Supplement to “Instrumental Variables Estimations for Infinite Order Autoregressive Processes”

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Abstract

This appendix contains the technical proofs that are not included in the main text. In particular, it provides the proofs of the lemmas and Theorem 10. For ease of reference, we restate the assumptions and the lemmas' statements.

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1 Assumptions

For ease of reference we restate the assumptions made in the paper.

Assumption 1. *(i) $\{\epsilon_{it}\}$ are independently and identically distributed (i.i.d.) over time and across individuals with mean zero, $0 < E(\epsilon_{it}^2) = \sigma^2 < \infty$ and $E|\epsilon_{it}|^{2r} \leq C_1, r > 2$ for some constant $C_1 > 0$ and $\{\mu_i\}$ are i.i.d. across individuals with $E(\mu_i^2) < C_2$ for some constant $C_2 > 0$; (ii) ϵ_{it} is independent of μ_i for all i and t; (iii) $\sum_{k=1}^{\infty} |\alpha_k| < \infty$ and $1 - \sum_{k=1}^{\infty} \alpha_k z^k \neq 0$ for any $|z| \leq 1$; (iv) $y_{i,1-s}$, $s = 0, 1, 2, \ldots,$ are generated from the stationary distribution given μ_i.*

Assumption 2. *Γ_Δ is invertible for any p. The square root of the largest eigenvalue of $\Gamma_\Delta^{-1\prime}\Gamma_\Delta^{-1}$, denoted by λ_p, is finite for any p.*

Assumption 3. *$E(\mu_i^4)$ is finite.*

Assumption 4. $\sum_{k=1}^{\infty} k|\psi_k| < \infty$.

Assumption 5. *(i) $\{(g_{it}, \epsilon_{it})\}$ are independently and identically distributed (i.i.d.) across individuals, and ϵ_{it} is i.i.d. across over t. (ii) $E(\epsilon_{it} \mid \{g_{is}\}_{s=-\infty}^{\infty}, \mu_i) = 0$. (iii) $\{g_{it}\}$ is strictly stationary and of short memory over t. (iv) The long-run variance of g_{it} is uniformly bounded across i.*

2 Notation

Throughout the supplemental materials, $C \in (1, \infty)$ denotes a generic bounded constant, which does not depend on any index and whose actual value varies across occasions. Given a matrix A, we let $||A||$ denote the Euclidean matrix norm defined by $||A||^2 = tr(A'A)$. Additionally, let $||A||_1$ denote the Banach norm so that $||A||_1 = \sup_{x \neq 0}\{||Ax||/||x||\}$, using the Euclidean norm for the vector l, $||l|| = (l'l)^{1/2}$. For any symmetric matrix A, we let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ be the minimum and the maximum eigenvalues of A, respectively. We note that $||A||_1 = \sqrt{\lambda_{\max}(A'A)}$. When A is symmetric and positive definite, $||A||_1 = \lambda_{\max}(A)$. Define $\gamma_k = E(w_{it}w_{i,t-k})$. We let

$$\bar{w}_{i,t,\tau} = \frac{1}{\tau - t + 1}(w_{it} + \cdots + w_{i,\tau}).$$

We also define $\bar{w}_{i,t,\tau}(p) = (\bar{w}_{i,t,\tau}, \ldots, \bar{w}_{i,t-p+1,\tau-p+1})'$, $\bar{w}_{t,\tau} = (\bar{w}_{1,t,\tau}, \ldots, \bar{w}_{N,t,\tau})'$ and $\bar{w}_{t,\tau}(p) = (\bar{w}_{t,\tau}, \ldots, \bar{w}_{t-p+1,\tau-p+1})$. Similarly, define $\bar{\epsilon}_{i,t,\tau} = (\epsilon_{it} + \cdots + \epsilon_{i,\tau})/(\tau - t + 1)$ and $\bar{\epsilon}_{t,\tau} = (\bar{\epsilon}_{1,t,\tau}, \ldots, \bar{\epsilon}_{N,t,\tau})'$.

The following inequalities will be used below: $||A||_1 = \sqrt{\lambda_{\max}(A'A)} \leq (tr(A'A))^{1/2} = ||A||$. $||AB||^2 \leq ||A||_1^2||B||^2$ and $||AB||^2 \leq ||A||^2||B||_1^2$ (see Lewis and Reinsel (1985) and Wiener and Masani (1958)). For any conformable matrices A and D and any square matrix B, $||A'BD|| \leq \|B\|_1||A|| \cdot ||D||$.

We repeatedly use the result that there exists $C_1 > 0$ such that the minimum eigenvalue of Γ_p is greater than C_1 for any p and there exists $C_2 < \infty$ such that the maximum eigenvalue of Γ_p is smaller than C_2 for any p. This result holds under Assumption 1(iv) by Corollary 3.3 (i) and (ii) of Davies (1973).

3 Proofs of the lemmas for the AH estimator

Recall that the AH type estimator is:

$$\hat{\alpha}_{AH}(p) = \left(\sum_{i=1}^{N} \sum_{t=p+2}^{T} Z_{it}(p) \Delta x_{it}(p)' \right)^{-1} \sum_{i=1}^{N} \sum_{t=p+3}^{T} Z_{it}(p) \Delta y_{it}.$$

Define

$$\begin{aligned} \hat{\Gamma}_\Delta &= \frac{1}{N(T-p)} \sum_{i=1}^{N} \sum_{t=p+2}^{T} Z_{it}(p) \Delta x_{it}(p)', \\ U &= \frac{1}{N(T-p)} \sum_{i=1}^{N} \sum_{t=p+2}^{T} Z_{it}(p) \Delta u_{it,p}. \end{aligned}$$

Then, we write:

$$\hat{\alpha}_{AH}(p) - \alpha(p) = \hat{\Gamma}_\Delta^{-1} U.$$

We also write

$$U = U_1 + U_2,$$

where

$$\begin{aligned} U_1 &= \frac{1}{N(T-p)} \sum_{i=1}^{N} \sum_{t=p+2}^{T} Z_{it}(p) \Delta b_{it,p}, \\ U_2 &= \frac{1}{N(T-p)} \sum_{i=1}^{N} \sum_{t=p+2}^{T} Z_{it}(p) \Delta \epsilon_{it}. \end{aligned}$$

We have the following lemmas.

Lemma 1. *Suppose that Assumptions 1(i) and 1(iii) are satisfied. If $N \to \infty$ and $T \to \infty$, Then*

$$||\hat{\Gamma}_\Delta - \Gamma_\Delta|| = O_p\left(\frac{p}{\sqrt{N(T-p)}} \right).$$

Proof. We write that

$$||\hat{\Gamma}_\Delta - \Gamma_\Delta||^2 = \sum_{a=1}^{p} \sum_{b=1}^{p} \left(\frac{1}{N(T-p)} \sum_{i=1}^{N} \sum_{t=p+2}^{T} (y_{i,t-a-1}(y_{i,t-b} - y_{i,t-b-1}) - (\gamma_{|a-b+1|} - \gamma_{|a-b|})) \right)^2.$$

As $E(y_{i,t-a-1}(y_{i,t-b}-y_{i,t-b-1})-(\gamma_{|a-b+1|}-\gamma_{|a-b|}))=0$, we have

$$\begin{aligned}
& E\left(\frac{1}{N(T-p)}\sum_{i=1}^{N}\sum_{t=p+2}^{T}(y_{i,t-a-1}(y_{i,t-b}-y_{i,t-b-1})-(\gamma_{|a-b+1|}-\gamma_{|a-b|}))\right)^2 \\
= & \frac{1}{N(T-p)}E(y_{i,t-a-1}(y_{i,t-b}-y_{i,t-b-1})-(\gamma_{|a-b+1|}-\gamma_{|a-b|}))^2.
\end{aligned}$$

Given that $y_{it}=\eta_i+w_{it}$, we write

$$\begin{aligned}
& E(y_{i,t-a-1}(y_{i,t-b}-y_{i,t-b-1})-(\gamma_{|a-b+1|}-\gamma_{|a-b|}))^2 \\
= & E((w_{i,t-a-1}+\eta_i)(w_{i,t-b}-w_{i,t-b-1})-\gamma_{|a-b+1|}-\gamma_{|a-b|}))^2 \\
\leq & 2E(((w_{i,t-a-1}+\eta_i)(w_{i,t-b}-w_{i,t-b-1}))^2)+2(\gamma_{|a-b+1|}-\gamma_{|a-b|})^2 \\
= & 4E((w_{i,t-a-1}w_{i,t-b}-w_{i,t-a-1}w_{i,t-b-1})^2)+4E(\eta_i^2(w_{i,t-b}-w_{i,t-b-1})^2)+2(\gamma_{|a-b+1|}-\gamma_{|a-b|})^2 \\
\leq & 4E((w_{i,t-a-1}w_{i,t-b})^2)+4E((w_{i,t-a-1}w_{i,t-b-1})^2)+4E(\eta_i^2)(2\gamma_0-2\gamma_1)+2(\gamma_{|a-b+1|}-\gamma_{|a-b|})^2 \\
\leq & C,
\end{aligned}$$

by Assumption 1 where C does not depend on a, b, t, and s. It follows that

$$E||\hat{\Gamma}_\Delta-\Gamma_\Delta||^2\leq\frac{1}{N(T-p)}\sum_{a=1}^{p}\sum_{b=1}^{p}C=O\left(\frac{p^2}{N(T-p)}\right).$$

By the Markov inequality, we have the desired result. □

Lemma 2. *Suppose that Assumptions 1 and 2 are satisfied. If $N\to\infty$ and $T\to\infty$, then*

$$||\hat{\Gamma}_\Delta-\Gamma_\Delta||_1=O_p\left(\frac{p}{\sqrt{N(T-p)}}\right).$$

Proof. Given $||A||_1\leq||A||$ for any matrix A, Lemma 1 implies the desired result.

□

Lemma 3. *Suppose that Assumptions 1 and 2 are satisfied. If $N\to\infty$ and $T\to\infty$ with $\lambda_p p/\sqrt{N(T-p)}\to 0$, then*

$$||\hat{\Gamma}_\Delta^{-1}-\Gamma_\Delta^{-1}||_1=O_p\left(\frac{\lambda_p^2 p}{\sqrt{N(T-p)}}\right).$$

Proof. First, we note that $||\Gamma_\Delta^{-1}||_1=\lambda_p$. Following an argument like the proof in Lewis and Reinsel (1985, Theorem 1) or Berk (1974, Lemma 3), we can write

$$\hat{\Gamma}_\Delta^{-1}-\Gamma_\Delta^{-1} \;=\; -\hat{\Gamma}_\Delta^{-1}(\hat{\Gamma}_\Delta-\Gamma_\Delta)\Gamma_\Delta^{-1}=-\left[\Gamma_\Delta^{-1}+\left(\hat{\Gamma}_\Delta^{-1}-\Gamma_\Delta^{-1}\right)\right](\hat{\Gamma}_\Delta-\Gamma_\Delta)\Gamma_\Delta^{-1},$$

and thus,

$$\begin{aligned}
||\hat{\Gamma}_{\Delta}^{-1} - \Gamma_{\Delta}^{-1}||_1 &\leq \left(||\Gamma_{\Delta}^{-1}||_1 + ||\hat{\Gamma}_{\Delta}^{-1} - \Gamma_{\Delta}^{-1}||_1\right) ||\hat{\Gamma}_{\Delta} - \Gamma_{\Delta}||_1 ||\Gamma_{\Delta}^{-1}||_1 \\
&= \left(\lambda_p + ||\hat{\Gamma}_{\Delta}^{-1} - \Gamma_{\Delta}^{-1}||_1\right) ||\hat{\Gamma}_{\Delta} - \Gamma_{\Delta}||_1 \lambda_p.
\end{aligned}$$

We note that $\lambda_p ||\hat{\Gamma}_{\Delta} - \Gamma_{\Delta}||_1 \to_p 0$ if $\lambda_p p/\sqrt{N(T-p)} \to 0$ by Lemma 2. Thus, with a probability approaching one, we have $\lambda_p ||\hat{\Gamma}_{\Delta} - \Gamma_{\Delta}||_1 < 1$ and

$$||\hat{\Gamma}_{\Delta}^{-1} - \Gamma_{\Delta}^{-1}||_1 \leq \frac{\lambda_p^2 ||\hat{\Gamma}_{\Delta} - \Gamma_{\Delta}||_1}{1 - \lambda_p ||\hat{\Gamma}_{\Delta} - \Gamma_{\Delta}||_1}.$$

The above inequality implies the first result by Lemma 2. □

Lemma 4. *Suppose that Assumptions 1, 2 are satisfied. If $N \to \infty$ and $T \to \infty$ with $T^2/N \to 0$, then*

$$||\hat{\Gamma}_{\Delta}^{-1}||_1 = O_p\left(\lambda_p + \frac{\lambda_p^2 p}{\sqrt{N(T-p)}}\right).$$

Proof. The result follows by observing that

$$||\hat{\Gamma}_{\Delta}^{-1}||_1 \leq ||\Gamma_{\Delta}^{-1}||_1 + ||\hat{\Gamma}_{\Delta}^{-1} - \hat{\Gamma}_{\Delta}^{-1}||_1 = O_p\left(\lambda_p + \frac{\lambda_p^2 p}{\sqrt{N(T-p)}}\right).$$

by Lemma 3 and Assumption 2. □

Lemma 5. *Suppose that Assumptions 1 and 2 are satisfied. If $N, T, p \to \infty$ with $p^{1/2} \sum_{k=p+1}^{\infty} |\alpha_k| \to 0$, then*

$$||U_1|| = O_p\left(p^{1/2} \sum_{k=p+1}^{\infty} |\alpha_k|\right).$$

Proof. We have

$$\begin{aligned}
E||U_1|| &= E\left\|\frac{1}{N(T-p)} \sum_{i=1}^{N} \sum_{t=p+2}^{T} Z_{it}(p) \Delta b_{it}\right\| \leq \frac{1}{N(T-p)} \sum_{i=1}^{N} \sum_{t=p+2}^{T} E\left(||Z_{it}(p) \Delta b_{it}||\right) \\
&\leq C\left(E||Z_{it}(p)||^2\right)^{1/2} \left(E||\Delta b_{it}||^2\right)^{1/2} = C\left(pE(y_{it}^2)\right)^{1/2} \left(E\left\|\sum_{k=p+1}^{\infty} \alpha_k \Delta y_{i,t-k}\right\|^2\right)^{1/2} \\
&\leq C \cdot p^{1/2} \left(\sum_{k=p+1}^{\infty} \sum_{k'=p+1}^{\infty} |\alpha_k| \cdot |\alpha_{k'}| \cdot \left|E(\Delta y_{i,t-k} \Delta y_{i,t-k'})\right|\right)^{1/2} \leq C \cdot p^{1/2} \sum_{k=p+1}^{\infty} |\alpha_k|, \qquad (1)
\end{aligned}$$

noting that $E(y_{i,t-k}^2)$ and $\left|E(y_{i,t-k} y_{i,t-k'})\right| = \left|\gamma_{|k-k'|}\right|$ are uniformly bounded given $\sum_{j=-\infty}^{\infty} |\gamma_j| < \infty$, which follows from the condition $\sum_{j=0}^{\infty} |\psi_j| < \infty$. The Markov inequality implies the desired result.

