Supplementary Material

On the relation between discrete and continuous-time affine option pricing models

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SM.A Convergence of Markov chains to diffusion processes

The procedure for obtaining weak diffusion limits for discrete-time Markov processes has been outlined in detail in Ethier and Kurtz (1986) and Stroock and Varadhan (1979). Nelson (1990) first applied this procedure in the context of GARCH processes to derive the weak limit of a standard GARCH(1,1) model.

Let $\boldsymbol{X}^{(n)} = \{\boldsymbol{X}^{(n)}_{k\tau}\}_{k=0}^{nT}$ be a discrete-time Markov process on $\mathbb{R}^p$ and define its càdlàg extension as in Equation (16). We denote by $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^p$. The weak convergence theorem is stated below.

Theorem SM.1. *(Nelson, 1990) Assume there exist continuous, measurable functions $\boldsymbol{A}$ and $\boldsymbol{\Sigma}$ defined on $\mathbb{R}^p$ such that for any $c > 0$ the following conditions hold:*

(i) $$\limsup_{\tau\to 0}\sup_{\|\boldsymbol{x}\|\le c}\left\|\tau^{-1}\mathrm{E}\left[\Delta\boldsymbol{X}^{(n)}_{(k+1)\tau}\,\middle|\,\boldsymbol{X}^{(n)}_{k\tau}=\boldsymbol{x}\right]-\boldsymbol{A}(\boldsymbol{x})\right\|=0.$$

(ii) $$\limsup_{\tau\to 0}\sup_{\|\boldsymbol{x}\|\le c}\left\|\tau^{-1}\mathrm{E}\left[\Delta\boldsymbol{X}^{(n)}_{(k+1)\tau}\,\Delta\boldsymbol{X}^{(n)\top}_{(k+1)\tau}\,\middle|\,\boldsymbol{X}^{(n)}_{k\tau}=\boldsymbol{x}\right]-\boldsymbol{\Sigma}(\boldsymbol{x})\,\boldsymbol{\Sigma}^{\top}(\boldsymbol{x})\right\|=0.$$

(iii) There exists $\delta > 0$ such that

$$\limsup_{\tau\to 0}\sup_{\|\boldsymbol{x}\|\le c}\left\|\tau^{-1}\mathrm{E}\left[\left|\Delta X^{(n)}_{(k+1)\tau,i}\right|^{2+\delta}\,\middle|\,\boldsymbol{X}^{(n)}_{k\tau}=\boldsymbol{x}\right]\right\|=0,$$

where $\Delta X^{(n)}_{(k+1)\tau,i}$ represents the i^{th} element of vector $\boldsymbol{X}^{(n)}_{(k+1)\tau}-\boldsymbol{X}^{(n)}_{k\tau}$.

If we assume that $\boldsymbol{X}^{(n)}_0$ converges in distribution to $\boldsymbol{X}_0$, and if the SDE

$$d\boldsymbol{X}_t = \boldsymbol{A}\left(\boldsymbol{X}_t\right)dt + \boldsymbol{\Sigma}\left(\boldsymbol{X}_t\right)d\boldsymbol{W}_t \tag{SM.1}$$

*We are grateful to the Editor, Liangjun Su, the former Editor, Eric Renault, and three anonymous referees for their extensive comments, which substantially improved the paper. We also acknowledge the helpful comments from the seminar participants at Technical University of Munich, University of Calgary, and Quantact Actuarial and Financial Mathematics Laboratory, as well as the participants at the SSC 2024, Samos 2024, CAIMS 2024, SSC 2025, and QFFE 2025 conferences.

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admits a solution which is unique in distribution and almost surely finite for any $t \in [0,T]$, then the process $\{\boldsymbol{X}_t^{(n)}\}_{t\in[0,T]}$ converges weakly to $\{\boldsymbol{X}_t\}_{t\in[0,T]}$ as τ approaches zero. Here, $\boldsymbol{W}$ is a p-dimensional standard Brownian motion on the filtered probability space, also independent of $\boldsymbol{X}_0$.

Note that condition (iii) ensures that the sample paths of the limiting process are continuous with probability one; that is,

$$\lim_{\tau\to 0} \tau^{-1}\mathbb{P}\left(\left\| \boldsymbol{X}_{(k+1)\tau}^{(n)} - \boldsymbol{X}_{k\tau}^{(n)} \right\| > \varepsilon \,\middle|\, \boldsymbol{X}_{k\tau}^{(n)} = \boldsymbol{x}\right) = 0, \quad \text{for all} \quad \varepsilon > 0.$$

For simplicity, we only consider the convergence to time-homogeneous SDEs. However, the result from Nelson (1990) also holds if the drift and diffusion terms $\boldsymbol{A}$ and $\boldsymbol{\Sigma}$ depend on t.

The existence and uniqueness of a (strong) solution of the above SDE are guaranteed under some regularity conditions on the drift and diffusion functions, e.g., Lipschitz and linear growth conditions.[1] However, the weak limits of our affine GARCH models are square-root processes whose diffusion coefficients do not satisfy the Lipschitz property at zero, and therefore Feller-type conditions need to be considered. These are illustrated and discussed in Appendix A.2.2 of the manuscript.

SM.B Characterization of affine diffusion processes and proof of Proposition 4

To establish an affine setting in continuous time, we introduce below the following assumption for the process $\boldsymbol{X}$, which satisfies the SDE of Equation (SM.1).

Assumption SM.1. *For any $\boldsymbol{u} \in \mathbb{R}^p$, we assume the $\mathcal{F}_t$-conditional characteristic function (cf) of $\boldsymbol{X}_T$ is exponentially affine in $\boldsymbol{X}_t$, for all $t \le T$; that is,*

$$\chi_{\boldsymbol{X}_T}(\boldsymbol{u} \,|\, \mathcal{F}_t) = \mathrm{E}\left[\exp\left(\mathrm{i}\boldsymbol{u}^\top \boldsymbol{X}_T\right) \,\middle|\, \mathcal{F}_t\right] = \exp\left(\Phi(\boldsymbol{u}; T-t) + \boldsymbol{\Psi}(\boldsymbol{u}; T-t)^\top \boldsymbol{X}_t\right), \tag{SM.2}$$

where $\Phi(\boldsymbol{u}; T-t)$ and $\boldsymbol{\Psi}(\boldsymbol{u}; T-t) = \begin{bmatrix} \Psi_1(\boldsymbol{u}; T-t) & \dots & \Psi_p(\boldsymbol{u}; T-t) \end{bmatrix}^\top$ are complex-valued functions with jointly continuous t-derivatives satisfying the initial conditions $\Phi(\boldsymbol{u}; 0) = 0$ and $\boldsymbol{\Psi}(\boldsymbol{u}; 0) = \mathrm{i}\boldsymbol{u}$.

The necessary and sufficient conditions for $\boldsymbol{X}$ to be affine are provided below.

