Supplement to “Optimal reinsurance design under distortion risk measures and reinsurer’s default risk with partial recovery”

Yaodi Yong[a], Ka Chun Cheung[b], Yiying Zhang[a,*]

Department of Mathematics, Southern University of Science and Technology, Shenzhen 518055, Guangdong Province, P.R. China.
Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam Road, Hong Kong, P.R. China.

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Abstract

In this supplementary material designed for “*Optimal reinsurance design under distortion risk measures and reinsurer’s default risk with partial recovery*”, we furnish proofs regarding the main results and illustrative examples with shifted Exponential distribution. We refer to the main paper for context, notation, and definitions.

*Corresponding author. E-mail: zhangyy3@sustech.edu.cn.

1 Proofs of main results

1.1 Proof of Theorem 3.3

Proof. By the assumption that Y is independent of X, we notice that

$$\pi_r(YI(X)) = (1+\theta)[p+(1-p)\gamma]\mathbb{E}[I(X)] = \pi_r(YI_d(X)).$$

Using Lemma 3.2, regarding $Y=1$ and $Y=\gamma$, we have

$$X - I(X) \geq_{\text{cx}} X - (X-d)_+, \quad X - \gamma I(X) \geq_{\text{cx}} X - \gamma(X-d)_+.$$

Therefore, for any convex function $u(\cdot)$, one has

$$\begin{aligned}\mathbb{E}[u(X - YI(X))] &= p\mathbb{E}[u(X - I(X))] + (1-p)\mathbb{E}[u(X-\gamma I(X))] \\ &\geq p\mathbb{E}[u(X-(X-d)_+)] + (1-p)\mathbb{E}[u(X-\gamma(X-d)_+)] \\ &= \mathbb{E}[u(X - Y(X-d)_+)],\end{aligned}$$

which immediately yields

$$X - YI(X) \geq_{\text{cx}} X - Y(X-d)_+.$$

It then follows from Theorem 5.2.1 of Dhaene et al. (2006) that for all (increasing) concave distortion functions $g_i \in \mathcal{G}_{cv}$,

$$\rho_{g_i}(X - YI(X)) \geq \rho_{g_i}(X - YI_d(X)).$$

As ρ_{g_i} is translative invariant, we arrive at

$$\rho_{g_i}(X - YI(X) + \pi_r(YI(X))) \geq \rho_{g_i}(X - YI_d(X) + \pi_r(YI_d(X))),$$

which completes the proof. □

1.2 Proof of Proposition 3.5

Proof. We note that $R_d(X,Y)$ has the following expression:

$$R_d(X,Y) = X - YI_d(X) = \begin{cases} \min\{X,d\}, & Y=1; \\ X - \gamma(X-d)_+, & Y=\gamma. \end{cases}$$

For any $x \in \mathbb{R}_+$, the distribution function of $R_d(X,Y)$ is given by

$$F_{R_d(X,Y)}(x) = p\mathbb{P}(\min\{X,d\} \leq x) + (1-p)\mathbb{P}(X - \gamma(X-d)_+ \leq x).$$

Note that

$$\mathbb{P}(\min\{X,d\} \leq x) = \begin{cases} \mathbb{P}(X \leq d, X \leq x) = F_X(x), & x < d; \\ \mathbb{P}(X \leq d, X \leq x) + \mathbb{P}(X > d, d \leq x) = 1, & x \geq d. \end{cases}$$

Besides, since

$$\begin{aligned}\mathbb{P}(X - \gamma(X-d)_+ \leq x) &= \mathbb{P}(X \leq d, X \leq x) + \mathbb{P}\left(X > d, (1-\gamma)X + \gamma d \leq x\right) \\ &= \mathbb{P}(X \leq d, X \leq x) + \mathbb{P}\left(X > d, X \leq \frac{x - \gamma d}{1-\gamma}\right),\end{aligned}$$

we have

$$
\begin{aligned}
&\mathbb{P}(X-\gamma(X-d)_+\le x)\\
&=\begin{cases}\mathbb{P}(X\le x)=F_X(x), & x<d;\\ \mathbb{P}(X\le d)+\mathbb{P}\left(d<X\le \dfrac{x-\gamma d}{1-\gamma}\right)=F_X\left(\dfrac{x-\gamma d}{1-\gamma}\right), & x\ge d.\end{cases}
\end{aligned}
$$

