1. The general linear mixed model and the linear mixed model hypothesis

It is common to define the general linear mixed model as a statement about only the observations for independent sampling unit $i \in \{1, \ldots, N\}$ (Muller and Stewart, 2006, section 6.7). Let p_i be the number of outcomes for independent sampling unit i, and let $j \in \{1, 2, \ldots, p_i\}$ index outcomes. For example, if there are 6 repeated measures of BMI for a person in the study, we would have $p_i = 6$. Let y_{ij} indicate the j^{th} outcome (dependent) variable for independent sampling unit i. For independent sampling unit i, define the $(p_i \times 1)$ vector of outcomes by $\boldsymbol{y}_i$, where

$$\underset{p_i \times 1}{\boldsymbol{y}_i} = \begin{bmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{ip_i} \end{bmatrix}. \tag{1}$$

Here, we define the total number of outcome observations by

$$n = \sum_{i=1}^{N} p_i \,. \tag{2}$$

The mixed model includes fixed-effect predictors $\boldsymbol{X}_i$ (fixed and known) and random-effect predictors $\boldsymbol{Z}_i$ (fixed and known):

$$\underset{p_i \times 1}{\boldsymbol{y}_i} = \underset{(p_i \times q)(q \times 1)}{\boldsymbol{X}_i \boldsymbol{\beta}} + \underset{(p_i \times r)(r \times 1)}{\boldsymbol{Z}_i \boldsymbol{d}_i} + \underset{(p_i \times 1)}{\boldsymbol{e}_i} \,. \tag{3}$$

Here $\boldsymbol{\beta}$ contains unknown fixed-effect parameters which represent slopes and intercepts, $\boldsymbol{X}_i$ includes intercepts and predictors, $\boldsymbol{Z}_i$ is the participant-specific matrix of random effects, $\boldsymbol{d}_i$ is the participant specific matrix of random slopes and intercepts, and $\boldsymbol{e}_i$ is the participant-specific matrix of errors. By assumption, both $\boldsymbol{d}_i$ and $\boldsymbol{e}_i$ are independently normally distributed, with mean zero. The distributions are

$$\begin{bmatrix} \boldsymbol{d}_i \\ \boldsymbol{e}_i \end{bmatrix} \sim \mathcal{N}_{r+p_i}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_d & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{e_i} \end{bmatrix} \right), \tag{4}$$

which implies $\mathrm{E}(\boldsymbol{y}_i) = \boldsymbol{X}_i \boldsymbol{\beta}$ and $\boldsymbol{\mathcal{V}}(\boldsymbol{y}_i) = \boldsymbol{\Sigma}_i = \boldsymbol{Z}_i \boldsymbol{\Sigma}_d \boldsymbol{Z}_i' + \boldsymbol{\Sigma}_{e_i}$.

For the entire study sample, construct stacked matrices, so that

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \vdots \\ \boldsymbol{y}_N \end{bmatrix}, \tag{5}$$

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \\ \vdots \\ \boldsymbol{X}_N \end{bmatrix}, \tag{6}$$

$$\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{Z}_1 \\ \boldsymbol{Z}_2 \\ \vdots \\ \boldsymbol{Z}_N \end{bmatrix}, \tag{7}$$

$$\boldsymbol{d} = \begin{bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \\ \vdots \\ \boldsymbol{d}_N \end{bmatrix}, \tag{8}$$

and

$$\boldsymbol{e} = \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \vdots \\ \boldsymbol{e}_N \end{bmatrix},. \tag{9}$$

The general linear mixed model is given by

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Zd} + \boldsymbol{e} \ . \tag{10}$$

A general linear mixed model hypothesis about fixed effects can be written in terms of $\boldsymbol{\theta} = \boldsymbol{C\beta}$ as

$$H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0 \ . \tag{11}$$

2. The three models: a simplified version

2.1 Definitions and notation

We start with thinking about the data for one participant in the study. The paired data for person i looks like this, where y_{ij} is the outcome for person i at time point j, and x_{ij} is the predictor for person i at time point j. To make the dataset labels short, write PID for participant ID, write $\boldsymbol{y}$ for the outcome, $\boldsymbol{a}$ for age in years, and $\boldsymbol{x}$ for the predictor. For the purposes of this explanation, we assume four time points, and ignore covariates.