Theorem SM.2. *(Filipović and Mayerhofer, 2009) Suppose $\boldsymbol{X}$ satisfies the SDE from Equation (SM.1) and has the affine property of Assumption SM.1. Then, the drift and diffusion coefficients $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{\Sigma}(\boldsymbol{x})\boldsymbol{\Sigma}(\boldsymbol{x})^\top$ are affine in $\boldsymbol{x}$; that is,*

$$\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{A}_0 + \sum_{i=1}^{p} x_i \boldsymbol{A}_{1,i} = \boldsymbol{A}_0 + \boldsymbol{A}_1 \boldsymbol{x}, \tag{SM.3}$$

$$\boldsymbol{\Sigma}(\boldsymbol{x})\boldsymbol{\Sigma}(\boldsymbol{x})^\top = \boldsymbol{B}_0 + \sum_{i=1}^{p} x_i \boldsymbol{B}_i, \tag{SM.4}$$

for some p-dimensional vectors $\boldsymbol{A}_0$ and $\boldsymbol{A}_{1,i}$ and $p \times p$ matrices $\boldsymbol{B}_0$ and $\boldsymbol{B}_i$, with $i = 1, \dots, p$. Note that $\boldsymbol{A}_{1,i}$ is the i^{th} column vector of the $p \times p$ matrix $\boldsymbol{A}_1$. Moreover, Φ and Ψ_i, with $1 \le i \le p$, are solutions of the

[1] The standard Lipschitz and linear growth conditions which ensure the existence and uniqueness of the solution of the above SDE are

$$\|\boldsymbol{A}(\boldsymbol{x}) - \boldsymbol{A}(\boldsymbol{y})\| + \|\boldsymbol{\Sigma}(\boldsymbol{x}) - \boldsymbol{\Sigma}(\boldsymbol{y})\| \le C\|\boldsymbol{x} - \boldsymbol{y}\|,$$
$$\|\boldsymbol{A}(\boldsymbol{x})\|^2 + \|\boldsymbol{\Sigma}(\boldsymbol{x})\|^2 \le C^2\left(1 + \|\boldsymbol{x}\|^2\right),$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^p$, where C is a positive constant.

following system of Riccati equations:

$$\partial_t \Phi(\boldsymbol{u};t) = \boldsymbol{A}_0^\top \boldsymbol{\Psi}(\boldsymbol{u};t) + \frac{1}{2}\boldsymbol{\Psi}(\boldsymbol{u};t)^\top \boldsymbol{B}_0 \boldsymbol{\Psi}(\boldsymbol{u};t), \qquad \Phi(\boldsymbol{u};0) = 0, \tag{SM.5}$$

$$\partial_t \Psi_i(\boldsymbol{u};t) = \boldsymbol{A}_{1,i}^\top \boldsymbol{\Psi}(\boldsymbol{u};t) + \frac{1}{2}\boldsymbol{\Psi}(\boldsymbol{u};t)^\top \boldsymbol{B}_i \boldsymbol{\Psi}(\boldsymbol{u};t), \qquad \boldsymbol{\Psi}(\boldsymbol{u};0) = \mathrm{i}\boldsymbol{u}. \tag{SM.6}$$

Conversely, suppose the drift and diffusion coefficients $\boldsymbol{A}(\boldsymbol{x})$ *and* $\boldsymbol{\Sigma}(\boldsymbol{x})\boldsymbol{\Sigma}(\boldsymbol{x})^\top$ *are affine in* $\boldsymbol{x}$ *as in Equations* (SM.3) *and* (SM.4)*. If there exists a solution* Φ *and* Ψ_i*, with* $1 \leq i \leq p$*, to the system of Riccati equations from* (SM.5) *and* (SM.6)*, such that* $\Phi(\boldsymbol{u};t) + \boldsymbol{\Psi}(\boldsymbol{u};t)^\top \boldsymbol{x}$ *has non-positive real part for all* $\boldsymbol{u}, \boldsymbol{x} \in \mathbb{R}^p$ *and* $t \geq 0$*, then* $\boldsymbol{X}$ *is affine with the conditional cf from Equation* (SM.2)*.*

Note that the weak diffusion limits of the HN(C) and IG(C) models under both physical and risk-neutral measures are affine processes. Since option pricing relies on the computation of the $\mathcal{F}_t$-conditional risk-neutral cf of the log-asset price process Y_T, we only focus on deriving this quantity for the diffusion limit of the HN(C) model from Proposition 3.

We now prove Proposition 4. The proof follows the prescriptions outlined in Theorem SM.2 from Filipović and Mayerhofer (2009). First, we identify the affine specifications from Equations (SM.3) and (SM.4) for the HN(C) diffusion limit. We have:

$$\boldsymbol{A}_0 = \begin{bmatrix} r \\ 2\alpha_s \boldsymbol{\alpha}^\top \boldsymbol{\xi} \\ \kappa_q \sigma^2 + 2\alpha_q \boldsymbol{\alpha}^\top \boldsymbol{\xi} \end{bmatrix}, \quad \boldsymbol{A}_{1,1} = \boldsymbol{0}, \quad \boldsymbol{A}_{1,2} = \begin{bmatrix} -\frac{1}{2} \\ -\kappa_s + 2\alpha_s\gamma_s\left(\lambda + \frac{1}{2}\right) \\ 2\alpha_q\gamma_q\left(\lambda + \frac{1}{2}\right) \end{bmatrix}, \quad \boldsymbol{A}_{1,3} = \begin{bmatrix} -\frac{1}{2} \\ 2\alpha_s\gamma_s\left(\lambda + \frac{1}{2}\right) \\ -\kappa_q + 2\alpha_q\gamma_q\left(\lambda + \frac{1}{2}\right) \end{bmatrix},$$

and

$$\boldsymbol{B}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\alpha_s^2 & 2\alpha_s\alpha_q \\ 0 & 2\alpha_s\alpha_q & 2\alpha_q^2 \end{bmatrix}, \quad \boldsymbol{B}_1 = \boldsymbol{0}, \quad \boldsymbol{B}_2 = \boldsymbol{B}_3 = \begin{bmatrix} 1 & -2\alpha_s\gamma_s & -2\alpha_q\gamma_q \\ -2\alpha_s\gamma_s & 4\alpha_s^2\gamma_s^2 & 4\alpha_s\alpha_q\gamma_s\gamma_q \\ -2\alpha_q\gamma_q & 4\alpha_s\alpha_q\gamma_s\gamma_q & 4\alpha_q^2\gamma_q^2 \end{bmatrix}.$$

Plugging in the above values into the system of Riccati equations from (SM.5) and (SM.6), we obtain after some algebraic manipulations the required ODEs. Note that the conditional cf of Y_T under the risk-neutral measure $\mathbb{Q}$, which is required for valuation purposes, is obtained by letting $\boldsymbol{u} = [\, u \quad 0 \quad 0 \,]^\top$.

SM.C Multi-step conditional cumulant generating functions for discretized affine GARCH models

As in the continuous-time case, the valuation of option prices at any time t requires the computation of the multi-step $\mathcal{F}_t^{(n)}$-conditional cgf of the log-asset price process $Y_T^{(n)}$ under the risk-neutral measure $\mathbb{Q}^{(n)}$. Proposition SM.1 describes the recursive procedure to achieve this for our discrete-time affine pricing framework.

Proposition SM.1. *Assume that the process* $\boldsymbol{X}^{(n)} = \{\boldsymbol{X}_{k\tau}^{(n)}\}_{k=0}^{nT}$ *with* $\boldsymbol{X}_{k\tau}^{(n)} = [\, Y_{k\tau}^{(n)} \quad s_{k\tau}^{(n)} \quad q_{k\tau}^{(n)} \,]^\top$ *follows the dynamics of Equations* (1) *to* (4) *and satisfies the affine property from Assumption 1. Then, for any* $t \in \mathcal{T}^{(1)}$ *and any real* w *and* $\boldsymbol{v} = [\, v_1 \quad v_2 \,]^\top$*, the multi-step* $\mathcal{F}_t^{(n)}$*-conditional joint cgf of* $\boldsymbol{X}_T^{(n)}$ *under the risk-neutral measure* $\mathbb{Q}^{(n)}$ *introduced in Assumption 2 is given by*

$$\mathcal{C}_{\boldsymbol{X}_T^{(n)}}^{\mathbb{Q}^{(n)}}\left(w, \boldsymbol{v} \mid \mathcal{F}_t^{(n)}\right) = wY_t^{(n)} + \tilde{\mathcal{A}}^{*\mathcal{D}}(w, \boldsymbol{v}; T-t) + \tilde{\mathcal{B}}_s^{*\mathcal{D}}(w, \boldsymbol{v}; T-t)\, s_t^{(n)} + \tilde{\mathcal{B}}_q^{*\mathcal{D}}(w, \boldsymbol{v}; T-t)\, q_t^{(n)}.$$

