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RESEARCH ARTICLE

Generic framework for a coherent integration of experience and exposure rating in reinsurance—Appendices

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A. Continuous distributions

Most elements within the set of reduced random variables (see Section 4.3) are nonnegative, i.e., they have a support $\mathbb{R}^+$ or $\mathbb{Z}^+$. The underlying models are thus calibrated with respective scaling factors e^c (see exponential linking in (5.4)). The incurred lag $\tau^I_{o,i}$ can, however, be positive or negative, i.e., it has a support $\mathbb{R}$, and the underlying model is calibrated with a translate parameter c (linear linking). There are also random variables, e.g., the number of payments per year, with a support $0, \ldots, n$ implying a logistic linking for a probability parameter $p \in [0, 1]$.

Continuous distributions can also be used to calibrate discrete processes (see Figure 6.1). An exponential linking is used in combination with the $log\mathcal{N}$ family and a linear linking is used in combination with the $\mathcal{N}$ family.

A.1. $log\mathcal{N}$– and $\mathcal{N}$–based scale and location calibration

All occurrence years $o \in \{-T, \ldots, -1\}$ are calibrated with a scalar c, i.e., $c_o = c$ for all o. A $log\mathcal{N}(\mu_o(c), \sigma_o)$ approximation is used to calibrate distributions $f_o(x \mid c)$ that need to be scaled (exponential linking), and a $\mathcal{N}(\mu_o(c), \sigma_o)$ approximation is used to calibrate distributions $f_o(x \mid c)$ that need to be translated (linear linking). The conditional model parameters $\mu_o(c)$ are obtained as $\mu_o(c) = \mu_o^0 + c$ in both cases. Plugging the pdfs (probability density functions) into (5.5), (5.6), and (5.7), and using normal priors $c \sim \mathcal{N}(0, \sigma_c)$ yields following solutions for the MAP estimate on a *claim*, an *annual*, and a *period* level, respectively:

$$c^c_{MAP} = \frac{\sum_{o,i} \frac{y_{o,i} - \mu_o^{c0}}{(\sigma_o^c)^2}}{\frac{1}{(\sigma_{c^c})^2} + \sum_{o,i} \frac{1}{(\sigma_o^c)^2}} \; ; c^a_{MAP} = \frac{\sum_o \frac{y_o - \mu_o^{a0}}{(\sigma_o^a)^2}}{\frac{1}{(\sigma_{c^a})^2} + \sum_o \frac{1}{(\sigma_o^a)^2}} \; ; c^p_{MAP} = \frac{\frac{y - \mu^{p0}}{(\sigma^p)^2}}{\frac{1}{(\sigma_{c^p})^2} + \frac{1}{(\sigma^p)^2}}$$

where:

$$y_{o,i} \;, y_o \;, y := \begin{cases} Y_{o,i} & , Y_o & , Y & \text{if } f_o = \mathcal{N} \\ \ln(Y_{o,i}) & , \ln(Y_o) & , \ln(Y) & \text{if } f_o = log\mathcal{N} \end{cases} \tag{A.1}$$

A.2. $log\mathcal{N}$– and $\mathcal{N}$–based linear-trend calibration

The calibration of a linear trend is performed by replacing the calibration parameters $c_o = c$ with $c_o = a + b \cdot (o - o_0)$, i.e., the scalar calibration parameter c is replaced with the vector $\boldsymbol{c} = (a, b)$ (see Figure 6.1). The conditional model parameters $\mu_o(\boldsymbol{c})$ are obtained as $\mu_o(\boldsymbol{c}) = \mu_o^0 + a + b \cdot (o - o_0)$ in both cases. Plugging the pdfs into (5.8) and (5.9), and using normal priors $a \sim \mathcal{N}(0, \sigma_a)$ and $b \sim \mathcal{N}(0, \sigma_b)$ yields following equations to be fulfilled by the MAP estimates on a *claim* and an *annual* level, respectively:

$$\begin{bmatrix} \frac{1}{(\sigma_{a^c})^2} + \sum_{o,i} \frac{1}{(\sigma_o^c)^2} & \sum_{o,i} \frac{o - o_0}{(\sigma_o^c)^2} \\ \sum_{o,i} \frac{o - o_0}{(\sigma_o^c)^2} & \frac{1}{(\sigma_{b^c})^2} + \sum_{o,i} \frac{(o - o_0)^2}{(\sigma_o^c)^2} \end{bmatrix} \times \begin{bmatrix} a^c_{MAP} \\ b^c_{MAP} \end{bmatrix} = \begin{bmatrix} \sum_{o,i} \frac{y_{o,i} - \mu_o^{c0}}{(\sigma_o^c)^2} \\ \sum_{o,i} \frac{(y_{o,i} - \mu_o^{c0}) \cdot (o - o_0)}{(\sigma_o^c)^2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{(\sigma_{a^a})^2} + \sum_o \frac{1}{(\sigma_o^a)^2} & \sum_o \frac{o - o_0}{(\sigma_o^a)^2} \\ \sum_o \frac{o - o_0}{(\sigma_o^a)^2} & \frac{1}{(\sigma_{b^a})^2} + \sum_o \frac{(o - o_0)^2}{(\sigma_o^a)^2} \end{bmatrix} \times \begin{bmatrix} a^a_{MAP} \\ b^a_{MAP} \end{bmatrix} = \begin{bmatrix} \sum_o \frac{y_o - \mu_o^{a0}}{(\sigma_o^a)^2} \\ \sum_o \frac{(y_o - \mu_o^{a0}) \cdot (o - o_0)}{(\sigma_o^a)^2} \end{bmatrix}$$

where $y_{o,i}$ and y_o are defined in (A.1).