PID	$\boldsymbol{y}$	$\boldsymbol{a}$	$\boldsymbol{x}$
1	y_{i1}	1	x_{i1}
1	y_{i2}	2	x_{i2}
1	y_{i3}	3	x_{i3}
1	y_{i4}	4	x_{i4}

(12)

2.2 Fixed effect predictor matrix for the all-times-before model

The fixed effect predictor matrix for participant i is

$$\boldsymbol{X}_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & x_{i1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_{i1} & x_{i2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_{i1} & x_{i2} & x_{i3} \end{bmatrix}, \tag{13}$$

with $\boldsymbol{\beta}$

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \\ \beta_{10} \end{bmatrix}. \tag{14}$$

2.3 Fixed effect predictor matrix for the immediately-before model

The fixed effect predictor matrix for participant i is

$$\boldsymbol{X}_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & x_{i1} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & x_{i2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & x_{i3} \end{bmatrix}. \tag{15}$$

The contrast matrix to test the difference between all-times-before model and the immediately before model is

$$\boldsymbol{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \tag{16}$$

which tests

$$H_0 : \beta_5 = 0 \quad \beta_8 = 0 \quad \beta_9 = 0. \tag{17}$$

If these coefficients are zero, the all-times-before model reduces to the immediately-before model.

2.4 Fixed effect predictor matrix for the differential-sensitive-period model

The fixed effect predictor matrix for participant i is

$$\boldsymbol{X}_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & x_{i1} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & x_{i1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & x_{i1} \end{bmatrix}. \tag{18}$$

The contrast matrix to test the difference between all-times-before model and the differential-sensitive period model is

$$\boldsymbol{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{19}$$

which tests

$$H_0 : \beta_6 = 0 \quad \beta_9 = 0 \quad \beta_{10} = 0. \tag{20}$$

If these coefficients are zero, the all-times-before model reduces to the differential sensitive period model.

The fixed-effect predictor model for the differential sensitive period is as follows

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_7 \\ \beta_8 \end{bmatrix}. \tag{21}$$

2.5 Fixed effect predictor matrix for the stable-sensitive-period model

The fixed effect predictor matrix for participant i is

$$\boldsymbol{X}_i = \begin{bmatrix} 1 & 0 \\ 1 & x_{i1} \\ 1 & x_{i1} \\ 1 & x_{i1} \end{bmatrix}. \tag{22}$$

The contrast matrix to test the difference between the differential-sensitive period model and the stable-sensitive period model is

$$\boldsymbol{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}, \tag{23}$$

which tests

$$H_0 : \ \beta_5 - \beta_7 = 0 \quad \beta_5 - \beta_8 = 0 \tag{24}$$

If these hypotheses are true, the differential sensitive period model reduces to the stable sensitive period model.

3. Organizing the dataset for the models

3.1 Data set-up for all-times-before model

Reformat the data for person i so it looks like this, where xj is the predictor value for the participant at age j.

PID	$\boldsymbol{y}$	$\boldsymbol{a}$	$\boldsymbol{x}$	$x1$	$x2$	$x3$	$x4$
1	y_{i1}	1	x_{i1}	x_{i1}	x_{i2}	x_{i3}	x_{i4}
1	y_{i2}	2	x_{i2}	x_{i1}	x_{i2}	x_{i3}	x_{i4}
1	y_{i3}	3	x_{i3}	x_{i1}	x_{i2}	x_{i3}	x_{i4}
1	y_{i4}	4	x_{i4}	x_{i1}	x_{i2}	x_{i3}	x_{i4}

(25)

The goal is reformat the predictor matrix to define the all-times-before model. For this purpose, define indicator variables. Let

$$b1 = \begin{cases} 0 & \text{age} \leq 1 \\ 1 & \text{otherwise .} \end{cases} \tag{26}$$

Let

$$b2 = \begin{cases} 0 & \text{age} \leq 2 \\ 1 & \text{otherwise .} \end{cases} \tag{27}$$

In general, define

$$bj = \begin{cases} 0 & \text{age} \leq j \\ 1 & \text{otherwise .} \end{cases} \tag{28}$$

We do not define an indicator variable $b4$, because we are getting set up for a model which uses only predictors measured before the outcome.