Here, the coefficients $\tilde{\mathcal{A}}^{*\mathcal{D}}(w, \boldsymbol{v}; T-t)$ *and* $\tilde{\boldsymbol{\mathcal{B}}}^{*\mathcal{D}}(w, \boldsymbol{v}; T-t) = \left[\, \tilde{\mathcal{B}}_s^{*\mathcal{D}}(w, \boldsymbol{v}; T-t) \quad \tilde{\mathcal{B}}_q^{*\mathcal{D}}(w, \boldsymbol{v}; T-t) \,\right]$ *sat-*

isfy the following recursions:

$$\tilde{\mathcal{A}}^{*\mathcal{D}}(w,\boldsymbol{v};T-t)=\tilde{\mathcal{A}}^{*\mathcal{D}}(w,\boldsymbol{v};T-t-\tau)+\tilde{\mathcal{A}}^{*\mathcal{D}}\left(w,\tilde{\boldsymbol{\mathcal{B}}}^{*\mathcal{D}}(w,\boldsymbol{v};T-t-\tau);\tau\right),$$
$$\tilde{\boldsymbol{\mathcal{B}}}^{*\mathcal{D}}(w,\boldsymbol{v};T-t)=\tilde{\boldsymbol{\mathcal{B}}}^{*\mathcal{D}}\left(w,\tilde{\boldsymbol{\mathcal{B}}}^{*\mathcal{D}}(w,\boldsymbol{v};T-t-\tau);\tau\right),$$

with the terminal conditions $\tilde{\mathcal{A}}^{*\mathcal{D}}(w,\boldsymbol{v};0)=0$, *and* $\tilde{\boldsymbol{\mathcal{B}}}^{*\mathcal{D}}(w,\boldsymbol{v};0)=\boldsymbol{v}$. *The one-step risk-neutral coefficients* $\tilde{\mathcal{A}}^{*\mathcal{D}}(w,\boldsymbol{v};\tau)$ *and* $\tilde{\boldsymbol{\mathcal{B}}}^{*\mathcal{D}}(w,\boldsymbol{v};\tau)$ *are related to the coefficients of the one-step* $\mathcal{F}_t^{(n)}$*-conditional cgf of* $\Delta\boldsymbol{X}_{t+\tau}^{(n)}$ *given in Assumption 1 via the following relationships:*

$$\tilde{\mathcal{A}}^{*\mathcal{D}}(w,\boldsymbol{v};\tau)=\mathcal{A}^{\mathcal{D}}(w+\theta(\tau),\boldsymbol{v}+\boldsymbol{\xi}(\tau);\tau)-\mathcal{A}^{\mathcal{D}}(\theta(\tau),\boldsymbol{\xi}(\tau);\tau),$$
$$\tilde{\mathcal{B}}_s^{*\mathcal{D}}(w,\boldsymbol{v};\tau)=\mathcal{B}_s^{\mathcal{D}}(w+\theta(\tau),\boldsymbol{v}+\boldsymbol{\xi}(\tau);\tau)-\mathcal{B}_s^{\mathcal{D}}(\theta(\tau),\boldsymbol{\xi}(\tau);\tau)+v_1,$$
$$\tilde{\mathcal{B}}_q^{*\mathcal{D}}(w,\boldsymbol{v};\tau)=\mathcal{B}_q^{\mathcal{D}}(w+\theta(\tau),\boldsymbol{v}+\boldsymbol{\xi}(\tau);\tau)-\mathcal{B}_q^{\mathcal{D}}(\theta(\tau),\boldsymbol{\xi}(\tau);\tau)+v_2,$$

where $\theta(\tau)$ *and* $\boldsymbol{\xi}(\tau)$ *are assumed to satisfy the no-arbitrage constraints of Equations* (8) *and* (9).

Proof. This result follows in a similar way as the proof of Proposition 1 of Augustyniak et al. (2023). □

SM.D Additional proofs

SM.D.1 Proof of Proposition 1

First, we introduce the following notations for each distributional assumption:

$$\boldsymbol{K}_s^{\mathcal{N}}(\tau)=\begin{bmatrix}-(1-\phi_s(\tau))\\0\end{bmatrix},\qquad \boldsymbol{K}_q^{\mathcal{N}}(\tau)=\begin{bmatrix}0\\-\left(1-\phi_q(\tau)\right)\end{bmatrix},$$
$$\boldsymbol{K}_s^{\mathcal{JG}}(\tau)=\begin{bmatrix}-(1-\phi_s(\tau))\\0\end{bmatrix}-\eta^2\boldsymbol{\alpha}^{\mathcal{JG}}(\tau)-\frac{1}{\eta^2}\left(\boldsymbol{\alpha}^{\mathcal{JG}}(\tau)\odot\boldsymbol{\gamma}^{\mathcal{JG}}(\tau)\right),$$
$$\boldsymbol{K}_q^{\mathcal{JG}}(\tau)=\begin{bmatrix}0\\-\left(1-\phi_q(\tau)\right)\end{bmatrix}-\eta^2\boldsymbol{\alpha}^{\mathcal{JG}}(\tau)-\frac{1}{\eta^2}\left(\boldsymbol{\alpha}^{\mathcal{JG}}(\tau)\odot\boldsymbol{\gamma}^{\mathcal{JG}}(\tau)\right).$$

Using the constraints of Equation (15) and the above notations, we have

$$\boldsymbol{\alpha}^{\mathcal{JG}}(\tau)=\eta^{-4}\boldsymbol{\alpha}^{\mathcal{N}}(\tau),\qquad \left(\boldsymbol{\alpha}^{\mathcal{JG}}(\tau)\odot\boldsymbol{\gamma}^{\mathcal{JG}}(\tau)\right)=\boldsymbol{\alpha}^{\mathcal{N}}(\tau)-2\eta\left(\boldsymbol{\alpha}^{\mathcal{N}}(\tau)\odot\boldsymbol{\gamma}^{\mathcal{N}}(\tau)\right),$$
$$\boldsymbol{K}_u^{\mathcal{JG}}(\tau)=\boldsymbol{K}_u^{\mathcal{N}}(\tau)-\frac{2\left(\boldsymbol{\alpha}^{\mathcal{N}}(\tau)-\eta\left(\boldsymbol{\alpha}^{\mathcal{N}}(\tau)\odot\boldsymbol{\gamma}^{\mathcal{N}}(\tau)\right)\right)}{\eta^2},$$

for $u\in\{s,q\}$. We proceed by showing the convergence of the coefficients of the $\mathcal{F}_{k\tau}^{(n)}$-conditional joint cgf of $\Delta Y_{(k+1)\tau}^{(n)}$, $\Delta s_{(k+1)\tau}^{(n)}$, and $\Delta q_{(k+1)\tau}^{(n)}$ of the IG(C) model to the HN(C) counterparts, for any real numbers w and $\boldsymbol{v}=[\,v_1\quad v_2\,]^\top$. Using the first relationship, we can verify that $\mathcal{A}^{\mathcal{JG}}(w,\boldsymbol{v};\tau)=\mathcal{A}^{\mathcal{N}}(w,\boldsymbol{v};\tau)$. Next, we show the convergence of $\mathcal{B}_s^{\mathcal{JG}}(w,\boldsymbol{v};\tau)$. Using the above relationships and applying l'Hôpital's rule twice, we have

$$\begin{aligned}
&\lim_{\eta\to0}\mathcal{B}_s^{\mathcal{JG}}(w,\boldsymbol{v};\tau)\\
&=\lim_{\eta\to0}\left(w\left(\lambda^{\mathcal{JG}}(\tau)-\frac{\tau}{\eta}\right)+\boldsymbol{v}^\top\boldsymbol{K}_s^{\mathcal{JG}}(\tau)+\frac{1}{\eta^2}\left(1-\sqrt{\left(1-2\eta^4\boldsymbol{v}^\top\boldsymbol{\alpha}^{\mathcal{JG}}(\tau)\right)\left(1-2w\eta\sqrt{\tau}-2\boldsymbol{v}^\top\left(\boldsymbol{\alpha}^{\mathcal{JG}}(\tau)\odot\boldsymbol{\gamma}^{\mathcal{JG}}(\tau)\right)\right)}\right)\right)\\
&=\lim_{\eta\to0}\left(w\left(\lambda^{\mathcal{N}}(\tau)-\frac{\sqrt{\tau}}{\eta}\right)+\boldsymbol{v}^\top\begin{bmatrix}-(1-\phi_s(\tau))\\0\end{bmatrix}-\frac{2\boldsymbol{v}^\top\left(\boldsymbol{\alpha}^{\mathcal{N}}(\tau)-\eta\left(\boldsymbol{\alpha}^{\mathcal{N}}(\tau)\odot\boldsymbol{\gamma}^{\mathcal{N}}(\tau)\right)\right)}{\eta^2}\right)
\end{aligned}$$

