On the study of the cumulative residual extropy of mixed used systems and their complexity

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Abstract

In this paper, we use an information theoretic approach called cumulative residual extropy to compare mixed used systems. We establish mixture representations for the cumulative residual extropy of mixed used systems and then explore the measure and comparison results among these systems. We compare the mixed used systems based on stochastic orders and stochastically ordered conditional coefficients vectors. Additionally, we derive bounds for the cumulative residual extropy of mixed used systems with independent and identically distributed components. We also propose the Jensen-cumulative residual extropy divergence to calculate the complexity of systems. To demonstrate the utility of these results, we calculate and compare the cumulative residual extropy and Jensen-cumulative residual extropy divergence of mixed used systems in the Exponential model. Furthermore, we determine the optimal system configuration based on signature under a criterion function derived from Jensen-cumulative residual extropy in the exponential model.

Keywords and Phrases: Cumulative residual extropy, Jensen-cumulative residual extropy, Mixed system, Order statistics, Residual lifetime, Stochastic order, System signature, Used system.

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1 Introduction

A system is considered coherent if all of its components are relevant and its structure-function is monotone. The family of all stochastic mixtures of coherent systems of a given size is called a

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mixed system. For further details, refer to Barlow and Proschan (1981). System signatures are useful tools for studying and comparing engineering systems. The theory of system signature has been explained in Samaniego (2007). Additionally, Samaniego (1985) demonstrated that the lifetime distribution of a coherent system, comprising n independent and identically distributed (iid) components, can be expressed as a function that depends solely on the system design. The signature of a system of order n with component lifetimes Y_i which are iid random variables with the common distribution is an n-dimensional probability vector $\boldsymbol{s} = (s_1, \cdots, s_n)$ whose ith element is $s_i = P(T(n) = Y_{i:n})$, $i = 1, \cdots, n$ and $\sum_{i=1}^{n} s_i = 1$, where $T(n)$ denotes the system's lifetime and $Y_{i:n}$ stands for the lifetime of the ith weakest component. Let $\bar{G}_{T(n)}(t)$ is the survival function of the system with lifetime $T(n)$, then (see, Samaniego, 2007)

$$\bar{G}_{T(n)}(t) = P(T(n) > t) = \sum_{i=1}^{n} s_i P(Y_{i:n} > t). \tag{1}$$

In recent years, researchers have shown increasing interest in studying the reliability properties of coherent systems using a signature vector. Kochar et al. (1999) proposed some applications of the signature vector to compare coherent systems with independent and identically distributed (iid) components. Additionally, Navarro et al. (2005, 2007) utilized the concept of the signature vector to compare coherent systems when the components are not necessarily independent. The concept of residual lifetime is important in reliability theory, and it is widely used in engineering reliability, survival analysis, and economic issues. Let's consider a system with a lifetime of $T(n)$ that starts at time $t = 0$ and is still functioning at time $t > 0$. In this case, the residual lifetime of the system, given that $T(n) > t$, is defined as $T(n) - t | T(n) > t$. In reliability theory, this type of system is referred to as a "used system". The residual lifetime represents the remaining lifetime of a system that has reached an age of t. Navarro et al. (2008) established the representation similar to Equation (1) for the survival function of the residual lifetime $T(n) - t | T(n) > t$ as follows

$$P(T(n) - t > x | T(n) > t) = \sum_{i=1}^{n} p_i(t) P(Y_{i:n} - t > x | Y_{i:n} > t), \tag{2}$$

where $p_i(t) = \frac{s_i P(Y_{i:n} > t)}{P(T(n) > t)}$ for $i = 1, \ldots, n$ is conditional coefficient vector given $T(n) > t$. Navarro et al. (2008) studied the stochastic orders and various properties of Equation (2). One purpose of this paper is to investigate the information properties of mixed used systems from the viewpoint of cumulative residual extropy.

In the field of information theory, entropy measures the uncertainty associated with a random variable. It was introduced by Shannon in 1948. The Shannon entropy of a continuous random variable X with the probability density function (pdf) $f(x)$ is defined as $H(X) = -E(\log f(X))$ where "log" means the natural logarithm. In 2015, Lad et al. proposed a new way to measure the uncertainty of a random variable, which they called extropy. Extropy is used as an opposite concept to entropy and has been mentioned in academic literature before. It represents the intelligence, functional order, vitality, energy, life, experience, capacity, and drive for improvement and growth of a living or organizational system. The extropy of a random variable X with the pdf $f(x)$ is defined by Lad et al. (2015) in the following manner:

$$J(X) = -\frac{1}{2}\int_{-\infty}^{+\infty} f^2(x)dx = -\frac{1}{2}\int_0^1 f(F^{-1}(u))du.$$

Several studies have focused on extropy and its applications in investigating the information properties of reliability systems. Qiu and Jia (2018a) regarded the extropy of a system's residual lifetime as a measure of residual uncertainty. They also derived some properties of this measure in order statistics. Qiu (2017) provided characterization results and symmetric properties of the extropy of order statistics and record values. Additionally, Qiu and Jia (2018b) introduced two estimators for the extropy of an absolutely continuous random variable. For further studies on the information properties of a system's extropy, readers are encouraged to refer to Qiu (2017), Jose and Sathar (2019), Jahanshahi et al. (2019), and Chakraborty and Pradhan (2023).

Cumulative residual entropy (CRE) was first introduced by Rao et al. in 2004 in terms of the survival function of X. Cumulative residual extropy (CRJ) was introduced by Jahanshahi et al. in 2020, and it is defined as

$$\xi J(X) = -\frac{1}{2}\int_0^{\infty} \bar{F}^2(x)\,dx. \tag{3}$$

In their 2023 study, Chakraborty and Pradhan examined the CRJ of coherent and mixed systems using a signature-based approach. When monitoring the system at time t, the system may be in one mode, indicating that it is operational at that time (i.e., $T(n) > t$). If $T(n) > t$, the system's remaining lifetime is given by $T(n) - t|T(n) > t$. This paper investigates the CRJ of mixed used systems by utilizing Equation (2). In this paper, by substituting $t = 0$ into the results of Sections 2 and 3, we obtain some of the findings of Chakraborty and Pradhan's study. This paper aims to extend the findings of Chakraborty and Pradhan (2023) to the conditional system under a specific state condition.

The paper is structured as follows. In Section 2, an expression of the CRJ of the mixed used system is provided in terms of the conditional coefficients vector. Additionally, based on this representation, some stochastic comparisons and bounds have been obtained. A new divergence measure is developed to study the complexity of the mixed used system in Section 3. In Section

4, the CRJ of the mixed used systems are compared to each other under the exponential system consisting of three iid components for some configurations between the system components. Finally, in Section 5, the optimal signature vector has been obtained for simultaneously minimizing the Jensen-cumulative residual extropy divergence of mixed used systems and minimizing costs. Some concluding remarks are made in Section 6.

2 CRJ of mixed used systems

Let $Y_1, \dots, Y_n$ be iid component lifetimes with the cumulative distribution function (cdf) G of a coherent system of order n with the signature vector $\boldsymbol{s}$. Let the random variable $T(n)$ be the lifetime of the aforementioned system with the pdf $g_{T(n)}(.)$ and the cdf $G_{T(n)}(.)$. Also, assume that the system started at time $t = 0$ and it is alive at time $t > 0$. We are interested in computing the extropy of the used system or, equivalently, computing the extropy of residual lifetime $T(n) - t|T(n) > t$. For convenience of notation, let us denote $T(n) - t|T(n) > t$, $Y_{i:n} - t > x|Y_{i:n} > t$ and $Y_i - t|Y_i > t$, $i = 1, \dots, n$ by $T(n,t)$, $Y_{i:n,t}$ and $Y_{i,t}$, respectively in the rest of the paper. Also, assume that the pdf and cdf of $T(n,t)$ $\Big($or $Y_{i:n,t}\Big)$ are $g_{T(n,t)}(.)$ and $G_{T(n,t)}(.)$ $\Big($or $g_{i:n,t}(.)$ and $G_{i:n,t}(.)\Big)$, respectively. From (2) and Navarro et al. (2008), we have

$$\bar{G}_{T(n,t)}(x) = \sum_{i=1}^{n} p_i(t)\bar{G}_{i:n,t}(x), \tag{4}$$

and

$$g_{T(n,t)}(x) = \sum_{i=1}^{n} p_i(t) g_{i:n,t}(x),$$

where $p_i(t) = \frac{s_i P(Y_{i:n} > t)}{\bar{G}_{T(n)}(t)}$ for $i = 1, \dots, n$ and $\bar{G}_{T(n,t)}(x) = 1 - G_{T(n,t)}(x)$ and $\bar{G}_{T(n)}(t) = 1 - G_{T(n)}(t)$ are survival functions of systems with lifetimes $T(n,t)$ and $T(n)$, respectively. In this section, we obtain an expression for the CRJ of mixed used system's lifetime with the given signature vector $\boldsymbol{s}$. From (3) and (4), the CRJ of $T(n,t)$ is given by

