second columns address replacement events, while the third and fourth columns involve dissolution events. In each event type, the left and right columns correspond to the Gaussian copula model and the NM(1, 3) copula model, respectively. My Gaussian copula model (first and third columns) mostly succeeds in replicating CMS's results (the second and fourth columns of their Table 3). The absolute differences are, if ever, mostly less than 0.01. The maximum is 0.267 (the standard error for the intercept in the case of dissolution). If I compare my replication results not with Table 3 of CMS but with CMS's original output files (`biv.clw.1.rda` and `biv.clw.2.rda` in Chiba, Martin and Stevenson (2014)), the absolute

[S1] $\hat{\theta}_{12} = 0.300$ for the Gaussian copula (replacement event), $\hat{\theta}_{12} = 0.112$ for the Gaussian copula (dissolution event) and $\hat{\theta}_{12} = 1.000$ for the NM(1, 3) copula (either event). The corresponding Kendall rank correlations are 0.194, 0.071, and 0.036, respectively. Unlike the Gaussian copula model, the results of the NM(1, 3) copula model imply little linear correlation between formation and duration.




**(a)** Gaussian Copula **(b)** NM(1, 3) Copula

**Figure S1:** Scatter plots of $\overline{u}_{1,i}$'s and $\underline{u}_{2,i}$'s with contour plots of the estimated copula in the case of replacement events. $n = 432$.




**(a)** Gaussian Copula **(b)** NM(1, 3) Copula

**Figure S2:** Scatter plots of $\overline{u}_{1,i}$'s and $\underline{u}_{2,i}$'s with contour plots of the estimated copula in the case of dissolution events. $n = 432$.

differences almost vanish but still remain in three cells (e.g., the standard error for the intercept in the case of dissolution).

A few remarks are in order. First, as expected, all standard errors are smaller in the

**Table S2:** Competing risks analysis of government duration: models with Gaussian and NM(1, 3) copulas. Cell entries are estimates (with standard errors in parentheses). $n = 432$.

| | Replacement | | Dissolution | |
|---|---|---|---|---|
| | Gaussian | NM(1, 3) | Gaussian | NM(1, 3) |
| Minority Government | −0.200 | −0.257 | −0.325 | −0.175 |
| | ( 0.089 ) | ( 0.081 ) | ( 0.136 ) | ( 0.102 ) |
| Ideological Divisions in Coalition | −0.002 | −0.002 | 0.005 | 0.002 |
| | ( 0.002 ) | ( 0.002 ) | ( 0.004 ) | ( 0.003 ) |
| Returnability | −0.358 | −0.213 | −0.078 | −0.022 |
| | ( 0.103 ) | ( 0.091 ) | ( 0.141 ) | ( 0.100 ) |
| Effective Number of Legislative Parties | −0.006 | −0.012 | 0.107 | 0.009 |
| | ( 0.032 ) | ( 0.028 ) | ( 0.060 ) | ( 0.041 ) |
| Polarization Index | −0.022 | −0.028 | −0.064 | −0.036 |
| | ( 0.020 ) | ( 0.015 ) | ( 0.027 ) | ( 0.018 ) |
| Time Remaining in CIEP (Logged) | 0.895 | 0.878 | 0.753 | 0.852 |
| | ( 0.064 ) | ( 0.064 ) | ( 0.116 ) | ( 0.115 ) |
| Intercept | 1.334 | 1.340 | 2.018 | 1.373 |
| | ( 0.489 ) | ( 0.484 ) | ( 0.888 ) | ( 0.857 ) |
| Duration Dependence (Logged) | 0.683 | 0.505 | 0.543 | 0.568 |
| | ( 0.056 ) | ( 0.057 ) | ( 0.082 ) | ( 0.077 ) |

NM(1, 3) copula model than in the Gaussian copula model, except for that of Duration Dependence's parameter for replacement events. Second, we find nonnegligible differences in coefficients between the two models. For replacement events, in the NM(1, 3) copula model relative to the Gaussian copula model, the coefficient size of Minority Government is greater by 28% and the coefficient size of Returnability is less by 40%. For dissolution events, all coefficients are closer to zero except for that of Time Remaining in CIEP; in particular, those of Minority Government and Polarization Index cease to be significant. According to CMS, "the correlation between government formation and survival . . . is not statistically different from zero" and "[t]he effects of all variables are very similar across the models with and without selection" (pp. 55–56). However, given the performance of model fit, it would be natural to put more confidence in the coefficients of the NM(1, 3) copula model than in those of the Gaussian copula model.

Tables S3 and S4 show the estimation results of the formation parameters ($\boldsymbol{\theta}_1$) for replacement and dissolution events, respectively. I exclude 80 party fixed effects as Appendix Table 1 of CMS does. In each table, the left and right columns correspond to the Gaussian copula model and the NM(1, 3) copula model, respectively. As shown in the left column of Tables S3 and S4, I succeeded in replicating most of the point estimates in the second and fourth columns, respectively, of CMS's Appendix Table 1. If I compare my replication results not with Appendix Table 1 of CMS but with CMS's original output files, the absolute differences diminish but still remain in two and three cells for replacement and dissolution events, respectively. The maximum is 0.001. Roughly, the Gaussian copula model and the NM(1, 3) copula model lead to similar results. These results are reminiscent of the fact that a comparison of the results from the Gaussian copula model to those of the separate model "reveals no major differences in the effects of the independent variables" (CMS, their footnote 18). This is probably because the estimation of the formation parameters is not affected by selection bias.

