Supplementary material to "Nonparametric Identification and Estimation of Double Auctions with Bargaining"

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Abstract

The supplementary material includes three appendices. Appendix S.1 provides some asymptotic results when the private value densities are smoother than the continuously differentiable case. Appendix S.2 briefly discusses an extension of our identification and estimation to the case with auction-specific heterogeneity, unobserved heterogeneity, and estimation with only transacted bids. Appendix S.3 proves all results in the supplementary material.

S.1 Asymptotic results with smoother private value densities

This section provides supplementary asymptotic results for cases where the private value densities possess derivatives up to order R, with $R \geqslant 2$. Our estimators are defined as in Section 4.1, except that we employ a local polynomial method of order $p = R + 1$ to estimate the bid densities (g_B and g_S) and order $p = R$ for the value densities (f_V and f_C). These results establish the uniform convergence rates of the bid density, pseudo value, and value density estimates, assuming private value densities are smoother than those considered in Section 4. They are parallel to the asymptotic properties of Guerre, Perrigne, and Vuong (2000)'s two-step estimator of the private value density in first-price auctions.

We state the assumptions under which the supplementary asymptotic results are established.

Assumption F′. *F_V and F_C admit up to $R + 1$ continuous bounded derivatives on $[\underline{v}, \overline{v}]$ and $[\underline{c}, \overline{c}]$, respectively. In addition, $f_V(v) \geqslant \alpha_V > 0$ for all $v \in [\underline{v}, \overline{v}]$; $f_C(c) \geqslant \alpha_C > 0$ for all $c \in [\underline{c}, \overline{c}]$.*

Assumption G′. *The kernels are symmetric second-order kernels with support $[-1, 1]$ and have continuous bounded second order derivatives.*

Assumption H′. *The bandwidths h_B, h_S, h_V, h_C are of the form:*

$$h_B = \lambda_B\big(\log n/n\big)^{1/(2R+3)}, h_S = \lambda_S\big(\log n/n\big)^{1/(2R+3)}, h_V = \lambda_V\big(\log n/n\big)^{1/(2R+3)}, h_C = \lambda_C\big(\log n/n\big)^{1/(2R+3)},$$

where the λ's are positive constants.

Assumptions A, F and G can be viewed as a special case of Assumptions F′ to H′ with $R = 1$, respectively.

The first lemma generalizes Lemma 3 to the case with private value densities admitting up to R-th order derivatives.

Lemma S.1. *Given Assumptions B and C and Assumption F′, the distributions of regular equilibrium bids G_B and G_S satisfy:*

(i) for any $b \in [\underline{b}, \overline{b}]$ and any $s \in [\underline{s}, \overline{s}]$, $g_B(b) \geqslant \alpha_B > 0$, $g_S(s) \geqslant \alpha_S > 0$;

(ii) g_B and g_S admit up to $R+1$ continuous bounded derivatives on $[\underline{s}, \overline{b}]$.

The proof of Lemma S.1 is a straightforward extension of the proof for Lemma 3 in Appendix A.3, and is hence omitted here.

The second lemma studies the uniform convergence rate of bid density estimators and pseudo private values.

Lemma S.2. *Under Assumptions B to E and Assumptions F′ to H′,*

(i). $\sup_{b\in[\underline{s},\overline{b}]} |\hat{g}_B(b) - g_B(b)| = O_p\big((\log n/n)^{(R+1)/(2R+3)}\big)$, $\sup_{s\in[\underline{s},\overline{b}]} |\hat{g}_S(s) - g_S(s)| = O_p\big((\log n/n)^{(R+1)/(2R+3)}\big)$;

(ii). $\sup_i |\hat{V}_i - V_i| = O_p\big((\log n/n)^{(R+1)/(2R+3)}\big)$, $\sup_i |\hat{C}_i - C_i| = O_p\big((\log n/n)^{(R+1)/(2R+3)}\big)$.

Proof. See Appendix S.3.1. □

Lemma S.2 establishes that under the bandwidth conditions in Assumption H′ – where h_B and h_S are of order $(\log n/n)^{1/(2R+3)}$ – both the bid density estimators and pseudo private values achieve a uniform convergence rate of $(\log n/n)^{(R+1)/(2R+3)}$. Note that for the bid density estimators, this rate is established over the interval $[\underline{s}, \overline{b}]$.

The third result establishes the uniform convergence rate of the two-step estimators of value densities.

Proposition S.1. *Under Assumptions B to E and Assumptions F′ to H′, for any (fixed) closed inner subsets $\mathscr{C}_V$ of $[\underline{v}, \overline{v}]$ and $\mathscr{C}_C$ of $[\underline{c}, \overline{c}]$,*

$$\sup_{v\in\mathscr{C}_V} |\hat{f}_V(v) - f_V(v)| = O_p\big((\log n/n)^{R/(2R+3)}\big), \quad \sup_{c\in\mathscr{C}_C} |\hat{f}_C(c) - f_C(c)| = O_p\big((\log n/n)^{R/(2R+3)}\big).$$

Proof. See Appendix S.3.2. □

Proposition S.1 establishes the uniform consistency of our two-step estimators of the bidders' private value densities on any closed inner subset of value support. The rate of convergence coincides with the optimal convergence rate of Guerre, Perrigne, and Vuong (2000) for the first-price auctions.