□

Lemma 6. *Suppose that Assumptions 1 and 2 are satisfied. If $N \to \infty$ and $T \to \infty$, then*

$$||U_2|| = O_p\left(\frac{p}{\sqrt{N(T-p)}}\right)$$

Proof. We have

$$\begin{aligned} E(||U_2||^2) &= E\left[\left\|\frac{1}{N(T-p)}\sum_{i=1}^{N}\sum_{t=p+1}^{T} Z_{it}(p)\Delta\epsilon_{it}\right\|^2\right] \leq \frac{1}{N^2(T-p)^2}\sum_{i=1}^{N}\sum_{t=p+1}^{T} E\left(||Z_{it}(p)\Delta\epsilon_{it}||^2\right) \\ &\leq \frac{1}{N^2(T-p)^2}\sum_{i=1}^{N}\sum_{t=p+1}^{T}\left(E||\Delta\epsilon_{it}||^2\right)\left(E||Z_{it}(p)||^2\right) = C\frac{p}{N(T-p)}E(y_{it}^2) = O\left(\frac{p}{N(T-p)}\right). \end{aligned}$$

The desired result follows by the Chebyshev inequality.

□

Lemma 7. *Suppose that Assumptions 1 and 2 are satisfied. If $N \to \infty$ and $T \to \infty$ with $(T-p)\lambda_p^2 p/N^2 = O(1)$, then*

$$\sqrt{N(T-p)}\ell_p'\Gamma_\Delta^{-1}U_2/v_{AH,p} \to_d N(0,1),$$

where

$$v_{AH,p}^2 = \frac{1}{(T-p)}E\left(\sum_{s=p+2}^{T}\sum_{s'=p+2}^{T}\Delta\epsilon_{i,s}\Delta\epsilon_{i,s'}\ell_p'\Gamma_\Delta^{-1}Z_{is}(p)Z_{is'}(p)'\Gamma_\Delta^{-1\prime}\ell_p\right).$$

Proof. We observe that

$$\sqrt{N(T-p)}\ell_p'\Gamma_\Delta^{-1}U_2/v_{AH,p} = \frac{1}{\sqrt{N(T-p)}}\sum_{i=1}^{N}\sum_{t=p+1}^{T}\ell_p'\Gamma_\Delta^{-1}Z_{it}(p)\Delta\epsilon_{it}/v_{AH,p}.$$

Let

$$S_i = \frac{1}{\sqrt{N(T-p)}}\sum_{t=p+1}^{T}\ell_p'\Gamma_\Delta^{-1}Z_{it}(p)\Delta\epsilon_{it}/v_{AH,p}.$$

By construction, we have $\sqrt{N(T-p)}\ell_p'\Gamma_\Delta^{-1}U_2/v_{AH,p} = \sum_{i=1}^{N} S_i$. We apply the Lyapunov CLT to $\sum_{i=1}^{N} S_i$. First by Assumption 1, S_i is independent across i for any N. Next, by the definition of $v_{AH,p}$, we have $\sum_{i=1}^{N} E(S_i^2) = 1$.

Finally, we examine the moment condition of the theorem. Let $r > 2$. The Minkowski inequality implies that

$$(E(|S_i|^r))^{1/r} \leq \frac{1}{\sqrt{N(T-p)}}\sum_{t=p+1}^{T}(E(\ell_p'\Gamma_\Delta^{-1}Z_{it}(p)\Delta\epsilon_{it}/v_{AH,p})^r)^{1/r}.$$

We observe that

$$E(\ell_p'\Gamma_\Delta^{-1}Z_{it}(p)\Delta\epsilon_{it}/v_{AH,p})^r \leq \left(E|\ell_p'\Gamma_\Delta^{-1}Z_{it}(p)|^{2r}\right)^{1/2}\left(E|\Delta\varepsilon_{it}|^{2r}\right)^{1/2}|v_{AH,p}^{-1}|^r,$$

by the Cauchy–Schwarz inequality. We note that $E|\Delta\varepsilon_{it}|^{2r} = O(1)$. We also have $v_{AH,p}^{-1} = O(1)$, as we have $M_1\lambda_{\min}(\Gamma_\Delta^{-1\prime}\Gamma_\Delta^{-1}) \leq \lambda_{\min}(\Gamma_\Delta^{-1\prime}\Gamma_\Delta^{-1})||\ell_p||^2 \leq \ell'\Gamma_\Delta^{-1\prime}\Gamma_\Delta^{-1}\ell \leq \lambda_p M_2$, for fixed p, given that $||\ell_p||$ and $||B_p||_1$ are bounded.

We can write

$$\left|\ell_p'\Gamma_\Delta^{-1}Z_{it}(p)\right| \leq ||\ell_p||||\Gamma_\Delta^{-1}||_1||Z_{it}(p)||,$$

so that

$$\left|\ell_p'\Gamma_\Delta^{-1}Z_{it}(p)\right|^{2r} \leq M_2^r O(\lambda_p^{2r})\left|\sum_{j=2}^{p+1}(y_{i,t-j})^2\right|^r,$$

as $||\ell_p||^2 \leq M_2$ and $||\Gamma_\Delta^{-1}||_1 = O(\lambda_p)$. Thus, we have

$$E\left|\ell_p'\Gamma_\Delta^{-1}Z_{it}(p)\right|^{2r} \leq O(\lambda_p^{2r}p^r),$$

by Minkowski's inequality, and using $E|y_{it}|^{2r} \leq C < \infty$ by Assumption 1. This implies that $E(\ell_p'\Gamma_\Delta^{-1}Z_{it}(p)\Delta\epsilon_{it}/v_{AH,p})^r = O(\lambda_p^r p^{r/2})$. We thus have

$$E(|S_i|^r) = O\left((T-p)^{r/2}N^{-r/2}\lambda_p^r p^{r/2}\right)$$

uniformly over i. Note that $E(S_i^2) = 1/N$. Thus, the moment condition for the Lyapunov theorem holds when

$$\frac{(T-p)^{r/2}N^{-r/2}\lambda_p^r p^{r/2}}{N^{r/2}} = \frac{(T-p)^{r/2}\lambda_p^r p^{r/2}}{N^r} = \left(\frac{(T-p)\lambda_p^2 p}{N^2}\right)^{r/2}$$

is bounded.

□

4 Lemmas useful for the GMM and DFIV estimators

This section presents several lemmas that are commonly employed in the derivation of the asymptotic properties of the GMM and DFIV estimators. Some are presented in Lee, Okui and Shintani (2017) which is the supplemental appendix to Lee, Okui, and Shintani (2018).

Lemma 8 (Lee, Okui and Shintani (2017, Lemma B.1))**.** *Suppose that Assumption 1 is satisfied. Then,*

$$||\bar{w}_{t,\tau}(p)||^2 = O_p\left(\frac{Np}{\tau-t+1}\right) \text{ and } E||\bar{w}_{t,\tau}(p)||^2 = O\left(\frac{Np}{\tau-t+1}\right).$$

Lemma 9 (Lee, Okui and Shintani (2017, Lemma B.2)). *Suppose that Assumption 1 is satisfied. If $N \to \infty$, $T \to \infty$ and $p \to \infty$ with $p/T \to 0$, then*

$$\left\| \frac{1}{N(T-p)} \sum_{t=p+1}^{T} w_{t-1}(p)' w_{t-1}(p) - \Gamma_p \right\| = O_p\left(\frac{p}{\sqrt{NT}}\right).$$

Lemma 10 (Lee, Okui and Shintani (2018, Lemma A.1)). *Suppose that Assumption 1 is satisfied. Let $\hat{\Gamma}_p$ be an estimator of Γ_p such that $\left\|\hat{\Gamma}_p - \Gamma_p\right\| = O_p(\rho_{N,T,p})$ where $\rho_{N,T,p} = o(1)$ as $N \to \infty$, $T \to \infty$ and $p \to \infty$. Then, as $N \to \infty$, $T \to \infty$ and $p \to \infty$, we have*

$$\begin{aligned} ||\hat{\Gamma}_p - \Gamma_p||_1 &= O_p(\rho_{N,T,p}), \\ ||(\hat{\Gamma}_p)^{-1} - \Gamma_p^{-1}||_1 &= O_p(\rho_{N,T,p}), \\ \textit{and } ||(\hat{\Gamma}_p)^{-1}||_1 &= O_p(1). \end{aligned}$$

Lemma 11 (Lee, Okui and Shintani (2017, Lemma B.3)). *Suppose that Assumption 1 is satisfied. If $N \to \infty$, $T \to \infty$ and $p \to \infty$ with $p^3/(NT) \to 0$ and $p/T \to 0$, then*

$$\frac{1}{\sqrt{N(T-p)}} \ell_p' \Gamma_p^{-1} \sum_{t=p+1}^{T} w_{t-1}(p)' \epsilon_t / v_p \to_d N(0,1).$$

5 Proofs of the lemmas for the GMM estimator

Note that variables with superscript '*' are transformed by the forward filter so that $b_{t,p}^* = \sqrt{(T-t)/(T-t+1)}(b_{t,p} - \sum_{\tau=t+1}^{T} b_{\tau,p}/(T-t))$ and $\epsilon_t^* = \sqrt{(T-t)/(T-t+1)}(\epsilon_t - \sum_{\tau=t+1}^{T} \epsilon_t/(T-t))$. The estimation error of the GMM estimator can be decomposed as

$$\hat{\alpha}(p) - \alpha(p) = (\hat{\Gamma}_p^G)^{-1} G_1 + (\hat{\Gamma}_p^G)^{-1} G_2$$

where

$$\hat{\Gamma}_p^G = \frac{1}{NT} \sum_{t=p+1}^{T-1} x_t^*(p)' M_t x_t^*(p), \quad G_1 = \frac{1}{NT} \sum_{t=p+1}^{T-1} x_t^*(p)' M_t b_{t,p}^* \text{ and } G_2 = \frac{1}{NT} \sum_{t=p+1}^{T-1} x_t^*(p)' M_t \epsilon_t^*.$$

Note that we can write

$$x_t^*(p) = \sqrt{\frac{T-t}{T-t+1}} \left(w_{t-1}(p) - \bar{w}_{t,T-1}(p)\right).$$

Lemma 12. *Suppose that Assumptions 1 and 3 are satisfied. If $N \to \infty$, $T \to \infty$ and $p \to \infty$ with $p^2/T \to 0$, then*

$$||\hat{\Gamma}_p^G - \Gamma_p|| = O_p\left(\frac{p}{\sqrt{T}}\right).$$

Proof. We observe that

$$\begin{aligned}\hat{\Gamma}_p^G &= \frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'M_t w_{t-1}(p) - \frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'M_t\bar{w}_{t,T-1}(p)\\ &\quad -\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}\bar{w}_{t,T-1}(p)'M_t w_{t-1}(p) + \frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}\bar{w}_{t,T-1}(p)'M_t\bar{w}_{t,T-1}(p).\end{aligned}$$

The first term in the decomposition is that

$$\begin{aligned}&\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'M_t w_{t-1}(p)\\ =\ &\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'w_{t-1}(p) + \frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'(I-M_t)w_{t-1}(p).\end{aligned}$$

Lemma 9 gives

$$\left\|\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'w_{t-1}(p) - \Gamma_p\right\| = O_p\left(\frac{p}{\sqrt{NT}} + \frac{p\log T}{T}\right).$$

We consider the second term. Let η_t^* is the $N\times 1$ vector of the errors of the population linear projection of η on Z_t so that $\eta_t^* = \eta - Z_t'\delta_t$, where

$$\delta_t = E((z_{it}z_{it}'))^{-1}E(z_{it}\eta_i) = (\sigma_\eta^2\iota_t\iota_t + \Gamma_t)^2\sigma_\eta^2\iota_t = \frac{\sigma_\eta^2}{1+\sigma_\eta^2\iota_t'\Gamma_t^{-1}\iota_t}\Gamma_t^{-1}\iota.$$

We have that

$$E(\eta_t^{*\prime}\eta_t) = N(\sigma_\eta^2 - \delta_t' E(z_{it}\eta_i)) = N\left(\frac{\sigma_\eta^2}{1+\sigma_\eta^2\iota_t'\Gamma_t^{-1}\iota_t}\right) = \left(\frac{N}{t}\right),$$

where $1/(1+\iota_t'\Gamma_t^{-1}\iota_t) = O(1/t)$ by Assumption 1. Given $w_{t-1}(p) = x_{t-1}(p) - \eta\iota_p'$ and $x_{t-1}(p)(I-M_t) = 0$, we have that

$$\begin{aligned}\left\|\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'(I-M_t)w_{t-1}(p)\right\| &= \left\|\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}\iota_p\eta_t^{*\prime}(I-M_t)\eta_t^*\iota_p\right\|\\ &\le \frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}\lambda_{\max}(I-M_t)\left\|\eta_t^*\iota_p'\right\|^2.\end{aligned}$$