PID	$\boldsymbol{y}$	$\boldsymbol{a}$	$\boldsymbol{x}$	$b1$	$b2$	$b3$	$b4$
1	y_{i1}	1	x_{i1}	0	0	0	0
1	y_{i2}	2	x_{i2}	1	0	0	0
1	y_{i3}	3	x_{i3}	1	1	0	0
1	y_{i4}	4	x_{i4}	1	1	1	0

(29)

Form $i1 = b1{*}x1$, $i2 = b2{*}x2$, and so forth.

PID	$\boldsymbol{y}$	$\boldsymbol{a}$	$\boldsymbol{x}$	$i1$	$i2$	$i3$
1	y_{i1}	1	x_{i1}	0	0	0
1	y_{i2}	2	x_{i2}	x_{i1}	0	0
1	y_{i3}	3	x_{i3}	x_{i1}	x_{i2}	0
1	y_{i4}	4	x_{i4}	x_{i1}	x_{i2}	x_{i3}

(30)

Define indicator variables for ages so that

$$aj = \begin{cases} 1 & \text{age} = j \\ 0 & \text{otherwise .} \end{cases} \tag{31}$$

PID	$\boldsymbol{y}$	$\boldsymbol{a}$	$\boldsymbol{x}$	$i1$	$i2$	$i3$	$a1$	$a2$	$a3$	$a4$
1	y_{i1}	1	x_{i1}	0	0	0	1	0	0	0
1	y_{i2}	2	x_{i2}	x_{i1}	0	0	0	1	0	0
1	y_{i3}	3	x_{i3}	x_{i1}	x_{i2}	0	0	0	1	0
1	y_{i4}	4	x_{i4}	x_{i1}	x_{i2}	x_{i3}	0	0	0	1

(32)

Form $h21 = a2{*}i1$, $h31 = a3{*}i1$, $h32 = a3{*}i2$, $h41 = a4{*}i1$, $h42 = a4{*}i2$, and $h43 = a4{*}i3$. The columns of the dataset have been re-sorted to show the sets of predictors for y_{i1}, y_{i2} and so on in order.

PID	$\boldsymbol{y}$	$\boldsymbol{a}$	$\boldsymbol{x}$	$a1$	$a2$	$h21$	$a3$	$h31$	$h32$	$a4$	$h41$	$h42$	$h43$
1	y_{i1}	1	x_{i1}	1	0	0	0	0	0	0	0	0	0
1	y_{i2}	2	x_{i2}	0	1	x_{i1}	0	0	0	0	0	0	0
1	y_{i3}	3	x_{i3}	0	0	0	1	x_{i1}	x_{i2}	0	0	0	0
1	y_{i4}	4	x_{i4}	0	0	0	0	0	0	1	x_{i1}	x_{i2}	x_{i3}

(33)

The fixed effect predictor matrix for participant i is then

$$\boldsymbol{X}_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & x_{i1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_{i1} & x_{i2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_{i1} & x_{i2} & x_{i3} \end{bmatrix} \tag{34}$$

3.2 Data set-up for immediately before model

Start with the data formatted as shown below.

PID	$\boldsymbol{y}$	$\boldsymbol{a}$	$\boldsymbol{x}$	$i1$	$i2$	$i3$	$a1$	$a2$	$a3$	$a4$
1	y_{i1}	1	x_{i1}	0	0	0	1	0	0	0
1	y_{i2}	2	x_{i2}	x_{i1}	0	0	0	1	0	0
1	y_{i3}	3	x_{i3}	x_{i1}	x_{i2}	0	0	0	1	0
1	y_{i4}	4	x_{i4}	x_{i1}	x_{i2}	x_{i3}	0	0	0	1

(35)

Define variables $k21 = a2{*}i1$, $k32 = a2{*}i2$, $k42 = a4{*}i3$.