S.2 Extensions

S.2.1 Auction-Specific Heterogeneity

We now briefly discuss how to generalize our identification and estimation approach to allow auction-specific heterogeneity.[1] Let $X \in \mathbb{R}^d$ be a random vector that characterizes the heterogeneity of auctions. For auctions with $X = x$, let $F_{V|X}(\cdot|x)$ and $F_{C|X}(\cdot|x)$ be the private value distributions of the buyers and sellers, and $G_{B|X}(\cdot|x)$ and $G_{S|X}(\cdot|x)$ be their respective bid distribution functions with densities $g_{B|X}(\cdot|x)$ and $g_{S|X}(\cdot|x)$. Let all of our previous assumptions hold for every x in the support of X wherever it applies. The buyer's and the seller's inverse bidding functions in an auction with characteristic $X = x$ are, respectively,

$$v = \begin{cases} b + k(x) \cdot \dfrac{G_{S|X}(b \mid x)}{g_{S|X}(b \mid x)}, & \text{if } b \geqslant \underline{s}(x), \\ b, & \text{otherwise,} \end{cases} \qquad c = \begin{cases} s - (1 - k(x)) \cdot \dfrac{1 - G_{B|X}(s \mid x)}{g_{B|X}(s \mid x)}, & \text{if } s \leqslant \overline{b}(x), \\ s, & \text{otherwise,} \end{cases} \tag{S.2.1}$$

where $\underline{s}(x)$ is the lower bound of the support of $G_{S|X}(\cdot \mid x)$, $\overline{b}(x)$ is the upper bound of the support of $G_{B|X}(\cdot \mid x)$. Note that the weight $k(x)$ depends on the realization of heterogeneity X in this case.

We can then generalize most of our identification and estimation results to auctions with heterogeneity. Specifically, our identification and model restrictions results (Theorems 1 to 4 and Lemma 4) still hold as long as the value and bid distributions are simply replaced by the corresponding conditional distributions given X and all relevant conditions hold for every realization of X.

For estimation, our two-step procedure can be generalized to incorporate auction-specific heterogeneity. In the first step, for each auction, we use (S.2.1) to recover both the buyers' and the sellers' pseudo private values. Notice that, in (S.2.1), the estimation of conditional bid densities $g_{S|X}$ and $g_{B|X}$ needs to first recover the joint densities g_{SX} and g_{BX} of the bids and the covariates (as well as the marginal density f_X of the covariates), since

[1] The existence of auction-specific heterogeneity allows for correlation between the buyer's and the seller's private values. However, such a correlation exists only through the auction-specific heterogeneity.

$g_{S|X}(s \mid x) = g_{SX}(s, x)/f_X(x)$ and $g_{B|X}(b \mid x) = g_{BX}(b, x)/f_X(x)$. In the second step, we use the covariate data $\{X_1, \ldots, X_n\}$ and pseudo private values recovered previously to estimate the conditional value densities $f_{C|X}$ and $f_{V|X}$. Again, this needs the estimation of joint densities of valuation and covariates f_{CX} and f_{VX}. It is then possible to extend our estimation results in Section 4 to this new two-step estimator. However, the new estimator will suffer the "curse of dimensionality" with the introduction of auction-specific heterogeneity $X \in \mathbb{R}^d$. Moreover, for $d \geqslant 1$, the (interior and boundary) bias correction in kernel estimation of bid densities g_{SX} and g_{BX} will be an issue in a multi-dimensional scenario.[2] This issue is challenging, in that, to our knowledge, little is known in the existing literature regarding the boundary bias correction of kernel density estimators in a multi-dimensional setting.

Alternatively, we can deal with the auction-specific heterogeneity by a bid homogenizing procedure as described in Haile, Hong, and Shum (2003) and Liu and Luo (2017). Suppose that the bidders have a bidding strategy of $s_1(\cdot)$. They have shown that, under CRRA utility function, in another auction where value is a random variable $\widetilde{v}_i = \delta v_i$, the equilibrium bidding strategy is $\widetilde{b}_i = \delta s_1(v_i)$. They eventually obtained some so called "homogenized bid" which is the bid a bidder would submit in an "average" auction having $\mathbb{I}$ bidders:

$$b^*(\epsilon|\mathbb{I}) \equiv \exp(\delta_0 + b_0(\mathbb{I}) + \widetilde{b}(\epsilon|\mathbb{I})).$$

S.2.2 Unobserved heterogeneity

Our framework can incorporate an auction-level unobserved heterogeneity. We omit the conditioning on the observed covariates X to simplify our discussion. Let $\tilde{X}$ represent the unobserved heterogeneity, namely $\tilde{X}$ is observed by all bidders but unobserved by researchers. All bidders can hence condition on it when bidding.