$$\begin{aligned}
\xi J(T(n,t)) &= -\frac{1}{2}\int_0^{+\infty} \bar{G}^2_{T(n,t)}(x)dx \\
&= -\frac{1}{2}\int_t^{+\infty} \left(\sum_{i=1}^{n} p_i(t)\frac{\sum_{j=0}^{i-1}\binom{n}{j}G(x)^j\bar{G}(x)^{n-j}}{\bar{G}_{i:n}(t)}\right)^2 dx \\
&= -\frac{1}{2}\int_{G(t)}^{1} \frac{\left(\sum_{i=1}^{n} p_i(t)\frac{\sum_{j=0}^{i-1}\binom{n}{j}v^j(1-v)^{n-j}}{\bar{G}_{i:n}(t)}\right)^2}{g(G^{-1}(v))} dv \\
&= -\frac{1}{2}\int_{G(t)}^{1} \frac{\bar{G}^2_{V_t}(v)}{g(G^{-1}(v))}dv, \qquad (5)
\end{aligned}$$

where

$$\bar{G}_{V_t}(v) = \sum_{i=1}^{n} p_i(t)\bar{G}_{V_{t,i}}(v),$$

and

$$\bar{G}_{V_{t,i}}(v) = \sum_{j=0}^{i-1}\binom{n}{j}\frac{v^i(1-v)^{n-i}}{\bar{G}_{i:n}(t)}, \quad G(t) < v < 1.$$

In fact, for $i = 1, \ldots, n$, if we use the probability integral transformation $U_i = G(Y_{i,t})$, then the associated order statistics $V_{t,i} = G(Y_{i:n,t})$ has the pdf $g_{V_{t,i}}(v)$. Therefore, the survival function of $V_t = G(T(n,t))$ is $\bar{G}_{V_t}(v)$. In the following, we present alternative representations of $\xi J(T(n,t))$, which will be used throughout the paper. Equation (5) can be expressed as

$$\xi J(T(n,t)) = -\frac{1}{2\bar{G}^2_{T(n)}(t)}\int_{G(t)}^{1} \frac{\bar{G}^2_V(v)}{g(G^{-1}(v))}dv, \qquad (6)$$

where,

$$\bar{G}_V(v) = \sum_{i=1}^{n} s_i\bar{G}_{V_i}(v),$$

and

$$\bar{G}_{V_i}(v) = \sum_{j=0}^{i-1}\binom{n}{j}v^j(1-v)^{n-j}.$$

Also, expression (5) can be rewritten as

$$\xi J(T(n,t)) = -\frac{1}{2\bar{G}(t)\bar{G}^2_{T(n)}(t)} E\Big[\frac{\bar{G}^2_V(U_t)}{g(G^{-1}(U_t))}\Big], \tag{7}$$

where, U_t be a random variable uniformly distributed on $(G(t), 1)$ with the pdf $g_t(v) = \frac{1}{\bar{G}(t)}$ and the cdf $G_t(v) = \frac{v - G(t)}{\bar{G}(t)}$. To compare the uncertainty between two systems, we can use the measure $\xi J(T(n,t))$ in Equations (5), (6), and (7). For two mixed used systems with lifetimes $T_1(n,t)$ and $T_2(n,t)$, if $\xi J(T_1(n,t))$ is less than or equal to $\xi J(T_2(n,t))$, then $T_1(n,t)$ is more uncertain than $T_2(n,t)$ (see Qiu et al., 2019). Equation (5) can be used to compare the CRJ of two mixed used systems using the vector of coefficient $\boldsymbol{p}(t) = (p_1(t), \cdots, p_n(t))$. Equation (6) can be used to compare the CRJ of two mixed used systems using the signature vector $\boldsymbol{s} = (s_1, \cdots, s_n)$. In the following, we provide example to illustrate how to calculate CRJ for mixed used systems using Equation (6). In Section 4, we utilize the exponential model to demonstrate how to calculate CRJ for mixed used systems using Equation (5).

Example 1 *Suppose we have a mixed used system with the signature $\boldsymbol{s} = (\frac{1}{2}, 0, \cdots, 0, \frac{1}{2})$ and it consists of n gamma distributed components with pdf $g_Y(y) = \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y}$, $y > 0$ and cdf $G_Y(y) = \frac{\gamma(\alpha, \lambda y)}{\Gamma(\alpha)}$, where $\Gamma(.)$ is the complete gamma function and $\gamma(\alpha, \lambda y)$ is the incomplete gamma function. This system represents a uniform mixture of n-component series and parallel systems. The system is created by selecting n-component series and parallel systems with a probability of $\frac{1}{2}$. We have*

$$\begin{aligned}
\bar{G}_V(v) &= \sum_{i=1}^{n} s_i \bar{G}_{V_i}(v) \\
&= \frac{1}{2}\left[1 - v^n + (1-v)^n\right], \\
\bar{G}_{T(n)}(t) &= \sum_{i=1}^{n} s_i P(Y_{i:n} > t) \\
&= \frac{1}{2}\left[1 + \left(1 - \frac{\gamma(\alpha, \lambda t)}{\Gamma(\alpha)}\right)^n - \left(\frac{\gamma(\alpha, \lambda t)}{\Gamma(\alpha)}\right)^n\right],
\end{aligned}$$

and $g(G^{-1}(v)) = \frac{\lambda}{\Gamma(\alpha)} \left(\gamma^{-1}(\alpha, \Gamma(\alpha)v)\right)^{\alpha-1} e^{-\gamma^{-1}(\alpha, \Gamma(\alpha)v)}$, where $\gamma^{-1}(\alpha, \Gamma(\alpha)v)$ is the inverse of the incomplete gamma function $\gamma(\alpha, \Gamma(\alpha)v)$. Therefore, from Equation (6), we obtain

$$\xi J(T(n,t)) = -\frac{\Gamma(\alpha)}{2\lambda\left[1 + \left(1 - \frac{\gamma(\alpha,\lambda t)}{\Gamma(\alpha)}\right)^n - \left(\frac{\gamma(\alpha,\lambda t)}{\Gamma(\alpha)}\right)^n\right]^2} \int_{\frac{\gamma(\alpha,\lambda t)}{\Gamma(\alpha)}}^{1} \frac{\left[1 - v^n + (1-v)^n\right]^2}{\left(\gamma^{-1}(\alpha, \Gamma(\alpha)v)\right)^{\alpha-1} e^{-\gamma^{-1}(\alpha, \Gamma(\alpha)v)}} dv.$$

(8)

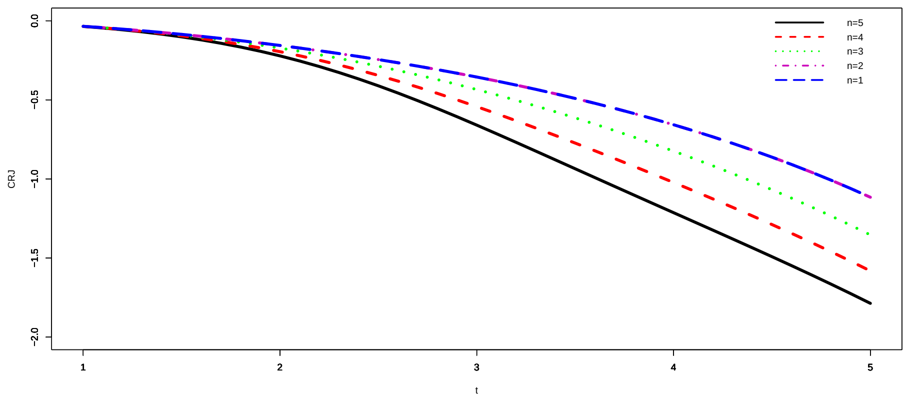

Figure 1: Plot of $\xi J(T(n,t))$ in Example 1.

In Figure 1, we plot the CRJ of Example 1 for $\lambda = \frac{2}{3}$, $\alpha = 3$, and different values of n. We observe that the CRJ of the mixed used system decreases as the number of components increases. This is natural because a system with minimum CRJ is better. According to the monotonicity of coherent systems, increasing the number of components will improve the reliability of the system.

Based on Equations (5), (6), and (7), we can analyze the CRJ measure for coherent and mixed used systems and develop some comparison results among systems. First, we recall the definition of some stochastic orders; see Shaked and Shanthikumar (2007) or Belzunce et al. (2015).