To summarize, the distribution function of $R_d(X,Y)$ is given by

$$
F_{R_d(X,Y)}(x)=\begin{cases}F_X(x), & x<d;\\ p+(1-p)F_X\left(\dfrac{x-\gamma d}{1-\gamma}\right), & x\ge d,\end{cases} \tag{1}
$$

from which the survival function of $R_d(X,Y)$ immediately follows. □

1.3 Proof of Theorem 3.8

Proof. In light of the form of $g_i(\cdot)$ for the TVaR measure, the discussion depends on the relative size of $\bar{F}_X(0)$ and α:

Case 1: If $\bar{F}_X(0)\le\alpha$, we know $(1-p)\bar{F}_X(d)<\alpha$ for $d\in\overline{\mathbb{R}}_+$ and $p\in(0,1]$. In this case, we have

$$\phi'(d)=(p+(1-p)\gamma)\left(\frac{1}{\alpha}-(1+\theta)\right)\bar{F}_X(d).$$

(a) If $1/\alpha>(1+\theta)$, $\phi'(d)>0$ and $\phi(d)$ is increasing on $d\in\mathbb{R}_+$, and so $\phi(d)$ attains its minimum at 0, that is, $d^*=0$.

(b) If $1/\alpha<(1+\theta)$, then $\phi(d)$ is decreasing on $d\in\mathbb{R}_+$, which implies that $d^*=\infty$.

In the special case where $1/\alpha=1+\theta$, we have $\phi(d)=1/\alpha\mathbb{E}[X]$. As $\bar{F}_X(0)\le\alpha$, it can be calculated that $\phi(\infty)=\mathrm{TVaR}_\alpha(X)=1/\alpha\mathbb{E}[X]$. In this case, it is favorable to buy full reinsurance, that is, $d^*=0$.

Case 2: Now we consider the case where $\bar{F}_X(0)>\alpha$.

(a) If $p\ge 1-\alpha/\bar{F}_X(0)$, which is equivalent to $\bar{F}_X(0)\le\alpha/(1-p)$, we determine the optimal value of d^* by considering possible relations between κ, α, and p.

(i) If $\alpha<\kappa<\bar{F}_X(0)$, when $d\in[0,\bar{F}_X^{-1}(\alpha))$, we have

$$\phi'(d)=1-\frac{\bar{F}_X(d)}{\kappa}.$$

Setting the above equation equal to zero yields $d=\bar{F}_X^{-1}(\kappa)$. It is straightforward to see that $d^*=\bar{F}_X^{-1}(\kappa)$.

(ii) If $\alpha<\bar{F}_X(0)\le\kappa$, when $d\in\left[0,\bar{F}_X^{-1}(\alpha)\right)$, $\alpha<\bar{F}_X(d)\le\bar{F}_X(0)$, we have

$$\phi'(d)=1-\frac{\bar{F}_X(d)}{\kappa}>0,$$

so $\phi(d)$ is increasing on that interval. Moreover, note that $\alpha<\kappa$ implies $1/\alpha>1+\theta$, and *vice versa*. When $d\in\left[\bar{F}_X^{-1}(\alpha),\infty\right)$, $0<\bar{F}_X(d)\le\alpha$, we have

$$\phi'(d)=(p+(1-p)\gamma)\left(\frac{1}{\alpha}-(1+\theta)\right)\bar{F}_X(d)\ge 0.$$

Hence, $\phi(d)$ is increasing on $\mathbb{R}_+$. Consequently, $d^*=0$.

(iii) If $\kappa = \alpha$, which is equivalent to $1/\alpha = 1+\theta$, we see that d is decreasing on the interval $\big[0, \bar{F}_X^{-1}(\alpha)\big)$. On the other hand, for $d \in \big[\bar{F}_X^{-1}(\alpha), \infty\big)$, $\phi'(d) = 0$ and $\phi(d) = \mathrm{TVaR}_\alpha(X)$. In this special case, d^* can be any number that is equal to or greater than $\bar{F}_X^{-1}(\alpha)$. The insurer is indifferent between any two different $d_1^*,\ d_2^* \in \big[\bar{F}_X^{-1}(\alpha), \infty\big)$.

(iv) If $\kappa < \alpha$, that is, $1/\alpha < 1+\theta$, we infer that $\phi(d)$ is decreasing on $\big[\bar{F}_X^{-1}(\alpha), \infty\big)$. Besides, when $d \in \big[0, \bar{F}_X^{-1}(\alpha)\big)$, $\alpha < \bar{F}_X(d)$, hence we have

$$\phi'(d) = 1 - \frac{\bar{F}_X(d)}{\kappa} < 0.$$

Therefore, $\phi(d)$ decreases on $\mathbb{R}_+$, and it must hold that $d^* = \infty$.