$$+\lim_{\eta\to 0}\frac{\left(1-\sqrt{\left(1-2\boldsymbol{v}^\top\boldsymbol{\alpha}^{\mathcal{N}}(\tau)\right)\left(1-2\eta\left(w\sqrt{\tau}-2\boldsymbol{v}^\top\left(\boldsymbol{\alpha}^{\mathcal{N}}(\tau)\odot\boldsymbol{\gamma}^{\mathcal{N}}(\tau)\right)\right)-2\boldsymbol{v}^\top\boldsymbol{\alpha}^{\mathcal{N}}(\tau)\right)}\right)}{\eta^2}$$

$$=w\lambda^{\mathcal{N}}(\tau)+\boldsymbol{v}^\top\begin{bmatrix}-(1-\phi_s(\tau))\\0\end{bmatrix}$$

$$+\lim_{\eta\to 0}\frac{\begin{pmatrix}1-w\eta\sqrt{\tau}-2\boldsymbol{v}^\top\left(\boldsymbol{\alpha}^{\mathcal{N}}(\tau)-\eta\left(\boldsymbol{\alpha}^{\mathcal{N}}(\tau)\odot\boldsymbol{\gamma}^{\mathcal{N}}(\tau)\right)\right)\\-\sqrt{\left(1-2\boldsymbol{v}^\top\boldsymbol{\alpha}^{\mathcal{N}}(\tau)\right)\left(1-2\eta\left(w\sqrt{\tau}-2\boldsymbol{v}^\top\left(\boldsymbol{\alpha}^{\mathcal{N}}(\tau)\odot\boldsymbol{\gamma}^{\mathcal{N}}(\tau)\right)\right)-2\boldsymbol{v}^\top\boldsymbol{\alpha}^{\mathcal{N}}(\tau)\right)}\end{pmatrix}}{\eta^2}$$

$$=w\lambda^{\mathcal{N}}(\tau)+\boldsymbol{v}^\top\boldsymbol{K}_s^{\mathcal{N}}(\tau)$$

$$-\lim_{\eta\to 0}\frac{\left(w\sqrt{\tau}-2\boldsymbol{v}^\top\left(\boldsymbol{\alpha}^{\mathcal{N}}(\tau)\odot\boldsymbol{\gamma}^{\mathcal{N}}(\tau)\right)\right)\left(1-\sqrt{\dfrac{1-2\boldsymbol{v}^\top\boldsymbol{\alpha}^{\mathcal{N}}(\tau)}{1-2\eta\left(w\sqrt{\tau}-2\boldsymbol{v}^\top\left(\boldsymbol{\alpha}^{\mathcal{N}}(\tau)\odot\boldsymbol{\gamma}^{\mathcal{N}}(\tau)\right)\right)-2\boldsymbol{v}^\top\boldsymbol{\alpha}^{\mathcal{N}}(\tau)}}\right)}{2\eta}$$

$$=w\lambda^{\mathcal{N}}(\tau)-v_1\left(1-\phi_s(\tau)\right)+\frac{\left(w\sqrt{\tau}-2\boldsymbol{v}^\top\left(\boldsymbol{\alpha}^{\mathcal{N}}(\tau)\odot\boldsymbol{\gamma}^{\mathcal{N}}(\tau)\right)\right)^2}{2\left(1-2\boldsymbol{v}^\top\boldsymbol{\alpha}^{\mathcal{N}}(\tau)\right)}=\mathcal{B}_s^{\mathcal{N}}(w,\boldsymbol{v};\tau).$$

The convergence of $\mathcal{B}_q^{\mathcal{IG}}(w,\boldsymbol{v};\tau)$ is obtained in a similar fashion.

SM.D.2 Proof of Proposition 2 for the IG(C) case

The weak limit for the IG(C) model is computed similarly to the HN(C) case. We therefore only summarize the main quantities needed for this evaluation.

The first-order derivatives evaluated at $w=0$ and $\boldsymbol{v}=\boldsymbol{0}$ of the step dependent coefficients of the IG(C) model provided in Equations (A.4) to (A.6) are almost identical to those obtained in the HN(C) case. The only exceptions are the coefficients $\mathcal{B}_{u,1,0,0}^{\mathcal{IG}}(\tau)$, which are now given by

$$\mathcal{B}_{u,1,0,0}^{\mathcal{IG}}(\tau)=\lambda(\tau)+\frac{\sqrt{\tau}}{\eta}-\frac{\tau}{\eta},$$

for $u\in\{s,q\}$. It follows that, based on the convergence rates of Assumption 3.A, we recover the same limiting coefficients as in the HN(C) case with the following difference:

$$\mathcal{B}_{u,1,0,0}^{\mathcal{IG}}=\lim_{\tau\to 0}\frac{\lambda(\tau)+\dfrac{\sqrt{\tau}}{\eta}-\dfrac{\tau}{\eta}}{\tau}=\lambda,$$

which consequently leads to the drift term of Equation (A.10):

$$\boldsymbol{A}^{\mathcal{IG}}(\boldsymbol{X}_t)=\begin{bmatrix}r+\lambda h_t\\-\kappa_s s_t\\-\kappa_q\left(q_t-\sigma^2\right)\end{bmatrix}.$$

For any $u\in\{s,q\}$, we compute below the second-order limits for the IG(C) model based on the convergence rates from Assumption 3.B:

$$\mathcal{A}_{2,0,0}^{\mathcal{IG}}=\lim_{\tau\to 0}\frac{\mathcal{A}_{2,0,0}^{\mathcal{IG}}(\tau)}{\tau}=\lim_{\tau\to 0}\frac{0}{\tau}=0,$$

$$\mathcal{A}_{1,1,0}^{\mathcal{IG}}=\lim_{\tau\to 0}\frac{\mathcal{A}_{1,1,0}^{\mathcal{IG}}(\tau)}{\tau}=\lim_{\tau\to 0}\frac{0}{\tau}=0,$$