We observe that $\lambda_{\max}(I-M_t) = 1$ because $I-M_t$ is an idempotent matrix and that $\left\|\eta_t^*\iota_p'\right\|^2 \le \|\eta_t^*\|^2\cdot\|\iota_p\|^2 = O_p(Np/t)$. Thus, it holds that

$$\begin{aligned}\left\|\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'(I-M_t)w_{t-1}(p)\right\| &= O_p\left(\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}\frac{Np}{t}\right)\\ &= O_p\left(\frac{p\log T}{T}\right).\end{aligned}$$

Next, we have

$$\begin{aligned}
&\left\| \frac{1}{NT} \sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' M_t \bar{w}_{t,T-1}(p) \right\| \\
\leq \;& \frac{1}{NT} \sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} \lambda_{\max}(M_t) \left\| w_{t-1}(p) \right\| \cdot \left\| \bar{w}_{t,T-1}(p) \right\| .
\end{aligned}$$

Given M_t is an idempotent matrix, we have $\lambda_{\max}(M_t) = 1$. Lemma 8 gives that $\|w_{t-1}(p)\| = O_p((Np)^{1/2})$ and $\|\bar{w}_{t,T-1}(p)\| = O_p(\sqrt{Np/(T-t)})$. Thus, we have

$$\begin{aligned}
&\left\| \frac{1}{NT} \sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' M_t \bar{w}_{t,T-1}(p) \right\| \\
=\;& O_p\left(\frac{1}{NT} \sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} \sqrt{Np} \sqrt{Np \frac{1}{T-t}} \right) = O_p\left(\frac{p}{\sqrt{T}} \right).
\end{aligned}$$

Similarly, we have

$$\left\| \frac{1}{NT} \sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{t,T-1}(p)' M_t w_{t-1}(p) \right\| = O_p\left(\frac{p}{\sqrt{T}} \right).$$

Finally, we have

$$\left\| \frac{1}{NT} \sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{t,T-1}(p)' M_t \bar{w}_{t,T-1}(p) \right\| \leq \frac{1}{NT} \sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} \lambda_{\max}(M_t) \left\| \bar{w}_{t,T-1}(p) \right\|^2 .$$

As M_t is an idempotent matrix, we have $\lambda_{\max}(M_t) = 1$. Lemma 8 implies that $\|\bar{w}_{t,T-1}(p)\|^2 = O_p(Np/(T-t))$. Thus, we have

$$\begin{aligned}
&\left\| \frac{1}{NT} \sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{t-1,T-1}(p)' M_t \bar{w}_{t-1,T-1}(p) \right\| \\
=\;& O_p\left(\frac{1}{NT} \sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} Np \frac{1}{T-t} \right) = O_p\left(\frac{p \log T}{T} \right).
\end{aligned}$$

To sum up, noting that $p \log T/T = o(p/\sqrt{T})$, it holds that

$$\left\| \hat{\Gamma}_p^G - \Gamma_p \right\| = O_p\left(\frac{p}{\sqrt{NT}} + \frac{p \log T}{T} + \frac{p \log T}{T} + \frac{p}{\sqrt{T}} + \frac{p \log T}{T} \right) = O_p\left(\frac{p}{\sqrt{T}} \right).$$

$\square$

Lemma 13. *Suppose that Assumption 1 is satisfied. If $N, T, p \to \infty$ with $p^{1/2} \sum_{k=p+1}^{\infty} |\alpha_k| \to 0$, then*

$$||G_1|| = O_p\left(\sqrt{p}\sum_{k=p+1}^{\infty} |\alpha_k|\right) = o_p(1).$$

Suppose that Assumptions 1 and 3 are satisfied. If $N, T, p \to \infty$ with $p^{1/2} \sum_{k=p+1}^{\infty} |\alpha_k| \to 0$ and $p^2/T \to 0$, then

$$||\ell_p'(\hat{\Gamma}_p^G)^{-1} G_1|| = O_p\left(\sum_{k=p+1}^{\infty} |\alpha_k|\right).$$

Proof. We have that

$$||G_1|| \quad = \quad \left\| \frac{1}{NT}\sum_{t=p+1}^{T-1} x_t^*(p)' M_t b_{t,p}^* \right\| \leq \frac{1}{NT}\sum_{t=p+1}^{T-1} \lambda_{\max}(M_t) \left\| x_t^*(p) \right\| \cdot \left\| b_{t,p}^* \right\|.$$

Given that M_t is an idempotent matrix, $\lambda_{\max}(M_t) = 1$. We also let

$$w_{i,t-k,t}^{\dagger} = \sqrt{\frac{T-t}{T-t+1}}\left(w_{i,t-k} - \bar{w}_{i,t-k+1,T-k}\right).$$

Then, we write $x_{it}^*(p) = (w_{i,t-1,1}^{\dagger}, \ldots, w_{i,t-p,1}^{\dagger})'$ and $b_{i,t,p}^* = \sum_{k=p+1}^{\infty} \alpha_k w_{i,t-k,t}^{\dagger}$. We have that

$$E\left\| x_t^*(p) \right\|^2 = NE(tr(x_{it}^*(p) x_{it}^*(p)')) = N\sum_{k=1}^{p} E\left(\left(w_{i,t-k,t}^{\dagger}\right)^2\right) = NpE((w_{i,t-k,t}^{\dagger})^2)$$

by the stationarity of $y_{i,t-k,t}^{\dagger}$. As $E((y_{i,t-k,t}^{\dagger})^2)$ is uniformly bounded, we have that

$$\left\| x_t^*(p) \right\| = O_p(\sqrt{Np})$$

uniformly in t. We also have

$$\begin{aligned} E\left\| b_{t,p}^* \right\|^2 &= NE((b_{it,p}^*)^2) = NE\left(\left(\sum_{k=p+1}^{\infty} \alpha_k w_{i,t-k,t}^{\dagger}\right)^2\right) \\ &\leq N\sum_{k=p+1}^{\infty}\sum_{k'=p+1}^{\infty} |\alpha_k| \cdot |\alpha_{k'}| \cdot |E(w_{i,t-k,t}^{\dagger} w_{i,t-k',t}^{\dagger})| \leq CN\left(\sum_{k=p+1}^{\infty} |\alpha_k|\right)^2, \end{aligned}$$

by observing that $|E(w_{i,t-k,t}^{\dagger} w_{i,t-k',t}^{\dagger})|$ is uniformly bounded. Therefore, we have

$$\left\| b_{t,p}^* \right\| = O_p\left(\sqrt{N}\sum_{k=p+1}^{\infty} |\alpha_k|\right). \tag{2}$$

To sum up, we have that

$$||G_1|| = \frac{1}{NT}\sum_{t=p+1}^{T-1} O_p\left(\sqrt{Np}\right) O_p\left(\sqrt{N}\sum_{k=p+1}^{\infty}|\alpha_k|\right) = O_p\left(\sqrt{p}\sum_{k=p+1}^{\infty}|\alpha_k|\right).$$

The proof concerning the order of $||\ell_p'(\hat{\Gamma}_p^G)^{-1}G_1||$ is analogous to that of the corresponding part of Lee, Okui and Shintani (2018, Lemma A.3) and is thus omitted. $\square$

Lemma 14. *Suppose that Assumptions 1, 3 and 4 are satisfied. If $N \to \infty$, $T \to \infty$ and $p \to \infty$ with $p \log T/T \to 0$ and $T/N \to 0$, then*

$$||G_2|| = O_p\left(\sqrt{\frac{p}{NT}} + \frac{\sqrt{p}\log T}{N} + \frac{\sqrt{p\log T}}{N^{3/4}\sqrt{T}} + \frac{p^{3/2}\sqrt{\log T}}{\sqrt{N}T}\right) = o_p(1).$$

The proof of Lemma 14 uses the following lemmas.

Lemma A.1. *Suppose that Assumption 1 is satisfied. Let d_t be the $N \times 1$ vector containing the diagonal elements of M_t. Define κ_3 and κ_4 be the third and fourth cumulants of ϵ_{it}, respectively. Suppose that $r \geq t$, $q \geq s$ and $t \geq s$. Then, $cov(\epsilon_l' M_t \epsilon_r, \epsilon_p' M_s \epsilon_q)$ is equal to*

$$\begin{aligned}
2\sigma^4 s + \kappa_4 E(d_t' d_s) \leq (2\sigma^4 + \kappa_4)s \quad & \textit{if} \quad l = r = p = q, \\
\kappa_3 E(d_t' M_s \epsilon_q) \quad & \textit{if} \quad l = r = p \neq q < t, \\
\kappa_3 E(d_s' M_t \epsilon_l) \quad & \textit{if} \quad r = p = q \neq l < t, \\
\kappa_3 E(d_t' M_s \epsilon_p) \quad & \textit{if} \quad l = r = q \neq p < t, \\
\sigma^4 s \quad & \textit{if} \quad s \leq l = p \neq r = q \textit{ or } l = q \neq r = p, \\
\sigma^2 E(\epsilon_l' M_s \epsilon_p) \quad & \textit{if} \quad r = q, l < s \textit{ and } p < s,
\end{aligned}$$

where $|E(d_t' M_s \epsilon_q)| < \sqrt{st}\sigma$ if $q < t$, $|E(d_s' M_t \epsilon_l)| \leq (sN)^{1/2}\sigma$, $|E(d_t' M_s \epsilon_p)| < \sqrt{st}\sigma$ if $s \leq p < t$ and $|E(d_t' M_s \epsilon_p)| < \sqrt{sN}\sigma$ if $p < s$ and $|E(\epsilon_l' M_s \epsilon_p)| \leq N\sigma$ if $l < s$ and $p < s$.

Proof. Alvarez and Arellano (2003, Lemma C1) show that for $l \geq r$ and $p \geq q$,

$$cov(\epsilon_l' M_t \epsilon_r, \epsilon_p' M_s \epsilon_q) = \begin{cases} 2\sigma^4 s + \kappa_4 E(d_t' d_s) \leq (2\sigma^4 + \kappa_4)s & \text{if } l = r = p = q, \\ \kappa_3 E(d_t' M_s \epsilon_q) & \text{if } l = r = p \neq q < t, \\ \sigma^4 s & \text{if } l = p \neq r = q, \\ 0 & \text{otherwise,} \end{cases}$$

where $|E(d_t' M_s \epsilon_q)| \leq \sqrt{st}\sigma$.

Therefore, we consider cases with $l < r$ or $p < q$. We note that in these cases $cov(\epsilon_l' M_t \epsilon_r, \epsilon_p' M_s \epsilon_q) = E(\epsilon_l' M_t \epsilon_r \epsilon_p' M_s \epsilon_q)$ because $E(\epsilon_l' M_t \epsilon_r) = 0$ if $l < r$, and $E(\epsilon_p' M_t \epsilon_q) = 0$ if $p < q$.

Let E_t be the conditional expectation operator given $(\epsilon_{t-1}, \ldots, \epsilon_0, \ldots)$.

First, we consider the case in which $l < r = p = q$. Noting that $r \geq t$, it follows that

$$E(\epsilon_l' M_t \epsilon_r \epsilon_p' M_s \epsilon_q) = E(\epsilon_l' M_t \epsilon_r \epsilon_r' M_s \epsilon_r) = E\left(tr\left(\epsilon_l M_t E_t(\epsilon_r \epsilon_r' M_s \epsilon_r)\right)\right) = \kappa_3 E(\epsilon_l' M_t d_s).$$

We have that $E(\epsilon_l' M_t d_s) = 0$ if $l \geq t$. If $l < t$, then, we have that

$$(E(\epsilon_l' M_t d_s))^2 \leq E((\epsilon_l' M_t d_s)^2) \leq E(d_s' d_s \epsilon_l' M_t \epsilon_l) = sE(\epsilon_l' M_t \epsilon_l) \leq \sigma^2 sN.$$

Next, we suppose that $p < l = r = q$. Noting that $r \geq t$, it follows that

$$E(\epsilon_l' M_t \epsilon_r \epsilon_p' M_s \epsilon_q) = E(\epsilon_r' M_t \epsilon_r \epsilon_p' M_s \epsilon_r) = E\left(tr\left(E_t(\epsilon_r M_t \epsilon_r \epsilon_r') M_s \epsilon_p\right)\right) = \kappa_3 E(d_t M_s \epsilon_p).$$

We have that $E(d_t' M_s \epsilon_p) = 0$ if $p \geq t$. If $p < t$, then, we have that

$$(E(d_t' M_s \epsilon_p))^2 \leq E((d_t' M_s \epsilon_p)^2) \leq E(d_t' d_t \epsilon_p' M_s \epsilon_p) = tE(\epsilon_p' M_s \epsilon_p) \leq \begin{cases} \sigma^2 st & \text{if } p \geq s, \\ \sigma^2 sN & \text{if } p < s. \end{cases}$$

For the case with $l = p = q < r$ and the case with $l = r = p < q$, it is obvious that

$$E(\epsilon_l' M_t \epsilon_r \epsilon_p' M_s \epsilon_q) = 0.$$

We now suppose that $l = p < r = q$. We observe that

$$E(\epsilon_l' M_t \epsilon_r \epsilon_p' M_s \epsilon_q) = E(\epsilon_l' M_t \epsilon_r \epsilon_l' M_s \epsilon_r) = E\left(tr\left(M_t E_r(\epsilon_r \epsilon_r') M_s \epsilon_l \epsilon_l'\right)\right) = \sigma^2 E\left(tr(M_s \epsilon_l \epsilon_l')\right)$$

If $l \geq s$, then $\sigma^2 E\left(tr(M_s \epsilon_l \epsilon_l')\right) = \sigma^4 s$. If $l < s$, then $\sigma^2 E\left(tr(M_s \epsilon_l \epsilon_l')\right) = \sigma^2 E(\epsilon_l' M_s \epsilon_l) \leq \sigma^2 E(\epsilon_l' \epsilon_l) \leq \sigma^4 N$.