Definition 1 (Stochastic orders) *Let X and Y be two random variables with cdfs F and G and pdfs f and g, respectively. Then X is said to be smaller than Y in the sense of:*
(i) usual stochastic order (denoted by $X \leqslant_{st} Y$ or $F \leqslant_{st} G$) if $\bar{F}(x) \leq \bar{G}(x)$ for all x;
(ii) hazard rate order (denoted by $X \leqslant_{hr} Y$ or $F \leqslant_{hr} G$) if $\dfrac{\bar{G}(x)}{\bar{F}(x)}$ is increasing in x.
(iii) disperse order (denoted by $X \leqslant_{disp} Y$ or $F \leqslant_{disp} G$) if $g(G^{-1}(v)) \leq f(F^{-1}(v))$ for all $0 < v < 1$.
Also, let $\boldsymbol{p}$ and $\boldsymbol{q}$ be two discrete distributions on the integers $\{1, ..., n\}$. Then, it is said that (see, for example, Kochar et al. 1999)
(iv) $\mathbf{p} \leqslant_{st} \mathbf{q}$ if and only if $\sum_{i=j}^{n} p_i \leq \sum_{i=j}^{n} q_i$, for $j = 1, \cdots, n$.
(v) $\mathbf{p} \leqslant_{hr} \mathbf{q}$ if and only if $\sum_{i=j}^{n} p_i \Big/ \sum_{i=j}^{n} q_i$ is decreasing in j, for $j = 1, \cdots, n$.
(vi) $\mathbf{p} \leqslant_{lr} \mathbf{q}$ if and only if p_i/q_i is decreasing in i, for $i = 1, \cdots, n$ when $p_i, q_i > 0$.

In what follows, let us denote $\mathcal{M}_{T(n,t),Y,G,\boldsymbol{p}(t)} = \{T(n,t), Y, G(.), \boldsymbol{p}(t)\}$ as the model function of the system associated with the iid components lifetimes $Y_1, \ldots, Y_n$ with the common cdf G of the system with lifetime $T(n,t)$ where, $\boldsymbol{p}(t) = (p_1(t), p_2(t), \ldots, p_n(t))$ is the vector of coefficients in (4). Also, assume that the random variable Y is one of $Y_1, \ldots, Y_n$. In the next result, we aim to explore the behavior of CRJ based on t, and we want to obtain the conditions under which the uncertainty of the mixed used system decreases over time.

Result 1 *Let $\mathcal{M}_{T(n,t),Y,G,\boldsymbol{p}(t)}$ be the model function of a mixed used system. If $\frac{\bar{G}_V^2(v)}{g(G^{-1}(v))}$ is increasing in $v \geq 0$, then $\xi J(T(n,t))$ is decreasing in $t \geq 0$.*

Proof. Assume that $0 \leq t_1 \leq t_2$. For $G(t_1) \leq v \leq G(t_2)$ and $G(t_2) \leq v \leq 1$, we have $\frac{g_{t_1}(v)}{g_{t_2}(v)} = \infty$, $i = 1, \ldots, n$ and $\frac{g_{t_1}(v)}{g_{t_2}(v)} = \frac{\bar{G}(t_2)}{\bar{G}(t_1)}$, $i = 1, \ldots, n$, respectively. Therefore, we conclude that $U_{t_1} \leqslant_{lr} U_{t_2}$. Accordingly, we have $U_{t_1} \leqslant_{st} U_{t_2}$. Since $\frac{\bar{G}_V^2(v)}{g(G^{-1}(v))}$ is increasing, we have $E\left[\frac{\bar{G}_V^2(U_{t_1})}{g(G^{-1}(U_{t_1}))}\right] \leq E\left[\frac{\bar{G}_V^2(U_{t_2})}{g(G^{-1}(U_{t_2}))}\right]$. Since $0 < \frac{1}{\bar{G}(t_1)} \leq \frac{1}{\bar{G}(t_2)}$ and $0 < \frac{1}{\bar{G}_{T(n)}(t_1)} \leq \frac{1}{\bar{G}_{T(n)}(t_2)}$, we conclude the desired result by noting that

$$\begin{aligned}
\xi J(T(n,t_1)) &= -\frac{1}{2\bar{G}(t_1)\bar{G}^2_{T(n)}(t_1)} E\Big[\frac{\bar{G}_V^2(U_{t_1})}{g(G^{-1}(U_{t_1}))}\Big] \\
&\geq -\frac{1}{2\bar{G}(t_2)\bar{G}^2_{T(n)}(t_2)} E\Big[\frac{\bar{G}_V^2(U_{t_2})}{g(G^{-1}(U_{t_2}))}\Big] = \xi J(T(n,t_2)).
\end{aligned}$$

□

The condition in Result 1 that $\frac{\bar{G}_V^2(v)}{g(G^{-1}(v))}$ is increasing in v is sufficient but not necessary. We show this problem in the following example.

Example 2 *Under the assumption of Example 1 and aforementioned parameter values, we plot $H(v) = \frac{\bar{G}_V^2(v)}{g(G^{-1}(v))}$ with respect to v in Figure 2. We see that the function $H(v)$ is decreasing in $v \in (0,1)$. However, we know from Figure 1 that $\xi J(T(n,t))$ is decreasing in t. So, the condition in Result 1 that $\frac{\bar{G}_V^2(v)}{g(G^{-1}(v))}$ is increasing in v is sufficient but not necessary.*

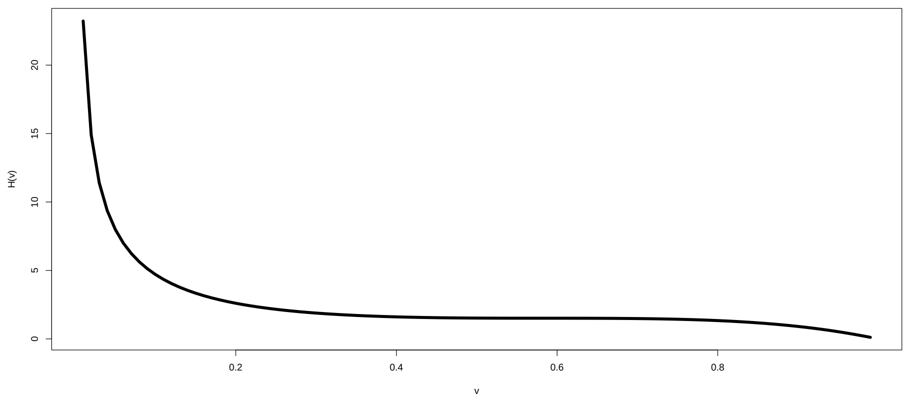

Figure 2: Plot of $H(v) = \frac{\bar{G}_V^2(v)}{g(G^{-1}(v))}$ in Example 2.

In the following, we give an example to show that the condition "$\frac{\bar{G}_V^2(v)}{g(G^{-1}(v))}$ is increasing in v" in Result 1 can be satisfied.

Example 3 *Under the assumption of Example 1 for parameter values $\alpha = 0.01$, $\lambda = 0.1$, and $n = 10$, we plot $H(v) = \frac{\bar{G}_V^2(v)}{g(G^{-1}(v))}$ with respect to v in Figure 3. We see that the function $H(v)$ is increasing in $v \in (0, 1)$. However, from Figure 4, we see that $\xi J(T(n,t))$ is decreasing in t. So, the condition in Result 1 is satisfied.*

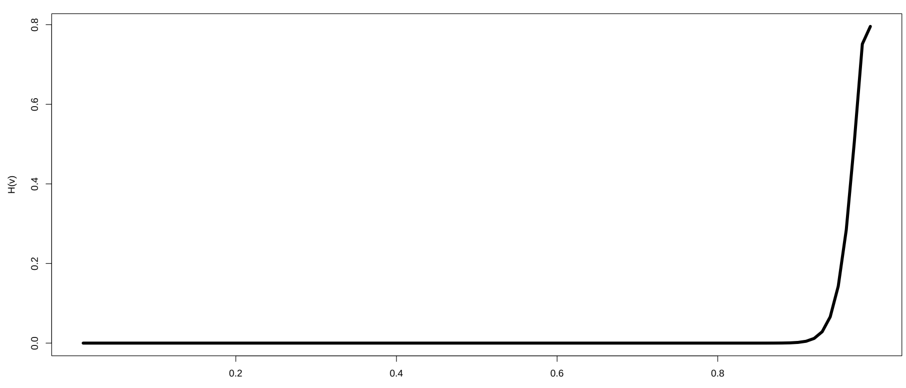

Figure 3: Plot of $H(v) = \frac{\bar{G}_V^2(v)}{g(G^{-1}(v))}$ in Example 3.