**Table S3:** Conditional logit analysis of government formation (replacement events).

| | Gaussian | NM(1, 3) |
|---|---|---|
| Minority Government | −0.777 | −0.701 |
| | (0.362) | (0.353) |
| Minimal Winning Coalition | 0.961 | 0.919 |
| | (0.191) | (0.188) |
| Number of Parties in Coalition | −0.945 | −0.935 |
| | (0.115) | (0.113) |
| Largest Party in Coalition | 2.291 | 2.218 |
| | (0.197) | (0.194) |
| Median Party in Coalition | 0.387 | 0.332 |
| | (0.170) | (0.169) |
| Ideological Divisions in Coalition | −0.019 | −0.019 |
| | (0.005) | (0.005) |
| Ideological Divisions within Majority Opposition | 0.007 | 0.005 |
| | (0.005) | (0.004) |
| Minority Government with Investiture Requirement | −0.179 | −0.133 |
| | (0.285) | (0.279) |
| Anti-System Presence in Coalition | −0.024 | −0.019 |
| | (0.022) | (0.022) |
| Pre-Electoral Pact associated with Coalition | 2.810 | 2.798) |
| | (0.487) | (0.407 |
| Anti-Pact associated with Coalition | −5.199 | −5.176) |
| | (0.451) | (0.426 |
| Previous PM Party in Coalition | −0.924 | −0.912) |
| | (0.307) | (0.308 |
| Status Quo Government | 2.124 | 1.963) |
| | (0.244) | (0.229) |
| Familiarity | 1.158 | 1.200 |
| | (0.506) | (0.501) |
| Similarity to Status Quo Government | 0.286 | 0.384 |
| | (0.459) | (0.454) |
| Intra-Cabinet Conflict × SQ Government | −2.079 | −1.999 |
| | (0.406) | (0.401) |
| Intra-Cabinet Conflict × Previous PM Party | 0.292 | 0.244 |
| | (0.385) | (0.384) |
| Post-Election Continuation Rule × SQ Government (No Conflict) | 1.005 | 1.092 |
| | (0.393) | (0.367) |
| Average Seat Change | −0.027 | −0.026 |
| | (0.022) | (0.021) |
| Average Seat Change × Post-Election Bargaining | 0.094 | 0.091 |
| | (0.029) | (0.028) |
| Average Seat Change × SQ Government | 0.044 | 0.048 |
| | (0.029) | (0.026) |
| Average Seat Change × Post-Election Bargaining | 0.043 | 0.052 |
| × SQ Government | (0.050) | (0.048) |

**Table S4:** Conditional logit analysis of government formation (dissolution events).

| | Gaussian | NM(1, 3) |
|---|---|---|
| Minority Government | −0.859 | −0.895 |
| | (0.365) | (0.371) |
| Minimal Winning Coalition | 0.923 | 0.913 |
| | (0.192) | (0.190) |
| Number of Parties in Coalition | −0.931 | −0.922 |
| | (0.115) | (0.114) |
| Largest Party in Coalition | 2.227 | 2.185 |
| | (0.199) | (0.198) |
| Median Party in Coalition | 0.338 | 0.322 |
| | (0.173) | (0.173) |
| Ideological Divisions in Coalition | −0.020 | −0.019 |
| | (0.005) | (0.005) |
| Ideological Divisions within Majority Opposition | 0.007 | 0.008 |
| | (0.005) | (0.005) |
| Minority Government with Investiture Requirement | −0.051 | −0.038 |
| | (0.286) | (0.284) |
| Anti-System Presence in Coalition | −0.022 | −0.020 |
| | (0.022) | (0.022) |
| Pre-Electoral Pact associated with Coalition | 2.921 | 2.801 |
| | (0.482) | (0.447) |
| Anti-Pact associated with Coalition | −5.396 | −5.216 |
| | (0.452) | (0.447) |
| Previous PM Party in Coalition | −0.932 | −0.932 |
| | (0.312) | (0.311) |
| Status Quo Government | 2.035 | 1.990 |
| | (0.245) | (0.238) |
| Familiarity | 1.257 | 1.306 |
| | (0.511) | (0.501) |
| Similarity to Status Quo Government | 0.372 | 0.412 |
| | (0.463) | (0.455) |
| Intra-Cabinet Conflict × SQ Government | −2.261 | −2.304 |
| | (0.421) | (0.399) |
| Intra-Cabinet Conflict × Previous PM Party | 0.256 | 0.209 |
| | (0.387) | (0.381 |
| Post-Election Continuation Rule × SQ Government (No Conflict) | 1.036 | 1.054 |
| | (0.395) | (0.376) |
| Average Seat Change | −0.026 | −0.021 |
| | (0.021) | (0.021) |
| Average Seat Change × Post-Election Bargaining | 0.088 | 0.083 |
| | (0.029) | (0.028) |
| Average Seat Change × SQ Government | 0.035 | 0.034 |
| | (0.029) | (0.029) |
| Average Seat Change × Post-Election Bargaining | 0.054 | 0.059 |
| × SQ Government | (0.050) | (0.049) |

# References

Chiba, Daina, Lanny W. Martin and Randolph T. Stevenson. 2014. "Replication data for: A Copula Approach to the Problem of Selection Bias in Models of Government Survival." Harvard Dataverse, V2, https://doi.org/10.7910/DVN/26966.

Martin, Lanny W. and Randolph T. Stevenson. 2010. "The Conditional Impact of Incumbency on Government Formation." *American Political Science Review* 104(3):503–518.