ONLINE APPENDIX TO MANUSCRIPT "MULTI-ASSET RETURN RISK MEASURES"

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Appendix A. Additions to the empirical study

Here, we provide additional information and extend the empirical study in the manuscript. In Section A.1, we explain why, for large volatilities, the optimal payoff does not permit a short position in the bank account. In Section A.2, we present another example, in which MARRM and MARM are significantly different. Finally, in Section A.3, we test MARRMs for financial market data, namely the time series of the DAX. We find that the MARRM suggests larger stock positions in times of crisis.

A.1. **VaR-case: objective of MARRM.** Here, we illustrate the objective function to calculate the MARRM for the VaR-case at level $\lambda = 0.95$ and the parameters used in the empirical study. Recall that the VaR-based MARRM acceptability criterion is given by

$$q^-_{\log\left(X/Z_T^{x_0,\xi}\right)}(\lambda) \leq 0.$$

The lognormal distribution of $\frac{X}{Z_T^{x_0,\xi}}$ (see Equation (3.2) in the manuscript) then gives us

$$\log(x_0) \geq \mu - rT - \xi^\intercal(b - r\mathbf{1})T + \frac{\|\xi^\intercal\Sigma\|^2}{2}T + \sqrt{\sigma^2 + \|\xi^\intercal\Sigma\|^2 T}\,\Phi^{-1}(\lambda) =: f(\xi), \tag{A.1}$$

where Φ^{-1} denotes the inverse of the cumulative distribution function of the standard normal distribution. Hence, the logarithmic MARRM value is obtained by minimizing function f. This function is plotted in Figure A.1.

Analyzing the function f in (A.1) leads to the following observations: Since all entries of $b - r\mathbf{1}$ are positive, the minimum is not attained if all values of ξ are negative. A mixture of positive and negative entries of ξ gives us the largest values in Figure A.1, due to the negative correlation in Σ. Hence, the smallest values of f are achieved by admitting only positive entries of ξ. Now, the large value of σ (recall $\sigma > 1$) means that the term $\sqrt{\sigma^2 + \|\xi^\intercal\Sigma\|^2 T}$ is dominated by σ^2 if $\|\xi^\intercal\Sigma\|$ is small. Note that $\sqrt{\sigma^2 + \|\xi^\intercal\Sigma\|^2 T}$ is minimized at zero and the slope of the function $x \mapsto \sqrt{\sigma^2 + x^2 T}$ is smaller for larger values of σ. Now, the remaining term $\mu - rT - \xi^\intercal(b - r\mathbf{1})T + \frac{\|\xi^\intercal\Sigma\|^2}{2}T$ is minimized for at least one strictly positive entry of

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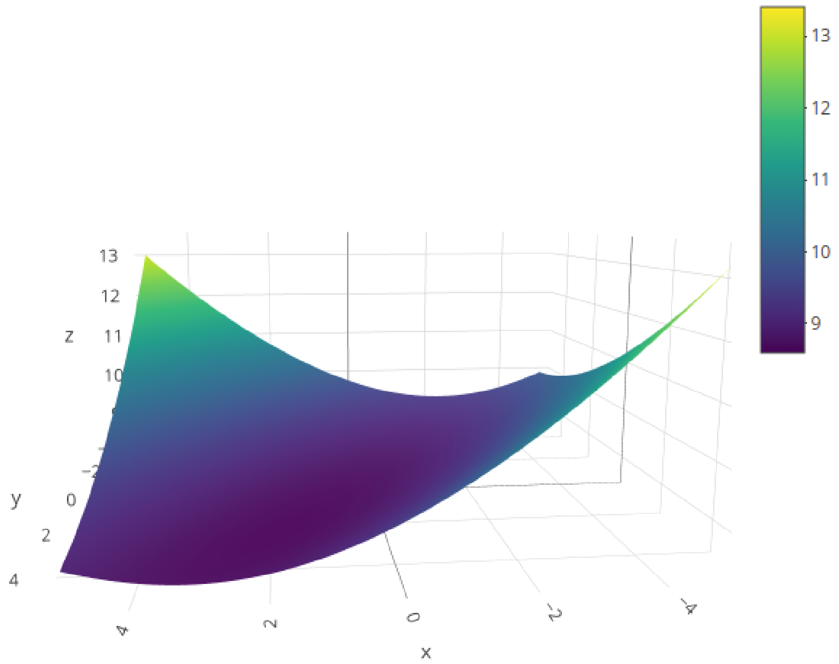


FIGURE A.1. Minimized value of this function gives us the logarithmic value of the MARRM with VaR acceptability criterion. The x-and y-axes are the possible values of ξ_1 and ξ_2.