A.3. Continuous modeling of discrete processes

A robust calibration of discrete processes is only possible if the annual frequencies $\lambda_o = \mathbb{E}[N_o]$ are sufficiently large, i.e., if $\Pr(N_o = 0) \ll 1$ for all o. The first two moments $\mathbb{E}[N_o]$ and $\mathbb{V}[N_o]$ are determined by the highest density region(s) of a probability distribution (see HDR definition in Section 4.1), and most of the observed random variables N_o are expected to be found within the HDR. The calibration is derived from the logarithm of the pmf (or pdf, respectively), and it thus only depends on *potentials*, i.e., relative probabilities that characterize the shape of the distributions.

A discrete model can thus be calibrated by fitting an 'easy to manage' continuous distribution family to the simulated discrete distributions. The following approach is used to convert a continuous pdf $f_X(x)$ with support $\mathbb{R}^+$ into a respective discrete pmf $f_N(k) \propto \phi_N(k)$ with support $\mathbb{Z}^+$ where the potential $\phi_N(k)$ is defined as follows:

$$\begin{aligned}
\phi_N(k) &\triangleq \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} f_X(x) \cdot dx \approx \begin{cases} f_X(\frac{1}{4}) \cdot \frac{1}{2} & \text{if } k = 0 \\ f_X(k) \cdot 1 & \text{if } k > 0 \end{cases} \\
&= f_X(k + \tfrac{\mathbb{1}_{k=0}}{4}) \cdot \tfrac{1+\mathbb{1}_{k>0}}{2} \\
&\approx f_X(k) \text{ if } f_X(\tfrac{1}{4}) \ll 1
\end{aligned}$$

B. Discrete: The $\mathcal{P}anjer$ class

B.1. Exponential linking

The three distribution families within the $\mathcal{P}anjer$ class are suitable to fit a broad variety of simulated discrete random variables. The class comprises the *Poisson* family $\mathcal{P}$, the *negative binomial* family $\mathcal{NB}$, and the *binomial* family $\mathcal{B}$. The variance-over-mean ratio (Panjer factor) $f_P = \mathbb{V}[N]/\mathbb{E}[N]$ is used to discriminate between the three families: $f_P = 1$ for $\mathcal{P}$, $f_P > 1$ for $\mathcal{NB}$, and $f_P < 1$ for $\mathcal{B}$.

A specific solution for the scale calibration is derived for each of the three families, and a generic solution—applicable to all three families—is derived with an approximation. Solutions for the calibration of features that require a logistic linking are derived for the $\mathcal{NB}$ and the $\mathcal{B}$ families.

The calibration for the families in the $\mathcal{P}anjer$ class is summarized in Table B.1: $\mathcal{P}$ for Poisson, $\mathcal{NB}$ for negative binomial, $\mathcal{B}$ for binomial, and $\mathcal{G}a$ for the gamma approximation.

The first column indicates the family, and the respective conditional probability mass function (pmf) $f_N(k \mid c)$ is shown in the second column. The third column indicates the linking, and the fourth column contains the equation to be fulfilled by the root c_{MAP} as obtained with (5.6) and the prior $\pi(c) = \mathcal{N}(c; 0, \sigma_c^2)$. The calibration of the probability parameter $p_o(c)$ via a logistic linking for fixed parameters r_o (in the $\mathcal{NB}$ case) and n_o (in the $\mathcal{B}$ case), respectively, is shown in the two bottom rows.

Remark B.1. *The root c_{MAP} corresponds to the intersection of the two graphs defined by the functions $c \mapsto L(c)$ and $c \mapsto R(c)$ as shown in Figure 6.4.*

B.2. The Poisson family $\mathcal{P}$

A $\mathcal{P}$ process is postulated by assuming that the simulated random variables N_o are mutually independent and that the simulated variance-over-mean ratios $f_{P,o} = \mathbb{V}[N_o]/\mathbb{E}[N_o]$ are characterized by $f_{P,o} \approx 1$ for all o. Another assumption is that all frequencies $\lambda_o(c)$ scale with e^c and that the prior distribution of the calibration parameter c is given by $\pi(c) = \mathcal{N}(c; 0, \sigma_c^2)$.

The equation to be fulfilled by the root $c = c_{MAP}$ is obtained by feeding the derivatives into (5.6). The resulting equation is expressed in the form $L(c) \overset{!}{=} R(c)$ where the expression on the left side is set to $L(c) = c/e^c$ and the expression $R(c)$ on the right side is shown in Table B.1.

The solution obtained for the $\mathcal{P}$ case depends on the sufficient statistics $\sum N_o$, the aggregate frequency $\sum \lambda_o^0$ and on the prior parameter σ_c^2. The calibration parameter c^a_{MAP} derived from the *annual* observations N_o is thus identical to the calibration parameter c^p_{MAP} derived from aggregate *period* statistics N (see Table 6.3 and Figure 6.4).

B.3. The negative binomial family $\mathcal{NB}$

A $\mathcal{NB}$ process is characterized by a variance-over-mean ratio $f_P = 1/p > 1$. The implicit factors might vary from year to year and change with the scaling of the frequencies via $f_{P,o}(c) = 1 + (f_{P,o}^0 - 1) \cdot e^c$. It is, however, difficult to derive robust estimates for annual factors $f_{P,o}$, and these factors have only a minor impact on the calibration (see Table 6.3 and Figure 6.4). The annual Panjer factors $f_{P,o}(c)$ are thus assumed to be equal to $f_P = \sum \mathbb{V}[N_o]/\sum \mathbb{E}[N_o]$ and f_P is assumed to be unaffected by the scaling. The model parameters $\{(r_o^0, p_o^0)\}_{-T \le o \le -1}$ are thus fully determined via $p_o^0 = 1/f_P$ and $r_o^0 = \mu_o^0/(f_P - 1)$.

