Supplementary Material

Dynamic tonuity: Adapting retirement benefits to a changing environment

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Abstract

This is the Supplementary Material for the paper "Dynamic tonuity: Adapting retirement benefits to a changing environment". Section 1 gives a review of the regular tonuity. Section 2 details the stochastic mortality model as well as the revelant data used in the paper. And Section 3 provides an exemplified key information document.

1 A Review of the Regular Tonuity

The tonuity is first introduced by Chen et al. (2019). It functions as a tontine up to a specified switching time (τ) and transitions into an annuity thereafter. As shown by Chen et al. (2020) in an expected lifetime utility framework, the optimal way to invest in tontines and annuities is to put part of the wealth in tontines and annuities respectively. In particular, they show that the optimal tonuity performs rather well and delivers just a slightly lower utility level than the optimal portfolio of annuities and tontines. However, the fixed switching time τ may not be desirable when some unexpected mortality shock (e.g., the Covid-19 pandemic) takes place, resulting in significant deviations of survival probabilities from the assumptions used to determine τ at the beginning. Detailed discussions have been presented in the paper, and here we review the construction of a regular tonuity.

Mathematically, the payoff of the regular tonuity can be expressed as

$$b_{[\tau]}(t) = \mathbb{1}_{\{0 \le t < \min\{\tau, T^{(b)}\}\}} \frac{n}{N^{(b)}(t)} d_{[\tau]}(t) + \mathbb{1}_{\{\tau \le t < T^{(b)}\}} c_{[\tau]}(t), \tag{1.1}$$

where the tontine-specific payoff $d_{[\tau]}(t)$ and annuity-specific payoff $c_{[\tau]}(t)$ are determined at the beginning of the contract. Further, $\mathbb{1}_B$ denotes an indicator function which equals 1 if B occurs and zero otherwise, and $N^{(b)}(t)$ denotes the number of policyholders in the book population alive after t years, which is at least 1. Assuming the conditional independence of pool members, this random variable is distributed as $\left(N^{(b)}(t) \mid \{\mu^{(b)}_{x+s,h+s}\}_{s\in[0,t]}\right) \sim \text{Bin}(n, e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds})$. Then, the premium under $\mathbb{Q}$ can be written as

$$\begin{aligned} P_0^{(RT,\mathbb{Q})} &= \int_0^\tau e^{-r_f t} \mathbb{E}^{\mathbb{Q}} \left[1 - \left(1 - e^{-\int_0^t \mu^{(b,\mathbb{Q})}_{x+s,h+s} ds} \right)^n \right] d_{[\tau]}(t) \mathrm{d}t \\ &\quad + \int_\tau^\infty e^{-r_f t} \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^t \mu^{(b,\mathbb{Q})}_{x+s,h+s} ds} \right] c_{[\tau]}(t) \mathrm{d}t, \end{aligned} \tag{1.2}$$

where r_f is a risk-free interest rate.

The policyholder chooses the optimal tonuity by addressing the optimization problem below:

$$\begin{aligned} &\max_{d_{[\tau]}(t), c_{[\tau]}(t)} \mathbb{E}^{\mathbb{P}} \left[\int_0^\infty e^{-\eta t} \left(\mathbb{1}_{\{0 \le t < \min\{\tau, T^{(b)}\}\}} u\left(\frac{n d_{[\tau]}(t)}{N^{(b)}(t)} \right) + \mathbb{1}_{\{\tau \le t < T^{(b)}\}} u\left(c_{[\tau]}(t) \right) \right) \mathrm{d}t \right] \\ &\quad s.t. \quad P_0^{(RT,\mathbb{Q})} = v. \end{aligned} \tag{1.3}$$

Theorem 1.1 presents the solution to the optimization problem (1.3).