ξ. So, the decrease of the term

$$\xi \mapsto \mu - rT - \xi^\intercal(b - r\mathbf{1})T + \frac{\|\xi^\intercal\Sigma\|^2}{2}T$$

is less compensated by an increase of

$$\xi \mapsto \sqrt{\sigma^2 + \|\xi^\intercal\Sigma\|^2 T}\Phi^{-1}(\lambda), \tag{A.2}$$

if σ is large. Hence, the minimum is achieved for large values of the portfolio process ξ.

In the situation of the empirical study, Figure A.2 shows that short positions in the bank account can be avoided, if the loss X admits a smaller value for σ than the one used in the manuscript.

Finally, note that the factor $\Phi^{-1}(\lambda)$ increases if λ does. Hence, the aforementioned influence of the term is amplified and the minimum of the function f is shifted closer to the origin.

A.2. **L^γ-norm and entropic acceptability criteria.** Here, we present another example, in which MARRM and MARM are significantly different. To do so, we use the classical Lebesgue norm to define an acceptability criterion for a MARRMs. For a strictly positive loss X and a security payoff Z modeled as a positive random variable itself, the relative loss X/Z is acceptable if and only if

$$\|\tfrac{X}{Z}\|_{L^\gamma} = \mathbb{E}\left[\left|\tfrac{X}{Z}\right|^\gamma\right]^{1/\gamma} \leq 1.$$

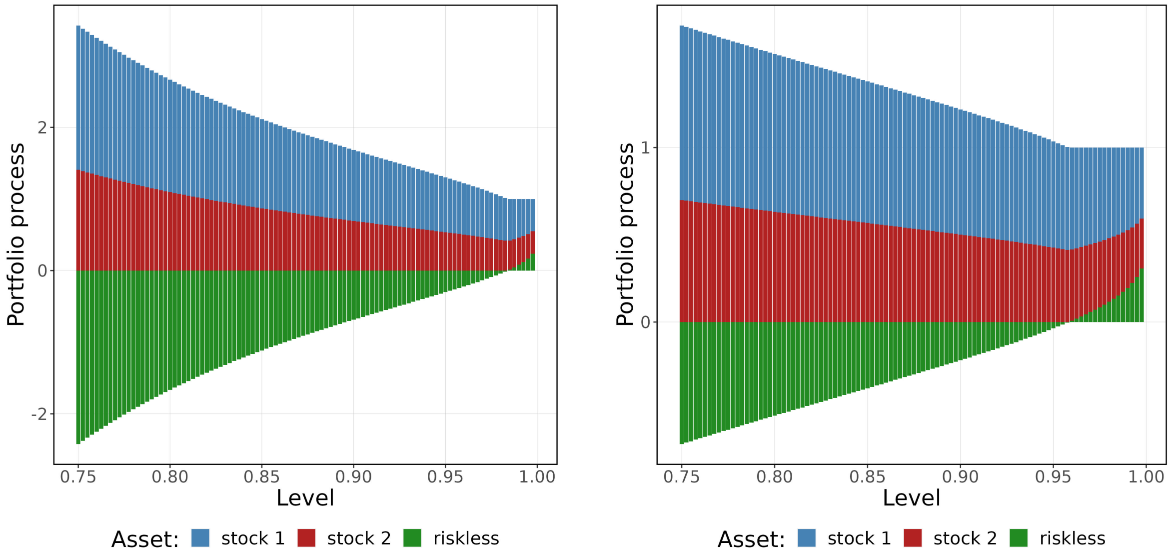


FIGURE A.2. Situation of $X \sim LN(\mu, \sigma^2)$ with $\mu = 1.5$ and $\sigma = 0.2$. *LHS:* Optimal portfolio process for MARRM with VaR acceptance set. *RHS:* Optimal portfolio process for MARRM with ARaR acceptance set.

Here, $\gamma > 0$ is a risk aversion parameter we shall vary. This is especially important, since values like $\gamma \in \{1, 2, 3, 4\}$ echo well-known statistics of the distribution of the relative loss random variable $\frac{X}{Z}$, namely mean, variance, skewness and kurtosis.