PID	$\boldsymbol{y}$	$\boldsymbol{a}$	$\boldsymbol{x}$	$i1$	$i2$	$i3$	$a1$	$a2$	$a3$	$a4$	$k21$	$k32$	$k42$
1	y_{i1}	1	x_{i1}	0	0	0	1	0	0	0	0	0	0
1	y_{i2}	2	x_{i2}	x_{i1}	0	0	0	1	0	0	x_{i1}	0	0
1	y_{i3}	3	x_{i3}	x_{i1}	x_{i2}	0	0	0	1	0	0	x_{i2}	0
1	y_{i4}	4	x_{i4}	x_{i1}	x_{i2}	x_{i3}	0	0	0	1	0	0	x_{i3}

(36)

The fixed effect predictor matrix for participant i is then

$$\boldsymbol{X}_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & x_{i1} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & x_{i2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & x_{i3} \end{bmatrix}. \tag{37}$$

3.3 Data set-up for differential sensitive periods model

Assume that the sensitive period is at age 1. Start with the data formatted as follows.

PID	$\boldsymbol{y}$	$\boldsymbol{a}$	$\boldsymbol{x}$	$x1$	$a1$	$a2$	$a3$	$a4$
1	y_{i1}	1	x_{i1}	x_{i1}	1	0	0	0
1	y_{i2}	2	x_{i2}	x_{i1}	0	1	0	0
1	y_{i3}	3	x_{i3}	x_{i1}	0	0	1	0
1	y_{i4}	4	x_{i4}	x_{i1}	0	0	0	1

(38)

Compute $k21 = a2{*}x1$, $k31 = a3{*}x1$, and $k41 = a4{*}x1$.

PID	$\boldsymbol{y}$	$\boldsymbol{a}$	$\boldsymbol{x}$	$x1$	$a1$	$a2$	$a3$	$a4$	$k21$	$k31$	$k41$
1	y_{i1}	1	x_{i1}	x_{i1}	1	0	0	0	0	0	0
1	y_{i2}	2	x_{i2}	x_{i1}	0	1	0	0	x_{i1}	0	0
1	y_{i3}	3	x_{i3}	x_{i1}	0	0	1	0	0	x_{i1}	0
1	y_{i4}	4	x_{i4}	x_{i1}	0	0	0	1	0	0	x_{i1}

(39)

The fixed effect predictor matrix for participant i is then

$$\boldsymbol{X}_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & x_{i1} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & x_{i1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & x_{i1} \end{bmatrix}. \tag{40}$$

3.4 Data set-up for stable sensitive periods model

Again assuming that the sensitive period is period 1, start with the data formatted as below.

PID	$\boldsymbol{y}$	$\boldsymbol{a}$	$\boldsymbol{x}$	$x1$
1	y_{i1}	1	x_{i1}	x_{i1}
1	y_{i2}	2	x_{i2}	x_{i1}
1	y_{i3}	3	x_{i3}	x_{i1}
1	y_{i4}	4	x_{i4}	x_{i1}

(41)

As before, let

$$b1 = \begin{cases} 0 & \text{age} \leq 1 \\ 1 & \text{otherwise}\ , \end{cases} \tag{42}$$

and define an intercept as a column where every value is 1. Form $i1 = b1 {*} x1$, yielding a the dataset which follows

PID	$\boldsymbol{y}$	$\boldsymbol{a}$	$\boldsymbol{x}$	intercept	$x1$	$b1$	$i1$
1	y_{i1}	1	x_{i1}	1	x_{i1}	0	0
1	y_{i2}	2	x_{i2}	1	x_{i1}	1	x_{i1}
1	y_{i3}	3	x_{i3}	1	x_{i1}	1	x_{i1}
1	y_{i4}	4	x_{i4}	1	x_{i1}	1	x_{i1}

(43)

The fixed effect predictor matrix for participant i is then

$$\boldsymbol{X}_i = \begin{bmatrix} 1 & 0 \\ 1 & x_{i1} \\ 1 & x_{i1} \\ 1 & x_{i1} \end{bmatrix} . \tag{44}$$