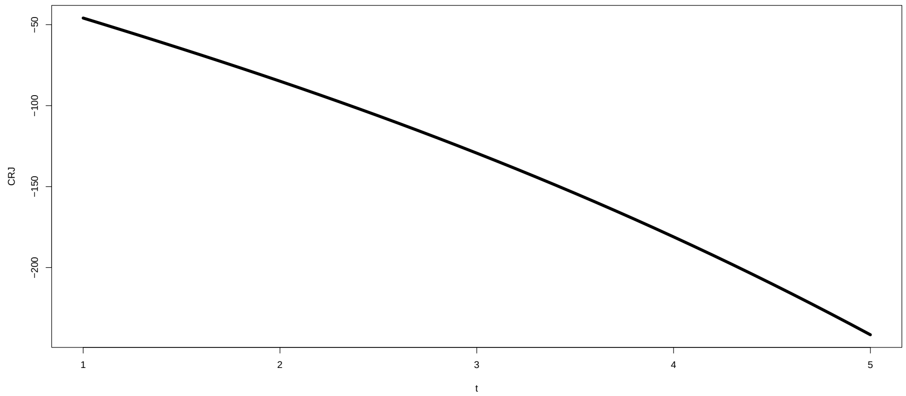

Figure 4: Plot of $\xi J(T(n,t))$ in Example 3.

The most important and common stochastic order that is studied the "variability" or the "dispersion" of random variables is the dispersive order. In fact, dispersive order is used to compare the variability in probability distributions (Jeon et al., 2006; Kochar, 2012; Shaked and Shanthikumar, 2007). It is connected to the concept of log-concavity, and several authors have studied the log-concavity properties of ordered random variables. For a comprehensive survey, we refer the reader to Kochar and Korwar (1996), Dykstra et al. (1997), and Khaledi and Kochar (2000). In the following result, we demonstrate that increasing the variability of the baseline cdf of the components in a mixed used system leads to make inequality between the CRJ in mixed used systems.

Result 2 *Let $\mathcal{M}_{T^Y(n,t),Y,G,\boldsymbol{p}^{\boldsymbol{Y}}(t)}$ and $\mathcal{M}_{T^X(n,t),X,F,\boldsymbol{p}^X(t)}$ be two model functions of two mixed used systems with the same signature vector $\boldsymbol{s} = (s_1, ..., s_n)$. Denote the supports of X and Y by S_X and S_Y, respectively. Let $l_X = \inf\{x : x \in S_X\}$ and $u_X = \sup\{x : x \in S_X\}$. Define l_Y and u_Y similarly for random variable Y. If $X \leqslant_{disp} Y$ and $l_X = l_Y > -\infty$, then $\frac{\bar{G}^2_{T^X(n)}(t)}{\bar{G}^2_{T^Y(n)}(t)}\xi J(T^X(n,t)) \geq \xi J(T^Y(n,t))$.*

Proof. Since $X \leqslant_{disp} Y$ from Theorem 3.B.13 (a) of Shaked and Shanthikumar (2007), we have $X \leqslant_{st} Y$. Therefore, we concluded that

$$\int_{G(t)}^1 \frac{\bar{G}^2_V(v)}{g(G^{-1}(v))}dv \geq \int_{F(t)}^1 \frac{\bar{G}^2_V(v)}{f(F^{-1}(v))}dv.$$

Therefore, we have

$$\frac{1}{\bar{G}^2_{T^Y(n)}(t)} \int_{G(t)}^1 \frac{\bar{G}^2_V(v)}{g(G^{-1}(v))}dv \geq \frac{\bar{G}^2_{T^X(n)}(t)}{\bar{G}^2_{T^Y(n)}(t)} \cdot \frac{1}{\bar{G}^2_{T^X(n)}(t)} \int_{F(t)}^1 \frac{\bar{G}^2_V(v)}{f(F^{-1}(v))}dv.$$

By using Equation (6), the proof is complete. It should be noted that, if $X \leqslant_{disp} Y$, Theorem 3.B.26 in Shaked and Shanthikumar (2007) claims that $X_{i:n} \leqslant_{disp} Y_{i:n}$, so from Theorem 3.B.13 (a) of Shaked and Shanthikumar (2007), we have $X_{i:n} \leqslant_{st} Y_{i:n}$. Therefore, we have $\frac{\bar{G}^2_{T^X(n)}(t)}{\bar{G}^2_{T^Y(n)}(t)} \leq 1$.
□

In the following theorems, we compare the CRJ of two mixed used system based on vector of coefficient $\boldsymbol{p}(t) = (p_1(t), \cdots, p_n(t))$ and properties of order statistics of their components.

Theorem 1 *Let $\mathcal{M}_{T^Y(n,t),Y,G,\boldsymbol{p}^Y(t)}$ and $\mathcal{M}_{T^X(n,t),X,F,\boldsymbol{p}^X(t)}$ be the two model functions of the two mixed used systems.*

(i) If $G \leqslant_{hr} F$ and $\boldsymbol{p}^Y(t) \leqslant_{st} \boldsymbol{p}^X(t)$ then $\xi J(T^Y(n,t)) \geq \xi J(T^X(n,t))$.

(ii) If $G \leqslant_{hr} F$ and $\boldsymbol{p}^Y(t) \leqslant_{hr} \boldsymbol{p}^X(t)$ then $\xi J(T^Y(n,t)) \geq \xi J(T^X(n,t))$.

(iii) If F and G are absolutely continuous, $\boldsymbol{p}^Y(t) \leqslant_{lr} \boldsymbol{p}^X(t)$ then $\xi J(T^Y(n,t)) \geq \xi J(T^X(n,t))$.

Proof. The results can be immediately obtained from Theorem 2.3 (a), (b), and (c) of Navarro et al. (2008). □

Theorem 2 *Let $\mathcal{M}_{T(n,t),Y,G,\boldsymbol{p}(t)}$ be a model functions of the mixed used system.*

(i) If $Y_{i:n} \leqslant_{st} (Y_{i:n} - t | Y_{i:n} > t)$, then $\xi J(T(n)) \geq \xi J(T(n,t))$.

(ii) If $(Y_{i:n} - t_1 | Y_{i:n} > t_1) \leqslant_{st} (Y_{i:n} - t_2) | Y_{i:n} > t_2)$, then $\xi J(T(n,t_1)) \geq \xi J(T(n,t_2))$.

Proof. The results follow immediately from Theorem 2.4. (a) and (b) of Navarro et al. (2008).
□

Result 3 *Let $\mathcal{M}_{T(n,t),Y,G,\boldsymbol{p}(t)}$ be the model function of a mixed used system. If Y is DFR, then $\frac{\bar{G}^{2n}(t)}{\bar{G}^2_{T(n)}(t)} \xi J(Y_{1:n,t}) \geq \xi J(T(n,t))$.*

Proof. Bagai and Kochar (1986) stated that $X \leqslant_{disp} Y$ if $X \leqslant_{hr} Y$ and either X or Y is DFR. From this fact and Result 2, it is easy to prove Result 3. It is easy to see that $Y_{1:n,t} \leqslant_{hr} T(n,t)$ by Theorem 2.1 (b) of Navarro et al. (2008). Since $Y_{1:n,t}$ is DFR provided that Y is DFR, $\frac{\bar{G}^{2n}(t)}{\bar{G}^2_{T(n)}(t)} \xi J(Y_{1:n,t}) \geq \xi J(T(n,t))$ by Result 2. □

Computing the exact information of the system can be challenging due to the large number of components and its complicated structure. Therefore, it is important to have bounds for the CRJ of the mixed used system to approximate its behaviour in complex situations. Following theorems establish bounds on the CRJ of the mixed used system.

Theorem 3 *Let $\xi J(T(n,t))$ be the CRJ of a mixed used system with the model function $\mathcal{M}_{T(n,t),Y,G,\boldsymbol{p}(t)}$. Then*

$$\xi J(Y_t)\frac{\bar{G}^2(t)}{\bar{G}^2_{T(n)}(t)}M^2_Y(t) < \xi J(T(n,t)) < \xi J(Y_t)\frac{\bar{G}^2(t)}{\bar{G}^2_{T(n)}(t)}m^2_Y(t),$$

where $m_Y(t) = \inf_{G(t)<v<1} \frac{G_V(v)}{(1-v)^2}$ and $M_Y(t) = \sup_{G(t)<v<1} \frac{G_V(v)}{(1-v)^2}$.