Each L^γ-norm is an RRM. As mentioned in Example 3 in [1], the corresponding monetary risk measure is the entropic risk measure. Hence, we now aim to compare the performance of the MARRM based on the L^γ-norm and the MARM based on the entropic risk measure. More precisely, the MARM deems a random variable Y acceptable if $\log(\mathbb{E}[\exp(\gamma Y)]) \le 0$. The parameter $\gamma > 0$ is the same as for the L^γ-norm. Speaking from the perspective of expected utility, where the entropic risk measure arises as certainty equivalent, $\gamma > 0$ is the parameter of absolute risk aversion.

To ensure a finite expectation in case of the entropic risk measure, we use a light-tailed distribution for X, namely an exponential distribution with mean $\frac{1}{r}$ and $r > 0$. The acceptability criterion in case of the MARM for initial capital x_0 and a portfolio process ξ is then given as $\log(\mathbb{E}[\exp(\gamma(X - Z_T^{x_0,\xi}))]) \le 0$, which is equivalent to $\log\left(\frac{r}{r-\gamma}\right) + \log(\mathbb{E}[\exp(-\gamma Z_T^{x_0,\xi})]) \le 0$. The summand $\log(\mathbb{E}[\exp(-\gamma Z_T^{x_0,\xi})])$ is obtained by Monte Carlo simulation. Further, we see that it is only meaningful for $r > \gamma$ and therefore, we set $r = 4$. This is a sufficiently large value to illustrate the effect of the approach for different values of the level γ. On the left-hand side in Figure A.3, we plot the MARRM and the MARM for different levels of γ. The MARM function for values of γ greater than 1.5 looks strictly convex. The MARRM function looks linear. As for the ARaR/ES-case, we see on the right-hand side in Figure A.3 that the difference between MARRM and MARM is significant. So, in this example it is important which risk measure an agent chooses, because the deviations for some levels are above 30%.

For completeness, in Figure A.4, we illustrate the underlying optimal portfolio processes. We see that the short position in the bank account seems to converge to zero if $\gamma \to r = 4$.

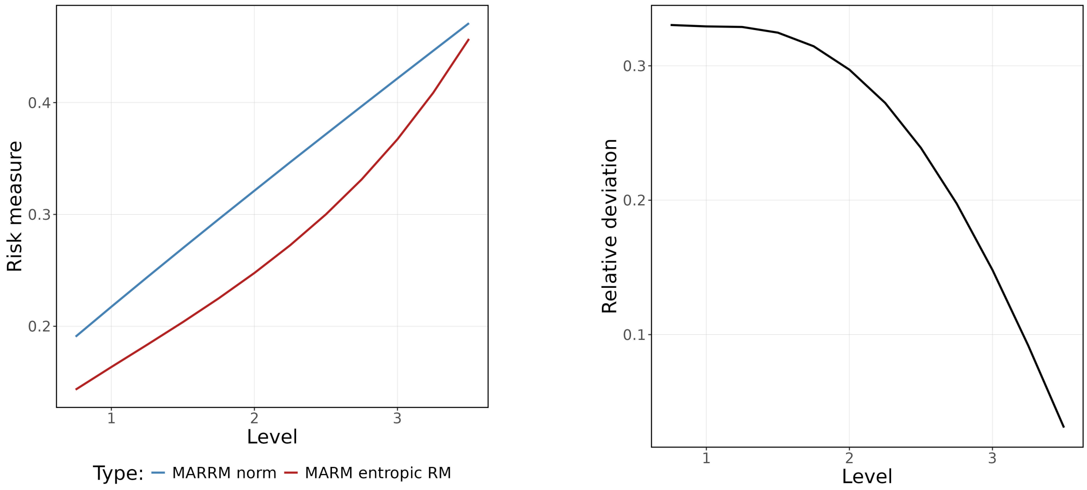
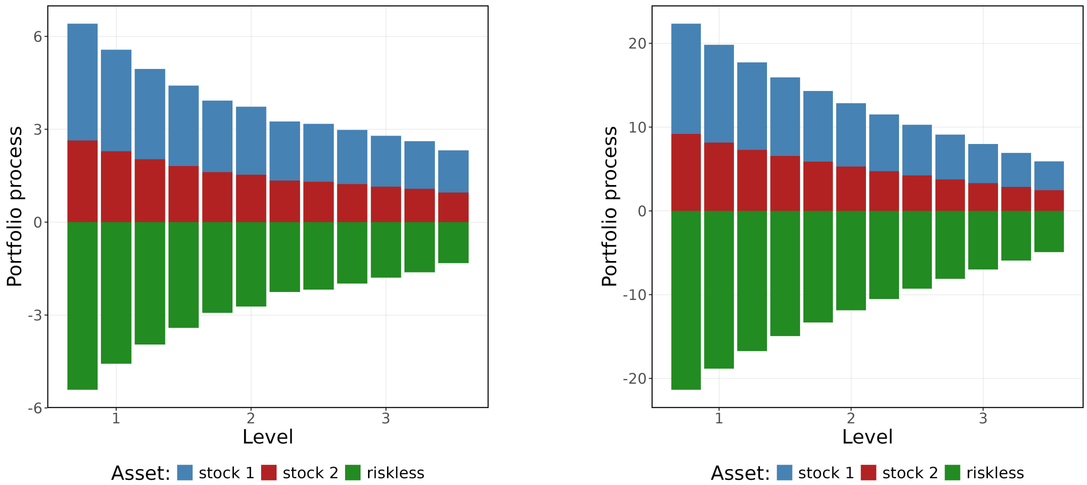