We examine the case in which $l = q < r = p$. Noting that $l = q \geq s$, we have

$$E(\epsilon_l' M_t \epsilon_r \epsilon_r' M_s \epsilon_l) = E\left(tr\left(M_t E_r(\epsilon_r \epsilon_r') M_s \epsilon_l \epsilon_l'\right)\right) = \sigma^2 E\left(tr(M_s \epsilon_l \epsilon_l')\right) = \sigma^4 s.$$

The case in which $l = q > r = p$ can be considered similarly. Noting that $l > t$, we have

$$E(\epsilon_l' M_t \epsilon_r \epsilon_r' M_s \epsilon_l) = E\left(tr\left(M_t \epsilon_r \epsilon_r' M_s E_l(\epsilon_l \epsilon_l')\right)\right) = \sigma^2 E\left(tr(M_s \epsilon_r \epsilon_r')\right) = \sigma^4 s.$$

Consider the case in which $l = r$, $l \neq p$, $l \neq q$ and $p < q$. If $q > l$, it is easy to see that

$$E(\epsilon_l' M_t \epsilon_r \epsilon_p' M_s \epsilon_q) = 0.$$

On the other hand, if $l > q$, then

$$E(\epsilon_l' M_t \epsilon_r \epsilon_p' M_s \epsilon_q) = trE(E_t(\epsilon_l' M_t \epsilon_l) \epsilon_p' M_s \epsilon_q) = \sigma^2 tE(\epsilon_p' M_s \epsilon_q) = 0,$$

where the third equality follows because $l = r \geq t$.

Similarly, if $p = q$, $l < r$, $l \neq p$ and $r \neq p$, then $E(\epsilon_l' M_t \epsilon_r \epsilon_p' M_s \epsilon_q) = 0$.

For the case with $l = p$, $l \neq r$, $l \neq q$ and $r \neq q$, we have

$$\begin{aligned} E(\epsilon_l' M_t \epsilon_r \epsilon_p' M_s \epsilon_q) = E(\epsilon_l' M_t \epsilon_r \epsilon_l' M_s \epsilon_q) &= \begin{cases} 0 & \text{if } r > l \text{ or } q > l, \\ tr(E(M_t \epsilon_r \epsilon_q' M_s \epsilon_l \epsilon_l')) & \text{if } l > r \text{ and } l > q, \end{cases} \\ &= \begin{cases} 0 & \text{if } r > l \text{ or } q > l, \\ \sigma^2 tr(E(M_s \epsilon_r \epsilon_q')) & \text{if } l > r \text{ and } l > q, \end{cases} \\ &= 0, \end{aligned}$$

where the third equality follows by noting $l > t$ if $l > r$.

Similarly, we have $E(\epsilon_l' M_t \epsilon_r \epsilon_p' M_s \epsilon_q) = 0$ for the case with $l = q$, $l \neq r$, $l \neq p$ and $r \neq p$, and for the case $r = p$, $l \neq p$, $l \neq q$ and $r \neq q$, $E(\epsilon_l' M_t \epsilon_r \epsilon_p' M_s \epsilon_q) = 0$.

For the case that $r = q$, $l \neq r$, $l \neq p$ and $r \neq p$, we have that

$$\begin{aligned} E(\epsilon_l' M_t \epsilon_r \epsilon_p' M_s \epsilon_q) = E(\epsilon_l' M_t \epsilon_r \epsilon_p' M_s \epsilon_r) &= \begin{cases} 0 & \text{if } l > r \text{ or } p > r, \\ tr(E(M_t \epsilon_l \epsilon_p' M_s \epsilon_r \epsilon_r')) & \text{if } r > l \text{ and } r > p, \end{cases} \\ &= \begin{cases} 0 & \text{if } l > r \text{ or } p > r, \\ \sigma^2 tr(E(M_s \epsilon_l \epsilon_p')) & \text{if } r > l \text{ and } r > p. \end{cases} \end{aligned}$$

If $l \geq s$ or $p \geq s$, then $tr(E(M_s \epsilon_l \epsilon_p')) = E(\epsilon_l' M_s \epsilon_p) = 0$. However, if $l < s$ and $p < s$, then $tr(E(M_s \epsilon_l \epsilon_p')) = E(\epsilon_l' M_s \epsilon_p)$ may not be zero but $|E(\epsilon_l' M_s \epsilon_p)| \leq \sigma^2 N$.

Lastly, when l, q, r, p are all different, it holds that

$$E(\epsilon_l' M_t \epsilon_r \epsilon_p' M_s \epsilon_q) = 0.$$

□

Lemma A.2. *Suppose that Assumption 1 is satisfied. Suppose that $r \geq t$, $q \geq s$ and $t \geq s$. Then, it holds that, if $r = q$,*

$$\begin{aligned} & cov(w_l' M_t \epsilon_r, w_p' M_s \epsilon_q) \\ = \ & O\left(\sqrt{st} + \sqrt{sN}\right) + \sigma^2 \sum_{k=l-s+1}^{\infty} \sum_{m=p-s+1}^{\infty} \psi_k \psi_m E(\epsilon_{l-k}' M_s \epsilon_{p-m}), \end{aligned}$$

where $E(\epsilon_{l-k}' M_s \epsilon_{p-m}) < \sigma N$, if $r \neq q$ and $q \geq t$,

$$cov(w_l' M_t \epsilon_r, w_p' M_s \epsilon_q) = \psi_{l-q} \psi_{p-r} (2\sigma^4 s + \kappa_4 E(d_t' d_s)) = O(s),$$

and, if $q < t$,

$$cov(w_l' M_t \epsilon_r, w_p' M_s \epsilon_q) = \psi_{l-r} \psi_{p-r} \kappa_3 E(d_t' M_s \epsilon_q) + s\sigma^4 \psi_{l-q} \psi_{p-r} = O(\sqrt{st}).$$

Proof. We first consider cases with $r = q$. Lemma A.1 implies that

$$\begin{aligned}
& cov(w_l' M_t \epsilon_r, w_p' M_s \epsilon_r) \\
= \;& \psi_{l-r}\psi_{p-r} cov(\epsilon_r' M_t \epsilon_r, \epsilon_r' M_s \epsilon_r) + \sum_{k=0}^{\infty} \psi_k \psi_{p-r} cov(\epsilon_{l-k}' M_t \epsilon_r, \epsilon_r' M_s \epsilon_r) \\
& + \psi_{l-r} \sum_{m=0}^{\infty} \psi_m cov(\epsilon_r' M_t \epsilon_r, \epsilon_{p-m}' M_s \epsilon_r) + \sum_{k=0, k \neq l-r}^{\infty} \psi_k \psi_{p-l+k} cov(\epsilon_{l-k}' M_t \epsilon_r, \epsilon_{l-k}' M_s \epsilon_r) \\
& + \sum_{k=l-s+1}^{\infty} \sum_{m=p-s+1}^{\infty} \psi_k \psi_m cov(\epsilon_{l-k}' M_t \epsilon_r, \epsilon_{p-m}' M_s \epsilon_r).
\end{aligned}$$

We have that

$$\begin{aligned}
& \psi_{l-r}\psi_{p-r} cov(\epsilon_r' M_t \epsilon_r, \epsilon_r' M_s \epsilon_r) + \sum_{k=0}^{\infty} \psi_k \psi_{p-r} cov(\epsilon_{l-k}' M_t \epsilon_r, \epsilon_r' M_s \epsilon_r) \\
& + \psi_{l-r} \sum_{m=0}^{\infty} \psi_m cov(\epsilon_r' M_t \epsilon_r, \epsilon_{p-m}' M_s \epsilon_r) + \sum_{k=0, k \neq l-r}^{\infty} \psi_k \psi_{p-l+k} cov(\epsilon_{l-k}' M_t \epsilon_r, \epsilon_{l-k}' M_s \epsilon_r) \\
= \;& \psi_{l-r}\psi_{p-r}(2\sigma^4 + \kappa_4 E(d_t' d_s)) + \sum_{k=0}^{\infty} \psi_k \psi_{p-r} \kappa_3 E(d_s' M_t \epsilon_{l-k}) \\
& + \psi_{l-r} \sum_{m=0}^{\infty} \psi_{mk} E(d_t' M_s \epsilon_{p-m}) + s\sigma^4 \sum_{k=0, k \neq l-r}^{\infty} \psi_k \psi_{p-l+k} \\
= \;& O\left(s + (sN)^{1/2} + \sqrt{st} + (sN)^{1/2} + s\right) = O\left(\sqrt{st} + (sN)^{1/2}\right),
\end{aligned}$$

noting that $t \geq s$. We also have

$$\sum_{k=l-s+1}^{\infty} \sum_{m=p-s+1}^{\infty} \psi_k \psi_m cov(\epsilon_{l-k}' M_t \epsilon_r, \epsilon_{p-m}' M_s \epsilon_r) = \sigma^2 \sum_{k=l-s+1}^{\infty} \sum_{m=p-s+1}^{\infty} \psi_k \psi_m E(\epsilon_{l-k}' M_s \epsilon_{p-m}).$$

Next, we suppose that $r \neq q$ and $q \geq t$. Lemma A.1 gives that

$$\begin{aligned}
cov(w_l' M_t \epsilon_r, w_p' M_s \epsilon_r) &= \psi_{l-r}\psi_{p-r} cov(\epsilon_r' M_t \epsilon_r, \epsilon_r' M_s \epsilon_r) \\
&= \psi_{l-q}\psi_{p-r}(2\sigma^4 s + \kappa_4 E(d_t' d_s)) = O(s).
\end{aligned}$$

For cases with $q < t$ (note that in this case $r \neq q$), Lemma A.1 provides that

$$\begin{aligned}
cov(w_l' M_t \epsilon_r, w_p' M_s \epsilon_r) &= \psi_{l-r}\psi_{p-r}\kappa_3 E(d_t' M_s \epsilon_q) + s\sigma^4 \psi_{l-q}\psi_{p-r} \\
&= O\left(\sqrt{st} + s\right) = O(\sqrt{st}).
\end{aligned}$$

$\square$

We now present the proof of Lemma 14.

Proof. We observe that

$$
\begin{aligned}
G_2 &= \frac{1}{NT}\sum_{t=p+1}^{T-1} x_t^*(p)' M_t \epsilon_t^* \\
&= \frac{1}{NT}\sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' M_t \epsilon_t - \frac{1}{NT}\sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' M_t \bar{\epsilon}_{t+1,T} \\
&\quad - \frac{1}{NT}\sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{t-1,T-1}(p)' M_t \epsilon_t + \frac{1}{NT}\sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{t-1,T-1}(p)' M_t \bar{\epsilon}_{t+1,T}.
\end{aligned}
$$

Noting that $w_{t-1}(p)' M_t \epsilon_t$ is a martingale difference sequence, we have

$$
\begin{aligned}
& E\left\|\frac{1}{NT}\sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' M_t \epsilon_t\right\|^2 \\
&= \frac{1}{N^2T^2}\sum_{t=p+1}^{T-1} \left(\frac{T-t}{T-t+1}\right)^2 tr(E(w_{t-1}(p)' M_t \epsilon_t \epsilon_t' M_t w_{t-1}(p))) \\
&= \frac{1}{N^2T^2}\sum_{t=p+1}^{T-1} \left(\frac{T-t}{T-t+1}\right)^2 \sigma^2 tr(E(w_{t-1}(p)' M_t w_{t-1}(p))).
\end{aligned}
$$

Since

$$
tr(E(w_{t-1}(p)' M_t w_{t-1}(p))) \leq tr(E(w_{t-1}(p)' w_{t-1}(p))) = E||w_{t-1}(p)||^2 = O(Np),
$$

where the last equality follows from Lemma 1, and $(T-t)^2/(T-t+1)^2 < 1$, we have

$$
E\left\|\frac{1}{NT}\sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' M_t \epsilon_t\right\|^2 = O\left(\frac{1}{N^2T^2}\sum_{t=p+1}^{T-1} Np\right) = O\left(\frac{p}{NT}\right).
$$

Therefore, the Markov inequality gives

$$
\left\|\frac{1}{NT}\sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' M_t \epsilon_t\right\| = O_p\left(\sqrt{\frac{p}{NT}}\right).
$$

We consider the term

$$
\frac{1}{NT}\sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' M_t \bar{\epsilon}_{t+1,T} = \frac{1}{NT}\sum_{t=p+1}^{T-1}\sum_{m=t+1}^{T} \frac{1}{T-t+1} w_{t-1}(p)' M_t \epsilon_m.
$$

Now, we have that

$$
\begin{aligned}
& E\left\|\frac{1}{NT}\sum_{t=p+1}^{T-1}\sum_{m=t+1}^{T} \frac{1}{T-t+1} w_{t-1}(p)' M_t \epsilon_m\right\|^2 \\
&= \frac{1}{N^2T^2}\sum_{t=p+1}^{T-1}\sum_{t'=p+1}^{T-1} \frac{1}{T-t+1}\frac{1}{T-t'+1}\sum_{m=t+1}^{T}\sum_{m'=t'+1}^{T} tr\left(E\left(w_{t-1}(p)' M_t \epsilon_m \epsilon_{m'}' M_{t'} w_{t'-1}(p)\right)\right).
\end{aligned}
$$

Now, we have that

$$E\left(w_{t-1}(p)'M_t\epsilon_m\epsilon_{m'}'M_{t'}w_{t'-1}(p)\right) = \begin{cases} 0, & \text{if } m \neq m', \\ \sigma^2 E\left(w_{t-1}(p)'M_{\min(t,t')}w_{t'-1}(p)\right), & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\begin{aligned} & E\left\|\frac{1}{NT}\sum_{t=p+1}^{T-1}\sum_{m=t+1}^{T}\frac{1}{T-t+1}w_{t-1}(p)'M_t\epsilon_m\right\|^2 \\ = & \frac{\sigma^2}{N^2T^2}\sum_{t=p+1}^{T-1}\sum_{t'=p+1}^{T-1}\frac{T-\max(t,t')}{(T-t+1)(T-t'+1)}tr\left(E\left(w_{t-1}(p)'M_{\min(t,t')}w_{t'-1}(p)\right)\right). \end{aligned}$$