The partial derivatives $\partial/\partial c$ are evaluated with the Stirling approximation

$$\begin{aligned} \ln(\Gamma(x)) &\approx (x-1) \cdot \ln(x-1) - (x-1) + \tfrac{1}{2} \cdot \ln(\sqrt{2\pi} \cdot (x-1)) \\ \tfrac{\partial}{\partial x} \ln(\Gamma(x)) &\approx \ln(x-1) + \tfrac{1}{2} \cdot \tfrac{1}{x-1} \end{aligned}$$

leading to the function $R(c)$ for the $\mathcal{NB}$ case shown in Table B.1. The statistics entering $R(c)$ depends on c, and the calibration parameter c^p_{MAP} derived from aggregated *period* data thus varies from the parameter c^a_{MAP} derived from *annual* data.

Table B.1: Calibration of the families in the $\mathcal{P}$anjer class

Fam.	Conditional pmf $f_N(k \mid c)$	Linking	Equation $L(c) \stackrel{!}{=} R(c)$ for the root $c = c_{MAP}$
$\mathcal{P}$	$= e^{-\lambda_o(c)} \cdot \frac{(\lambda_o(c))^k}{k!}$	$\lambda_o(c) = \lambda_o^0 \cdot e^c$ $\lambda_o^0 = \mu_o^0$ $(f_P = 1)$	$\frac{c}{e^c} = \sigma_c^2 \cdot \sum_{o=-T}^{-1} \lambda_o^0 \cdot \left(\frac{N_o}{\lambda_o^0 \cdot e^c} - 1\right)$ $c^{(0)} = 0$ $c^{(i)} = \ln\left(\sum_{o=-T}^{-1} N_o - c^{(i-1)}/\sigma_c^2\right) - \ln\left(\sum_{o=-T}^{-1} \lambda_o^0\right)$ for $i > 0$
$\mathcal{NB}$	$= \frac{\Gamma(k+r_o(c)) \cdot (1-p_o)^k \cdot p_o{}^{r_o(c)}}{\Gamma(k+1) \cdot \Gamma(r_o(c))}$	$r_o(c) = r_o^0 \cdot e^c$ $r_o^0 = \frac{\mu_o^0}{f_P - 1}$ $p_o = p = \frac{1}{f_P}$	$\frac{c}{e^c} = \sigma_c^2 \cdot \sum_{o=-T}^{-1} r_o^0 \cdot \left[\ln\left(1 + \frac{N_o}{r_o^0 \cdot e^c - 1}\right) - \frac{1}{2} \cdot \frac{N_o}{(N_o + r_o^0 \cdot e^c - 1) \cdot (r_o^0 \cdot e^c - 1)} - \ln(f_P)\right]$
$\mathcal{B}$	$= \frac{\Gamma(n_o(c)+1) \cdot p_o{}^k \cdot (1-p_o)^{n_o(c)-k}}{\Gamma(k+1) \cdot \Gamma(n_o(c)-k+1)}$	$n_o(c) = n_o^0 \cdot e^c$ $n_o^0 = \lceil \frac{\mu_o^0}{1-f_P} \rceil$ $p_o = \frac{\mu_o^0}{n_o^0}$	$\frac{c}{e^c} = \sigma_c^2 \cdot \sum_{o=-T}^{-1} n_o^0 \cdot \left[\ln\left(1 + \frac{N_o}{n_o^0 \cdot e^c - N_o}\right) - \frac{1}{2} \cdot \frac{N_o}{n_o^0 \cdot e^c \cdot (n_o^0 \cdot e^c - N_o)} + \ln(f_P)\right]$
$\mathcal{G}a$	$\propto \frac{e^{(\alpha_o(c) \cdot \ln(\beta_0 \cdot k) - \beta_0 \cdot k)}}{k \cdot \Gamma(\alpha_o(c))}$	$\alpha_o(c) = \alpha_o^0 \cdot e^c$ $\alpha_o^0 = \frac{\mu_o^0}{f_P}$ $\beta = \frac{1}{f_P}$	$\frac{c}{e^c} = \sigma_c^2 \cdot \sum_{o=-T}^{-1} \alpha_o^0 \cdot \left[\ln\left(\frac{N_o}{\alpha_o^0 \cdot e^c - 1}\right) - \frac{1}{2} \cdot \frac{1}{\alpha_o^0 \cdot e^c - 1} - \ln(f_P)\right]$ $c^{(0)} = c_{\mathcal{P}}$ $c^{(i)} = \frac{\sum_{o=-T}^{-1} \alpha_o^0 \cdot \ln\left(\frac{N_o}{\alpha_o^0}\right) - \frac{1}{e^{c^{(i-1)}}} \cdot \left(\frac{T}{2} + \frac{c^{(i-1)}}{\sigma_c^2}\right)}{\sum_{o=-T}^{-1} \alpha_o^0} - \ln(f_P)$ if $\alpha_o^0 \gg 1 \; \forall o$
$\mathcal{NB}$	$\propto (1 - p_o(c))^k \cdot p_o(c)^{r_o}$	$p_o(c) = \frac{p_o^0 \cdot e^c}{p_o^0 \cdot e^c + (1-p_o^0) \cdot e^{-c}}$	$c = 2 \cdot \sigma_c^2 \cdot \sum_{o=-T}^{-1} \frac{r_o \cdot (1-p_o^0) \cdot e^{-c} - N_o \cdot p_o^0 \cdot e^c}{p_o^0 \cdot e^c + (1-p_o^0) \cdot e^{-c}}$
$\mathcal{B}$	$\propto p_o(c)^k \cdot (1 - p_o(c))^{n_o - k}$	$p_o(c) = \frac{p_o^0 \cdot e^c}{p_o^0 \cdot e^c + (1-p_o^0) \cdot e^{-c}}$	$c = 2 \cdot \sigma_c^2 \cdot \sum_{o=-T}^{-1} \frac{N_o \cdot (1-p_o^0) \cdot e^{-c} - (n_o - N_o) \cdot p_o^0 \cdot e^c}{p_o^0 \cdot e^c + (1-p_o^0) \cdot e^{-c}}$