FIGURE A.3. *LHS:* MARRM for L^γ-acceptability and MARM for acceptability based on the entropic risk measure. *RHS:* Relative deviation between the two risk measures.



(A) MARRM for norm.

(B) MARM for entropic risk measure.

FIGURE A.4. Portfolio processes for different levels.

Concluding, we find a significant difference between MARRM and MARM in this example. Increasing the level shrinks the proportion invested in the bank account.

Comparing this with the empirical study in the manuscript, the VaR acceptability criteria suggest little difference between MARRM and MARM. However, the ARaR/ES and L^γ-norm/entropic criteria reveal that this conclusion is not generally valid, and the similarity between MARRM and MARM depends heavily on the chosen acceptability criteria.

A.3. **Financial market data.** In the empirical study, we used insurance losses which are part of the liabilities of the insurer in question. Now, we analyze the value of an MARRM for the assets of an insurer. Here, we focus on financial market models as it is done, for instance, in [2, 3]. While these two contributions describe assets by a Black-Scholes type model, we shall adopt the approach of [5] to be able to calculate the MARRM dynamically over time.

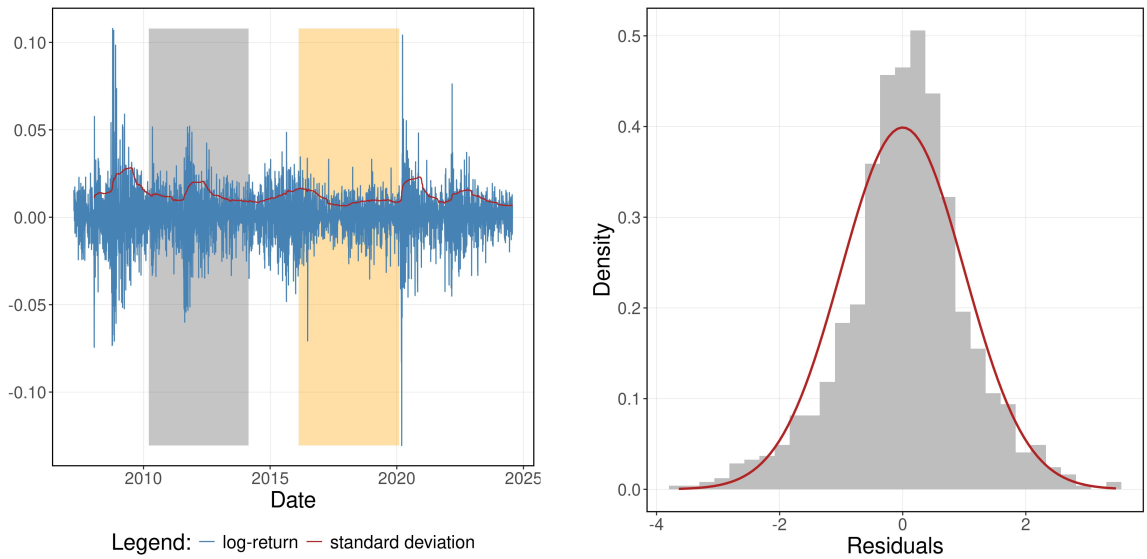
More precisely, we fit a classical time series model to real-world data (DAX) and calculate the MARRM for one-day-ahead market values.