Theorem 1.1 *The solution to problem* (1.3) *is given by*

$$d^*_{[\tau]}(t) = \left(\lambda_{[\tau]} e^{(\eta - r_f)t} \frac{\mathbb{E}^{\mathbb{Q}}\left[1 - \left(1 - e^{-\int_0^t \mu^{(b,\mathbb{Q})}_{x+s,h+s} ds}\right)^n\right]}{\mathbb{E}^{\mathbb{P}}\left[\mathcal{K}_{n,\gamma}\left(e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds}\right)\right]}\right)^{-\frac{1}{\gamma}}, \quad t \in [0, \tau), \tag{1.4}$$

$$c^*_{[\tau]}(t) = \left(\lambda_{[\tau]} e^{(\eta - r_f)t} \frac{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t \mu^{(b,\mathbb{Q})}_{x+s,h+s} ds}\right]}{\mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds}\right]}\right)^{-\frac{1}{\gamma}}, \quad t \in [\tau, \infty). \tag{1.5}$$

The optimal Lagrangian multiplier is given by

$$\lambda_{[\tau]} = \left(\frac{1}{v}\int_0^\tau e^{(\frac{r_f-\eta}{\gamma} - r_f)t} \frac{\mathbb{E}^{\mathbb{P}}\left[\mathcal{K}_{n,\gamma}\left(e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds}\right)\right]^{\frac{1}{\gamma}}}{\mathbb{E}^{\mathbb{Q}}\left[1 - \left(1 - e^{-\int_0^t \mu^{(b,\mathbb{Q})}_{x+s,h+s} ds}\right)^n\right]^{\frac{1}{\gamma}-1}} \mathrm{d}t \right.$$
$$\left. + \frac{1}{v}\int_\tau^\infty e^{(\frac{r_f-\eta}{\gamma} - r_f)t} \frac{\mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds}\right]^{\frac{1}{\gamma}}}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t \mu^{(b,\mathbb{Q})}_{x+s,h+s} ds}\right]^{\frac{1}{\gamma}-1}} \mathrm{d}t\right)^\gamma,$$

where

$$\mathcal{K}_{n,\gamma}\left(e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds}\right) := \sum_{k=1}^n \binom{n}{k}\left(\frac{k}{n}\right)^\gamma \left(e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds}\right)^k \left(1 - e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds}\right)^{n-k}.$$

Then the policyholder's optimal expected utility is

$$EU_{[\tau]} = \frac{\lambda_{[\tau]}}{1-\gamma} \cdot v.$$

Proof: In order to obtain the optimal payoff stream functions to the regular tonuity, we need to maximize the policyholder's expected utility subject to the condition $P_0^{\mathrm{RT}}(\mathbb{Q}) = v$. Using $\left(N^{(b)}(t) - 1 \mid T^{(b)} > t, \{\mu^{(b)}_{x+s,h+s}\}_{s\in[0,t]}\right) \sim \mathrm{Bin}(n-1, e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds})$, we can write down the Lagrangian function as

$$L = \mathbb{E}^{\mathbb{P}}\left[\int_0^\infty e^{-\eta t}\left(\mathbb{1}_{\{0\le t<\min\{\tau, T^{(b)}\}\}} u\left(\frac{n d_{[\tau]}(t)}{N^{(b)}(t)}\right) + \mathbb{1}_{\{\tau \le t < T^{(b)}\}} u\left(c_{[\tau]}(t)\right)\right)\mathrm{d}t\right]$$
$$+ \lambda_{[\tau]}\left(v - \int_0^\tau e^{-r_f t}\mathbb{E}^{\mathbb{Q}}\left[1 - \left(1 - e^{-\int_0^t \mu^{(b,\mathbb{Q})}_{x+s,h+s} ds}\right)^n\right] d_{[\tau]}(t)\mathrm{d}t\right.$$
$$\left. - \int_\tau^\infty e^{-r_f t}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t \mu^{(b,\mathbb{Q})}_{x+s,h+s} ds}\right] c_{[\tau]}(t)\mathrm{d}t\right)$$