The condition $r_o^0 \cdot e^c > 1$ for all o leads to a lower bound $c > c_{lo} = -\ln\left(\min\{r_o^0\}_{-T\le o\le -1}\right)$ for the calibration parameter c. An upper bound c_{hi} for the solution can also be identified since $L(c) > 0$ and $R(c) < 0$ for large c. The equation $L(c) \stackrel{!}{=} R(c)$ has two roots $c_{lo} < c_1 < c_2 < c_{hi}$. The solution c_1 is found near c_{lo} (where the Stirling approximation is no longer valid) and corresponds to a minimum of $L(c)$. The relevant solution c_2 is easily found near the $\mathcal{P}$ solution (see Table 6.3 and Figure 6.4) with the Newton-Raphson method.

B.4. The binomial family $\mathcal{B}$

The solution for the $\mathcal{B}$ case is like the solution derived for the $\mathcal{NB}$ case. The Panjer factor is constrained by $0 < f_P = (1-p) < 1$, and the maximum number of observations $n_o^0 = \mu_o^0/(1-f_P) \in \mathbb{Z}^+$ must be a positive integer. The simulated mean μ_o^0 and the variance $\mu_o^0 \cdot f_P$ can thus not be exactly preserved when fitting a $\mathcal{B}$ distribution to a simulated distribution with $f_P < 1$.

This constraint can be neglected when evaluating the root of the equation $L(c) \stackrel{!}{=} R(c)$ for the $\mathcal{B}$ case (see Table B.1). Depending on the process implemented in the generative model, this constraint must be considered when adjusting the model. This is done by first setting $n_o = \lceil \mu_o/(1-f_P) \rceil$, and then setting $p_o = \mu_o/n_o$.

The equation $L(c) \stackrel{!}{=} R(c)$ for the $\mathcal{B}$ case has a single root near the $\mathcal{P}$ solution (see Table 6.3 and Figure 6.4) with a lower bound $c > c_{lo} = \ln(\max\{N_o/n_o^0\}_{-T\le o\le -1})$. The statistics entering the right-side term $R(c)$ depends on c, and the calibration parameter c_{MAP}^p derived from aggregated *period* data will thus be different from the parameter c_{MAP}^a derived from *annual* data.

B.5. The gamma family $\mathcal{G}a$

Discrete probability distributions can also be calibrated with continuous approximations (see Section A.3), and the $\mathcal{G}a(\alpha,\beta)$ family with support $\mathbb{R}^+$ is used as an approximation for the $\mathcal{P}anjer$ class. The parameters (α,β) are easily derived from the first two moments $\mathbb{E}[X] = \alpha/\beta$ and $\mathbb{V}[X] = \mathbb{E}[X]/\beta$.

An exponential coupling $\mathbb{E}[X_o \mid c] = \alpha_o(c)/\beta_o \propto e^c$ is postulated by assuming that the variance-over-mean ratios $1/\beta_o$ do not depend on the calibration parameter c. The scaling behavior is thus the same as for the discrete distributions in the $\mathcal{P}anjer$ class. These ratios are assumed to be identical for all years, i.e., $1/\beta_o = f_P$ for all o. The resulting equations to be fulfilled by the root $c = c_{MAP}$ are listed in Table B.1.

The $\mathcal{G}a$ family can thus be used as an approximation for the $\mathcal{P}anjer$ class without differentiating between the cases $f_P < 1$, $f_P = 1$, and $f_P > 1$ (see Table 6.3 and Figure 6.4). Another advantage of the $\mathcal{G}a$ family (as, e.g., compared to the $log\mathcal{N}$ family) is that the pdf has its finite maximum at $x = 0$ for $1 < \alpha < 2$. The $\mathcal{G}a$ family can thus also be used to approximate discrete distributions in a lower frequency regime.

B.6. Logistic linking

The $\mathcal{NB}$ and $\mathcal{B}$ families

The above calibration is obtained by scaling the parameters r_o (in the $\mathcal{NB}$ case) and n_o (in the $\mathcal{B}$ case) while keeping parameters p_o fixed. There are, however, also situations where the parameters r_o or n_o are fixed. The respective solutions for the calibration of the parameters $p_o(c)$ via the logistic linking (as defined in (5.4)) are listed at the bottom of Table B.1.

C. Description of the examples

C.1. Example 1: Claims count process

C.1.1. Model characteristics

This example describes the framework with the help of the claims count process N_o (see Section C.1.2, Table C.1, and Example 6.1 in the printed document):