The proof of Case 2(b) is similar and is thus omitted. □

1.4 Proof of Proposition 4.1

Proof. For any $x \in \mathbb{R}_+$,

$$F_{I_d(X,Y)}(x) = p\mathbb{P}((X-d)_+ \le x) + (1-p)\mathbb{P}(\gamma(X-d)_+ \le x).$$

Note that

$$\mathbb{P}((X-d)_+ \le x) = \begin{cases} \mathbb{P}((X-d)_+ \le 0) = F_X(d), & x = 0; \\ \mathbb{P}((X-d)_+ \le x) = F_X(x+d), & x > 0. \end{cases}$$

Besides, we have

$$\mathbb{P}(\gamma(X-d)_+ \le x) = \begin{cases} \mathbb{P}(\gamma(X-d)_+ \le 0) = F_X(d), & x = 0; \\ \mathbb{P}(\gamma(X-d)_+ \le x) = F_X(x/\gamma+d), & x > 0. \end{cases}$$

Putting these together, we obtain the distribution function of $I_d(X,Y)$ as

$$F_{I_d(X,Y)}(x) = pF_X(x+d) + (1-p)F_X(x/\gamma+d), \quad x \ge 0.$$

The survival function of $I_d(X,Y)$ stated in (10) then follows immediately. □

2 Supplement numerical examples

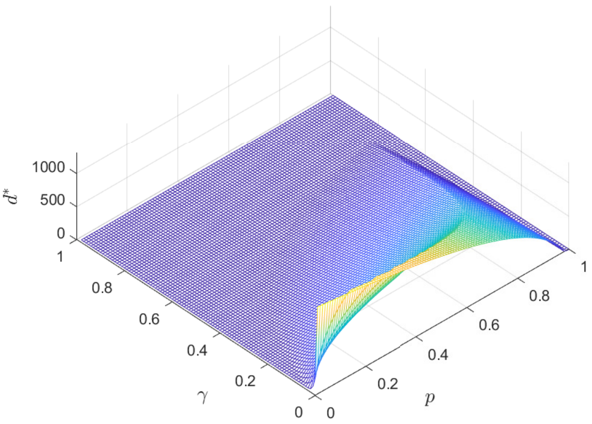
We assume the aggregate loss X follows a shifted Exponential distribution, given by

$$F_X(x) = \begin{cases} F_0, & x = 0; \\ 1-(1-F_0)e^{-\lambda x}, & x > 0. \end{cases}$$

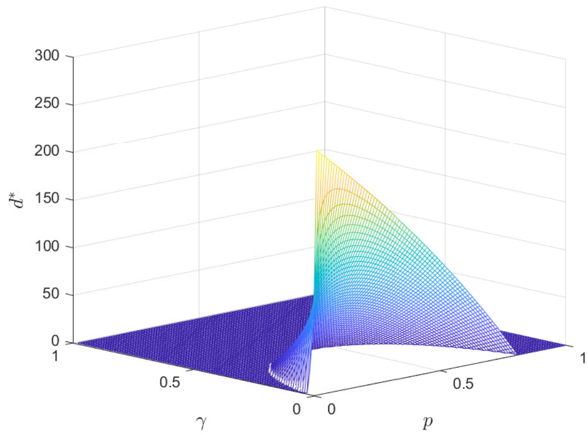
For illustration purposes, we let $F_0 = 0.3$ and $\lambda = 0.002$.

We first conduct sensitivity analysis on the insurer's optimal deductible level. To address the value of d^* under the effect between the performance probability p and the recovery rate γ, we present Figure 1 with a fixed loading. When γ increases, the insurer cares more about the loading compared to the default risk. It, again, shows that the optimal deductible decreases in general.

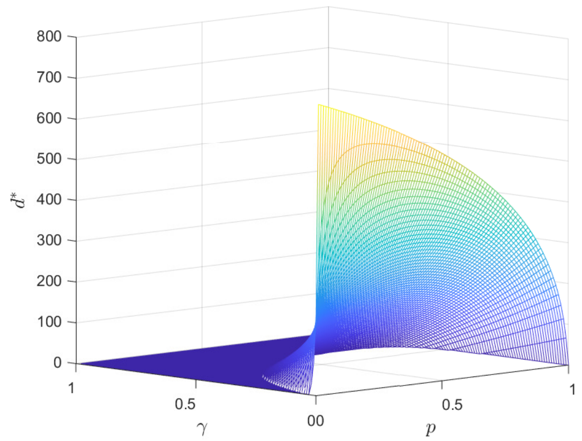
Subject to a fixed recovery rate γ, we examine how the performance probability p and safety loading θ affect d^* in Figure 2. The optimal deductible increases as θ increases, showing that



(a) TVaR with $\alpha = 0.05$.