We consider that the buyer's value (resp. seller's cost) has a multiplicatively separable form of $V = \tilde{X} \cdot \epsilon$ (resp. $C = \tilde{X} \cdot \delta$) where ϵ and δ are private information to the buyer and the seller, respectively. Let $\beta_{B,\tilde{x}}(\cdot)$ and $\beta_{S,\tilde{x}}(\cdot)$ be the buyer and seller bidding strategies under $\tilde{X} = \tilde{x}$. Suppose that $\beta_{B,1}(\cdot)$ and $\beta_{S,1}(\cdot)$ are the equilibrium bidding strategies under $\tilde{X} = 1$, that is, they satisfy the first-order conditions (3.5) and (3.6) when $\tilde{X} = 1$. It can be verified that $\beta_{B,\tilde{x}}(\cdot) = \tilde{x} \cdot \beta_{B,1}(\cdot/\tilde{x})$ and $\beta_{S,\tilde{x}}(\cdot) = \tilde{x} \cdot \beta_{S,1}(\cdot/\tilde{x})$ also satisfy (3.5) and (3.6) and are therefore equilibrium bidding strategies when $\tilde{X} = \tilde{x}$.

Let $B = \beta_{B,\tilde{X}}(V)$ and $S = \beta_{S,\tilde{X}}(C)$. The above discussion yields $B = \tilde{X} \cdot \beta_{B,1}(V/\tilde{X}) = \tilde{X} \cdot \beta_{B,1}(\epsilon)$ and $S = \tilde{X} \cdot \beta_{S,1}(C/\tilde{X}) = \tilde{X} \cdot \beta_{S,1}(\delta)$. Taking logarithms gives

$$\log(B) = \log(\tilde{X}) + \log(\beta_{B,1}(\epsilon))$$

[2] Notice that the supports of S and B are finite. In addition, the bid densities can have discontinuity points in the interior of the supports.

$$\log(S) = \log(\tilde{X}) + \log(\beta_{S,1}(\delta)).$$

We can then apply the deconvolution approach of Krasnokutskaya (2011) to identify the distributions of $\tilde{X}$, ϵ, and δ under some scale normalizations.

S.2.3 Estimation with transacted bids

We consider an estimation procedure closely following the identification strategy proposed in Section 3.3.

We first recover the marginal bid distributions $G_B(\cdot)$ and $G_S(\cdot)$ in the transacted bids interval $[\underline{s}, \overline{b}]$ as

$$\hat{G}_B(b) = 1 - \widehat{\Pr}(B > b | S = \underline{s}), \quad \hat{G}_S(s) = \widehat{\Pr}(S \leqslant s | B = \overline{b}),$$

where $\widehat{\Pr}(B > b|S = \underline{s})$ and $\widehat{\Pr}(S \leqslant s|B = \overline{b})$ are some nonparametric smoothing estimators. The densities are then estimated by the derivatives as $\hat{g}_B(b) = \hat{G}'_B(b)$ and $\hat{g}_S(s) = \hat{G}'_S(s)$.

In the second step, we recover the corresponding private values $\hat{V}_i$'s (resp. $\hat{C}_i$'s) for the buyer (resp. the seller) using the inverse bidding strategy of (3.9) (resp. (3.10)) for the bids on $[\underline{s}, \overline{b}]$. We can then estimate the conditional value densities $f_{V|V \geqslant \underline{s}}(\cdot)$ and $f_{C|C \leqslant \overline{b}}(\cdot)$ given the successful transaction. We can also estimate the conditional value distribution functions $\Pr\left(V \leqslant v \,|\, V \geqslant \underline{s}\right)$ and $\Pr\left(C \leqslant c \,|\, C \leqslant \overline{b}\right)$ according to (3.13). Their asymptotic properties are left for future research.

S.3 Proofs of Lemma and Proposition in Supplementary Material

S.3.1 Proof of Lemma S.2

Part (i) is established by following the proof of Lemma 7 except that (i) the uniform convergence rate of $\tilde{g}_S$ on $[\underline{s}, \overline{b}]$ becomes $O_p(h_S^{R+1} + \sqrt{\log(n)/(nh_S)})$ from Lemma 6 with a polynomial order of $p = R + 1$; and (ii) the bandwidth $h_S = \lambda_S \cdot (\log(n)/n)^{1/(2R+3)}$.

Similarly, part (ii) follows from the proof of Lemma 5-(ii), given that the uniform convergence rates of $\hat{g}_S$ and $\hat{g}_B$ on $[\underline{s}, \overline{b}]$ are $O_p((\log n/n)^{(R+1)/(2R+3)})$ as established in part (i). □

S.3.2 Proof of Proposition S.1

The desired results are established by adapting the arguments in the proof of Theorem 6, using: (i) a polynomial order of $p = R$ for the value density estimates, and (ii) bandwidths $h_V = \lambda_V (\log n/n)^{1/(2R+3)}$ and $h_C = \lambda_C (\log n/n)^{1/(2R+3)}$. □

References

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