1. Annual claims count process:
   (a) The frequencies $\lambda_o = \mu_o$ are specified for the occurrence years $o = -T$ and $o = -1$ (with $T = 12$).
   (b) A geometric series is used to specify the frequencies for the occurrence years $o \in \{-T + 1, \ldots, -2\}$ and to extrapolate to the cover year $o = +1$.
   (c) A single $\mathcal{P}anjer$ factor $f_P = \mathbb{V}[N]/\mathbb{E}[N]$ is specified for all occurrence years o.
   (d) Annual claims count: $N_o \sim f_N(\lambda_o, f_P)$ where f_N is a distribution family in the $\mathcal{P}anjer$ class with $f_N = \mathcal{B}$ if $f_P < 1$, $f_N = \mathcal{P}$ if $f_P = 1$, and $f_N = \mathcal{NB}$ if $f_P > 1$
   (e) The expected average frequency for the period is $\hat{\mu} = \sum \mu_o / T$
2. Bayesian inference: The Bayesian calibration is performed with normal priors for the scale calibration scalar c and the linear-trend calibration vector $\boldsymbol{c} = (a, b)$: $\pi_c(c) = \mathcal{N}(c; 0, \phi_c)$, $\pi_a(a) = \mathcal{N}(a; 0, \phi_a)$, and $\pi_b(b) = \mathcal{N}(b; 0, \phi_b)$ where $\phi = \sigma^2$ is the respective variance. The reference year $o_0 = -(T + 1)/2$ is set to the middle of the observation period.
3. Generative process: A $\mathcal{NB}$ process ($f_P = 1.25$) is imposed, and the frequency is specified to increase by a factor of 3 during the observation period. A single random variable N_o (see 'Statistics' and Table C.1) is drawn for each observation year o.
4. Prior model: A $\mathcal{P}$ process ($f_P = 1$) is postulated for the generative model. The average frequency is overstated, and the trend is understated. The simulated annual frequencies μ_o^0 are derived from the simulated pmfs defined by the $K \gg 1$ random variables drawn for each year o. The simulated $\mathcal{P}anjer$ factor is derived from the random variables aggregated on a *period* level. The extrapolation of the prior models leads to a projected frequency μ_1 for the observation year $o = +1$.
5. Scale calibration: The calibration parameters c are derived from the respective mean frequencies and combined with the prior assumption in the Bayesian case (see Remark C.1). The overstated scale assumption in the prior model is adjusted but the understated trend assumption remains unaffected by the calibration. Thus, the projected estimates μ_1 are too low.
6. Linear trend calibration: The calibration parameters (a, b) are derived from the respective linear regression parameters (see 'Statistics') and combined with prior assumptions in the Bayesian case (see Remark C.1). The observations are used to adjust the overstated scale assumption and the understated trend assumption. The projected estimate μ_1 is, however, quite sensitive to the trend parameter b which is derived from the volatile slope of the observations Y_o.
7. Random variables: The annual random variable N_o are drawn from the respective $\mathcal{NB}$ processes used to characterize the generative process. The annual random variable μ_o^0 are the means from the $K \gg 1$ random variables drawn from the respective $\mathcal{P}$ distributions used to characterize the Prior model.

C.1.2. Statistics

The upper part of following summary (see also Table C.1) contains the specified frequencies, the simulated observations ($K = 1$) for the generative process, and the simulated frequencies ($K \gg 1$) for the generative model ($o \in \{-T, \ldots, -1\}$ and `mu_o`$\equiv \mu_o$, `Y_o`$\equiv Y_o$, and `o_0`$\equiv o_0$). The mean (`mean`$\equiv \hat{\mu}$) of the series, the logarithms (`ln`$\equiv \ln(.)$) of the series, the mean of the logarithms (`mean`$\equiv \widehat{\ln(\mu)}$), and the linear regression (LR) of the logarithms (with the intersection evaluated at o_0) are listed.

The lower part of the summary contains the projections to $o = +1$ (`mu_1`$\equiv \mu_1$) derived from the model specifications, the calibration parameters c and (a, b), and the calibrated projections to $o = +1$

derived: from the model parameters ('model'), from the simulated series via a frequentist approach ('freq'), and from the simulated series via a Bayesian approach ('Bayes'):

```
o:    [-12, -11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1]
Generative process:
- Specified mu_o:
     [5.0, 5.53, 6.11, 6.75, 7.46, 8.24, 9.1, 10.06, 11.12, 12.28, 13.57, 15.0]
      mean = 9.184
 ln: [1.61, 1.71, 1.81, 1.91, 2.01, 2.11, 2.21, 2.31, 2.41, 2.51, 2.61, 2.71]
      mean = 2.159
      LR (o_0 = -6.5): slope = 0.100 / intersection = 2.159
- Observed Y_o (K=1):
     [3, 4, 6, 7, 10, 10, 7, 8, 12, 8, 18, 13]
      mean = 8.833
 ln: [1.1, 1.39, 1.79, 1.95, 2.3, 2.3, 1.95, 2.08, 2.48, 2.08, 2.89, 2.56]
      mean = 2.073
      LR (o_0 = -6.5): slope = 0.117 / intersection = 2.073
Generative model:
- Specified mu_o:
     [10.0, 10.65, 11.34, 12.08, 12.87, 13.7, 14.59, 15.54, 16.56, 17.63, 18.78, 20.0]
      mean = 14.479
 ln: [2.3, 2.37, 2.43, 2.49, 2.55, 2.62, 2.68, 2.74, 2.81, 2.87, 2.93, 3.0]
      mean = 2.649
      LR (o_0 = -6.5): slope = 0.063 / intersection = 2.649
- Simulated mu_o (K=100):
     [10.46, 10.53, 11.71, 11.87, 12.77, 14.2, 14.8, 15.21, 16.18, 16.9, 18.81, 19.91]
      mean = 14.446
 ln: [2.35, 2.35, 2.46, 2.47, 2.55, 2.65, 2.69, 2.72, 2.78, 2.83, 2.93, 2.99]
      mean = 2.649
      LR (o_0 = -6.5): slope = 0.059 / intersection = 2.649
Generative process and prior model:
- gen process: mu_1 = 18.316
- gen model  : mu_1 = 22.686
Scale calibration and projection:
- model: c = ln(9.184 / 14.479) = -0.455 / mu_1 = 14.390
- freq : c = ln(8.833 / 14.446) = -0.492 / mu_1 = 13.872
- Bayes: c =                      -0.486 / mu_1 = 13.957
Linear-trend calibration and projection (o_0 = -6.5):
- model: a = 2.159 - 2.649 = -0.490 / b = 0.0999 - 0.0630 = 0.0369 / mu_1 = 18.316
- freq : a = 2.073 - 2.649 = -0.576 / b = 0.1166 - 0.0594 = 0.0572 / mu_1 = 19.579
- Bayes: a =                 -0.537 / b =                   0.0569 / mu_1 = 20.317
```