Proof. By assumptions, we obtain

$$\frac{(1-v)^2 m^2_Y(t)}{g(G^{-1}(v))} \leq \frac{G^2_V(v)}{g(G^{-1}(v))} \leq \frac{(1-v)^2 M^2_Y(t)}{g(G^{-1}(v))}.$$

By using some computations, we conclude that

$$-\frac{M^2_Y(t)}{2\bar{G}^2_{T(n)}(t)}\int_{G(t)}^1 \frac{(1-v)^2}{g(G^{-1}(v))}dv \leq -\frac{1}{2\bar{G}^2_{T(n)}(t)}\int_{G(t)}^1 \frac{G^2_V(v)}{g(G^{-1}(v))}dv \leq -\frac{m^2_Y(t)}{2\bar{G}^2_{T(n)}(t)}\int_{G(t)}^1 \frac{(1-v)^2}{g(G^{-1}(v))}dv.$$

Hence, from Equation (6), the proof is complete. □

Next, we will derive a different lower bound for the CRJ of the mixed used system's lifetime compared to Theorem 3. We obtain the lower bound in the following theorem in terms of the CRJ of the r-out-of-n system's lifetimes.

Theorem 4 *Let $\xi J(T(n,t))$ be the extropy of a mixed used system with the model function $\mathcal{M}_{T(n,t),Y,G,\boldsymbol{p}(t)}$. Then*

$$\xi J(T(n,t)) \geq \sum_{i=1}^{n} p_i(t)\xi J(Y_{i:n,t}).$$

The equality holds for i-out-of-n used systems, $i = 1, \ldots, n$.

Proof. Applying Jensen's inequality in (5), we get

$$\begin{aligned} \xi J(T(n,t)) &\geq -\frac{1}{2}\int_{G(t)}^1 \frac{\sum_{i=1}^{n} p_i(t)G^2_{V_{t,i}}(v)}{g(G^{-1}(v))}dv \\ &= \sum_{i=1}^{n} p_i(t)\xi J(Y_{i:n,t}). \end{aligned}$$

Equality holds for a k-out-of-n systems since $p_i(t) = 0$ for all $i \neq n-k+1$ and $p_i(t) = 1$ for $i = n-k+1$. □

In the following example, the comparisons of lower bounds in Theorems 3 and 4 are carried out under the exponential model.

Example 4 *Consider a system with lifetime* $T(4) = \min(Y_{2:3}, Y_4)$ *and with the signature vector* $\boldsymbol{s} = (\frac{1}{4}, \frac{3}{4}, 0, 0)$. *Suppose that the components of the system follow the exponential distribution with mean* 0.25. *It is easy to show that,* $\frac{G_V(v)}{v^2} = \frac{1-(1-v)^4-3v(1-v)^3}{v^2}$ *is decreasing function of* v. *Hence,* $M_Y(t) = \frac{G_V(1-e^{-4t})}{(1-e^{-4t})^2}$. *The lower bounds for* $\xi J(T(4,t))$ *from Theorems 3 and 4 are given by* $L_1 = -\frac{(2-(1-e^{-4t})^2+2-e^{-4t})^2}{16e^{-16t}(3-2e^{-4t})^2}$ *and* $L_2 = p_1(t)\xi J(Y_{1:4,t}) + p_2(t)\xi J(Y_{2:4,t})$, *respectively, where* $p_1(t) = \frac{e^{-4t}}{4(3-2e^{-4t})}$ *and* $p_2(t) = \frac{3(4-3e^{-4t}}{4(3-2e^{-4t})}$. *We study the behaviour of* L_1, L_2, *and* $\xi J(T(4,t))$ *based on* t *in Figure 5. We observe that* L_2 *performs well than* L_1.

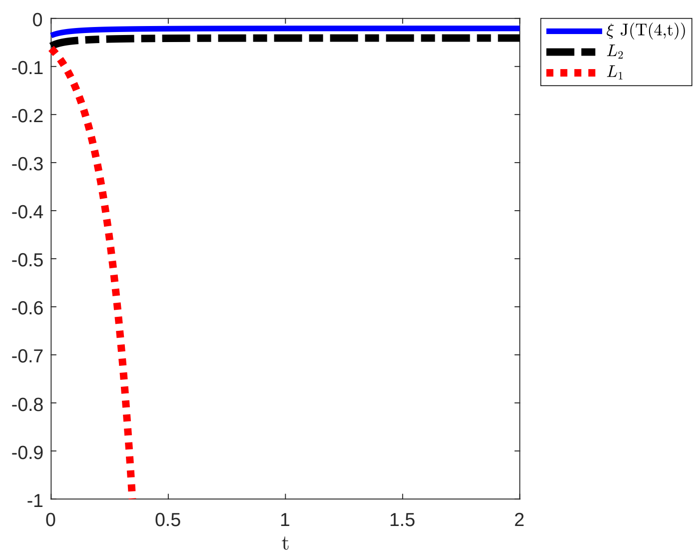

Figure 5: Exact value of $\xi J(T(4,t))$ and lower bounds L_1 and L_2 for the system with lifetime $T(4) = \min(Y_{2:3}, Y_4)$ in exponential model.

3 Jensen-cumulative residual extropy divergence and complexity of mixed used systems

We propose a new divergence measure using the CRJ of a mixed-use system consisting of iid components. As mentioned earlier, one important application of information measures in reliability engineering is to measure the complexity of the system. To address this issue, Asadi et al. (2016) proposed the Jensen-Shannon (JS) divergence between the system with lifetime T and $Y_{1:n}, \ldots, Y_{n:n}$ as

$$JS(T : Y_{1:n}, \ldots, Y_{n:n}) = H(T) - \sum_{i=1}^{n} s_i H(Y_{i:n}), \tag{9}$$

where $H(T)$ is the Shannon entropy of T. This measure compares the system entropy with its components entropies and it is zero for the k-out-of-n systems. This property helps us to study the complexity of systems as higher values of $JS(T : Y_{1:n}, \ldots, Y_{n:n})$ will imply that the n-component system T is more complex than the k-out-of-n consisting of same type of components. Analogous to JS divergence, Qiu et al. (2019) defined the Jensen–Extropy (JJ) divergence between system T and $Y_{1:n}, \ldots, Y_{n:n}$ as

$$JJ(T : Y_{1:n}, \ldots, Y_{n:n}) = J(T) - \sum_{i=1}^{n} s_i J(Y_{i:n}), \tag{10}$$

where $J(T)$ is the extropy of T. By analogy of (9) and (10), Chakrabortyand and Pradhan (2023) proposed the Jensen-cumulative residual extropy (JCRJ) divergence in terms of the CRJ function. Chakrabortyand and Pradhan (2023) defined the JCRJ divergence between T and $Y_{1:n}, \ldots, Y_{n:n}$ as

$$JCRJ(T : Y_{1:n}, \ldots, Y_{n:n}) = \xi J(T) - \sum_{i=1}^{n} s_i \xi J(Y_{i:n}). \tag{11}$$

Analogous to (11), we propose the JCRJ divergence between a mixed used system with lifetime $T(n, t)$ and $Y_{1:n,t}, \ldots, Y_{n:n,t}$ as

$$JCRJ(T(n, t) : Y_{1:n,t}, \ldots, Y_{n:n,t}) = \xi J(T(n, t)) - \sum_{i=1}^{n} p_i(t) \xi J(Y_{i:n,t}). \tag{12}$$

We can present (12) as

$$JCRJ(T(n, t) : Y_{1:n,t}, \ldots, Y_{n:n,t}) = -\frac{1}{2} \int_{G(t)}^{1} \frac{\bar{G}^2_{V_t}(v) - \sum\limits_{i=1}^{n} p_i(t) \bar{G}^2_{V_{t,i}}(v)}{g(G^{-1}(v))} dv. \tag{13}$$

Also, we can rewrite (13) as

$$JCRJ(T(n, t) : Y_{1:n,t}, \ldots, Y_{n:n,t}) = -\frac{1}{2\bar{G}^2_{T(n)}(t)} \int_{G(t)}^{1} \frac{\bar{G}^2_{V}(v) - \sum\limits_{i=1}^{n} \frac{\bar{G}^2_{V_i}(v)}{p_i(t)}}{g(G^{-1}(v))} dv.$$

Like JS and JJ divergence measures, JCRJ divergence is non-negative and from Theorem 4, we see that $JCRJ(T(n, t) : Y_{1:n,t}, \ldots, Y_{n:n,t}) = 0$ for k-out-of-n used systems. JCRJ divergence measures the complexity of a mixed used system in comparison with k-out-of-n system having used same iid components with cdf $\frac{\bar{F}(x+t)}{\bar{F}(t)}$.