4. Definitions and notation for the example study

Let $\mathbf{1}_N$ be an $(N \times 1)$ vector containing the entry 1 in every spot, and $\mathbf{0}_N$ similarly. Let $N = 191$ be the number of study participants. Let i index study participant, for $i \in \{1, 2, \ldots, N\}$. Let j index outcome measurement within study participant, for $j \in \{1, 2, \ldots, p_i\}$ indexing the number of measurements. Here, p_i is the number of outcome measures for study participant i. The total number of observations (i.e., the total number of repeated measured per person, summed over all the people in the study) is given by

$$n = \sum_{i=1}^{N} p_i. \quad (45)$$

The maximum number of observations per participant is 8, so $N_i \leq 8$. For participant i at measurement time j, let

$$\begin{aligned} a_{ij} &= \text{age (years) for participant } i \text{ at measurement time } j, \\ a_{ij}^2 &= \text{age}^2 \text{ (years) for participant } i \text{ at measurement time } j, \\ s_{i6} &= \text{income to needs ratio for participant } i \text{ at 6 months of age, and} \\ c_{i6} &= \text{predictor for participant } i \text{ at 6 months of age.} \end{aligned} \quad (46)$$

For study participant i, form vectors of predictors, so that, for example

$$\boldsymbol{a}_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{iN_i} \end{bmatrix}. \quad (47)$$

Let y_{ij} be the outcome measured on participant i for measurement j. The outcome vector for person i is given by

$$\underset{(N_i \times 1)}{\boldsymbol{y}_i} = \begin{bmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{iN_i} \end{bmatrix}. \quad (48)$$

The population outcome vector stacks the participants-specific outcome vectors, so

$$\underset{(N \times 1)}{\boldsymbol{y}} = \begin{bmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{in} \end{bmatrix}. \quad (49)$$

The population random effect matrix is

$$\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{Z}_1 & & & \\ & \boldsymbol{Z}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{Z}_n \end{bmatrix} \quad (50)$$

with $\boldsymbol{Z}_i = [\,\mathbf{1}_{N_i} \quad \boldsymbol{a}_i\,]$, and with corresponding random effects $\boldsymbol{d}_i = [d_{0i}, d_{1i}]$, and independent error term $\boldsymbol{e}_i = [e_{0i}, e_{1i}]$.

5. Specification of four models from the paper

The model differ only in their predictor matrix, and estimates. The stable-sensitive period model (Equation 51) was the model used to produce the figures and estimates for the paper. The all-times-before model (Equation 54) was used for hypothesis testing, to compare to thethe immediately-before model (Equation 15), or the differntial sensitive period model (Equation 52) was the best.

Stable-sensitive period model

$$\underset{(N \times 4)}{\boldsymbol{X}} = \begin{bmatrix} \mathbf{1} & \boldsymbol{c}_6 & \boldsymbol{a} & \boldsymbol{a} \\ \mathbf{1} & \boldsymbol{c}_6 & \boldsymbol{a} & \boldsymbol{a} \\ \mathbf{1} & \boldsymbol{c}_6 & \boldsymbol{a} & \boldsymbol{a} \\ \mathbf{1} & \boldsymbol{c}_6 & \boldsymbol{a} & \boldsymbol{a} \end{bmatrix} \tag{51}$$

Differential-sensitive period model:

$$\underset{(17 \times N)}{\boldsymbol{X}'} = \tag{52}$$

$$\begin{bmatrix} \mathbf{1} & & & & & & & \\ & \mathbf{1} & & & & & & \\ & & \mathbf{1} & & & & & \\ & & & \mathbf{1} & & & & \\ & & & & \mathbf{1} & & & \\ & & & & & \mathbf{1} & & \\ & & & & & & \mathbf{1} & \\ & & & & & & & \mathbf{1} \\ & \boldsymbol{c}_6 & & & & & & \\ & & \boldsymbol{c}_6 & & & & & \\ & & & \boldsymbol{c}_6 & & & & \\ & & & & \boldsymbol{c}_6 & & & \\ & & & & & \boldsymbol{c}_6 & & \\ & & & & & & \boldsymbol{c}_6 & \\ & & & & & & & \boldsymbol{c}_6 \\ \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} \\ \boldsymbol{a}^2 & \boldsymbol{a}^2 & \boldsymbol{a}^2 & \boldsymbol{a}^2 & \boldsymbol{a}^2 & \boldsymbol{a}^2 & \boldsymbol{a}^2 & \boldsymbol{a}^2 \end{bmatrix}$$