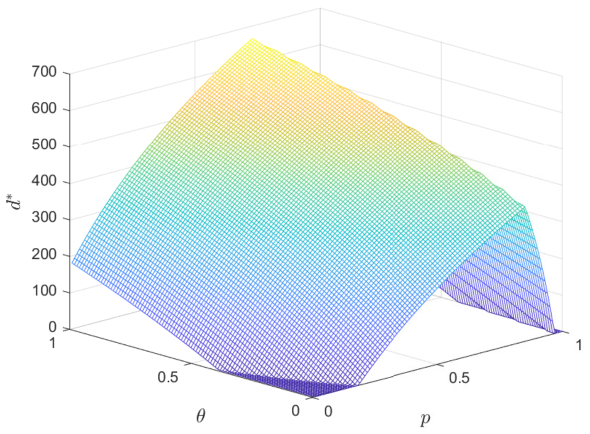


(b) Gini with $r = 0.6$.

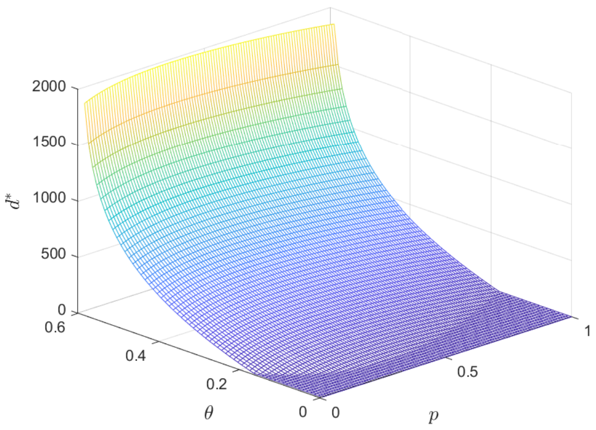


(c) PH with $k = 1/3$.

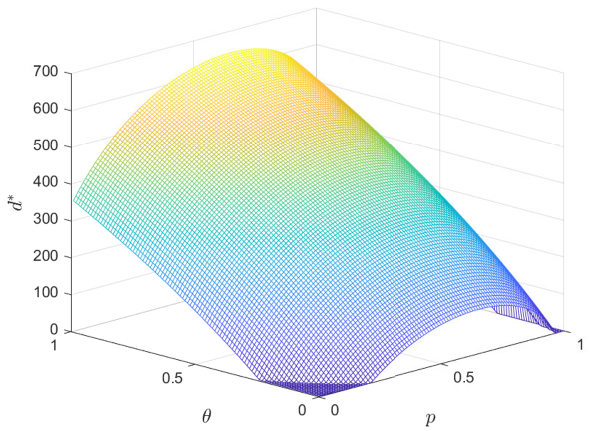
Figure 1: Effect of p and γ on d^* with $\theta = 0.1$.



(a) TVaR with $\alpha = 0.05$.



(b) Gini with $r = 0.6$.



(c) PH with $k = 1/3$.

Figure 2: Effect of p and θ on d^* with $\gamma = 0.3$.

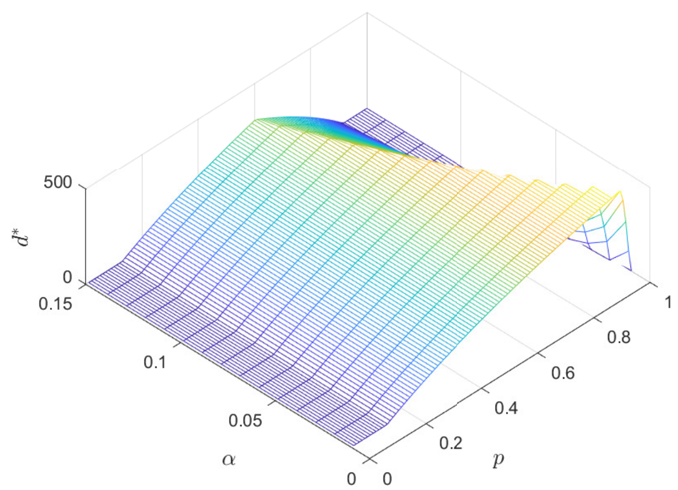


Figure 3: Effect of p and α on d^* for the TVaR measure with $\theta = 0.1$ and $\gamma = 0.3$.

the insurer's insurance demand is lowered. It is reasonable because the insurer becomes less willing to buy insurance as the contract premium becomes more expensive. Moreover, for the TVaR measure, we present Figure 3. When the default risk exists, a higher α reflects a less prudent attitude of the insurer, resulting in a decreasing trend of d^*. The insurer will be willing to cede more risk to the reinsurer.