Now, we introduce the relative CRJ measure, which is similar to the relative extropy measure introduced by Lad et al. (2015). This measure will be used to compare two systems. The relative

CRJ for two non-negative random variables X and Y with survival functions $\bar{F}(x)$ and $\bar{G}(x)$ is defined as follows

$$R(X:Y) = \frac{1}{2}\int_0^{\infty} (\bar{F}(x) - \bar{G}(x))^2 \, dx. \tag{14}$$

The formula of relative CRJ measure calculates the squared difference between the survival functions $\bar{F}(x)$ and $\bar{G}(x)$ over all time points from zero to infinity, integrating these differences and then normalizing by a factor of $\frac{1}{2}$. Relative CRJ measures the divergence between the two survival functions over time. This measure is used to compare the residual life distributions of two systems or components, providing insight into how much they differ in terms of their reliability or survival characteristics. The Kullback-Leibler divergence is focused on probability density functions rather than survival functions, which makes it more suitable for capturing overall distributional differences rather than tail behavior. Jensen-Shannon divergence is more sensitive to differences in the entire distribution but may not provide detailed insights into survival differences as relative CRJ does. Relative CRJ specifically measures the relative difference between two distributions, making it more suitable for comparative analysis. The relative CRJ can be viewed as a generalization or an extension of other distance measures between distributions (such as the Kullback-Leibler divergence or the Hellinger distance), but specifically tailored to survival functions. We refer the reader to Saranya and Sunoj (2024) and Kharazmi and Balakrishnan (2023). In risk analysis, particularly for financial systems or insurance, relative CRJ can help in assessing the risk of extreme events (e.g., rare but catastrophic failures) and comparing the residual risks between different scenarios or portfolios. It can be particularly useful in stress testing and scenario analysis, where the tail behavior of distributions is critical. See Artzner et al. (1999) for more details. Note that the relative CRJ measure is a scale transformation of the energy distance between two non-negative random variables. JCRJ divergence can be expressed in terms of the relative CRJ.

Proposition 1 *The JCRJ divergence between the system T and $Y_{1:n}, \ldots, Y_{n:n}$ is given by*

$$JCRJ(T(n,t) : Y_{1:n,t}, \ldots, Y_{n:n,t}) = \sum_{i=1}^{n} p_i(t) R(T(n,t) : Y_{i:n,t}). \tag{15}$$

Proof. From (14), we have

$$\begin{aligned}
\sum_{i=1}^{n} p_i(t) R(T(n,t) : Y_{i:n,t}) &= \frac{1}{2}\sum_{i=1}^{n} p_i(t) \int_t^{\infty} \left(\bar{G}_{T(n,t)}(x) - \bar{G}_{i:n,t}(x)\right)^2 dx \\
&= \sum_{i=1}^{n} p_i(t) \int_t^{\infty} \left(\sum_{i=1}^{n} p_i(t)\bar{G}_{i:n,t}(x) - \bar{G}_{i:n,t}(x)\right)^2 dx \\
&= \sum_{i=1}^{n} p_i(t) \int_{G(t)}^{1} \frac{\left(\bar{G}_{V_t}(v) - \bar{G}_{V_{t,i}}(v)\right)^2}{g(G^{-1}(v)} dv \\
&= \sum_{i=1}^{n} p_i(t) \int_{G(t)}^{1} \frac{\left(\bar{G}^2_{V_t}(v) - 2\bar{G}_{V_t}(v)\bar{G}_{V_{t,i}}(v) + \bar{G}^2_{V_{t,i}}(v)\right)}{g(G^{-1}(v)} dv \\
&= -\frac{1}{2}\int_{G(t)}^{1} \frac{\bar{G}^2_{V_t}(v) - \sum_{i=1}^{n} p_i(t)\bar{G}^2_{V_{t,i}}(v)}{g(G^{-1}(v)} dv \\
&= JCRJ(T(n,t) : Y_{1:n,t}, \ldots, Y_{n:n,t}).
\end{aligned}$$

□

The JCRJ divergence of a mixed used system is determined by its signature vector $\boldsymbol{s}$, vector of the coefficient $\boldsymbol{p}(t)$, and the common cdf of the lifetimes of its components. Proposition 1 establishes that the JCRJ divergence is always non-negative and that the minimum value is attained by k-out-of-n used systems. The inequality $JCRJ(T(n,t) : Y_{1:n,t}, \ldots, Y_{n:n,t}) \geq 0$ measures how much more complex a mixed used system with signature vector $\boldsymbol{s}$ and vector of the coefficient $\boldsymbol{p}(t)$ is in comparison to an k-out-of-n used system with homogeneous components. Therefore, the JCRJ divergence can be used as an alternative information criterion for comparing mixed used systems with homogeneous components.

4 Exponential Model

In this section, we investigate the CRJ of the exponential mixed used system of order n. Let the components of mixed used system have exponential distribution with survival function $\bar{G}(y) = e^{-\beta y}$, where β is positive constant. From the Corollary 2.1 of Navarro et al. (2008), for the mixed used system of order n with iid exponential components, we have

$$\bar{G}_{T(n,t)}(t) = \sum_{i=1}^{n} p_i^*(t)\bar{G}_{i:n}(x), \tag{16}$$

where $p_1^*(t), p_2^*(t), \ldots, p_n^*(t)$ are coefficients such that $\sum_{i=1}^{n} p_i^*(t) = 1$ and $p_i^*(t) = \frac{a_i \bar{G}_{1:i}(t)}{\bar{G}_{T(n)}(t)}$ for $i = 1, \cdots, n$, where $\boldsymbol{a} = (a_1, \cdots, a_n)$ is domination vector; see Navarro et al. (2007). This result follows directly from the lack of memory property of exponential distribution. By using (16) the CRJ of the exponential mixed used system of order n is given by

$$\xi J(T(n,t)) = -\frac{1}{2}\int_0^1 \frac{\left(\sum_{i=1}^{n} p_i^*(t)\bar{G}_{V_i}(v)\right)^2}{\beta(1-v)} dv. \tag{17}$$

Denote the vector of coefficient in (17) by $\boldsymbol{p}^*(t) = (p_1^*(t), p_2^*(t), \ldots, p_n^*(t))$. We compute the vectors of coefficients $\boldsymbol{p}^*(t)$ for the exponential mixed used system of order 3 with 1-3 iid components. The results are displayed in Table 1. Actually, we rewritten Table 2 in Navarro et al. (2008) for extropy of exponential mixed used systems with 1-3 iid components.

Table 1: The vectors of coefficients $\boldsymbol{p}^*(t)$ in (17) for coherent systems with 1-3 iid exponential components.

System	$T(3) = \phi(Y_1, Y_2, Y_3)$	$\boldsymbol{p}^*(t)$	CRJ
1	$T_1(3) = Y_1$	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	$\xi J(T_1(3,t))$
2	$T_2(3) = \min\{Y_1, Y_2\}$	$\left(\frac{2}{3}, \frac{1}{3}, 0\right)$	$\xi J(T_2(3,t))$
3	$T_3(3) = \max\{Y_1, Y_2\}$	$\left(\frac{2-2e^{-\beta t}}{6-3e^{-\beta t}}, \frac{2-e^{-\beta t}}{6-3e^{-\beta t}}, \frac{2}{6-3e^{-\beta t}}\right)$	$\xi J(T_3(3,t))$
4	$T_4(3) = \min\{Y_1, Y_2, Y_3\}$	$(1, 0, 0)$	$\xi J(T_4(3,t))$
5	$T_5(3) = \min\{Y_1, \max\{Y_2, Y_3\}\}$	$\left(\frac{4-3e^{-\beta t}}{6-3e^{-\beta t}}, \frac{2}{6-3e^{-\beta t}}, 0\right)$	$\xi J(T_5(3,t))$
6	$T_6(3) = Y_{2:3}$(2-out-of-3)	$\left(\frac{2-2e^{-\beta t}}{3-2e^{-\beta t}}, \frac{1}{3-2e^{-\beta t}}, 0\right)$	$\xi J(T_6(3,t))$
7	$T_7(3) = \max\{Y_1, \min\{Y_2, Y_3\}\}$	$\left(\frac{1+2e^{-\beta t}-3e^{-2\beta t}}{3+3e^{-\beta t}-3e^{-2\beta t}}, \frac{1+e^{-\beta t}}{3+3e^{-\beta t}-3e^{-2\beta t}}, \frac{1}{3+3e^{-\beta t}-3e^{-2\beta t}}\right)$	$\xi J(T_7(3,t))$
8	$T_8(3) = \max\{Y_1, Y_2, Y_3\}$	$\left(\frac{1-2e^{-\beta t}+e^{-2\beta t}}{3-3e^{-\beta t}+e^{-2\beta t}}, \frac{1-e^{-\beta t}}{3-3e^{-\beta t}+e^{-2\beta t}}, \frac{1}{3-3e^{-\beta t}+e^{-2\beta t}}\right)$	$\xi J(T_8(3,t))$