$$
\begin{aligned}
&= \int_0^\tau e^{-\eta t} u\left(d_{[\tau]}(t)\right) \\
&\cdot \mathbb{E}^{\mathbb{P}}\left[\underbrace{e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds} \sum_{k=0}^{n-1} \left(\frac{n}{k+1}\right)^{1-\gamma} \binom{n-1}{k} \left(e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds}\right)^k \left(1 - e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds}\right)^{n-1-k}}_{\mathcal{G}}\right] \mathrm{d}t \\
&+ \int_\tau^\infty e^{-\eta t} u\left(c_{[\tau]}(t)\right) \mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds}\right] \mathrm{d}t \\
&+ \lambda_{[\tau]}\Bigg(v - \int_0^\tau e^{-r_f t} \mathbb{E}^{\mathbb{Q}}\left[1 - \left(1 - e^{-\int_0^t \mu^{(b,\mathbb{Q})}_{x+s,h+s} ds}\right)^n\right] d_{[\tau]}(t) \mathrm{d}t \\
&- \int_\tau^\infty e^{-r_f t} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t \mu^{(b,\mathbb{Q})}_{x+s,h+s} ds}\right] c_{[\tau]}(t) \mathrm{d}t\Bigg),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{G} &= \sum_{k=0}^{n-1} \left(\frac{k+1}{n}\right)^\gamma \underbrace{\frac{n}{k+1}\binom{n-1}{k}}_{\binom{n}{k+1}} \left(e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds}\right)^{k+1} \left(1 - e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds}\right)^{n-(1+k)} \\
&= \sum_{k=1}^{n} \left(\frac{k}{n}\right)^\gamma \binom{n}{k} \left(e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds}\right)^k \left(1 - e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds}\right)^{n-k} \\
&:= \mathcal{K}_{n,\gamma}\left(e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds}\right).
\end{aligned}
$$

In order to obtain the maximum of Lagrangian function, we take derivatives with respect to $d_{[\tau]}(t)$ and $c_{[\tau]}(t)$:

$$
\begin{aligned}
\frac{\partial L}{\partial d_{[\tau]}(t)} &= e^{-\eta t} u'\left(d_{[\tau]}(t)\right) \mathbb{E}^{\mathbb{P}}\left[\mathcal{K}_{n,\gamma}\left(e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds}\right)\right] \\
&\quad - \lambda_{[\tau]} e^{-r_f t} \mathbb{E}^{\mathbb{Q}}\left[1 - \left(1 - e^{-\int_0^t \mu^{(b,\mathbb{Q})}_{x+s,h+s} ds}\right)^n\right] \stackrel{!}{=} 0. \\
\frac{\partial L}{\partial c_{[\tau]}(t)} &= e^{-\eta t} u'\left(c_{[\tau]}(t)\right) \mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds}\right] \\
&\quad - \lambda_{[\tau]} e^{-r_f t} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t \mu^{(b,\mathbb{Q})}_{x+s,h+s} ds}\right] \stackrel{!}{=} 0.
\end{aligned}
$$

Therefore, we have

$$
d^*_{[\tau]}(t) = \left(\lambda_{[\tau]} e^{(\eta - r_f)t} \frac{\mathbb{E}^{\mathbb{Q}}\left[1 - \left(1 - e^{-\int_0^t \mu^{(b,\mathbb{Q})}_{x+s,h+s} ds}\right)^n\right]}{\mathbb{E}^{\mathbb{P}}\left[\mathcal{K}_{n,\gamma}\left(e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds}\right)\right]}\right)^{-\frac{1}{\gamma}}, \forall t \in [0, \tau),
$$

$$c^*_{[\tau]}(t) = \left(\lambda_{[\tau]} e^{(\eta - r_f)t} \frac{\mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^t \mu^{(b,\mathbb{Q})}_{x+s,h+s} ds} \right]}{\mathbb{E}^{\mathbb{P}} \left[e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds} \right]} \right)^{-\frac{1}{\gamma}}, \forall t \in [\tau, \infty).$$