We observe that

$$\begin{aligned} & \frac{\sigma^2}{N^2T^2}\sum_{t=p+1}^{T-1}\sum_{t'=p+1}^{T-1}\frac{T-\max(t,t')}{(T-t+1)(T-t'+1)}tr\left(E\left(w_{t-1}(p)'M_{\min(t,t')}w_{t'-1}(p)\right)\right) \\ \leq & \frac{2\sigma^2}{N^2T^2}\sum_{t=p+1}^{T-1}\sum_{t'=t}^{T-1}\frac{T-t'}{(T-t+1)(T-t'+1)}tr\left(E\left(w_{t-1}(p)'M_t w_{t'-1}(p)\right)\right) \\ = & \frac{2\sigma^2}{N^2T^2}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}tr\left(E\left(w_{t-1}(p)'M_t\tilde{w}_{t-1,T-2}(p)\right)\right), \end{aligned}$$

where

$$\tilde{w}_{t-1,T-2}(p) = \frac{1}{T-t}\sum_{t'=t}^{T-1}\frac{T-t'}{T-t'+1}w_{t'-1}(p).$$

It holds that

$$\begin{aligned} tr(E\left(w_{t-1}(p)'M_t\tilde{w}_{t-1,T-2}(p)\right)) & \leq tr(E\left(w_{t-1}(p)'\tilde{w}_{t-1,T-2}(p)\right)) \\ & \leq N\sqrt{E(||w_{i,t-1}(p)||^2)}\sqrt{E(||\tilde{w}_{i,t-1,T-2}(p)||^2)} \\ & = O\left(\frac{Np}{\sqrt{T-t}}\right), \end{aligned}$$

where $E(||\tilde{w}_{i,t-1,T-2}(p)||^2 = O\left(Np/(T-t)\right)$ by the short memory assumption in Assumption 1. Thus, it holds that

$$\begin{aligned} E\left\|\frac{1}{NT}\sum_{t=p+1}^{T-1}\sum_{m=t+1}^{T}\frac{1}{T-t+1}w_{t-1}(p)'M_t\epsilon_m\right\|^2 & = \frac{2\sigma^2}{N^2T^2}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}O\left(\frac{Np}{\sqrt{T-t}}\right) \\ & = O\left(\frac{p}{NT^{3/2}}\right). \end{aligned}$$

The Chebyshev inequality gives that

$$\left\| \frac{1}{NT} \sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' M_t \bar{\epsilon}_{t+1,T} \right\| = O_p\left(\frac{\sqrt{p}}{\sqrt{N} T^{3/4}} \right).$$

Next, we consider the term

$$\frac{1}{NT} \sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{t,T-1}(p)' M_t \epsilon_t = \frac{1}{NT} \sum_{t=p+1}^{T-1} \sum_{m=t}^{T-1} \frac{1}{T-t+1} w_m(p)' M_t \epsilon_t.$$

We have

$$\begin{aligned} & E \left\| \frac{1}{NT} \sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{t,T-1}(p)' M_t \epsilon_t \right\|^2 \\ = \; & \frac{1}{N^2 T^2} \sum_{t=p+1}^{T-1} \sum_{t'=p+1}^{T-1} \frac{1}{T-t+1} \frac{1}{T-t'+1} \sum_{m=t}^{T-1} \sum_{m'=t'}^{T-1} tr\left(E\left(w_m(p)' M_t \epsilon_t \epsilon_{t'}' M_{t'} w_{m'}(p) \right) \right). \end{aligned}$$

We have that

$$\begin{aligned} & E\left(w_m(p)' M_t \epsilon_t \epsilon_{t'}' M_{t'} w_{m'}(p) \right) \\ = \; & cov\left(w_m(p)' M_t \epsilon_t, w_{m'}(p)' M_{t'} \epsilon_{t'} \right) + E\left(w_m(p)' M_t \epsilon_t \right) E\left(\epsilon_{t'}' M_{t'} w_{m'}(p) \right). \end{aligned}$$

Let $\psi_t(p) = (\psi_t, \ldots, \psi_{t-p})'$. Since $E\left(\epsilon_{t'}' M_t \epsilon_t\right) = 0$ for $t' \neq t$, we have that

$$E\left(w_m' M_t \epsilon_t \right) = E\left(\sum_{k=0}^{\infty} \psi_k \epsilon_{m-k}' M_t \epsilon_t \right) = \psi_{m-t} E(\epsilon_t' M_t \epsilon_t) = \psi_{m-t} \sigma^2 (t-1).$$

It therefore holds that

$$E\left(w_m(p)' M_t \epsilon_t \right) = \sigma^2 (t-1) \psi_{m-t}(p).$$

Thus, we can write

$$\begin{aligned} & \frac{1}{N^2 T^2} \sum_{t=p+1}^{T-1} \sum_{t'=p+1}^{T-1} \frac{1}{T-t+1} \frac{1}{T-t'+1} \sum_{m=t}^{T-1} \sum_{m'=t'}^{T-1} tr\left(E\left(w_m(p)' M_t \epsilon_t \right) E\left(\epsilon_{t'}' M_{t'} w_{m'}(p) \right) \right) \\ = \; & \frac{\sigma^4}{N^2 T^2} \sum_{t=p+1}^{T-1} \sum_{t'=p+1}^{T-1} \frac{t-1}{T-t+1} \frac{t'-1}{T-t'+1} \sum_{m=t}^{T-1} \sum_{m'=t'}^{T-1} tr\left(\psi_{m-t}(p) \psi_{m'-t'}(p)' \right) \\ = \; & \frac{\sigma^4}{N^2 T^2} \sum_{t=p+1}^{T-1} \sum_{t'=p+1}^{T-1} \frac{t-1}{T-t+1} \frac{t'-1}{T-t'+1} \sum_{m=t}^{T-1} \sum_{m'=t'}^{T-1} \psi_{m-t}(p)' \psi_{m'-t'}(p) \\ = \; & \frac{\sigma^4}{N^2 T^2} \sum_{t=p+1}^{T-1} \sum_{t'=p+1}^{T-1} \frac{t-1}{T-t+1} \frac{t'-1}{T-t'+1} \left(\sum_{m=t}^{T-1} \psi_{m-t}(p) \right)' \left(\sum_{m'=t'}^{T-1} \psi_{m'-t'}(p) \right). \end{aligned}$$

Since $\left\|\sum_{m=t}^{T-1}\psi_{m-t}(p)\right\| = O(\sqrt{p})$ uniformly in m, t, T and

$$\sum_{t=p+1}^{T-1}\frac{t-1}{T-t+1} = \sum_{s=2}^{T-p}\frac{T-s+1}{s} = O\left(T\log T\right),$$

we have

$$\begin{aligned}
&\frac{\sigma^4}{N^2T^2}\sum_{t=p+1}^{T-1}\sum_{t'=p+1}^{T-1}\frac{t-1}{T-t+1}\frac{t'-1}{T-t'+1}\left(\sum_{m=t}^{T-1}\psi_{m-t}(p)\right)'\left(\sum_{m'=t'}^{T-1}\psi_{m'-t'}(p)\right)\\
= \; & O\left(\frac{pT^2(\log T)^2}{N^2T^2}\right) = O\left(\frac{p(\log T)^2}{N^2}\right).
\end{aligned}$$

Next, we consider the covariances. Lemma A.2 gives

$$\begin{aligned}
&\frac{1}{N^2T^2}\sum_{t=p+1}^{T-1}\sum_{t'=p+1}^{T-1}\frac{1}{T-t+1}\frac{1}{T-t'+1}\sum_{m=t}^{T-1}\sum_{m'=t'}^{T-1} tr\left(cov\left(w_m(p)'M_t\epsilon_t, w_{m'}(p)'M_{t'}\epsilon_{t'}\right)\right)\\
= \; & \frac{1}{N^2T^2}\sum_{t=p+1}^{T-1}\sum_{t'=p+1}^{T-1}\frac{1}{T-t+1}\frac{1}{T-t'+1}\sum_{m=t}^{T-1}\sum_{m'=t'}^{T-1}\sum_{j=0}^{p-1} cov\left(w'_{m-j}M_t\epsilon_t, w'_{m'-j}M_{t'}\epsilon_{t'}\right)\\
= \; & \frac{1}{N^2T^2}\sum_{t=p+1}^{T-1}\frac{1}{(T-t+1)^2}\sum_{m=t}^{T-1}\sum_{m'=t}^{T-1}\sum_{j=0}^{p-1} O\left(t+\sqrt{tN}\right)\\
& + \frac{1}{N^2T^2}\sum_{t=p+1}^{T-1}\frac{1}{(T-t+1)^2}\sum_{m=t}^{T-1}\sum_{m'=t}^{T-1}\sum_{j=0}^{p-1}\sum_{k=m-j-t+1}^{\infty}\sum_{k=m'-j-t+1}^{\infty}\psi_k\psi'_kE(\epsilon'_{m-j-k}M_t\epsilon_{m'-j-k'})\\
& + \frac{2}{N^2T^2}\sum_{t'=p+1}^{T-1}\sum_{t=t'+1}^{T-1}\frac{1}{T-t+1}\frac{1}{T-t'+1}\\
& \times\sum_{m=t}^{T-1}\sum_{m'=t'}^{T-1}\sum_{j=0}^{p-1}(\psi_{m-j-t'}\psi_{m'-j-t'}\kappa_3E(d'_tM_{t'}\epsilon_{t'}) + \sigma^4t'\psi_{m-j-t'}\psi_{m'-j-t}).
\end{aligned}$$

We observe that

$$\frac{1}{N^2T^2}\sum_{t=p+1}^{T-1}\frac{1}{(T-t+1)^2}\sum_{m=t}^{T-1}\sum_{m'=t}^{T-1}\sum_{j=0}^{p-1} O\left(t+\sqrt{tN}\right) = O\left(\frac{p}{N^2}+\frac{p}{N^{3/2}\sqrt{T}}\right).$$

For the second term, we have

$$\begin{aligned}
&\frac{1}{N^2T^2}\sum_{t=p+1}^{T-1}\frac{1}{(T-t+1)^2}\sum_{m=t}^{T-1}\sum_{m'=t}^{T-1}\sum_{j=0}^{p-1}\sum_{k=m-j-t+1}^{\infty}\sum_{k=m'-j-t+1}^{\infty}\psi_k\psi'_kE(\epsilon'_{m-j-k}M_t\epsilon_{m'-j-k'})\\
= \; & O\left(\frac{1}{NT^2}\sum_{t=p+1}^{T-1}\frac{1}{(T-t+1)^2}\sum_{m=t}^{T-1}\sum_{m'=t}^{T-1}\sum_{j=0}^{p-1}\sum_{k=m-j-t+1}^{\infty}\sum_{k=m'-j-t+1}^{\infty}\psi_k\psi'_k\right)\\
= \; & O\left(\frac{1}{NT^2}\sum_{j=0}^{p-1}\sum_{t=p+1}^{T-1}\frac{1}{(T-t+1)^2}\left(\sum_{m=t}^{T-1}\sum_{k=m-j-t+1}^{\infty}\psi_k\right)^2\right).
\end{aligned}$$

Since $\left|\sum_{m=t}^{T-1}\sum_{k=m-j-t+1}^{\infty}\psi_k\right| \leq \sum_{k=1}^{\infty} k|\psi_{-j+k}|$, which exists for any j by Assumption 4.

$$O\left(\frac{1}{NT^2}\sum_{j=0}^{p-1}\sum_{t=p+1}^{T-1}\frac{1}{(T-t+1)^2}\left(\sum_{m=t}^{T-1}\sum_{k=m-j-t+1}^{\infty}\psi_k\right)^2\right) = O\left(\frac{1}{NT^2}\sum_{j=0}^{p-1}\left(\sum_{k=1}^{\infty}k|\psi_{-j+k}|\right)^2\right).$$

Now, we observe that

$$\sum_{j=0}^{p-1}\left(\sum_{k=1}^{\infty}k|\psi_{-j+k}|\right)^2 \leq p\left(\sum_{k=1}^{\infty}k|\psi_{-p+1+k}|\right)^2 = p\left(\sum_{k=0}^{\infty}(k+p-1)|\psi_k|\right)^2 = O(p^3),$$

which implies that

$$O\left(\frac{1}{NT^2}\sum_{j=0}^{p-1}\left(\sum_{k=1}^{\infty}k|\psi_{-j+k}|\right)^2\right) = O\left(\frac{p^3}{NT^2}\right).$$

Therefore, we have that

$$\begin{aligned}
&\frac{1}{N^2T^2}\sum_{t=p+1}^{T-1}\frac{1}{(T-t+1)^2}\sum_{m=t}^{T-1}\sum_{m'=t}^{T-1}\sum_{j=0}^{p-1}\sum_{k=m-j-t+1}^{\infty}\sum_{k=m'-j-t+1}^{\infty}\psi_k\psi_k' E(\epsilon_{m-j-k}' M_t \epsilon_{m'-j-k'}) \\
=\;& O\left(\frac{p^3}{NT^2}\right).
\end{aligned}$$

Since $\sum_{m=t}^{T-1}\psi_{m-j-t'} = O(1)$ by Assumption 1, and $|E(d_t M_{t'}\epsilon_{t'-1})| = O(\sqrt{tt'})$, we have

$$\begin{aligned}
&\frac{2}{N^2T^2}\sum_{t'=p+1}^{T-1}\sum_{t=t'+1}^{T-1}\frac{1}{T-t+1}\frac{1}{T-t'+1} \\
&\times\sum_{m=t}^{T-1}\sum_{m'=t'}^{T-1}\sum_{j=0}^{p-1}(\psi_{m-j-t'}\psi_{m'-j-t'}\kappa_3 E(d_t' M_{t'}\epsilon_{t'}) + \sigma^4 t'\psi_{m-j-t'}\psi_{m'-j-t}) \\
=\;& O\left(\frac{p}{N^2T^2}\sum_{t'=p+1}^{T-1}\sum_{t=t'+1}^{T-1}\frac{\sqrt{t}}{T-t+1}\frac{\sqrt{t'}}{T-t'+1}\right) = O\left(\frac{p}{N^2T}\right).
\end{aligned}$$

Therefore, it holds that

$$\begin{aligned}
\left\|\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}\bar{w}_{t,T-1}(p)' M_t\epsilon_t\right\| &= O_p\left(\frac{\sqrt{p}\log T}{N} + \frac{\sqrt{p}}{N} + \frac{\sqrt{p}}{N^{3/4}T^{1/4}} + \frac{p^{3/2}}{\sqrt{N}T} + \frac{\sqrt{p}}{N\sqrt{T}}\right) \\
&= O_p\left(\frac{\sqrt{p}\log T}{N} + \frac{\sqrt{p}}{N^{3/4}T^{1/4}} + \frac{p^{3/2}}{\sqrt{N}T}\right),
\end{aligned}$$

because $T/N \to 0$.