Let us denote the CRJ of systems with lifetimes $T_i(3,t)$ by $\xi J(T_i(3,t))$, for $i = 1, \ldots, 8$. For vectors of coefficients $\boldsymbol{p}^*(t)$ in Table 1, the CRJ of exponential mixed used systems of order 3 have been displayed in Figure 3 as a function of t. From Figure 6, it is observed that $\xi J(T_i(3,t))$ is a increasing function with respect to t for $i = 1, \ldots, 8$. For $t < 0.008$, Figure 6 shows that the following inequalities hold between the CRJ of the aforementioned systems as follows

$$\xi J(T_4(3,t)) \geq \xi J(T_2(3,t)) \geq \xi J(T_5(3,t)) \geq \xi J(T_1(3,t)) \geq \xi J(T_6(3,t)) \geq \xi J(T_7(3,t)) \geq \xi J(T_3(3,t)) \geq \xi J(T_8(3,t)), \tag{18}$$

for $0.008 < t < 0.048$,

$$\xi J(T_4(3,t)) \geq J(T_2(3,t)) \geq \xi J(T_5(3,t)) \geq \xi J(T_6(3,t)) \geq \xi J(T_1(3,t)) \geq \xi J(T_7(3,t)) \geq \xi J(T_3(3,t)) \geq \xi J(T_8(3,t)), \tag{19}$$

and for $t > 0.048$, we have

$$\xi J(T_4(3,t)) \geq \xi J(T_2(3,t)) \geq \xi J(T_5(3,t)) \geq \xi J(T_6(3,t)) \geq \xi J(T_7(3,t)) \geq \xi J(T_1(3,t)) \geq \xi J(T_3(3,t)) \geq \xi J(T_8(3,t)). \tag{20}$$

Also, inequalities (18), (19), and (20) confirm Theorem 1. For example $\boldsymbol{p}_2^*(t) \leqslant_{st} \boldsymbol{p}_5^*(t)$ and $\boldsymbol{p}_3^*(t) \leqslant_{st} \boldsymbol{p}_8^*(t)$, therefore from inequality (18), we have $\xi J(T_2(3,t)) \geq \xi J(T_5(3,t))$ and $\xi J(T_3(3,t)) \geq \xi J(T_8(3,t))$, respectively. Note that $\boldsymbol{p}_2^*(t)$, $\boldsymbol{p}_5^*(t)$, $\boldsymbol{p}_3^*(t)$, and $\boldsymbol{p}_8^*(t)$ are vectors of coefficients for systems 2, 5, 3, and 8 in Table 1, respectively.

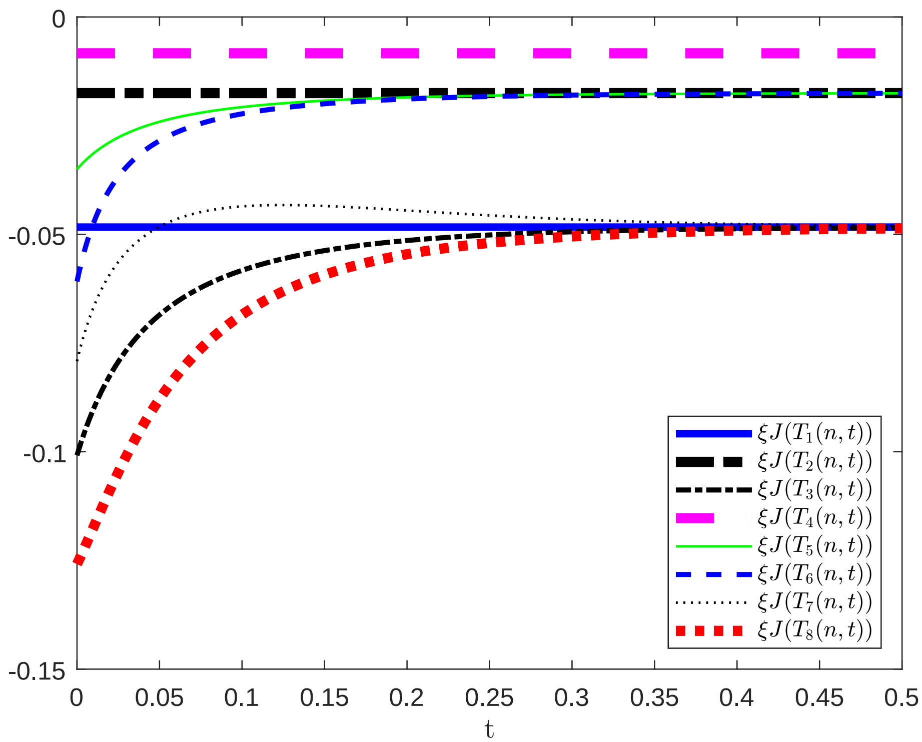

Figure 6: The CRJ of exponential mixed used systems of order 3 in Table 1.

The JCRJ divergence for the group of systems listed in Table 1 is depicted in Figure 3. This group of systems comprises three iid components, each with an exponential lifetime having a mean of 0.1. In Figure 3, it can be observed that system 4 is a 3-out-of-3 system with a JCRJ divergence of zero. System 4 is the least complex system among the systems listed in Table 1. Furthermore, it is evident from Figure 3 that as the complexity of the systems increases, the JCRJ divergence also increases. The JCRJ divergence of systems 5 and 6 are the same for $t > 0.048$. As t approaches infinity, the JCRJ divergence of systems 5 and 6 tends to the JCRJ divergence of system 2. Additionally, as t approaches infinity, the JCRJ divergence of systems 7, 3, and 8 tends to the JCRJ divergence of system 1. One advantage of the JCRJ divergence measure is that we can compare the complexity of systems consisting of different numbers of iid components.

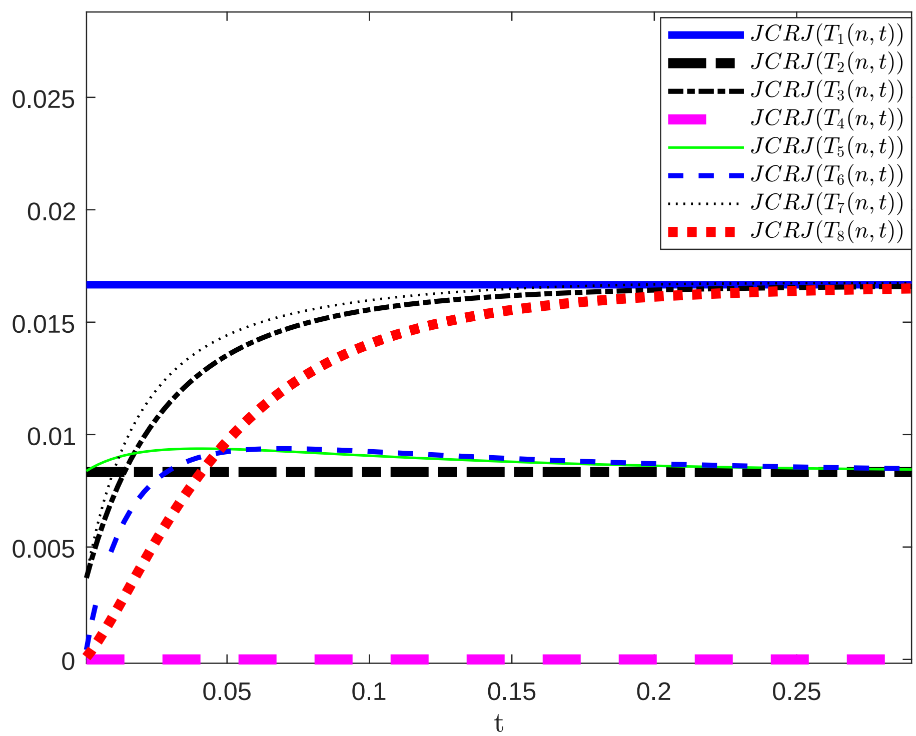

Figure 7: JCRJ of exponential mixed used systems of order 3 in Table 1.

5 Optimal signature for the JCRJ divergence and complexity of used systems

In this section, we consider the problem of finding the optimal signature system in the exponential model by maximizing a chosen criterion function over the class of all mixed used systems of order n at the fixed time t. The formulation of any reliability economics problem will always involve specifying a criterion function that quantifies how the performance and cost of the system or policy will be evaluated in relation to each other. It's important to note that choosing the criterion function is somewhat subjective and will have an impact on the outcome of the optimization process. The criterion function should vary proportionately with measures of system performance and inversely with measures of system cost. These properties are equivalent, for example, to the position that, given two systems with the same performance characteristics, we would prefer the one which costs less, and given two systems which cost the same, we would prefer the better-performing system. A mixed system with a signature $\boldsymbol{s}$ can be physically created using a randomization process that selects a k-out-of-n system with a probability of s_k, where k ranges from 1 to n. Since the class of mixed systems includes all coherent systems as special cases, considering this larger space does not result in any loss. Furthermore, the space of signatures of mixed systems, based on coherent systems of order n in iid components, is uncountably infinite, which leads to replacing a discrete search with a search over a much larger space. The larger problem is much more amenable to analytical treatment, as the tools of differential calculus become available in the maximization problem of interest. The optimization problem to which we now turn involves the maximization of a chosen criterion function over the $(n-1)$-dimensional simplex $\left\{\mathbf{s} \in [0,1]^n | \sum\limits_{i=1}^{n} s_i = 1\right\}$.