The $\lambda_{[\tau]} > 0$ is chosen to meet the budget constraint, such that

$$\lambda_{[\tau]} = \left(\frac{1}{v} \int_0^\tau e^{(\frac{r_f - \eta}{\gamma} - r_f)t} \frac{\mathbb{E}^{\mathbb{P}} \left[\mathcal{K}_{n,\gamma} \left(e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds} \right) \right]^{\frac{1}{\gamma}}}{\mathbb{E}^{\mathbb{Q}} \left[1 - \left(1 - e^{-\int_0^t \mu^{(b,\mathbb{Q})}_{x+s,h+s} ds} \right)^n \right]^{\frac{1}{\gamma} - 1}} \mathrm{d}t \right.$$
$$\left. + \frac{1}{v} \int_\tau^\infty e^{(\frac{r_f - \eta}{\gamma} - r_f)t} \frac{\mathbb{E}^{\mathbb{P}} \left[e^{-\int_0^t \mu^{(b)}_{x+s,h+s} ds} \right]^{\frac{1}{\gamma}}}{\mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^t \mu^{(b,\mathbb{Q})}_{x+s,h+s} ds} \right]^{\frac{1}{\gamma} - 1}} \mathrm{d}t \right)^{\gamma}.$$

□

2 Stochastic Mortality Model

In practice, the expected remaining lifetime and mortality rates can be estimated using the data of different populations, e.g., the general population (or the reference population), and the annuitant pool (or the book population). The book population is usually a proportion of the reference population, and survival probabilities among the book population are usually greater than those in the other one due to the adverse selection. To distinguish the estimates from different populations, we use $\mu_x^{(r)}$ and $\mu_x^{(b)}$ to represent the stochastic forces of mortality for the reference population and book population respectively. The mortality rates of these two populations are correlated. To obtain the estimates, we need to first introduce our stochastic mortality models. Here, we add an additional calendar year factor h to them. To be more concrete, we apply $\mu^{(i)}_{\{x,h\}}$, $i = r, b$ for population i, where the central death rate is marked as $m^{(i)}_{\{x,h\}}$.

2.1 Two-population Mortality Model

Due to its tractable form and inclusion of a cohort effect, the Age-Period-Cohort (APC) model is widely used to model the stochastic mortality rate (Osmond, 1985; Jacobsen et al., 2002; Renshaw and Haberman, 2006). We follow Dowd et al. (2011) and Chen et al. (2022) to establish the two-population stochastic mortality model. The key idea is to estimate the parameters for mortality models of the reference and book populations respectively, using the historical data; and based on the fitted models, when carrying out the mortality projections, we then connect the mortality rates of these two populations by adding specific components to the dynamics of the period effect and cohort effect of the book population. Further details are stated in the next Subsection 2.2. Here we start by introducing the mortality models of the two populations to fit the historical data.

According to Currie (2006), we set up our stochastic mortality model for the reference population by:

$$\log m^{(r)}_{\{x,h\}} = \beta^{(r)}_x + n_a^{-1}\kappa^{(r)}_h + n_a^{-1}\omega^{(r)}_{h-x}, \tag{2.1}$$

where r denotes the reference population, x represents the age, h means the calendar year of observation and $h-x$ is thus the cohort year of birth. $\beta^{(r)}_x$, $\kappa^{(r)}_h$ and $\omega^{(r)}_{h-x}$ are described as the age-specific, period-related and cohort-related stochastic factors for the reference population respectively. Correspondingly, n_a represents the number of ages in the sample under investigation.

Similarly, the mortality model for the book population is also assumed to be given by an APC model as well, i.e.

$$\log m^{(b)}_{\{x,h\}} = \beta^{(b)}_x + n_a^{-1}\kappa^{(b)}_h + n_a^{-1}\omega^{(b)}_{h-x}, \tag{2.2}$$

where b marks the book population, and $\beta^{(b)}_x$, $\kappa^{(b)}_h$ and $\omega^{(b)}_{h-x}$ are the age, period and cohort stochastic factors for the book population respectively.