Lastly, we consider the term

$$\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}\bar{w}_{t,T-1}(p)'M_t\bar{\epsilon}_{t+1,T}=\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{1}{(T-t)(T-t+1)}\sum_{m=t}^{T-1}\sum_{l=t+1}^{T}w_m(p)'M_t\epsilon_l.$$

By an argument similar to what shown above, we have

$$E(w_m'M_t\epsilon_l)=E\left(\sum_{k=0}^{\infty}\psi_k\epsilon_{m-k}'M_t\epsilon_l\right)=\psi_{m-l}E(\epsilon_l'M_t\epsilon_l)=\psi_{m-l}\sigma^2(t-1),$$

so that

$$E\left(w_m(p)'M_t\epsilon_l\right)=\sigma^2(t-1)\psi_{m-l}(p).$$

Therefore, we have that

$$\begin{aligned}
&\left\|E\left(\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{1}{(T-t)(T-t+1)}\sum_{m=t}^{T-1}\sum_{l=t+1}^{T}w_m(p)'M_t\epsilon_l\right)\right\|^2\\
=&\quad\frac{\sigma^4}{N^2T^2}\sum_{t=p+1}^{T-1}\sum_{t'=p+1}^{T-1}\frac{t-1}{(T-t)(T-t+1)}\frac{t-1}{(T-t')(T-t'+1)}\\
&\times\sum_{m=t}^{T-1}\sum_{l=t+1}^{T}\sum_{m'=t'}^{T-1}\sum_{l'=t'+1}^{T}tr\left(\psi_{m-l}(p)\psi_{m'-l'}(p)'\right).
\end{aligned}$$

Now, we observe that

$$\begin{aligned}
&\sum_{m=t}^{T-1}\sum_{l=t+1}^{T}\sum_{m'=t'}^{T-1}\sum_{l'=t'+1}^{T}tr\left(\psi_{m-l}(p)\psi_{m'-l'}(p)'\right)\\
=&\quad\left(\sum_{m=t}^{T-1}\sum_{l=t+1}^{T}\psi_{m-l}(p)\right)'\left(\sum_{m'=t'}^{T-1}\sum_{l'=t'+1}^{T}\psi_{m'-l'}(p)\right)=O\left(p\sqrt{(T-t)(T-t')}\right),
\end{aligned}$$

uniformly in m,m',l,l'. Since

$$\sum_{t=p+1}^{T-1}\frac{t-1}{\sqrt{T-t}(T-t+1)}=\sum_{s=1}^{T-p-1}\frac{T-s}{\sqrt{s}(s+1)}=O\left(T\right),$$

we have that

$$\left\|E\left(\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{1}{(T-t)(T-t+1)}\sum_{m=t}^{T-1}\sum_{l=t+1}^{T}w_m(p)'M_t\epsilon_l\right)\right\|^2=O\left(\frac{p}{N^2}\right).$$

We then consider the variance. Let $g_T(t) = (T-t)^{-1}(T-t+1)^{-1}$. We observe that

$$
\begin{aligned}
& tr\left(var\left(\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{1}{(T-t)(T-t+1)}\sum_{m=t}^{T-1}\sum_{l=t+1}^{T} w_m(p)'M_t\epsilon_l\right)\right) \\
= \;& \frac{\sigma^4}{N^2T^2}\sum_{t=p+1}^{T-1}\sum_{t'=p+1}^{T-1} g_T(t)g_T(t')\sum_{m=t}^{T-1}\sum_{l=t+1}^{T}\sum_{m'=t'}^{T-1}\sum_{l'=t'+1}^{T} tr\left(cov(w_m(p)'M_t\epsilon_l, w_{m'}(p)'M_{t'}\epsilon_{l'})\right) \\
= \;& \frac{\sigma^4}{N^2T^2}\sum_{t=p+1}^{T-1}\sum_{t'=p+1}^{T-1} g_T(t)g_T(t')\sum_{m=t}^{T-1}\sum_{l=t+1}^{T}\sum_{m'=t'}^{T-1}\sum_{l'=t'+1}^{T}\sum_{j=0}^{p-1} cov(w'_{m-j}M_t\epsilon_l, w'_{m'-j}M_{t'}\epsilon_{l'}).
\end{aligned}
$$

Noting that $O(\min(t,t')) = O(\sqrt{tt'})$, Lemma A.2 shows that

$$
\begin{aligned}
& tr\left(var\left(\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{1}{(T-t)(T-t+1)}\sum_{m=t}^{T-1}\sum_{l=t+1}^{T} w_m(p)'M_t\epsilon_l\right)\right) \\
= \;& \frac{\sigma^4}{N^2T^2}\sum_{t=p+1}^{T-1}\sum_{t'=p+1}^{T-1} g_T(t)g_T(t')\sum_{m=t}^{T-1}\sum_{l=\max(t,t')+1}^{T}\sum_{m'=t'}^{T-1}\sum_{j=0}^{p-1} O(\sqrt{tt'} + \sqrt{\min(t,t')N}) \\
& + \frac{\sigma^4}{N^2T^2}\sum_{t=p+1}^{T-1}\sum_{t'=p+1}^{T-1} g_T(t)g_T(t')\sum_{m=t}^{T-1}\sum_{l=\max(t,t')+1}^{T}\sum_{m'=t'}^{T-1} \\
& \times\sum_{j=0}^{p-1}\sum_{k=m-j-\min(t,t')+1}^{\infty}\sum_{k=m'-j-\min(t,t')+1}^{\infty}\psi_k\psi_{k'}E(\epsilon'_{m-j-k}M_{\min(t,t')}\epsilon_{m'-j-k'}) \\
& + \frac{\sigma^4}{N^2T^2}\sum_{t=p+1}^{T-1}\sum_{t'=p+1}^{T-1} g_T(t)g_T(t')\sum_{m=t}^{T-1}\sum_{l=t+1}^{T}\sum_{m'=t'}^{T-1}\sum_{l'=t'+1,l'\neq l}^{T}\sum_{j=0}^{p-1} O(\sqrt{tt'}).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \frac{\sigma^4}{N^2T^2}\sum_{t=p+1}^{T-1}\sum_{t'=p+1}^{T-1} g_T(t)g_T(t')\sum_{m=t}^{T-1}\sum_{l=\max(t,t')+1}^{T}\sum_{m'=t'}^{T-1}\sum_{j=0}^{p-1} O(\sqrt{tt'} + \sqrt{\min(t,t')N}) \\
= \;& O\left(\frac{p}{N^2}\right) + O\left(\frac{p}{N^2T^2}\sum_{t=p+1}^{T-1}\sum_{t'=p+1}^{T-1}\frac{T-\max(t,t')}{(T-t+1)(T-t'+1)}(\sqrt{\min(t,t')N}\right) \\
= \;& O\left(\frac{p}{N^2} + \frac{p\log T}{N^{3/2}T}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
& \sum_{t=p+1}^{T-1} \sum_{t'=p+1}^{T-1} \frac{T-\max(t,t')}{(T-t+1)(T-t'+1)} \sqrt{\min(t,t')N} \\
= \;& \sum_{t=p+1}^{T-1} \frac{T-t}{(T-t+1)^2} \sqrt{tN} + 2 \sum_{t=p+1}^{T-1} \sum_{t'=t+1}^{T-1} \frac{T-t'}{(T-t+1)(T-t'+1)} \sqrt{tN} \\
= \;& O\left(\sqrt{N} T \log T\right) + O\left(\sqrt{N} \sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} \sqrt{t}\right) \\
= \;& O\left(\sqrt{N} T \log T + \sqrt{NT}\right) = O\left(\sqrt{N} T \log T\right).
\end{aligned}
$$

For the second term, it holds that

$$
\begin{aligned}
& \Big| \frac{\sigma^4}{N^2T^2} \sum_{t=p+1}^{T-1} \sum_{t'=p+1}^{T-1} g_T(t) g_T(t') \sum_{m=t}^{T-1} \sum_{l=\max(t,t')+1}^{T} \sum_{m'=t'}^{T-1} \sum_{j=0}^{p-1} \\
& \times \sum_{k=m-j-\min(t,t')+1}^{\infty} \sum_{k=m'-j-\min(t,t')+1}^{\infty} \psi_k \psi_{k'} E(\epsilon'_{m-j-k} M_{\min(t,t')} \epsilon_{m'-j-k'}) \Big| \\
\leq \;& \frac{\sigma^6}{NT^2} \sum_{t=p+1}^{T-1} \sum_{t'=p+1}^{T-1} g_T(t) g_T(t') \sum_{m=t}^{T-1} \sum_{l=\max(t,t')+1}^{T} \sum_{m'=t'}^{T-1} \sum_{j=0}^{p-1} \sum_{k=m-j-\min(t,t')+1}^{\infty} \sum_{k'=m'-j-\min(t,t')+1}^{\infty} |\psi_k| \cdot |\psi_{k'}| \\
= \;& \frac{\sigma^6}{NT^2} \sum_{j=0}^{p-1} \sum_{t=p+1}^{T-1} g_T(t) \frac{1}{(T-t+1)} \left(\sum_{m=t}^{T-1} \sum_{k=m-j-t+1}^{\infty} |\psi_k| \right)^2 \\
& +2 \frac{\sigma^6}{NT^2} \sum_{j=0}^{p-1} \sum_{t=p+1}^{T-1} \sum_{t'=t+1}^{T-1} g_T(t) \frac{1}{(T-t'+1)} \left(\sum_{m=t}^{T-1} \sum_{k=m-j-t+1}^{\infty} |\psi_k| \right) \left(\sum_{m'=t'}^{T-1} \sum_{k'=m'-j-t+1}^{\infty} |\psi_{k'}| \right).
\end{aligned}
$$

Observing that

$$
\begin{aligned}
\sum_{m=t}^{T-1} \sum_{k=m-j-t+1}^{\infty} |\psi_k| \;&\leq\; \sum_{k=1}^{\infty} k |\psi_{-j+k}|, \\
\sum_{m'=t'}^{T-1} \sum_{k'=m'-j-t+1}^{\infty} |\psi_k| \;&\leq\; \sum_{k=1}^{\infty} k |\psi_{t'-t-j+k}|,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
& \Big| \frac{\sigma^4}{N^2T^2} \sum_{t=p+1}^{T-1} \sum_{t'=p+1}^{T-1} g_T(t) g_T(t') \sum_{m=t}^{T-1} \sum_{l=\max(t,t')+1}^{T} \sum_{m'=t'}^{T-1} \sum_{j=0}^{p-1} \\
& \times \sum_{k=m-j-\min(t,t')+1}^{\infty} \sum_{k=m'-j-\min(t,t')+1}^{\infty} \psi_k \psi_{k'} E(\epsilon'_{m-j-k} M_{\min(t,t')} \epsilon_{m'-j-k'}) \Big| \\
= \; & O\left(\frac{1}{NT^2} \sum_{j=0}^{p-1} \left(\sum_{k=1}^{\infty} k|\psi_{-j+k}| \right)^2 \right) \\
& + O\left(\frac{1}{NT^2} \sum_{j=0}^{p-1} \sum_{t=p+1}^{T-1} \sum_{t'=t+1}^{T-1} \frac{1}{(T-t)(T-t+1)} \frac{1}{(T-t'+1)} \left(\sum_{k=1}^{\infty} k|\psi_{-j+k}| \right) \left(\sum_{k=1}^{\infty} k|\psi_{t'-t-j+k}| \right) \right).
\end{aligned}
$$

Now, we observe that

$$
\sum_{j=0}^{p-1} \left(\sum_{k=1}^{\infty} k|\psi_{-j+k}| \right)^2 \leq p \left(\sum_{k=1}^{\infty} k|\psi_{-p+1+k}| \right)^2 = p \left(\sum_{k=0}^{\infty} (k+p-1)|\psi_k| \right)^2 = O(p^3),
$$

and

$$
\sum_{j=0}^{p-1} \left(\sum_{k=1}^{\infty} k|\psi_{-j+k}| \right) \left(\sum_{k=1}^{\infty} k|\psi_{t'-t-j+k}| \right) \leq \sum_{j=0}^{p-1} \left(\sum_{k=1}^{\infty} k|\psi_{-j+k}| \right)^2 = O(p^3).
$$

It therefore follows that

$$
\begin{aligned}
& \Big| \frac{\sigma^4}{N^2T^2} \sum_{t=p+1}^{T-1} \sum_{t'=p+1}^{T-1} g_T(t) g_T(t') \sum_{m=t}^{T-1} \sum_{l=\max(t,t')+1}^{T} \sum_{m'=t'}^{T-1} \sum_{j=0}^{p-1} \\
& \times \sum_{k=m-j-\min(t,t')+1}^{\infty} \sum_{k=m'-j-\min(t,t')+1}^{\infty} \psi_k \psi_{k'} E(\epsilon'_{m-j-k} M_{\min(t,t')} \epsilon_{m'-j-k'}) \Big| \\
= \; & O\left(\frac{p^3}{NT^2} \right) + O\left(\frac{p^3 \log T}{NT^2} \right) = O\left(\frac{p^3 \log T}{NT^2} \right).
\end{aligned}
$$

For the third term, we have

$$
\frac{\sigma^4}{N^2T^2} \sum_{t=p+1}^{T-1} \sum_{t'=p+1}^{T-1} g_T(t) g_T(t') \sum_{m=t}^{T-1} \sum_{l=t+1}^{T} \sum_{m'=t'}^{T-1} \sum_{l'=t'+1, l' \neq l}^{T} \sum_{j=0}^{p-1} O((tt')^{1/2}) = O\left(\frac{p}{N^2} \right).
$$

Therefore, we have that

$$
\left\| \frac{1}{NT} \sum_{t=p+1}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{t,T-1}(p)' M_t \bar{\epsilon}_{t+1,T} \right\| = O_p\left(\frac{\sqrt{p}}{N} + \frac{\sqrt{p \log T}}{N^{3/4}\sqrt{T}} + \frac{p^{3/2}\sqrt{\log T}}{\sqrt{N}T} \right).
$$

To sum up, we have

$$\begin{aligned}
||G_2|| &= O_p\left(\sqrt{\frac{p}{NT}}\right) + O_p\left(\frac{\sqrt{p}}{N^{1/2}T^{3/4}}\right) + O_p\left(\frac{\sqrt{p}\log T}{N} + \frac{\sqrt{p}}{N^{3/4}T^{1/4}} + \frac{p^{3/2}}{\sqrt{N}T}\right) \\
&\quad + O_p\left(\frac{\sqrt{p}}{N} + \frac{\sqrt{p\log T}}{N^{3/4}\sqrt{T}} + \frac{p^{3/2}\sqrt{\log T}}{\sqrt{N}T}\right) \\
&= O_p\left(\sqrt{\frac{p}{NT}} + \frac{\sqrt{p}\log T}{N} + \frac{\sqrt{p\log T}}{N^{3/4}\sqrt{T}} + \frac{p^{3/2}\sqrt{\log T}}{\sqrt{N}T}\right)
\end{aligned}$$

by $T/N \to 0$.