Remark C.1. *The corrections implied by the calibration parameters c and (a, b) are small compared to the width $\sigma = \sqrt{\phi}$ of the respective prior distributions. A low credibility is thus given to priors and the MAP estimates are found near the 'frequentist' estimates.*

C.2. Example 2: Sensitivity analysis

This example analyzes the sensitivity of the calibration parameters c^a_{MAP} and c^p_{MAP} on the distribution family used to emulate the claims-count generative process (Table C.2) and on the selection of the distribution family fitted to the simulated pmfs. The random annual frequencies $\Lambda_o \sim \mathcal{G}a$ are used as parameters in the claims-count processes $N_o \sim f_N(\Lambda_o, f_P)$.

1. Annual loss count processes:
   (a) Reference frequencies λ^{ref}_{-T} and λ^{ref}_{-1} are specified for the beginning and the end of the observation period, and a geometric series is used to specify the reference frequencies λ^{ref}_o for the intermediate years $o \in \{-T + 1, \ldots, -2\}$.

Table C.1: Parameters and statistics for the generative process and the generative models shown in Figure 6.1

	Calibration			Annual			Average		Projected
Model	a or c	b	o_0	μ_{-12}	μ_{-1}	f_P	$\hat{N}$	$\hat{\mu}$	μ_1
Generative process				5.0	15.0	1.25		9.2	18.3
Actual pmf				5.0	15.0	1.25	8.8	9.2	18.3
Prior model				10.0	20.0	1.00		14.5	22.7
Simulated pmf				10.5	19.9	1.08		14.4	20.8
Level calibration	-0.49			6.2	12.3	1.00		8.9	14.0
Simulated pmf				6.3	12.9	0.99		9.0	14.1
Linear-trend cal.	-0.54	0.057	-6.5	4.3	16.0	1.00		9.0	20.3
Simulated pmf				4.3	16.2	1.02		9.0	20.0
Random variables used	N_o	(3 , 4 , 6 , 7 ,10 ,10 , 7 , 8 ,12 , 8 ,18 ,13)							
for the 'toy model'	μ_o^0	(10.5, 10.5, 11.7, 11.9, 12.8, 14.2, 14.8, 15.2, 16.2, 16.9, 18.8, 19.9)							

(b) Annual $\mathcal{G}a(\alpha_o, \beta$ distributions are used to draw the random annual frequencies Λ_o. The joint parameter $\beta = \mathbb{E}[\Lambda]/\mathbb{V}[\Lambda]$ is specified, and the annual parameters $\alpha_o = \lambda_o^{ref} \cdot \beta$ are derived from $\mathbb{E}[\Lambda_o] = \lambda_o^{ref} = \alpha_o/\beta$.

(c) The annual random frequencies Λ_o and the selected $\mathcal{P}anjer$ factor f_P define the generative process and the generative model, respectively (i.e., $\mathcal{NB}$ in the case $f_P = f_P{}^{\mathcal{NB}}$, $\mathcal{P}$ in the case $f_P = f_P{}^{\mathcal{P}}$, and $\mathcal{B}$ in the case $f_P = f_P{}^{\mathcal{B}}$).

2. Generative process and generative model: Different scale and trend assumptions (see Table C.2) are specified via the respective reference frequencies λ_o^{ref} for the generative process and the generative model. A single parameter $\beta = 5$ is used in all models and the same case-dependent parameter f_P is used in the generative process and in the generative model.
3. Scale calibration: The calibration of the three generative process cases (defined by f_P) is performed with the help of $\mathcal{NB}$ (if applicable), $\mathcal{P}$, $\mathcal{B}$ (if applicable), $\mathcal{G}a$, and $log\mathcal{N}$ distributions fitted to the simulated pmfs (where $K = 50$), respectively. The calibration is performed five times for each case, and the resulting calibration parameters c^a_{MAP} and c^p_{MAP} are shown in Table C.3.

Table C.2: Parameters used for the model comparison

Model	T	λ_{-T}^{ref}	λ_{-1}^{ref}	$\widehat{\ln \lambda_o^{ref}}$	β	K	$f_P^{\mathcal{NB}}$	$f_P^{\mathcal{P}}$	$f_P^{\mathcal{B}}$
Generative process	10	10.0	12.5	2.414	5.0	1	1.5	1.0	0.6
Generative model	10	5.0	10.0	1.956	5.0	50	1.5	1.0	0.6

C.3. Example 3: Multi-feature calibration

This example is used to demonstrate the calibration of multiple partly interdependent reduced variables (see Table C.4) with the help of a simplified claims generator.