The time series model that we are using is an AR(1)-GARCH(1,1) model for log-returns. The model description is as follows: We denote by Y_t the index value at time t. Then, for the log-returns it holds that

$$\log\left(\frac{Y_t}{Y_{t-1}}\right) = \phi \log\left(\frac{Y_{t-1}}{Y_{t-2}}\right) + \sigma_t z_t \quad \text{and} \quad \sigma_t^2 = \alpha_0 + \alpha_1(\sigma_{t-1} z_{t-1})^2 + \beta\sigma_{t-1}^2,$$

where $\phi, \alpha_0, \alpha_1, \beta \in \mathbb{R}$ are constant parameters and $\{z_t\}$ is a sequence of independent standard Gaussians. For further details we refer to [4]. The parameters are calibrated via a pseudo-maximum-likelihood approach with the function `garchFit` from the `R`-package `fGarch`. As data for the calibration, we use the DAX index.

We calibrate the AR(1)-GARCH(1,1) for two different time periods, illustrated by the gray and yellow boxes on the left-hand side in Figure A.5. The calculation of risk measures starts then always from the end of the corresponding time period used for calibration, see Figure A.6. Both time periods consist of 1 000 log-returns. The reason to choose two different time periods is the behavior of the log-returns after these intervals. In case of the first interval, the log-returns after this time period behave moderately. Instead, after the second interval, we have huge fluctuations of the log-returns due to the COVID crisis.



(A) DAX log-returns.

(B) Histogram of residuals.

FIGURE A.5. *LHS:* Log-returns for DAX. The red line is the estimated (rolling) standard deviation based on 200 log-returns. The colored boxes are the log-returns used to calibrate the AR(1)-GARCH(1,1) model. *RHS:* Histogram of residuals of the calibrated AR(1)-GARCH(1,1) with respect to log-returns from the gray box. The red line is the density function of a standard Gaussian.

The calibrated parameters are given in Table 1. The corresponding residuals for the first time interval are illustrated on the right-hand side in Figure A.5.

Time interval (YYYY/MM/DD)	$\hat{\phi}$	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}$
2010/03/29 - 2014/03/28	0.0340	$2.8718 \cdot 10^{-6}$	0.0746	0.9067
2016/03/31 - 2020/03/26	−0.0003	$6.4917 \cdot 10^{-6}$	0.1069	0.8228

TABLE 1. Calibrated model parameters of the log-returns of the DAX.

The calibrated model allows us to calculate an RRM for Y_t based on the information at time $t-1$. Note that in this situation, we have that

$$Y_t|_{Y_{t-1},Y_{t-2}} \sim \mathrm{LN}\left((1+\phi)\log(Y_{t-1}) - \phi\log(Y_{t-2}), \sigma_t^2\right).$$

Initial values for σ_t are obtained from the R-function `volatility` applied to the output of `fGarch`. For the gray box this gives 0.0088 and for the yellow box it is 0.0108.

Our aim is to compare the RRM with the MARRM based on the Black-Scholes model already used in the empirical study, using the same parameter values as in there. Motivated by the results of the empirical case study, we omit the VaR-case and focus solely on the case of the ARaR.

We plot the following relative differences between RRM, MARRM and DAX on the left-hand side in Figure A.6:

$$\frac{\text{RRM}-\text{index value}}{\text{index value}} \text{ (red)}, \quad \frac{\text{MARRM}-\text{index value}}{\text{index value}} \text{ (blue)}, \quad \frac{\text{RRM}-\text{MARRM}}{\text{MARRM}} \text{ (yellow)}.$$

We see that the relative differences between risk measures and index values (red and blue lines) are quite large at the beginning of the COVID crisis. In the aftermath, their behaviour is much more similar to the relative differences observed between April 2014 and April 2015.

At first glance, the relative difference between RRM and MARRM (yellow line) looks nearly constant. But, by a closer look, during the COVID crisis there is a peak of 1.64% at April 12, 2020. In comparison to 1.12% at February 21, 2020, this is 0.42% larger, which is a significant increase by noting that we only perform daily forecasts.

This is also visible in the underlying portfolio process, i.e., the percentages of the capital invested in specific assets, see the right-hand side in Figure A.6. During the COVID crisis, the MARRM reduces the investment in the riskless asset, which means that the hedging strategy is given by a more diversified portfolio between the riskless asset and the two risky stocks.