Furthermore, by taking the assumptions that the forces of mortality ($\mu^{(i)}_{\{x,h\}}$, $i = r, b$) remain constant over each year of integer age and over each calendar year, and the size of these two populations at all ages remains constant over time (Cairns et al., 2009), we get

$$m^{(i)}_{\{x,h\}} = \mu^{(i)}_{\{x,h\}}, \tag{2.3}$$

$$q^{(i)}_{\{x,h\}} = 1 - \exp\left(-\mu^{(i)}_{\{x,h\}}\right) = 1 - \exp\left(-m^{(i)}_{\{x,h\}}\right), \quad i = r, b. \tag{2.4}$$

And then the survival curve can be written as:

$$p^{(i)}_{\{x,h\}} = \exp\left(-m^{(i)}_{\{x,h\}}\right), \tag{2.5}$$

$${}_tp^{(i)}_{\{x,h\}} = \exp\left(-\sum_{s=0}^{t}\mu^{(i)}_{\{x,h+s\}}\right), \quad i = r, b. \tag{2.6}$$

2.2 Mortality Projection under the Real-world Measure $\mathbb{P}$

In order to make projections to the mortality rates, for the reference population, we assume the dynamics of the period effect ($\kappa^{(r)}_h$) can be described by a random walk with drift $\alpha^{(\kappa,r)}$, and the cohort effect ($\omega^{(r)}_{h-x}$) follows $ARIMA(1,1,0)$ with drift $\alpha^{(\omega,r)}$

$$\kappa^{(r)}_h - \kappa^{(r)}_{h-1} = \alpha^{(\kappa,r)} + \Theta^{(\kappa,r)} Z^{(\kappa,r)}_h, \tag{2.7}$$

$$\omega^{(r)}_{h-x} - \omega^{(r)}_{h-x-1} = \phi^{(\omega,r)}(\omega^{(r)}_{h-x-1} - \omega^{(r)}_{h-x-2}) + (1-\phi^{(\omega,r)})\alpha^{(\omega,r)} + \Theta^{(\omega,r)} Z^{(\omega,r)}_{h-x}. \tag{2.8}$$

Here, $Z_h^{(\kappa,r)}$ and $Z_{h-x}^{(\omega,r)}$ are standard normal random variables under the real-world measure $\mathbb{P}$.

It is reasonable to assume that the book population (e.g., the insurer's annuity portfolio) has lower mortality rates than the reference population (e.g., the national population) (Cairns et al., 2011b). Furthermore, the mortality rates of the reference population and book population are correlated. We incorporate these features via additional specific spread patterns in the predictive distributions of the period factor ($\kappa_h^{(b)}$) and cohort factor ($\omega_{h-x}^{(b)}$) for the book population's mortality model. With reference to Dowd et al. (2011), the dynamics of the period-specific effects can be written as:

$$\kappa_h^{(b)} - \kappa_{h-1}^{(b)} = \psi^{(\kappa)}(\kappa_{h-1}^{(r)} - \kappa_{h-1}^{(b)}) + \alpha^{(\kappa,b)} + \Theta^{(\kappa,b1)} Z_h^{(\kappa,r)} + \Theta^{(\kappa,b2)} Z_h^{(\kappa,b)}, \tag{2.9}$$

$$\begin{aligned}\omega_{h-x}^{(b)} - \omega_{h-x-1}^{(b)} = {}& \phi^{(\omega,b)}(\omega_{h-x-1}^{(b)} - \omega_{h-x-2}^{(b)}) + \psi^{(\omega)}(\omega_{h-x-1}^{(r)} - \omega_{h-x-1}^{(b)}) \\ & + (1-\phi^{(\omega,b)})\alpha^{(\omega,b)} + \Theta^{(\omega,b1)} Z_{h-x}^{(\omega,r)} + \Theta^{(\omega,b2)} Z_{h-x}^{(\omega,b)}. \end{aligned} \tag{2.10}$$