Table C.3: Comparison of the calibration parameters derived on an annual *and a* period *level with different approximations for the three model cases $\mathcal{NB}$, $\mathcal{P}$, and $\mathcal{B}$*

Model		Calibration parameters c^a_{MAP} and c^p_{MAP}								
Case	f_P^{sim}	$\mathcal{NB}^a$	$\mathcal{NB}^p$	$\mathcal{P}^a = \mathcal{P}^p$	$\mathcal{B}^a$	$\mathcal{B}^p$	$\mathcal{G}a^a$	$\mathcal{G}a^p$	$log\mathcal{N}^a$	$log\mathcal{N}^p$
	1.527	**0.423**	**0.447**	0.447	-	-	0.322	0.451	0.345	0.452
$\mathcal{NB}$	1.452	**0.342**	**0.344**	0.343	-	-	0.342	0.349	0.406	0.350
f_P: 1.5	1.598	**0.404**	**0.386**	0.385	-	-	0.437	0.391	0.465	0.391
c^{mod}: 0.458	1.431	**0.304**	**0.312**	0.311	-	-	0.286	0.316	0.387	0.317
	1.510	**0.559**	**0.569**	0.569	-	-	0.547	0.573	0.691	0.572
$\mathcal{NB}$ mean	1.504	**0.406**	**0.412**	0.411	-	-	0.387	0.416	0.459	0.416
	0.961	-	-	**0.386**	0.387	0.386	0.361	0.391	0.485	0.392
$\mathcal{P}$	0.921	-	-	**0.543**	0.547	0.543	0.498	0.547	0.596	0.546
f_P: 1.0	0.906	-	-	**0.357**	0.355	0.357	0.380	0.362	0.419	0.363
c^{mod}: 0.458	1.140	0.433	0.436	**0.436**	-	-	0.398	0.441	0.490	0.441
	0.989	-	-	**0.294**	0.294	0.293	0.207	0.298	0.268	0.299
$\mathcal{P}$ mean	0.983	-	-	**0.403**	-	-	0.369	0.408	0.452	0.408
	0.579	-	-	0.333	**0.331**	**0.333**	0.330	0.338	0.380	0.338
$\mathcal{B}$	0.639	-	-	0.418	**0.429**	**0.418**	0.380	0.422	0.408	0.422
f_P: 0.6	0.629	-	-	0.528	**0.577**	**0.528**	0.469	0.533	0.561	0.533
c^{mod}: 0.458	0.622	-	-	0.408	**0.411**	**0.407**	0.403	0.411	0.479	0.412
	0.629	-	-	0.527	**0.533**	**0.527**	0.522	0.531	0.542	0.531
$\mathcal{B}$ mean	0.619	-	-	0.443	**0.456**	**0.443**	0.421	0.447	0.474	0.447

1. Claims representation: Claims are represented by the nine reduced random variables $N^{inc}_{o,i}$, $N^{clo}_{o,i}$, $I_{o,i}$, $P_{o,i}$, $\tau^{rep}_{o,i}$, $\tau^{I}_{o,i}$, $\tau^{P}_{o,i}$, $N^{I}_{o,i}$, and $N^{P}_{o,i}$. The count of closed claims $N^{clo}_{o,i}$ and the cumulative paid amounts $P_{o,i}(o = 0)$ at the submission time are determined by the other variables. They are thus evaluated and monitored but not independently calibrated.
2. Claims generator:
   (a) Different scale and trend assumptions are imposed via the parameters. The imposed deviations of the generative model from the generative process are supposed to be adjusted by the calibration procedure.
   (b) Specification: The mean values at $o = -T$ and $o = -1$ are specified for the seven variables to be calibrated and geometric series are used to specify the respective means for the intermediate years $o \in \{-T + 1, \ldots, -2\}$.
   (c) Stochastic model: A parametric distribution family is assigned to each of the seven model features. The distributions are initialized with the respective mean (default values are assigned to all other parameters).
3. Prior model:
   (a) The calibration of the seven model features ℓ is performed on an *annual* level.
   (b) Independent normal priors $\mathcal{N}(0, \phi)$ are specified for all calibration parameters $a_\ell{}^a$, $b_\ell{}^a$, (and $c_\ell{}^a$).
   (c) The initial All prior values for the calibration parameters are set equal to 0 and the variances are defined in Table C.4
4. Iterative calibration:
   0^{th} step: Initialize the models by setting the *current* parameters equal to the a priori model parameters (Table C.4), derive a simulated set of 'observed' claims $Y_{\ell,o,j}$ from the generative process and aggregate to $Y_{\ell,o}$ on an *annual* level, specify the number of iterations n and initialize the iteration count by setting $i = 0$, and draw K sets with simulated claims $X_{\ell,o,j,k}{}^{(0)}$ from the generative model and aggregate to $X_{\ell,o,k}{}^{(0)}$ on an *annual* level.
   1^{st} step: Fit parametric probability distributions $f^{a}_{\ell,o}{}^{(i)}$ to the simulated distributions $g\left(\left\{X_{\ell,o,k}{}^{(i)}\right\}_{k\in\{1,\ldots,K\}}\right)$.
   2^{nd} step: Use Bayesian inference to evaluate the calibration parameters $a_\ell{}^a_{,MAP}{}^{(i+1)}$, $b_\ell{}^a_{,MAP}{}^{(i+1)}$, and $c_\ell{}^a_{,MAP}{}^{(i+1)}$ for each model feature ℓ to be calibrated.
   3^{rd} step: Update the model parameters $\vartheta^{(i+1)}_{\ell,o}$ by applying the respective calibration parameters $c_{\ell,o}{}^{(i+1)}$, where $c_{\ell,o}{}^{(i+1)} = c_\ell{}^a_{,MAP}{}^{(i+1)}$ in the case of a scale-only calibration, and $c_{\ell,o}{}^{(i+1)} = a_\ell{}^a_{,MAP}{}^{(i+1)} + b_\ell{}^a_{,MAP}{}^{(i+1)} \cdot (o - o_0)$ in the case of a linear-trend calibration.
   4^{th} step: Draw K sets with simulated claims $X_{\ell,o,j,k}{}^{(i+1)}$ from the generative model and aggregate to $X_{\ell,o,k}{}^{(i+1)}$ on an *annual* level.
   5^{th} step: Increase the iteration count $i \rightarrow i + 1$ and shrink the prior parameters ϕ.
   6^{th} step: If $i < n$ repeat steps 1–5 else proceed.
5. Analyze: Compare the simulated distributions $g\left(\left\{X_{\ell,o,k}{}^{(i)}\right\}_{k\in\{1,\ldots,K\}}\right)$ and the fitted distributions $f^{a}_{\ell,o}{}^{(i)}$ with the respective distributions specified for the generative process.