We consider the following criterion function

$$m_d(\mathbf{s}, \mathbf{a}, \mathbf{c}) = \frac{\sum\limits_{i=1}^{n} a_i s_i}{(\sum\limits_{i=1}^{n} c_i s_i)^d}, \tag{21}$$

where $\mathbf{s}$ is the signature vector, $d > 0$ and vectors $\mathbf{a}$ and $\mathbf{c}$ can be chosen arbitrary within the context of two natural constraints: $0 < a_1 < \ldots < a_n$ and $0 < c_1 < \ldots < c_n$. The criterion function (21) has been presented in Chapter 7 of Samaniego (2007) which is dedicated to a particular problem in the area of reliability economics. The optimization problem considered in Samaniego (2007) is divided into two mutually exclusive cases, and the precise nature of the optimal design is obtained for each and it is showed that the criterion function can be maximized by a given coherent system. The cost vector $\mathbf{c} = (c_1, \cdots, c_n)$ and the calibration parameter d involve assessments on the part of the experimenter (the producer, customer or both), and it seem reasonable to assume that the value of $\mathbf{c}$ can be determined, in most applications of interest, with the assistant of engineering judgement and other expert advice. Furthermore, the value of d can be chosen to fit a given application, perhaps with the help of suitable sensitivity analysis.

Equation (15) in a mixed used system assesses the complexity of the system, which indicates how much more complex the system is in comparison to a k-out-of-n system with a baseline CDF of $\frac{\bar{F}(x+t)}{\bar{F}(t)}$. The objective is to find a system that minimizes the JCRJ divergence and reduces costs simultaneously. To achieve this, we use the criterion function in Equation (21) and reformulated Equation (14) as

$$JCRJ(T(n,t) : Y_{1:n,t}, \ldots, Y_{n:n,t}) = \sum_{i=1}^{n} s_i a_i, \tag{22}$$

where $a_i = \frac{P(Y_{i:n}>t)}{P(T(n)>t)} R(T(n,t) : Y_{i:n,t})$. By substituting (22) into the criterion function (20), we can achieve our goal.

In Equation (21), we assume that $a_i = \frac{P(Y_{i:n}>t)}{P(T(n)>t)} R(T(n,t) : Y_{i:n,t})$ and $c_i = C_I + n(A-B) + iB$ for $i = 1, \ldots, n$, where C_I is an initial fixed cost of manufacturing the systems of interest, A is cost of an individual component, B is salvage value of a used but working component removed after system failure, and t is a fixed time. For additional information about the values of c_i, refer to Page 95 of chapter 7 in Samaniego (2007).

Let the components of mixed used system have exponential distribution with survival function $\bar{G}(y) = e^{-\beta y}$, where β is positive constant. Let $C_I = 0$ and $B = 1$. Consider the class of all mixed used systems based on n components with iid lifetimes. The goal is to identify a system that simultaneously minimizes the JCRJ (maximizes $(-1)\times$JCRJ) divergence as well decreases the costs. Now the question come up is, "What is the optimal signature vector subject to the criterion function (21)?". This problem can be formulated as

$$\text{maximized}: \; m_d(\mathbf{s}, \mathbf{a}, \mathbf{c}) = \frac{-\frac{1}{2} \sum\limits_{i=1}^{n} p_i^*(t) \int_0^1 \left(\sum\limits_{i=1}^{n} \sum\limits_{j=0}^{i-1} p_i^*(t) \binom{n}{j} u^i (1-u)^{n-i} - \sum\limits_{j=0}^{i-1} \binom{n}{j} u^i (1-u)^{n-i} \right)^2 dx}{\left(\sum\limits_{i=1}^{n} (i + n(A-1)) s_i\right)^d},$$

$$\text{subject to}: \begin{cases} \sum\limits_{i=1}^{n} s_i = 1, \\ 0 \le s_i \le 1. \end{cases}$$

Numerical computations are needed to find the solutions. We have done the computations to obtain the optimal signature of mixed used system with 5 independent exponentially distributed components for some selected values of A, d, and β. The numerical results are presented in Table 3 which shows that, there is no continuity in nature of the optimal signature. From Table 2, we observe that the optimal signature system is a series system for $d \le 0.13$ and with increasing the mean lifetime of components, the optimal signature system is a series system for $d \le 0.5$. Also, we observe that for $d \ge 12$, the optimal system is a mixture system with uniform signature $\mathbf{s} = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$.

Table 2: Optimal signatures of mixed used system of order 5 with independent exponentially distributed components at time $t = 0.1$.

		$\beta = 2$					$\beta = 0.1$				
A	d	s_1	s_2	s_3	s_4	s_5	s_1	s_2	s_3	s_4	s_5
1.5	≤ 0.1	1	0	0	0	0	1	0	0	0	0
1.5	0.13	1	0	0	0	0	1	0	0	0	0
1.5	0.2	1	0	0	0	0	1	0	0	0	0
1.5	0.5	1	0	0	0	0	1	0	0	0	0
1.5	0.9	0.9998	0	0	0	0	0.9231	0.0769	0	0	0
1.5	1	0.9970	0.0002	0.0001	0	0	0.5820	0.0761	0.3355	0.0051	0.0013
1.5	1.5	0.3130	0.2772	0.2270	0.1530	0.0298	0.6142	0.2602	0.0784	0.0319	0.0153
1.5	2	0.3053	0.2770	0.2329	0.1612	0.0236	0.5938	0.2896	0.0708	0.0306	0.0153
1.5	5.5	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
1.5	10	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
1.5	≥ 12	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
2	≤ 0.1	1	0	0	0	0	1	0	0	0	0
2	0.13	1	0	0	0	0	1	0	0	0	0
2	0.2	0.9999	0.0001	0	0	0	1	0	0	0	0
2	0.5	0.9998	0.0002	0	0	0	1	0	0	0	0
2	0.9	0.9997	0.0003	0	0	0	1	0	0	0	0
2	1	0.9999	0.0001	0	0	0	1	0	0	0	0
2	1.5	0	0	0	0.0001	0.9999	09224	0.0763	0.0009	0.0002	0.0001
2	2	0	0	0.0002	0.0001	0.9997	0.5622	0.1246	0.3006	0.0100	0.0026
2	5.5	0.0088	0.0112	0.0168	0.0399	0.9233	0.5936	0.2897	0.0708	0.0306	0.0153
2	10	0.1970	0.1984	0.1999	0.2015	0.2032	0.2	0.2	0.2	0.2	0.2
2	≥ 12	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
3	≤ 0.1	1	0	0	0	0	1	0	0	0	0
3	0.13	1	0	0	0	0	1	0	0	0	0
3	0.2	1	0	0	0	0	1	0	0	0	0
3	0.5	0.8701	0.1299	0	0	0	1	0	0	0	0
3	0.9	0.8701	0.1299	0	0	0	1	0	0	0	0
3	1	0.8704	0.1295	0.0001	0	0	0.9230	0.0765	0.0002	0.0001	0.0001
3	1.5	0.8712	0.1283	0.0002	0.0002	0.0001	0.5448	0.2199	0.2026	0.0254	0.0073
3	2	0.5936	0.2897	0.0708	0.3060	0.1520	0.5937	0.2896	0.0708	0.0306	0.0153
3	5.5	0.5772	0.3050	0.6770	0.3160	0.0185	0.5838	0.2896	0.0708	0.0306	0.0153
3	10	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
3	≥ 12	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2

6 Conclusion and future remarks

The purpose of this paper is to develop representations for the cumulative residual extropy (CRJ) of systems with an order of n, under specific conditions related to the system state at time t. We focused on one form of conditioning, which is that the system is operational at time t (i.e. $T(n) > t$). When $T(n) > t$, the system is considered a used system with a lifetime of $T(n) - t|T(n) > t$. The representations of the CRJ for these conditional systems are valuable for analyzing the information properties of conditional mixed systems. By using these representations, we established some stochastic comparisons and boundaries for the CRJ of conditional systems. We computed the CRJ of coherent systems with 3 iid components with exponentially distributed lifetimes, and discovered that the series system has the highest CRJ, while the parallel system has the lowest CRJ. Additionally, we introduced a divergence measure based on the CRJ of a mixed used system that evaluates the complexity of the system, i.e., how much more complex the system is compared to a k-out-of-n system with baseline cdf $\frac{\bar{F}(x+t)}{\bar{F}(t)}$. For our future work, we aim to explore the information properties and complexity of conditional mixed systems from the perspective of other information theoretic approaches.

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