$$
\begin{aligned}
\mathcal{A}^{\mathcal{JG}}_{1,0,1} &= \lim_{\tau\to 0}\frac{\mathcal{A}^{\mathcal{JG}}_{1,0,1}(\tau)}{\tau} = \lim_{\tau\to 0}\frac{0}{\tau} = 0,\\
\mathcal{A}^{\mathcal{JG}}_{0,2,0} &= \lim_{\tau\to 0}\frac{\mathcal{A}^{\mathcal{JG}}_{0,2,0}(\tau)}{\tau} = \lim_{\tau\to 0}\frac{2\eta^8\alpha_s^2(\tau)}{\tau} = 2\eta^8\alpha_s^2,\\
\mathcal{A}^{\mathcal{JG}}_{0,1,1} &= \lim_{\tau\to 0}\frac{\mathcal{A}^{\mathcal{JG}}_{0,1,1}(\tau)}{\tau} = \lim_{\tau\to 0}\frac{2\eta^8\alpha_s(\tau)\alpha_q(\tau)}{\tau} = 2\eta^8\alpha_s\alpha_q,\\
\mathcal{A}^{\mathcal{JG}}_{0,0,2} &= \lim_{\tau\to 0}\frac{\mathcal{A}^{\mathcal{JG}}_{0,0,2}(\tau)}{\tau} = \lim_{\tau\to 0}\frac{2\eta^8\alpha_q^2(\tau)}{\tau} = 2\eta^8\alpha_q^2,\\
\mathcal{B}^{\mathcal{JG}}_{u,2,0,0} &= \lim_{\tau\to 0}\frac{\mathcal{B}^{\mathcal{JG}}_{u,2,0,0}(\tau)}{\tau} = \lim_{\tau\to 0}\frac{\tau}{\tau} = 1,\\
\mathcal{B}^{\mathcal{JG}}_{u,1,1,0} &= \lim_{\tau\to 0}\frac{\mathcal{B}^{\mathcal{JG}}_{u,1,1,0}(\tau)}{\tau} = \lim_{\tau\to 0}\frac{\sqrt{\tau}\alpha_s(\tau)\left(\gamma_s(\tau)-\eta^4\right)/\eta}{\tau} = \frac{\alpha_s\left(\gamma_s-\eta^4\right)}{\eta},\\
\mathcal{B}^{\mathcal{JG}}_{u,1,0,1} &= \lim_{\tau\to 0}\frac{\mathcal{B}^{\mathcal{JG}}_{u,1,0,1}(\tau)}{\tau} = \lim_{\tau\to 0}\frac{\sqrt{\tau}\alpha_q(\tau)\left(\gamma_q(\tau)-\eta^4\right)/\eta}{\tau} = \frac{\alpha_q\left(\gamma_q-\eta^4\right)}{\eta},\\
\mathcal{B}^{\mathcal{JG}}_{u,0,2,0} &= \lim_{\tau\to 0}\frac{\mathcal{B}^{\mathcal{JG}}_{u,0,2,0}(\tau)}{\tau} = \lim_{\tau\to 0}\frac{\alpha_s^2(\tau)\left(\gamma_s(\tau)-\eta^4\right)^2/\eta^2}{\tau} = \frac{\alpha_s^2\left(\gamma_s-\eta^4\right)^2}{\eta^2},\\
\mathcal{B}^{\mathcal{JG}}_{u,0,1,1} &= \lim_{\tau\to 0}\frac{\mathcal{B}^{\mathcal{JG}}_{u,0,1,1}(\tau)}{\tau} = \lim_{\tau\to 0}\frac{\alpha_s(\tau)\alpha_q(\tau)\left(\gamma_s(\tau)-\eta^4\right)\left(\gamma_q(\tau)-\eta^4\right)/\eta^2}{\tau} = \frac{\alpha_s\alpha_q\left(\gamma_s-\eta^4\right)\left(\gamma_q-\eta^4\right)}{\eta^2},\\
\mathcal{B}^{\mathcal{JG}}_{u,0,0,2} &= \lim_{\tau\to 0}\frac{\mathcal{B}^{\mathcal{JG}}_{u,0,0,2}(\tau)}{\tau} = \lim_{\tau\to 0}\frac{\alpha_q^2(\tau)\left(\gamma_q(\tau)-\eta^4\right)^2/\eta^2}{\tau} = \frac{\alpha_q^2\left(\gamma_q-\eta^4\right)^2}{\eta^2}.
\end{aligned}
$$

Plugging these values into the matrix of asymptotic second-order cross-cumulants yields

$$
\boldsymbol{B}^{\mathcal{JG}}(\boldsymbol{X}_t) = \begin{bmatrix}
h_t & \frac{\alpha_s\left(\gamma_s-\eta^4\right)}{\eta}h_t & \frac{\alpha_q\left(\gamma_q-\eta^4\right)}{\eta}h_t\\
\frac{\alpha_s\left(\gamma_s-\eta^4\right)}{\eta}h_t & 2\eta^8\alpha_s^2+\frac{(\alpha_s)^2\left(\gamma_s-\eta^4\right)^2}{\eta^2}h_t & 2\eta^8\alpha_s\alpha_q+\frac{\alpha_s\alpha_q\left(\gamma_s-\eta^4\right)\left(\gamma_q-\eta^4\right)}{\eta^2}h_t\\
\frac{\alpha_q\left(\gamma_q-\eta^4\right)}{\eta}h_t & 2\eta^8\alpha_s\alpha_q+\frac{\alpha_s\alpha_q\left(\gamma_s-\eta^4\right)\left(\gamma_q-\eta^4\right)}{\eta^2}h_t & 2\eta^8\alpha_q^2+\frac{\left(\alpha_q\right)^2\left(\gamma_q-\eta^4\right)^2}{\eta^2}h_t
\end{bmatrix}.
$$

Performing a Cholesky decomposition of the above covariance matrix leads to the following diffusion term:

$$
\boldsymbol{\Sigma}^{\mathcal{JG}}(\boldsymbol{X}_t) = \begin{bmatrix}
\sqrt{h_t} & 0 & 0\\
\frac{\alpha_s\left(\gamma_s-\eta^4\right)}{\eta}\sqrt{h_t} & \sqrt{2}\eta^4\alpha_s & 0\\
\frac{\alpha_q\left(\gamma_q-\eta^4\right)}{\eta}\sqrt{h_t} & \sqrt{2}\eta^4\alpha_q & 0
\end{bmatrix}.
$$

The resulting limit is again a degenerate diffusion driven by two mutually independent Brownian motions W_1 and W_2.

We verify now Condition (iii) of Theorem SM.1 in the context of IG(C). Letting $\delta = 2$ and taking the fourth-order derivatives of the moment generating function of $\Delta\boldsymbol{X}^{(n)}_{(k+1)\tau}$ conditionally on $\mathcal{F}^{(n)}_{k\tau}$ leads to

$$
\tau^{-1}\mathrm{E}\left[\left|\Delta Y^{(n)}_{(k+1)\tau}\right|^4 \mid \mathcal{F}^{(n)}_{k\tau}\right] = L^{\mathcal{JG}}_Y\tau + \mathcal{O}(\tau),
$$

$$\tau^{-1}\mathrm{E}\left[\left|\Delta s^{(n)}_{(k+1)\tau}\right|^4 \mid \mathcal{F}^{(n)}_{k\tau}\right] = L^{\mathcal{IG}}_s\tau + \mathcal{O}(\tau),$$
$$\tau^{-1}\mathrm{E}\left[\left|\Delta q^{(n)}_{(k+1)\tau}\right|^4 \mid \mathcal{F}^{(n)}_{k\tau}\right] = L^{\mathcal{IG}}_q\tau + \mathcal{O}(\tau),$$

where $L^{\mathcal{IG}}_Y$, $L^{\mathcal{IG}}_s$ and $L^{\mathcal{IG}}_q$ are finite quantities that depend on $X^{(n)}_{k\tau}$ and the underlying diffusion limit parameters. Finally, the verification of the multidimensional Feller constraints from Duffie and Kan (1996) for the IG(C) diffusion limit is done in a similar way as for the HN(C) limit, see Appendix A.2.2 from the manuscript for details.

SM.D.3 Proof of Proposition 3 for the IG(C) case

The risk-neutral limit of the IG(C) model follows the same prescription as the proof for the HN(C) counterpart from Appendix A.2.3.

First, we solve the martingale constraints in Equations (8) and (9) based on the coefficients of the $\mathcal{F}^{(n)}_{k\tau}$-conditional joint cgf of $\Delta Y^{(n)}_{(k+1)\tau}$, $\Delta s^{(n)}_{(k+1)\tau}$, and $\Delta q^{(n)}_{(k+1)\tau}$ under $\mathbb{P}^{(n)}$ computed in Equations (A.4) to (A.6). We obtain the following relationship between $\theta(\tau)$ and $\boldsymbol{\xi}(\tau)$

$$\theta(\tau) = \frac{1 - 2\boldsymbol{\xi}(\tau)^\top\left(\boldsymbol{\alpha}(\tau)\odot\boldsymbol{\gamma}(\tau)\right)}{2\eta\sqrt{\tau}} - \frac{\left(\eta\left(\eta\lambda(\tau)-\tau\right)^2 + 2\sqrt{\tau}\left(1 - 2\eta^4\boldsymbol{\xi}(\tau)^\top\boldsymbol{\alpha}(\tau)\right)\right)^2}{8\eta\sqrt{\tau}\left(\eta\lambda(\tau)-\tau\right)^2\left(1 - 2\eta^4\boldsymbol{\xi}(\tau)^\top\boldsymbol{\alpha}(\tau)\right)}. \tag{SM.7}$$

Using the limiting parameter constraints of Assumptions 3 and 4, the limit of the equity risk preference parameter exists and is given by

$$\theta = \lim_{\tau\to 0}\theta(\tau) = -\frac{1}{2} - \lambda + \eta^3\boldsymbol{\alpha}^\top\boldsymbol{\xi} - \frac{\boldsymbol{\xi}^\top\left(\boldsymbol{\alpha}\odot\boldsymbol{\gamma}\right)}{\eta}. \tag{SM.8}$$

Note that under the parametric relationships between the HN(C) and IG(C) parameters from Equation (24) one can obtain the equity risk preference parameter for the HN(C) model from Equation (A.18) as a limit when η approaches zero of its counterpart from Equation (SM.8).