Table C.4: Parameters used for the full calibration.

	Period	$\mathbb{E}[N_o^{inc}]$	$\mathbb{E}[N_o^{clo}]$	$\mathbb{E}[I_o]$	$\mathbb{E}[P_o]$	$\mathbb{E}[\tau_o^{rep}]$	$\mathbb{E}[\tau_o^I]$	$\mathbb{E}[\tau_o^P]$	$\mathbb{E}[N_o^I]$	$\mathbb{E}[N_o^P]$
Generative process	o=-12	10.0		1.0×10^8		4.5	2.5	15.0	5.0	10.0
	o=-1	30.0		1.5×10^8		3.5	−1.5	10.0	8.0	5.0
Generative model	o=-12	20.0		2.0×10^8		3.0	0.0	18.0	10.0	12.0
	o=-1	40.0		2.5×10^8		3.0	0.0	18.0	10.0	12.0
Calibration item		T	F	T	F	T	T	T	T	T
Priors	$\phi_{\ell,a}$	1.00		1.00		1.00	1.00	1.00	1.00	1.00
	$\phi_{\ell,b}$	0.10		0.10		0.10	0.10	0.10	0.10	0.10
	$\phi_{\ell,c}$	1.00		1.00		1.00	1.00	1.00	1.00	1.00

Table D.1: GIRF library with python code and dependencies.

Python module	ID	1	2	3	4	5	6	7	8	9
`GIRF_main.py`	1	o	+	+				+		
`GIRF_reduced.py`	2		o		+			+		
`GIRF_calibrate.py`	3			o	+	+	+	+	+	+
`GIRF_plot.py`	4				o		+	+	+	+
`GIRF_claim.py`	5					o		+		
`GIRF_Bayes.py`	6						o		+	
`GIRF_models.py`	7							o		
`GIRF_stats.py`	8								o	
`Transscript.py`	9									o
External libraries:	sys, numpy, scipy, datetime, matplotlib									

D. Python library GIRF

The GIRF (Generic Integrated Rating Framework) python library comprises the modules listed in Table D.1 (dependencies are indicated by a '+' symbol). The library is used to generate most figures and tables in the printed document and in the online supplementary material.

D.1. Access to GIRF

The GIRF library can be downloaded from GitHub: https://github.com/Steivan/GIRF

D.2. Modules

The most relevant modules for a user are the main module `GIRF_main.py` used to run the various top-level routines and the module `GIRF_models.py` containing the parameters for the various example models.

GIRF_main.py
Following code extract from the `GIRF_main.py` module provides an overview of the routines used to generate the figures and tables:

```
if __name__ == "__main__":

# Generic Integrated Rating Framework (GIRF):
# *******************************************
# - Input : the parameters for the various models are defined in:
#           - GIRF_models.py
# - Output: the output file names (figures and tables) are defined in:
#           - GIRF_models.py / GIRF_fn_dict
#

    selection = [1, 2, 3, 4, 5, 6, 7]

    # Plots 'claims representation and reduced variables'
    # and 'patterns and lags' (Figures 4.1 and 4.2)
    if 1 in selection: patterns_and_red_var()
```

```
# Create default parameters: copy / paste from console to
# module GIRF_models.py
if 2 in selection: get_new_default_param(K_sim=100)

# Four plots 'calibration of the annual observations' and print
# parameters (Figure 6.1 (a)-(d), Table 6.1 = C.1, and stats file for App. C.1)
if 3 in selection: calibrate_claims_count()

# Plots 'conditional' and 'unconditional calibration statistics
# (Figures 6.2 and 6.3)
if 4 in selection: claims_count_stats(N_run=100, K_sim=200, use_default=True)

# Plot 'fitting comparison' (Figure 6.4)
if 5 in selection: calibration_comparison()

# Print model and calibration parameters (Tables 6.2 = C.2 and 6.3 = C.3)
if 6 in selection: model_comparison(N_run=5, K_sim=50, f_P_list=[1.5, 1.0, 0.6])

# Run full calibration model (Figures 6.5 and 6.6 and Table C.4)
if 7 in selection: full_calibration(Nr_iter=5, K_sim=200)
```

GIRF_reduced.py
This module is used to generate the chart depicting the reduced variables and the chart depicting the claims development patterns and the temporal evolution of the lags.

GIRF_calibrate.py
This module contains the routines used to calibrate the example models.

GIRF_plot.py
This module contains the routines used to generate the figures with the results of the simulations and the figures containing analytical results. The module is also used to generate the tables containing the respective parameters.

GIRF_claim.py
This module contains the classes used to represent the reduced variables on a *claim*, an *annual*, and a *period* level. It also contains a routine used to generate reduced claims.

GIRF_Bayes.py
This module contains the routines used to evaluate the calibration parameters with the help of various parametric distributions fitted to the simulated distributions.

GIRF_models.py
This module contains the global parameters and the specific parameters used in the sample models. The dictionary `Red_Fields` is used to assign a parametric distribution family to each reduced variable. The dictionary `GIRF_fn_dict` is used to specify the names and the formats of the output files (Figures and LaTeX tables).

GIRF_stats.py
This module contains some classes which are used as wrappers for the scipy library.

Transscript.py
Module used to redirect the output from the console to a text file (and the console).