□

Lemma 15. *Suppose that Assumptions 1, 3 and 4 are satisfied. If $N \to \infty$, $T \to \infty$ and $p \to \infty$ with $p^2T/N \to 0$ and $p^3 \log T/T \to 0$, then*

$$\sqrt{NT}\ell_p'\Gamma_p^{-1}G_2/v_p \to_d N(0,1).$$

Proof. We observe that

$$G_2 = \frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'\epsilon_t + \frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'(I-M_t)\epsilon_t + G_{22},$$

where

$$\begin{aligned}
G_{22} &= -\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'M_t\bar{\epsilon}_{t+1,T} - \frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}\bar{w}_{t,T-1}(p)'M_t\epsilon_t \\
&\quad + \frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}\bar{w}_{t,T-1}(p)'M_t\bar{\epsilon}_{t+1,T}.
\end{aligned}$$

The proof of Lemma 14 shows that

$$\begin{aligned}
\|G_{22}\| &= O_p\left(\frac{\sqrt{p}}{\sqrt{N}T^{3/4}}\right) + O_p\left(\frac{\sqrt{p}\log T}{N} + \frac{\sqrt{p}}{N^{3/4}T^{1/4}} + \frac{p^{3/2}}{\sqrt{N}T}\right) \\
&\quad + O_p\left(\frac{\sqrt{p}}{N} + \frac{\sqrt{p\log T}}{N^{3/4}\sqrt{T}} + \frac{p^{3/2}\sqrt{\log T}}{\sqrt{N}T}\right) \\
&= O_p\left(\frac{\sqrt{p}}{\sqrt{N}T^{3/4}} + \frac{\sqrt{p}\log T}{N} + \frac{\sqrt{p}}{N^{3/4}T^{1/4}} + \frac{\sqrt{p\log T}}{N^{3/4}\sqrt{T}} + \frac{p^{3/2}\sqrt{\log T}}{\sqrt{N}T}\right)
\end{aligned}$$

so that $\sqrt{NT}||G_{22}|| = o_p(1)$ if $p^2/T \to 0$, $pT(\log T)^2/N \to 0$, $p^2T/N \to 0$, $p^2\log T/N \to 0$ and $p^3\log T/T \to 0$, which are satisfied by the conditions of the lemma. It therefore follows that

$$\left\|\sqrt{NT}\ell_p'\Gamma_p^{-1}G_{22}\right\| \leq \|\ell_p\|_1 \cdot \left\|\Gamma_p^{-1}\right\|_1 \cdot \left\|\sqrt{NT}G_{22}\right\| = o_p(1),$$

by the assumption that $\|\ell_p\|_1 = O(1)$ and Assumption 1.

We consider the first term in the decomposition of G_2. Lemma 11 gives

$$\sqrt{NT}\ell_p'\Gamma_p^{-1}\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'\epsilon_t/v_p \to_d N(0,1).$$

Next, we consider the second term in the decomposition of G_2. As $w_{t-1}(p)'(I-M_t)\epsilon_t$ is a martingale difference sequence, we have that

$$\begin{aligned}
& E\left\|\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'(I-M_t)\epsilon_t\right\|^2 \\
= & \frac{1}{N^2T^2}\sum_{t=p+1}^{T-1}\left(\frac{T-t}{T-t+1}\right)^2\sigma^2 tr\left(E(w_{t-1}(p)'(I-M_t)w_{t-1}(p))\right).
\end{aligned}$$

Hence, as in the proof of Lemma 12, we have that

$$tr\left(E(w_{t-1}(p)'(I-M_t)w_{t-1}(p))\right) = O\left(\frac{Np^2}{t}\right).$$

It therefore follows that

$$\begin{aligned}
E\left\|\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'(I-M_t)\epsilon_t\right\|^2 &= O\left(\frac{1}{N^2T^2}\sum_{t=p+1}^{T-1}\left(\frac{T-t}{T-t+1}\right)^2\frac{Np^2}{t}\right) \\
&= O\left(\frac{p^2\log T}{NT^2}\right).
\end{aligned}$$

By the Chebyshev inequality, it holds that

$$\left\|\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'(I-M_t)\epsilon_t\right\| = O_p\left(\frac{p(\log T)^{1/2}}{N^{1/2}T}\right).$$

This result implies that

$$\begin{aligned}
& \left\|\sqrt{NT}\ell_p'\Gamma_p^{-1}\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'(I-M_t)\epsilon_t\right\| \\
\leq & \|\ell_p\|_1\cdot\left\|\Gamma_p^{-1}\right\|_1\cdot\left\|\sqrt{NT}\frac{1}{NT}\sum_{t=p+1}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'(I-M_t)\epsilon_t\right\| \\
= & O_p\left(\frac{p(\log T)^{1/2}}{T^{1/2}}\right) = o_p(1),
\end{aligned}$$

by $\|\ell_p\|_1 = O(1)$, Assumption 1 and $p^2\log T/T \to 0$.

□

6 Proofs of the lemmas for the DFIV estimator

Recall that the estimation error of the DFIV estimator can be decomposed as

$$\hat{\alpha}(p) - \alpha(p) = (\hat{\Gamma}_p^{DF})^{-1} D_1 + (\hat{\Gamma}_p^{DF})^{-1} D_2$$

where

$$\hat{\Gamma}_p^{DF} = \frac{1}{NT} \sum_{t=p+2}^{T-1} h_t(p)' x_t^*(p), \quad D_1 = \frac{1}{NT} \sum_{t=p+2}^{T-1} h_t(p)' b_{t,p}^* \text{ and } D_2 = \frac{1}{NT} \sum_{t=p+2}^{T-1} h_t(p)' \epsilon_t^*.$$

Note that we can write

$$x_t^*(p) = \sqrt{\frac{T-t}{T-t+1}} \left(w_{t-1}(p) - \bar{w}_{t,T-1}(p)\right) \text{ and } h_t(p) = \sqrt{\frac{T-t}{T-t+1}} \left(w_{t-1}(p) - \bar{w}_{p,t-2}(p)\right).$$

Lemma 16. *Suppose that Assumption 1 is satisfied. If $N \to \infty$, $T \to \infty$ and $p \to \infty$ with $p^2/T \to 0$, then*

$$||\hat{\Gamma}_p^{DF} - \Gamma_p|| = O_p\left(\frac{p}{\sqrt{T}}\right).$$

Proof. We observe that

$$\begin{aligned} \hat{\Gamma}_p^{DF} &= \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' w_{t-1}(p) - \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' \bar{w}_{t,T-1}(p) \\ &\quad - \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{p,t-2}(p)' w_{t-1}(p) + \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{p,t-2}(p)' \bar{w}_{t,T-1}(p). \end{aligned}$$

For the first term in the decomposition, Lemma 9 gives

$$\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' w_{t-1}(p) - \Gamma_p \right\| = O_p\left(\frac{p}{\sqrt{NT}} + \frac{p \log T}{T}\right).$$

The second term is

$$\begin{aligned} \left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' \bar{w}_{t,T-1}(p) \right\| &\leq \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \|w_{t-1}(p)\| \cdot \|\bar{w}_{t,T-1}(p)\| \\ &= O_p\left(\frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \sqrt{Np} \sqrt{Np \frac{1}{T-t}} \right) = O_p\left(\frac{p}{\sqrt{T}}\right), \end{aligned}$$

where the first equality follows from Lemma 8.

For the third term, we observe that

$$\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{p,t-2}(p)' w_{t-1}(p) \right\| \leq \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \|w_{t-1}(p)\| \cdot \|\bar{w}_{p,t-2}(p)\|$$

$$= O_p\left(\frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \sqrt{Np} \sqrt{Np \frac{1}{t-p}} \right) = O_p\left(\frac{p}{\sqrt{T}} \right),$$

where Lemma 8 provides the first equality.

Finally, again applying Lemma 8, we obtain

$$\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{p,t-2}(p)' \bar{w}_{t,T-1}(p) \right\| \leq \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \|\bar{w}_{p,t-2}(p)\| \cdot \|\bar{w}_{t,T-1}(p)\|$$

$$= O_p\left(\frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \sqrt{Np \frac{1}{T-t}} \sqrt{Np \frac{1}{t-p}} \right) = O_p\left(\frac{p}{\sqrt{T}} \right).$$

In sum, it holds that

$$\left\| \hat{\Gamma}_p^{DF} - \Gamma_p \right\| = O_p\left(\frac{p}{\sqrt{NT}} + \frac{p \log T}{T} + \frac{p}{\sqrt{T}} \right) = O_p\left(\frac{p}{\sqrt{T}} \right).$$

□

Lemma 17. *Suppose that Assumption 1 is satisfied. If $N, T, p \to \infty$ with $p^{1/2} \sum_{k=p+1}^{\infty} |\alpha_k| \to 0$, then*

$$||D_1|| = O_p\left(\sqrt{p} \sum_{k=p+1}^{\infty} |\alpha_k| \right) = o_p(1).$$

Suppose that Assumption 1 is satisfied. If $N, T, p \to \infty$ with $p^{1/2} \sum_{k=p+1}^{\infty} |\alpha_k| \to 0$ and $p^2/T \to 0$, then

$$||\ell_p' (\hat{\Gamma}_p^{DF})^{-1} D_1|| = O_p\left(\sum_{k=p+1}^{\infty} |\alpha_k| \right).$$

Proof. Similarly to the proof of Lemma 13, we have

$$E\|h_t(p)\|^2 = NE(tr(h_{it}(p) h_{it}(p)')) = N \sum_{k=1}^{p} E\left((h_{it,k})^2 \right) = NpE((h_{it,1})^2) = O_p(Np),$$

so that $\|h_t(p)\| = O_p(\sqrt{Np})$ and

$$\left\| b_{t,p}^* \right\| = O_p\left(\sqrt{N} \sum_{k=p+1}^{\infty} |\alpha_k| \right)$$

by (2). Thus, it holds that

$$
\begin{aligned}
||D_1|| &= \left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} h_t(p)' b^*_{t,p} \right\| \leq \frac{1}{NT} \sum_{t=p+2}^{T-1} \left\| h_t(p)' b^*_{t,p} \right\| \leq \frac{1}{NT} \sum_{t=p+2}^{T-1} \| h_t(p) \| \cdot \left\| b^*_{t,p} \right\| \\
&= \frac{1}{NT} \sum_{t=p+2}^{T-1} O_p\left(\sqrt{Np}\right) O_p\left(\sqrt{N} \sum_{k=p+1}^{\infty} |\alpha_k|\right) = O_p\left(\sqrt{p} \sum_{k=p+1}^{\infty} |\alpha_k|\right).
\end{aligned}
$$

The proof concerning the order of $||\ell_p'(\hat{\Gamma}_p^D)^{-1} D_1||$ is analogous to that of the corresponding part of Lee, Okui and Shintani (2018, Lemma A.3) and is thus omitted.

□

Lemma 18. *Suppose that Assumption 1 is satisfied. If $N \to \infty$, $T \to \infty$ and $p \to \infty$ with $p/T \to 0$, then*

$$
||D_2|| = O_p\left(\sqrt{\frac{p}{NT}}\right) = o_p(1).
$$

Proof. We observe that

$$
\begin{aligned}
D_2 &= \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' \epsilon_t - \frac{1}{NT} \sum_{t=p+2}^{T-1} \sqrt{\frac{T-t}{T-t+1}} h_t(p)' \bar{\epsilon}_{t+1,T} \\
&\quad - \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{p,t-2}(p)' \epsilon_t.
\end{aligned}
$$

As in the proof of Lemma 14, it holds that

$$
\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' \epsilon_t \right\| = O_p\left(\sqrt{\frac{p}{NT}}\right).
$$

Let $\dot{h}_t(p) = \sqrt{(T-t+1)/(T-t)} h_t(p) = w_{t-1}(p) - \bar{w}_{p,t-2}$. Then, the second term in the decomposition is

$$
\frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \dot{h}_t(p)' \bar{\epsilon}_{t+1,T} = \frac{1}{NT} \sum_{t=p+2}^{T-1} \sum_{m=t+1}^{T} \frac{1}{T-t+1} \dot{h}_t(p)' \epsilon_m.
$$