For any non-negative integers i, j, k with $1 \le i + j + k \le 2$, and $u \in \{s, q\}$, we introduce the first- and second-order derivatives of the risk-neutral coefficients evaluated at $w = 0$ and $\boldsymbol{\nu} = \mathbf{0}$:

$$\tilde{\mathcal{A}}^{\mathcal{IG}}_{i,j,k}(\tau) = \left.\frac{\partial^{i+j+k}\tilde{\mathcal{A}}^{\mathcal{IG}}(w,\boldsymbol{\nu};\tau)}{\partial w^i\partial\nu_1^j\partial\nu_2^k}\right|_{w=0,\boldsymbol{\nu}=\mathbf{0}} = \left.\frac{\partial^{i+j+k}\mathcal{A}^{\mathcal{IG}}(w,\boldsymbol{\nu};\tau)}{\partial w^i\partial\nu_1^j\partial\nu_2^k}\right|_{w=\theta(\tau),\boldsymbol{\nu}=\boldsymbol{\xi}(\tau)}, \tag{SM.9}$$

$$\tilde{\mathcal{B}}^{\mathcal{IG}}_{u,i,j,k}(\tau) = \left.\frac{\partial^{i+j+k}\tilde{\mathcal{B}}^{\mathcal{IG}}_u(w,\boldsymbol{\nu};\tau)}{\partial w^i\partial\nu_1^j\partial\nu_2^k}\right|_{w=0,\boldsymbol{\nu}=\mathbf{0}} = \left.\frac{\partial^{i+j+k}\mathcal{B}^{\mathcal{IG}}_u(w,\boldsymbol{\nu};\tau)}{\partial w^i\partial\nu_1^j\partial\nu_2^k}\right|_{w=\theta(\tau),\boldsymbol{\nu}=\boldsymbol{\xi}(\tau)}. \tag{SM.10}$$

Using Equations (SM.7) and (SM.8), together with the limiting parametric constraints from Assumptions 3 and 4, we compute the first-order derivative limits as follows

$$\begin{aligned}
\tilde{\mathcal{A}}^{\mathcal{IG}}_{1,0,0} &= r,\\
\tilde{\mathcal{A}}^{\mathcal{IG}}_{0,1,0} &= 2\eta^8\alpha_s\boldsymbol{\alpha}^\top\boldsymbol{\xi},\\
\tilde{\mathcal{A}}^{\mathcal{IG}}_{0,0,1} &= \kappa_q\sigma^2 + 2\eta^8\alpha_q\boldsymbol{\alpha}^\top\boldsymbol{\xi},\\
\tilde{\mathcal{B}}^{\mathcal{IG}}_{u,1,0,0} &= -\frac{1}{2},\\
\tilde{\mathcal{B}}^{\mathcal{IG}}_{s,0,1,0} &= -\kappa_s - \frac{\alpha_s\left(\gamma_s - \eta^4\right)}{\eta}\left(\lambda + \frac{1}{2}\right),
\end{aligned}$$

$$\tilde{\mathcal{B}}^{\mathcal{JG}}_{s,0,0,1} = -\frac{\alpha_q\left(\gamma_q - \eta^4\right)}{\eta}\left(\lambda + \frac{1}{2}\right),$$
$$\tilde{\mathcal{B}}^{\mathcal{JG}}_{q,0,1,0} = -\frac{\alpha_s\left(\gamma_s - \eta^4\right)}{\eta}\left(\lambda + \frac{1}{2}\right),$$
$$\tilde{\mathcal{B}}^{\mathcal{JG}}_{q,0,0,1} = -\kappa_q - \frac{\alpha_q\left(\gamma_q - \eta^4\right)}{\eta}\left(\lambda + \frac{1}{2}\right).$$

Replacing these quantities into Equation (A.10) allows us to obtain the risk-neutral drift coefficient from Equation (27). The limits of the second-order derivatives are the same as under the physical measure, which leads to the same covariance matrix $\tilde{\boldsymbol{\Sigma}}^{\mathcal{JG}}(\boldsymbol{X}_t) = \boldsymbol{\Sigma}^{\mathcal{JG}}(\boldsymbol{X}_t)$. Condition (iii) of Theorem SM.1 (see Section SM.A) and the multidimensional Feller constraints from Duffie and Kan (1996) are verified in the same way as in Appendix A.2.2.

SM.D.4 Affine property of the HNSV model and the characterization of its weak limits

We show that the HNSV model of Equations (41) to (42) satisfies the affine property of Assumption 1. Indeed, for any $w, v \in \mathbb{R}$, the $\mathcal{F}^{(n)}_{k\tau}$-conditional cgf of $\Delta Y^{(n)}_{(k+1)\tau}$ and $\Delta h^{(n)}_{(k+1)\tau}$ is computed as

$$\begin{aligned}
&\mathcal{C}^{\mathbb{P}^{(n)}}_{\Delta Y^{(n)}_{(k+1)\tau}, \Delta h^{(n)}_{(k+1)\tau}}\left(w, v \,\middle|\, \mathcal{F}^{(n)}_{k\tau}\right) = \log\left(\mathrm{E}^{\mathbb{P}^{(n)}}\left[\exp\left(w\,\Delta Y^{(n)}_{(k+1)\tau} + v\,\Delta h^{(n)}_{(k+1)\tau}\right)\middle|\, \mathcal{F}^{(n)}_{k\tau}\right]\right)\\
&= wr\tau + v\left(\left(1-\phi(\tau)\right)\sigma^2(\tau) - \alpha(\tau)\right) + \left(w\lambda(\tau) - v\left(1-\phi(\tau)\right)\right)h^{(n)}_{k\tau}\\
&\quad + \log\left(\mathrm{E}^{\mathbb{P}^{(n)}}\left[\exp\left(w\sqrt{\tau}\sqrt{h^{(n)}_{k\tau}}\,\epsilon^{(n)}_{(k+1)\tau} + v\alpha(\tau)\left(\psi^{(n)2}_{(k+1)\tau} - 2\gamma(\tau)\sqrt{h^{(n)}_{k\tau}}\,\psi^{(n)}_{(k+1)\tau}\right)\right)\middle|\, \mathcal{F}^{(n)}_{k\tau}\right]\right)\\
&= wr\tau + v\left(\left(1-\phi(\tau)\right)\sigma^2(\tau) - \alpha(\tau)\right) + \left(w\lambda(\tau) - v\left(1-\phi(\tau)\right)\right)h^{(n)}_{k\tau}\\
&\quad + \mathcal{C}^{\mathbb{P}^{(n)}}_{\epsilon^{(n)}_{(k+1)\tau}, \epsilon^{(n)2}_{(k+1)\tau}, U^{(n)}_{(k+1)\tau}, U^{(n)2}_{(k+1)\tau}, \epsilon^{(n)}_{(k+1)\tau}U^{(n)}_{(k+1)\tau}}\left(\delta_1(\tau)\sqrt{h^{(n)}_{k\tau}}, \delta_2(\tau), \delta_3(\tau)\sqrt{h^{(n)}_{k\tau}}, \delta_4(\tau), \delta_5(\tau)\,\middle|\, \mathcal{F}^{(n)}_{k\tau}\right).
\end{aligned}$$