The main conclusion from Figure A.6 is that the relative difference between the RRM and MARRM is nearly constant in times without crisis. In times of crisis, the larger security space of the MARRM leads to a better diversified hedging portfolio, which in turn gives us a stronger relative reduction compared to the RRM than in times without crisis.

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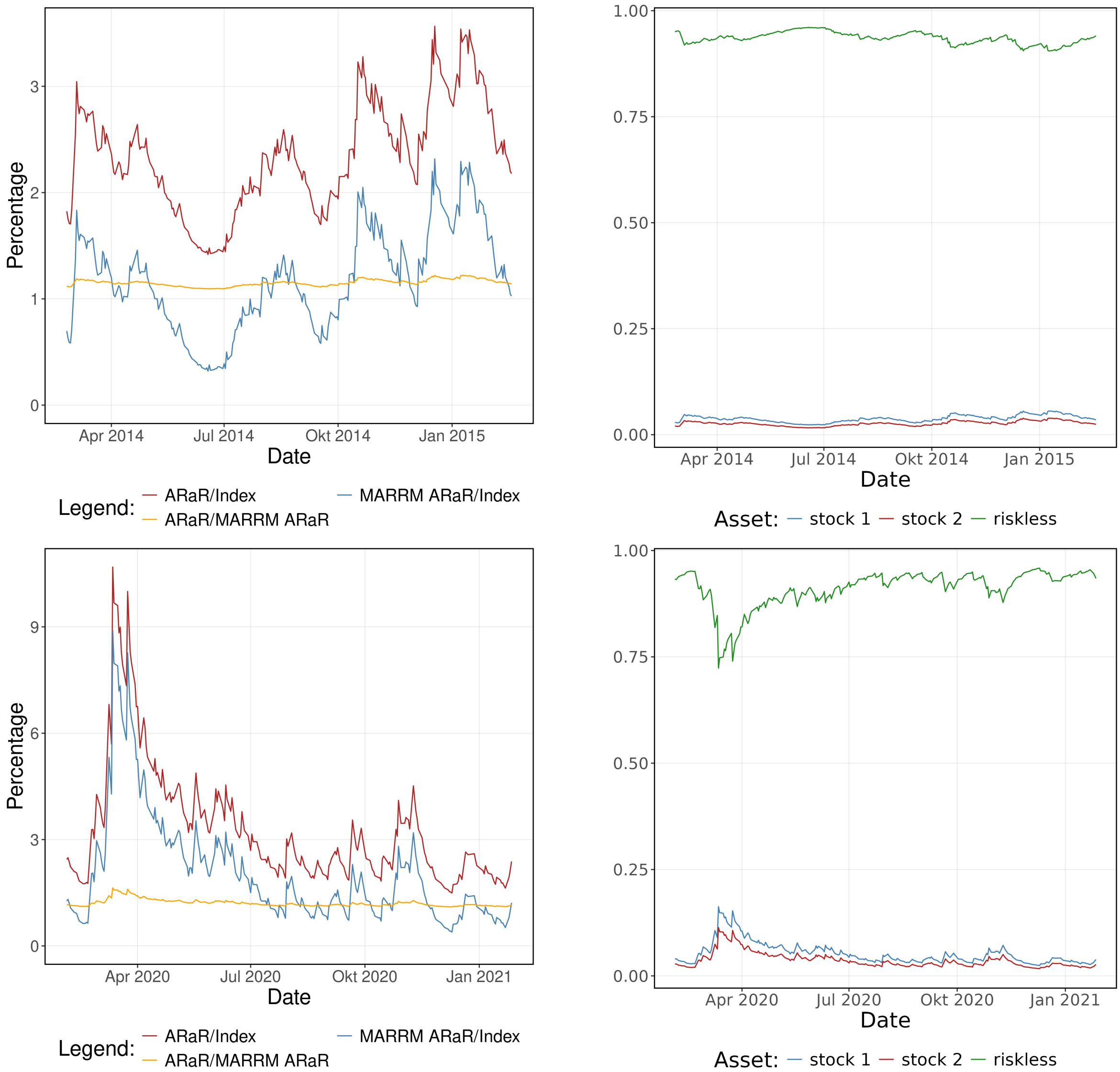

FIGURE A.6. *LHS:* Relative differences of risk measures and index, as well as relative differences between RRM and MARRM for DAX. *RHS:* Underlying portfolio processes.

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