Similarly, $Z_h^{(\kappa,b)}$ and $Z_{h-x}^{(\omega,b)}$ are standard normal random variables under $\mathbb{P}$ in terms of the book population. $(Z_h^{(\kappa,b)}, Z_h^{(\kappa,r)})$ are assumed to be independent and identically distributed, and so are the pair of $(Z_{h-x}^{(\omega,b)}, Z_{h-x}^{(\omega,r)})$. The spread terms are specified as $(\psi^{(\kappa)}(\kappa_{h-1}^{(r)} - \kappa_{h-1}^{(b)}))$ for the period effect and $(\psi^{(\omega)}(\omega_{h-x-1}^{(r)} - \omega_{h-x-1}^{(b)}))$ for the cohort effect. Positive values of $\psi^{(\kappa)}$ and $\psi^{(\omega)}$ will pull the spread towards 0. More detailed explanations can be found in Dowd et al. (2011).

2.3 Mortality Projection under the Risk Neutral Measure $\mathbb{Q}$

With reference to the existing literature (e.g., Cairns et al., 2006; Chen et al., 2022), we specify the risk-neutral measure by incorporating the longevity risk premiums to the mortality trends. More concretely, we add some constants to drift terms of period effects for both populations, i.e.

$$\widetilde{Z}_h^{(\kappa,r)} = g^{(r)} + Z_h^{(\kappa,r)}, \quad \widetilde{Z}_h^{(\kappa,b)} = g^{(b)} + Z_h^{(\kappa,b)}.$$

It is assumed that $(\widetilde{Z}_h^{(\kappa,r)}, \widetilde{Z}_h^{(\kappa,b)})$ are two dimensional i.i.d. normal random variables under the risk neutral measure $\mathbb{Q}$. $g^{(r)}$ and $g^{(b)}$ represent the market prices of the longevity risk related to $Z_h^{(\kappa,r)}$ and $Z_h^{(\kappa,b)}$ respectively. Correspondingly, the longevity risk premium for the reference population is given by $g^{(r)}$, and that for the book population is composed by $g^{(r)}$ and $g^{(b)}$. Then, we further write down the dynamics of the period effects of both populations under $\mathbb{Q}$:

$$\kappa_h^{(r)} - \kappa_{h-1}^{(r)} = \underbrace{\alpha^{(\kappa,r)} + \Theta^{(\kappa,r)} g^{(r)}}_{\widetilde{\alpha}^{(\kappa,r)}} + \Theta^{(\kappa,r)} Z_h^{(\kappa,r)}, \tag{2.11}$$

$$\kappa_h^{(b)} - \kappa_{h-1}^{(b)} = \psi^{(\kappa)}(\kappa_{h-1}^{(r)} - \kappa_{h-1}^{(b)}) + \underbrace{\alpha^{(\kappa,b)} + \Theta^{(\kappa,b1)}g^{(r)} + \Theta^{(\kappa,b2)}g^{(b)}}_{\widetilde{\alpha}^{(\kappa,b)}} + \Theta^{(\kappa,b1)}Z_h^{(\kappa,r)} + \Theta^{(\kappa,b2)}Z_h^{(\kappa,b)}. \tag{2.12}$$

2.4 Data Introduction

In this paper, we acquire the historical data of two populations from 1961 to 2021 for ages 56 to 98. England and Wales (EW) male experience is used as the reference population data, which is generated by the general population of England and Wales, and can be obtained from the Human Mortality Database (HMD). As for the book population, we use the datasets of male pensioners from UK Continuous Mortality Investigation (CMI). The pensioner datasets are composed of the ones that are able to participate in the pension scheme. In this sense, it is reasonable to infer that compared to males from HMD, males from CMI are possibly to live longer because of adverse selection. Moreover, the CMI population is (mostly) a sub-population of roughly 10% in size of the EW population.