Here, $U^{(n)}_{(k+1)\tau}$ is a standard Gaussian random variable independent of $\epsilon^{(n)}_{(k+1)\tau}$ conditionally on $\mathcal{F}^{(n)}_{k\tau}$ that satisfies the following representation:

$$\psi^{(n)}_{(k+1)\tau} = \rho\epsilon^{(n)}_{(k+1)\tau} + \sqrt{1-\rho^2}U^{(n)}_{(k+1)\tau},$$

and $\delta_1(\tau), \ldots, \delta_5(\tau)$ are given by

$$\delta_1(\tau) = w\sqrt{\tau} - 2v\rho\alpha(\tau)\gamma(\tau), \qquad \delta_2(\tau) = v\rho^2\alpha(\tau), \qquad \delta_3(\tau) = -2v\sqrt{1-\rho^2}\alpha(\tau)\gamma(\tau),$$
$$\delta_4(\tau) = v\left(1-\rho^2\right)\alpha(\tau), \qquad \delta_5(\tau) = 2v\rho\sqrt{1-\rho^2}\alpha(\tau).$$

Using the fact that for any two independent standard Gaussian random variables Z_1 and Z_2, the joint cgf of Z_1, Z_1^2, Z_2, Z_2^2, and Z_1Z_2 is given by

$$\begin{aligned}
\mathcal{C}_{(Z_1, Z_1^2, Z_2, Z_2^2, Z_1Z_2)}(u_1, u_2, u_3, u_4, u_5) &= \frac{u_1^2(1-2u_4) + u_3^2(1-2u_2) + 2u_1u_3u_5}{2\left[(1-2u_2)(1-2u_4) - u_5^2\right]}\\
&\quad - \frac{1}{2}\log\left(\left[(1-2u_2)(1-2u_4) - u_5^2\right]\right),
\end{aligned} \tag{SM.11}$$

we obtain after some straightforward algebraic manipulations the following affine representation for the $\mathcal{F}_{k\tau}^{(n)}$-conditional cgf of $\Delta Y_{(k+1)\tau}^{(n)}$ and $\Delta h_{(k+1)\tau}^{(n)}$

$$\mathcal{C}^{\mathbb{P}^{(n)}}_{\Delta Y_{(k+1)\tau}^{(n)},\Delta h_{(k+1)\tau}^{(n)}}\left(w,v\,\middle|\,\mathcal{F}_{k\tau}^{(n)}\right)=\mathcal{A}\left(w,v;\tau\right)+\mathcal{B}_h\left(w,v;\tau\right)h_{k\tau}^{(n)},$$

where the coefficients $\mathcal{A}\left(w,v;\tau\right)$ and $\mathcal{B}_h\left(w,v;\tau\right)$ are given by

$$\mathcal{A}\left(w,v;\tau\right)=wr\tau+v\left(\left(1-\phi(\tau)\right)\sigma^2(\tau)-\alpha(\tau)\right)-\frac{1}{2}\log\left(1-2v\alpha(\tau)\right),$$

$$\mathcal{B}_h\left(w,v;\tau\right)=w\lambda(\tau)-v\left(1-\phi(\tau)\right)+\frac{w^2\tau\left(1-\rho^2\right)}{2}+\frac{\left(w\rho\sqrt{\tau}-2v\alpha(\tau)\gamma(\tau)\right)^2}{2\left(1-2v\alpha(\tau)\right)}.$$

Since the HNSV model is affine, the proof of Proposition 5 follows by computations that are analogous to those used to derive the diffusion limits in Corollary 1. The new parametric convergence rates in Assumption 5 only affect the computation of the diffusion matrix of the weak limit of the HNSV model. For all cases, the drift coefficient $\boldsymbol{A}(\boldsymbol{X}_t)$ is identical to the one derived for the HN model in Corollary 1. However, the limits of the second-order derivatives per discretization step of Appendix A.2.2 must be recomputed under each of the convergence rate choices for the parameters $\alpha(\tau)$ and $\gamma(\tau)$. For all cases, we have $B_{11}\left(\boldsymbol{X}_t\right)=h_t$. When $\delta_1=0$ and $\delta_2=0$, we obtain $B_{12}\left(\boldsymbol{X}_t\right)=-2\alpha\gamma\rho h_t$ and $B_{22}\left(\boldsymbol{X}_t\right)=2\alpha^2+4\alpha^2\gamma^2h_t$. When $\delta_1>0$ and $\delta_2=-\delta_1$, we obtain $B_{12}\left(\boldsymbol{X}_t\right)=-2\alpha\gamma\rho h_t$ and $B_{22}\left(\boldsymbol{X}_t\right)=4\alpha^2\gamma^2h_t$. When $\delta_1=0$ and $\delta_2>0$, we obtain $B_{12}\left(\boldsymbol{X}_t\right)=0$ and $B_{22}\left(\boldsymbol{X}_t\right)=2\alpha^2$. Finally, when $\delta_1>0$ and $\delta_2>-\delta_1$, we obtain $B_{12}\left(\boldsymbol{X}_t\right)=0$ and $B_{22}\left(\boldsymbol{X}_t\right)=0$. For each case, the expressions for the diffusion matrices $\boldsymbol{\Sigma}\left(\boldsymbol{X}_t\right)$ follow from the Cholesky decomposition of the covariance matrices $\boldsymbol{B}\left(\boldsymbol{X}_t\right)$ with the above entries; that is,

$$\boldsymbol{\Sigma}\left(\boldsymbol{X}_t\right)=\begin{cases}\begin{pmatrix}\sqrt{h_t} & 0\\ -2\alpha\gamma\rho\sqrt{h_t} & \sqrt{2\alpha^2+4\alpha^2\gamma^2\left(1-\rho^2\right)h_t}\end{pmatrix} & \text{if } \delta_1=0 \text{ and } \delta_2=0\\ \begin{pmatrix}\sqrt{h_t} & 0\\ -2\alpha\gamma\rho\sqrt{h_t} & 2\alpha\gamma\sqrt{1-\rho^2}\sqrt{h_t}\end{pmatrix} & \text{if } \delta_1>0 \text{ and } \delta_2=-\delta_1\\ \begin{pmatrix}\sqrt{h_t} & 0\\ 0 & \sqrt{2}\alpha\end{pmatrix} & \text{if } \delta_1=0 \text{ and } \delta_2>0\\ \begin{pmatrix}\sqrt{h_t} & 0\\ 0 & 0\end{pmatrix} & \text{if } \delta_1>0 \text{ and } \delta_2>-\delta_1\end{cases}.$$

SM.D.5 Weak limits of the CEVSV model

The CEVSV model does not satisfy the affine property of Assumption 1 for any $p>0$. Indeed, using a similar approach as in Section SM.D.4, we can show that for any $w,v\in\mathbb{R}$, the joint cgf of $\Delta Y_{(k+1)\tau}^{(n)}$ and $\Delta h_{(k+1)\tau}^{(n)}$ conditional on $\mathcal{F}_{k\tau}^{(n)}$ is

$$\mathcal{C}^{\mathbb{P}^{(n)}}_{\Delta Y_{(k+1)\tau}^{(n)},\Delta h_{(k+1)\tau}^{(n)}}\left(w,v\,\middle|\,\mathcal{F}_{k\tau}^{(n)}\right)=wr\tau+v\left(1-\phi(\tau)\right)\sigma^2(\tau)+\left(w\lambda(\tau)-v\left(1-\phi(\tau)\right)\right)h_{k\tau}^{(n)}-v\alpha(\tau)\left(h_{k\tau}^{(n)}\right)^p$$

$$+ \mathcal{C}^{\mathbb{P}^{(n)}}_{\epsilon^{(n)}_{(k+1)\tau},\epsilon^{(n)2}_{(k+1)\tau},U^{(n)}_{(k+1)\tau},U^{(n)2}_{(k+1)\tau},\epsilon^{(n)}_{(k+1)\tau}U^{(n)}_{(k+1)\tau}} \left(\delta^{(n)}_{1,k\tau},\delta^{(n)}_{2,k\tau},\delta^{(n)}_{3,k\tau},\delta^{(n)}_{4,k\tau},\delta^{(n)}_{5,k\tau} \,\middle|\, \mathcal{F}^{(n)}_{k\tau}\right), \quad \text{(SM.12)}$$