We now pay attention to the scenario analysis on the Bowley solution. A discretization method on θ is applied. For brevity, we only present Bowley solutions for the TVaR-TVaR case. Other cases can be similarly discussed.

Example 2.1 (TVaR-TVaR Case). Suppose both the insurer and the reinsurer adopt the TVaR risk measure. The former sets a confidence level of $1-\alpha$, while the latter sets a confidence level $1-\beta$ where $\beta > \alpha$. We take $\alpha = 0.05$, $\beta = 0.1$, and $c = 0.35$.

(i) Suppose that $\gamma = 0.1$ and $p = 0.2 < 0.929$. For those θ such that $\kappa_\theta \leq \alpha/(1-p) = 0.0625$, $d^* > 0$ and $\varphi = 4.7143 > 0$. The optimal safety loading must belong to $\mathcal{A}^*_{1,\varphi}$. As $\psi(\theta)$ is increasing, $\theta^*_{1,\varphi} = 1/\alpha - 1 = 19$. On the other hand, for those θ such that $\kappa_\theta > 0.0625$, we have $\nu_\theta < \bar{F}_X(0)$, so $d^* > 0$. However, as $\xi = -0.4898 < 0$, the optimal safety loading must belong to $\mathcal{A}^*_{2,\xi,\varphi}$. In this case, θ^* does not exist. To sum up, the optimal safety loading is $\theta^* = \theta^*_{1,\varphi} = 19$ and the Bowley solution is $(\theta^*,\ d^*(\theta^*)) = (19,\ 1319.53)$.

(ii) Suppose that $\gamma = 0.35$ and $p = 0.2 < 0.929$. For those θ such that $\kappa_\theta \leq 0.0625$, $d^* > 0$ and $\varphi = 10.6667 > 0$. The optimal safety loading must belong to $\mathcal{A}^*_{1,\varphi}$. As $\psi(\theta)$ is increasing, we obtain $\theta^*_{1,\varphi} = 1/\alpha - 1 = 19$. On the other hand, for those θ such that $\kappa_\theta > 0.0625$ and $\nu_\theta < \bar{F}_X(0)$, $d^* > 0$. Since $\xi = 0.0417 > 0$, the optimal safety loading must belong to $\mathcal{A}^*_{1,\xi,\varphi}$. Furthermore, for those θ such that $\kappa_\theta > 0.0625$ and $\nu_\theta \geq \bar{F}_X(0)$, $d^* = 0$. In this case, the optimal safety loading is $\theta^*_\xi = 0.0417$. To sum up, as $\psi(\theta)$ is an increasing function, the optimal safety loading is $\theta^*_{1,\varphi} = 19$, and the Bowley solution equals $(\theta^*,\ d^*(\theta^*)) = (19,\ 1319.53)$.

(iii) Suppose that $\gamma = 0.35$ and $p = 0.95 > a$, $\iota = -0.1953 < 0$. The optimal deductible is always positive, so the optimal safety loading must belong to admissible set $\mathcal{A}^*_{2,\iota}$. As $\psi(\theta)$ is an increasing function, $\theta^* = 1/\alpha - 1 = 19$. The Bowley solution in this case is given by $(\theta^*,\ d^*(\theta^*)) = (19,\ 1319.53)$.

(iv) Suppose that $p = 1$. In this case, there is no default risk, and $\iota = 0.4286$. For those $d^* > 0$, the optimal safety loading must belong to admissible set $\mathcal{A}^*_{1,\iota}$. In this case, we obtain $\theta^*_{1,\iota} = 1/\alpha - 1 = 19$. For those $d^* = 0$, the maximum of $\psi(\theta)$ is given by $\theta_\iota = 0.4286$. Putting all these together, $\theta^* = \theta^*_{1,\iota}$, hence the Bowley solution is given by $(\theta^*,\ d^*(\theta^*)) = (19,\ 1319.53)$.

We again note that the same strategy $(\theta^*, d^*(\theta^*)) = (19, 1319.53)$ occurs under different parameter settings. This is due to the assumption of the insurer's indifferent attitude when $1+\theta = 1/\alpha$ and the insurer would purchase a stop-loss treaty with $\bar{F}_X^{-1}(\alpha)$ being the deductible. As the reinsurer is monopolistic and profit-seeking, the reinsurer would choose the largest loading $\theta^* = 19$ in the above cases.

References

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