2.5 Mortality Rates for the Base and Stress Case under the Real World Measure $\mathbb{P}$

To estimate the parameters of the mortality models, we apply a two-stage method, which is widely used by researchers (e.g., Lee and Carter, 1992; Cairns et al., 2009, 2011a). More concretely, at stage 1, age, period, and cohort effects regardless of their underlying dynamics are estimated; at stage 2, the mortality rates can be forecast by fitting appropriate stochastic processes to the period and cohort effects. Due to the fact that our obtained historical data is only for ages till 98, we extrapolate the age effects $\beta_x^{(r)}$ and $\beta_x^{(b)}$ to 120, and simulate the future mortality rates by extrapolating the period effect and cohort effect applying their corresponding time-series processes.

In the base case (i), where there is no unforeseen shock, we apply the data from 1961 to 2018 to estimate and forecast the mortality rates. In the stress case with an unexpected mortality shock (ii-M), we apply the entire data set from 1961 to 2021 for the mortality estimation and projection. And in the stress case with an unexpected longevity shock (ii-L), we add a safety buffer to mortality trends in the mortality model estimated under the base-case assumptions. To elaborate further, in the context of a longevity shock, we assume a 10% increase in the period parameter ($\kappa_h^{(r)}$) for the base-case reference mortality model. Simultaneously, the period parameter ($\kappa_h^{(b)}$) for the base-case book mortality model experiences larger inflation, specifically, a 25% increase. This adjustment accounts for the notion that the book population is generally wealthier than the reference population. Furthermore, it considers the possibility that a medical treatment breakthrough may initially benefit wealthier individuals more, given their ability to afford associated expenses.

In the following contents, we take the cohort 1949 as an example. To obtain a relatively stable prediction of the mortality rates, we have simulated 10000 mortality paths for both populations.

Obtained mortality data for the base case:

1. Reference population ($DATA_{\{refbase\}}$): 10000 simulated paths starting from 2014 for cohort-1949 males.
2. Book population ($DATA_{\{bookbase\}}$): 10000 simulated paths starting from 2014 for cohort-1949 males.

In the stress case (ii-M), where an unforeseen mortality shock strikes 5 years after the contract is made, the prediction of mortality starts in 2022. Here, we construct the data as: mortality rates for ages 70, 71, and 72 of cohort-1949 (in years 2019, 2020, and 2021) are calculated based on real data; and mortality rates over 73 (included) of the cohort-1949 (in 2022 and afterwards) are predicted.

Obtained mortality data for the stress case (ii-M):

1. Reference population ($DATA_{\{refstr(M)\}}$): Real mortality rates for years 2019-2021, and 10000 simulated paths starting from 2022 for cohort-1949 males. In other words, for 10000 paths of the mortality rates in the stress case, they have the same mortality rates for the years 2019-2021.
2. Book population ($DATA_{\{bookstr(M)\}}$): Real mortality rates for years 2019-2021, and 10000 simulated paths starting from 2022 for cohort-1949 males.

In the stress case (ii-L), shortly after the contract begins, an unforeseen longevity shock takes place. As elaborated above, we adjust the base-case mortality model to get the mortality rates under the longevity shock. The mortality projection starts from 2019.

Obtained mortality data for the stress case (ii-L):

1. Reference population ($DATA_{\{refstr(L)\}}$): 10000 (adjusted) simulated paths starting from 2019 for cohort-1949 males.
2. Book population ($DATA_{\{bookstr(L)\}}$): 10000 (adjusted) simulated paths starting from 2019 for cohort-1949 males.