where $\delta^{(n)}_{1,k\tau},\ldots,\delta^{(n)}_{5,k\tau}$ are $\mathcal{F}^{(n)}_{k\tau}$-measurable processes given by

$$\begin{aligned}
\delta^{(n)}_{1,k\tau} &= w\sqrt{\tau}\sqrt{h^{(n)}_{k\tau}} - 2v\rho\alpha(\tau)\gamma(\tau)\left(h^{(n)}_{k\tau}\right)^p, & \delta^{(n)}_{2,k\tau} &= v\rho^2\alpha(\tau)\left(h^{(n)}_{k\tau}\right)^p,\\
\delta^{(n)}_{3,k\tau} &= -2v\sqrt{1-\rho^2}\alpha(\tau)\gamma(\tau)\left(h^{(n)}_{k\tau}\right)^p, & \delta^{(n)}_{4,k\tau} &= v\left(1-\rho^2\right)\alpha(\tau)\left(h^{(n)}_{k\tau}\right)^p,\\
\delta^{(n)}_{5,k\tau} &= 2v\rho\sqrt{1-\rho^2}\alpha(\tau)\left(h^{(n)}_{k\tau}\right)^p.
\end{aligned}$$

Using the general expression of Equation (SM.11) and after some algebraic manipulations, we obtain

$$\begin{aligned}
\mathcal{C}^{\mathbb{P}^{(n)}}_{\Delta Y^{(n)}_{(k+1)\tau},\Delta h^{(n)}_{(k+1)\tau}}\left(w,v \,\middle|\, \mathcal{F}^{(n)}_{k\tau}\right) = {} & wr\tau + v\left(1-\phi(\tau)\right)\sigma^2(\tau) + \left(w\lambda(\tau) - v\left(1-\phi(\tau)\right) + \frac{w^2\tau\left(1-\rho^2\right)}{2}\right)h^{(n)}_{k\tau}\\
& - v\alpha(\tau)\left(h^{(n)}_{k\tau}\right)^p + \frac{\left(w\rho\sqrt{\tau}\sqrt{h^{(n)}_{k\tau}} - 2v\alpha(\tau)\gamma(\tau)\left(h^{(n)}_{k\tau}\right)^p\right)^2}{2\left(1-2v\alpha(\tau)\left(h^{(n)}_{k\tau}\right)^p\right)}\\
& - \frac{1}{2}\log\left(1-2v\alpha(\tau)\left(h^{(n)}_{k\tau}\right)^p\right).
\end{aligned}$$

For $p > 0$, the logarithmic term

$$-\frac{1}{2}\log\left(1-2v\alpha(\tau)\left(h^{(n)}_{k\tau}\right)^p\right)$$

is nonlinear in $h^{(n)}_{k\tau}$, so the conditional cgf is not affine. When $p = 0$, the denominator and logarithmic term are independent of $h^{(n)}_{k\tau}$. Expanding the squared term in the numerator, however, produces the term

$$-\frac{2wv\rho\alpha(\tau)\gamma(\tau)\sqrt{\tau}}{1-2v\alpha(\tau)}\sqrt{h^{(n)}_{k\tau}},$$

which is not affine in $h^{(n)}_{k\tau}$ unless $\rho\alpha(\tau)\gamma(\tau) = 0$. Apart from the trivial case $\alpha(\tau) = 0$, the CEVSV model is therefore affine when $p = 0$ only if $\rho = 0$ or $\gamma(\tau) = 0$, and is non-affine for every $p > 0$.

The proof of Proposition 6 follows in a similar way to that of Proposition 2 but relies on a direct computation of the first- and second-order limits of the conditional moments per discretization step instead of using the general methodology of Appendix A.2.1—only applicable to affine settings. Similarly to the HNSV setting, the drift coefficient $\boldsymbol{A}(\boldsymbol{X}_t)$ from the representation in Equation (17) is identical to the one derived for the HN model in Corollary 1, under all convergence rates considered in Assumption 5. The second-order limiting moments are computed as follows:

$$\begin{aligned}
\lim_{\tau\to 0}\frac{1}{\tau}\mathrm{Var}^{\mathbb{P}^{(n)}}\left[\Delta Y^{(n)}_{(k+1)\tau}\,\middle|\,\mathcal{F}^{(n)}_{k\tau}\right] &= h_t,\\
\lim_{\tau\to 0}\frac{1}{\tau}\mathrm{Var}^{\mathbb{P}^{(n)}}\left[\Delta h^{(n)}_{(k+1)\tau}\,\middle|\,\mathcal{F}^{(n)}_{k\tau}\right] &= \left(2\alpha^2\lim_{\tau\to 0}\tau^{2\delta_1} + 4\alpha^2\gamma^2\lim_{\tau\to 0}\tau^{2(\delta_1+\delta_2)}\right)h_t^{2p},\\
\lim_{\tau\to 0}\frac{1}{\tau}\mathrm{Cov}^{\mathbb{P}^{(n)}}\left[\Delta Y^{(n)}_{(k+1)\tau},\Delta h^{(n)}_{(k+1)\tau}\,\middle|\,\mathcal{F}^{(n)}_{k\tau}\right] &= -2\alpha\gamma\rho\left(\lim_{\tau\to 0}\tau^{\delta_1+\delta_2}\right)h_t^{p+1/2}.
\end{aligned}$$

These limits are evaluated for all possible choices of δ_1 and δ_2, and the diffusion matrices are then obtained

for each scenario using the Cholesky decomposition; that is,

$$\boldsymbol{\Sigma}(\boldsymbol{X}_t) = \begin{cases} \begin{pmatrix} \sqrt{h_t} & 0 \\ -2\alpha\gamma\rho h_t^p & \sqrt{2\alpha^2 + 4\alpha^2\gamma^2(1-\rho^2)}h_t^p \end{pmatrix} & \text{if } \delta_1 = 0 \text{ and } \delta_2 = 0 \\ \begin{pmatrix} \sqrt{h_t} & 0 \\ -2\alpha\gamma\rho h_t^p & 2\alpha\gamma\sqrt{1-\rho^2}h_t^p \end{pmatrix} & \text{if } \delta_1 > 0 \text{ and } \delta_2 = -\delta_1 \\ \begin{pmatrix} \sqrt{h_t} & 0 \\ 0 & \sqrt{2}\alpha h_t^p \end{pmatrix} & \text{if } \delta_1 = 0 \text{ and } \delta_2 > 0 \\ \begin{pmatrix} \sqrt{h_t} & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \delta_1 > 0 \text{ and } \delta_2 > -\delta_1 \end{cases}.$$

Note that Condition (iii) of Theorem SM.1 from Section SM.A of the Supplementary Material holds in a similar way as in the affine case presented in Appendix A.2.2.

To determine when these limiting diffusions are affine, recall that the drift coefficient is affine for every p, so it remains to examine $\boldsymbol{B}(\boldsymbol{X}_t) = \boldsymbol{\Sigma}(\boldsymbol{X}_t)\boldsymbol{\Sigma}(\boldsymbol{X}_t)^\top$. In the first two cases,

$$B_{12}(\boldsymbol{X}_t) = -2\alpha\gamma\rho h_t^{p+1/2},$$

while $B_{22}(\boldsymbol{X}_t)$ is proportional to h_t^{2p}. Hence, when the return-variance covariance is nonzero, affinity requires $p = 1/2$. If this covariance vanishes but the variance diffusion remains nonzero, affinity holds for $p \in \{0, 1/2\}$. In the third case, $B_{12}(\boldsymbol{X}_t) = 0$ and $B_{22}(\boldsymbol{X}_t) = 2\alpha^2 h_t^{2p}$, so the limit is affine for $p \in \{0, 1/2\}$. In the fourth case, $B_{12}(\boldsymbol{X}_t) = B_{22}(\boldsymbol{X}_t) = 0$, and the resulting deterministic-variance diffusion is affine for every p.

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