Immediately-before model Here, because $\boldsymbol{X}_{IB}$ has 17 columns, we present $\boldsymbol{X}'_{IB}$, with columns containing 0 suppressed for clarity.

$$\underset{(17 \times N)}{\boldsymbol{X}'} = \tag{53}$$

$$\begin{bmatrix} \mathbf{1} & & & & & & & \\ & \mathbf{1} & & & & & & \\ & & \mathbf{1} & & & & & \\ & & & \mathbf{1} & & & & \\ & & & & \mathbf{1} & & & \\ & & & & & \mathbf{1} & & \\ & & & & & & \mathbf{1} & \\ & & & & & & & \mathbf{1} \\ & \boldsymbol{c}_6 & & & & & & \\ & & \boldsymbol{c}_{12} & & & & & \\ & & & \boldsymbol{c}_{24} & & & & \\ & & & & \boldsymbol{c}_{60} & & & \\ & & & & & \boldsymbol{c}_{78} & & \\ & & & & & & \boldsymbol{c}_{120} & \\ & & & & & & & \boldsymbol{c}_{132} \\ \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} \\ \boldsymbol{a}^2 & \boldsymbol{a}^2 & \boldsymbol{a}^2 & \boldsymbol{a}^2 & \boldsymbol{a}^2 & \boldsymbol{a}^2 & \boldsymbol{a}^2 & \boldsymbol{a}^2 \end{bmatrix}$$

All-times-before model: Here, because $\boldsymbol{X}_b$ has 38 columns, we present $\boldsymbol{X}'_b$, with columns containing 0 suppressed for clarity.

$$\underset{(38 \times N)}{\boldsymbol{X}'} = \tag{54}$$

$$\begin{bmatrix}
\mathbf{1} & & & & & & & \\
& \mathbf{1} & & & & & & \\
& & \mathbf{1} & & & & & \\
& & & \mathbf{1} & & & & \\
& & & & \mathbf{1} & & & \\
& & & & & \mathbf{1} & & \\
& & & & & & \mathbf{1} & \\
& & & & & & & \mathbf{1} \\
& \boldsymbol{c}_6 & & & & & & \\
& & \boldsymbol{c}_6 & & & & & \\
& & & \boldsymbol{c}_6 & & & & \\
& & & & \boldsymbol{c}_6 & & & \\
& & & & & \boldsymbol{c}_6 & & \\
& & & & & & \boldsymbol{c}_6 & \\
& & & & & & & \boldsymbol{c}_6 \\
& & \boldsymbol{c}_{12} & & & & & \\
& & & \boldsymbol{c}_{12} & & & & \\
& & & & \boldsymbol{c}_{12} & & & \\
& & & & & \boldsymbol{c}_{12} & & \\
& & & & & & \boldsymbol{c}_{12} & \\
& & & & & & & \boldsymbol{c}_{12} \\
& & & \boldsymbol{c}_{24} & & & & \\
& & & & \boldsymbol{c}_{24} & & & \\
& & & & & \boldsymbol{c}_{24} & & \\
& & & & & & \boldsymbol{c}_{24} & \\
& & & & & & & \boldsymbol{c}_{24} \\
& & & & \boldsymbol{c}_{60} & & & \\
& & & & & \boldsymbol{c}_{60} & & \\
& & & & & & \boldsymbol{c}_{60} & \\
& & & & & & & \boldsymbol{c}_{60} \\
& & & & & \boldsymbol{c}_{78} & & \\
& & & & & & \boldsymbol{c}_{78} & \\
& & & & & & & \boldsymbol{c}_{78} \\
& & & & & & \boldsymbol{c}_{120} & \\
& & & & & & & \boldsymbol{c}_{120} \\
& & & & & & & \boldsymbol{c}_{132} \\
\boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} \\
\boldsymbol{a}^2 & \boldsymbol{a}^2 & \boldsymbol{a}^2 & \boldsymbol{a}^2 & \boldsymbol{a}^2 & \boldsymbol{a}^2 & \boldsymbol{a}^2 & \boldsymbol{a}^2
\end{bmatrix}$$