2.6 Parameter Calibration for the Risk Neutral Measure $\mathbb{Q}$

To calibrate the market prices of the longevity risk, we refer to the approach described in Chen et al. (2022), which relies on the Sharpe ratio approach (Milevsky et al., 2006). In this paper, we obtain the values for $g^{(r)}$ and $g^{(b)}$ by setting the Sharpe ratio of the insurer's annuity portfolio as 8% (Bauer et al., 2010)), where the derivations of the Sharpe ratio for the book and reference population are directly from Chen et al. (2022).

3 Key Information Document

Key Information Document
Purpose This document provides you with key information about the retirement product. It is not marketing material. The information is required by law to help you understand the nature, risks, costs, potential gains, and losses of this product and to help you compare it with other products.
Product Dynamic tonuity, Insurance Company ABC
What is this product? **Type**: Insurance-based investment product. **Objectives**: to provide incomes for retirement life. **Intended retail investor**: retirees. **Insurance benefits**: there are two types of insurance benefits: tontine-like payoffs and annuity-like payoffs. With tontine-like payoffs, your payouts increase as the number of members in the pool decreases, typically due to some members passing away. Annuity-like payoffs, on the other hand, provide consistent payments similar to purchasing an annuity. Initially, you will receive tontine-like payoffs, but these may transition to annuity-like payoffs at some point. This switch happens automatically when the one-year survival probability in the national population exceeds the threshold specified in your contract, ensuring that your benefits adjust favorably based on changing mortality rates. **Insurance costs**: the product cost involves a gross premium and other charges (e.g., administrative fees), which are paid in a lump sum at the start.
What are the risks and what could I get in return? **Risk Indicator**: Market risk value; Credit risk value; Mortality risk value. The market risk value is calculated by measuring the distance between the 2.5% value-at-risk (VaR) and the mean of the probability density function of the policyholder's return. The credit risk value is assessed based on credit ratings provided by certified rating agencies. The mortality risk value, which accounts for both idiosyncratic and systematic mortality risks, is determined and disclosed by Insurance Company ABC.

Table 3.1: Key Information Document

Key Information Document (continued)
What are the risks and what could I get in return? **Performance Scenarios**: Assume a policyholder's characteristics can be described using given parameters (see Table 2.1), you can check your potential payoffs under the following four scenarios by Figure 5.1: (a) Favourable scenario; (b) Moderate scenario; (c) Unfavourable scenario; (d) Stress scenario.
What happens if Insurance Company ABC is unable to pay out? **Protection by the Compensation Scheme:** Your policy may be covered by a national compensation scheme, which could provide compensation for some or all of your policy benefits if Insurance company ABC is unable to fulfill its commitments. The level and extent of compensation will depend on the scheme's regulations and the type of insurance policy you hold. **Continuation of Coverage:** Depending on the situation, there may be provisions for transferring your policy to another insurer, allowing your coverage to continue uninterrupted. This process would be managed by regulatory authorities to protect your interests. **Potential Reduction in Benefits:** If a transfer or compensation is not possible, there may be a reduction in the benefits or payouts from your policy. The extent of this reduction would depend on the assets remaining with the insurer and the terms of the insolvency proceedings. **Regulatory oversight:** Insurance Company ABC is subject to stringent regulatory oversight designed to minimize the risk of insolvency. This includes maintaining sufficient financial reserves and adhering to risk management practices to ensure the company can meet its obligations to policyholders.
What are the costs? **Costs over Time:** A single up-front premium and an up-front administration fee. **Composition of Costs:** The up-front gross premium includes the net premium and risk loading. And the administration fee is paid up-front.
How long should I hold it and can I take money out early? You will continue to receive payments for as long as you are alive, so it is advantageous to keep the policy active for as long as possible. You also have the option to surrender the policy at any time, in which case the present value will be calculated in accordance with the terms outlined in your contract.
How can I complain? If you have any concerns or complaints, you can reach out to our customer service team directly to file a complaint. The phone number is XXXXXXX, and the email address is XXXXXXX.
Other relevant information No other information.

Table 3.2: Key Information Document (continued)

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