Now, we have that

$$
\begin{aligned}
& E \left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \sum_{m=t+1}^{T} \frac{1}{T-t+1} \dot{h}_t(p)' \epsilon_m \right\|^2 \\
&= \frac{1}{N^2 T^2} \sum_{t=p+2}^{T-1} \sum_{t'=p+2}^{T-1} \frac{1}{T-t+1} \frac{1}{T-t'+1} \sum_{m=t+1}^{T} \sum_{m'=t'+1}^{T} tr\left(E\left(\dot{h}_t(p)' \epsilon_m \epsilon_{m'}' \dot{h}_{t'}(p)\right)\right) \\
&= \frac{\sigma^2}{N^2 T^2} \sum_{t=p+2}^{T-1} \sum_{t'=p+2}^{T-1} \frac{T - \max(t,t')}{(T-t+1)(T-t'+1)} tr\left(E\left(\dot{h}_t(p)' \dot{h}_{t'}(p)\right)\right)
\end{aligned}
$$

given $E\left(\dot{h}_t(p)'\epsilon_m\epsilon_{m'}'\dot{h}_{t'}(p)\right) = 0$ if $m \neq m'$ and $E\left(\dot{h}_t(p)'\epsilon_m\epsilon_m'\dot{h}_{t'}(p)\right) = \sigma^2 E\left(\dot{h}_t(p)'\dot{h}_{t'}(p)\right)$. We observe that

$$\begin{aligned} & \frac{\sigma^2}{N^2T^2}\sum_{t=p+2}^{T-1}\sum_{t'=p+2}^{T-1}\frac{T-\max(t,t')}{(T-t+1)(T-t'+1)}tr\left(E\left(\dot{h}_t(p)'\dot{h}_{t'}(p)\right)\right) \\ \leq \ & \frac{2\sigma^2}{N^2T^2}\sum_{t=p+2}^{T-1}\sum_{t'=t}^{T-1}\frac{T-t'}{(T-t+1)(T-t'+1)}tr\left(E\left(\dot{h}_t(p)'\dot{h}_{t'}(p)\right)\right) \\ = \ & \frac{2\sigma^2}{N^2T^2}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}tr\left(E\left(\dot{h}_t(p)'\tilde{\dot{h}}_{t,T-1}(p)\right)\right), \end{aligned}$$

where

$$\tilde{\dot{h}}_{t,T-1}(p) = \frac{1}{T-t}\sum_{t'=t}^{T-1}\frac{T-t'}{T-t'+1}\dot{h}_{t'}(p).$$

It holds that

$$tr(E\left(\dot{h}_t(p)'\tilde{\dot{h}}_{t,T-1}(p)\right)) \leq N(E(||\dot{h}_{it}(p)||^2))^{1/2}(E(||\tilde{\dot{h}}_{i,t,T-1}(p)||^2))^{1/2} = O\left(\frac{Np}{\sqrt{T-t}}\right),$$

where $E||\tilde{\dot{h}}_{i,t,T-1}(p)||^2 = O\left(Np/(T-t)\right)$ by the short memory assumption in Assumption 1. Thus, it holds that

$$E\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\sum_{m=t+1}^{T}\frac{1}{T-t+1}\dot{h}_t(p)'\epsilon_m\right\|^2 = O\left(\frac{1}{N^2T^2}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\frac{Np}{\sqrt{T-t}}\right) = O\left(\frac{p}{NT^{3/2}}\right).$$

The Chebyshev inequality gives that

$$\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\dot{h}_t(p)'\bar{\epsilon}_{t+1,T}\right\| = O_p\left(\frac{\sqrt{p}}{\sqrt{N}T^{3/4}}\right).$$

For the third term, given $\bar{w}_{p,t-2}(p)'\epsilon_t$ is a martingale difference sequence, we observe that

$$E\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\bar{w}_{p,t-2}(p)'\epsilon_t\right\|^2 = \frac{1}{N^2T^2}\sum_{t=p+2}^{T-1}\left(\frac{T-t}{T-t+1}\right)^2\sigma^2 tr(E(\bar{w}_{p,t-2}(p)'\bar{w}_{p,t-2}(p))).$$

As $tr(E(\bar{w}_{p,t-2}(p)'\bar{w}_{p,t-2}(p))) = O(Np/(t-p))$ by Lemma 8 and $(T-t)^2/(T-t+1)^2 < 1$, we have

$$E\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\bar{w}_{p,t-2}(p)'\epsilon_t\right\|^2 = O\left(\frac{1}{N^2T^2}\sum_{t=p+2}^{T-1}\left(\frac{T-t}{T-t+1}\right)^2\frac{Np}{t-p}\right) = O\left(\frac{p\log T}{NT^2}\right).$$

Therefore, the Chebyshev inequality gives

$$\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{p,t-2}(p)' \epsilon_t \right\| = O_p\left(\frac{\sqrt{p \log T}}{\sqrt{N} T} \right).$$

To sum up, we have

$$||D_2|| = O_p\left(\sqrt{\frac{p}{NT}} \right) + O_p\left(\frac{\sqrt{p}}{\sqrt{N} T^{3/4}} \right) + O\left(\frac{\sqrt{p \log T}}{\sqrt{N} T} \right) = O_p\left(\sqrt{\frac{p}{NT}} \right).$$

□

Lemma 19. *Suppose that Assumption 1 is satisfied. If $N \to \infty$, $T \to \infty$ and $p \to \infty$ with $p^3/(NT) \to 0$ and $p^2/T \to 0$, then*

$$\sqrt{NT} \ell_p' \Gamma_p^{-1} D_2 / v_p \to_d N(0,1).$$

Proof. We observe that

$$D_2 = \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' \epsilon_t + D_{22},$$

where

$$D_{22} = -\frac{1}{NT} \sum_{t=p+2}^{T-1} \sqrt{\frac{T-t}{T-t+1}} h_t(p)' \bar{\epsilon}_{t+1,T} - \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{p,t-2}(p)' \epsilon_t.$$

First, Lemma 11 gives

$$\sqrt{NT} \ell_p' \Gamma_p^{-1} \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' \epsilon_t / v_p \to_d N(0,1).$$

The proof of Lemma 18 shows that

$$\|D_{22}\| = O_p\left(\frac{\sqrt{p}}{\sqrt{N} T^{3/4}} \right) + O\left(\frac{\sqrt{p \log T}}{\sqrt{N} T} \right) = O_p\left(\frac{\sqrt{p}}{\sqrt{N} T^{3/4}} \right)$$

so that $\sqrt{NT} ||D_{22}|| = o_p(1)$ if $p^2/T \to 0$. It therefore follows that

$$\left\| \sqrt{NT} \ell_p' \Gamma_p^{-1} D_{22} \right\| \leq \|\ell_p\|_1 \cdot \left\| \Gamma_p^{-1} \right\|_1 \cdot \left\| \sqrt{NT} D_{22} \right\| = o_p(1),$$

by the assumption that $\|\ell_p\|_1 = O(1)$ and Assumption 1.

□

7 The DFIV estimator with exogenous regressors

In this section, we redefine w_{it} as $w_{it} = \sum_{k=0}^{\infty} \psi_k((g_{i,t-k} - E(g_{i,t-k}))'\beta + \epsilon_{i,t-k})$.

Lemma 20. *Suppose that Assumptions 1 and 5 are satisfied. Then,*

$$||\bar{w}_{t,\tau}(p)||^2 = O_p\left(\frac{Np}{\tau - t + 1}\right) \text{ and } E||\bar{w}_{t,\tau}(p)||^2 = O\left(\frac{Np}{\tau - t + 1}\right).$$

Similarly,

$$||\bar{g}_{t,\tau}||^2 = O_p\left(\frac{N}{\tau - t + 1}\right) \text{ and } E||\bar{g}_{t,\tau}||^2 = O\left(\frac{N}{\tau - t + 1}\right).$$

Proof. The proof is identical to that of Lemma 8 and is thus omitted. □

Let

$$\Gamma_p^x = E\begin{pmatrix} w_{i,t-1}(p)w_{i,t-1}(p)' & w_{i,t-1}(p)g_{it}' \\ g_{it}w_{i,t-1}(p)' & g_{it}g_{it}' \end{pmatrix}.$$

Lemma 21. *Suppose that Assumptions 1 and 5 are satisfied. If $N \to \infty$, $T \to \infty$ and $p \to \infty$ with $p/T \to 0$, then*

$$\left\| \frac{1}{N(T-p)} \sum_{t=p+1}^{T} \begin{pmatrix} w_{t-1}(p)'w_{t-1}(p) & w_{t-1}(p)'g_t \\ g_t'w_{t-1}(p) & g_t'g_t \end{pmatrix} - \Gamma_p^x \right\| = O_p\left(\frac{p}{\sqrt{NT}}\right).$$

Proof. The proof is similar to that of Lemma 9 and is thus omitted. □

Lemma 22. *Suppose that Assumptions 1 and 5 are satisfied. Let $\hat{\Gamma}_p^x$ be an estimator of Γ_p^x such that $\left\|\hat{\Gamma}_p^x - \Gamma_p^x\right\| = O_p(\rho_{N,T,p})$ where $\rho_{N,T,p} = o(1)$ as $N \to \infty$, $T \to \infty$ and $p \to \infty$. Then, as $N \to \infty$, $T \to \infty$ and $p \to \infty$, we have*

$$\begin{aligned} ||\hat{\Gamma}_p^x - \Gamma_p^x||_1 &= O_p(\rho_{N,T,p}), & (3) \\ ||(\hat{\Gamma}_p^x)^{-1} - (\Gamma_p^x)^{-1}||_1 &= O_p(\rho_{N,T,p}), & (4) \\ \text{and } ||(\hat{\Gamma}_p^x)^{-1}||_1 &= O_p(1). & (5) \end{aligned}$$

Proof. The proof is similar to that of Lemma 10 and is thus omitted. □

Lemma 23. *Suppose that Assumptions 1 and 5 are satisfied. If $N \to \infty$, $T \to \infty$ and $p \to \infty$ with $p^3/(NT) \to 0$ and $p/T \to 0$, then*

$$\frac{1}{\sqrt{N(T-p)}} \ell_p'(\Gamma_p^x)^{-1} \sum_{t=p+1}^{T} \begin{pmatrix} w_{t-1}(p)' \\ g_t' \end{pmatrix} \epsilon_t / v_p^x \to_d N(0,1),$$

where $v_p^x = \sigma^2 \ell_p'(\Gamma_p^x)^{-1}\ell_p$.

Proof. The proof is similar to that of Lemma 11 and is thus omitted. $\square$

The estimation error of the DFIV estimator can be decomposed as

$$\begin{pmatrix} \hat{\alpha}_{DF}(p) \\ \hat{\beta}_{DF} \end{pmatrix} - \begin{pmatrix} \alpha(p) \\ \beta \end{pmatrix} = (\hat{\Gamma}_p^{DFx})^{-1} D_1^x + (\hat{\Gamma}_p^{DFx})^{-1} D_2^x$$

where

$$\hat{\Gamma}_p^{DFx} = \frac{1}{NT} \sum_{t=p+2}^{T-1} \begin{pmatrix} h_t(p)' x_t^*(p) & h_t(p)' g_t^* \\ g_t^{*\prime} x_t^*(p) & g_t^{*\prime} g_t^* \end{pmatrix},$$

$$D_1^x = \frac{1}{NT} \sum_{t=p+2}^{T-1} \begin{pmatrix} h_t(p)' \\ g_t^{*\prime} \end{pmatrix} b_{t,p}^*, \text{ and } D_2^x = \frac{1}{NT} \sum_{t=p+2}^{T-1} \begin{pmatrix} h_t(p)' \\ g_t^{*\prime} \end{pmatrix} \epsilon_t^*.$$

Note that we can write

$$x_t^*(p) = \sqrt{\frac{T-t}{T-t+1}} \left(w_{t-1}(p) - \bar{w}_{t,T-1}(p) \right) \text{ and } h_t(p) = \sqrt{\frac{T-t}{T-t+1}} \left(w_{t-1}(p) - \bar{w}_{p,t-2}(p) \right).$$

Lemma 24. *Suppose that Assumptions 1 and 5 are satisfied. If $N \to \infty$, $T \to \infty$ and $p \to \infty$ with $p^2/T \to 0$, then*

$$||\hat{\Gamma}_p^{DFx} - \Gamma_p^x|| = O_p\left(\frac{p}{\sqrt{T}} \right).$$

Proof. We observe that

$$\begin{aligned} \hat{\Gamma}_p^{DF} &= \frac{1}{NT} \sum_{t=p+2}^{T-1} h_t(p)' x_t^*(p) \\ &= \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' w_{t-1}(p) - \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' \bar{w}_{t,T-1}(p) \\ &\quad - \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{p,t-2}(p)' w_{t-1}(p) + \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{p,t-2}(p)' \bar{w}_{t,T-1}(p). \end{aligned}$$

For the first term in the decomposition, Lemma 21 gives

$$\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' w_{t-1}(p) - \Gamma_p \right\| = O_p\left(\frac{p}{\sqrt{NT}} + \frac{p \log T}{T} \right).$$

The second term is

$$\begin{aligned} \left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' \bar{w}_{t,T-1}(p) \right\| &\leq \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \|w_{t-1}(p)\| \cdot \|\bar{w}_{t,T-1}(p)\| \\ = O_p\left(\frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \sqrt{Np} \sqrt{Np \frac{1}{T-t}} \right) &= O_p\left(\frac{p}{\sqrt{T}} \right), \end{aligned}$$

where the first equality follows from Lemma 20.

For the third term, we observe that

$$\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{p,t-2}(p)' w_{t-1}(p) \right\| \leq \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \|w_{t-1}(p)\| \cdot \|\bar{w}_{p,t-2}(p)\|$$

$$= O_p\left(\frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \sqrt{Np} \sqrt{Np \frac{1}{t-p}} \right) = O_p\left(\frac{p}{\sqrt{T}} \right),$$

where Lemma 20 provides the first equality.

Finally, again applying Lemma 20, we obtain

$$\begin{aligned} & \left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{p,t-2}(p)' \bar{w}_{t,T-1}(p) \right\| \\ \leq & \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \|\bar{w}_{p,t-2}(p)\| \cdot \|\bar{w}_{t,T-1}(p)\| \\ = & O_p\left(\frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \sqrt{Np \frac{1}{T-t}} \sqrt{Np \frac{1}{t-p}} \right) = O_p\left(\frac{p}{\sqrt{T}} \right). \end{aligned}$$

In sum, it holds that

$$\left\| \hat{\Gamma}_p^{DF} - \Gamma_p \right\| = O_p\left(\frac{p}{\sqrt{NT}} + \frac{p \log T}{T} + \frac{p}{\sqrt{T}} \right) = O_p\left(\frac{p}{\sqrt{T}} \right).$$

Next, we consider

$$\begin{aligned} & \frac{1}{NT} \sum_{t=p+2}^{T-1} h_t(p)' g_t^* \\ = & \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' g_t - \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' \bar{g}_{t,T-1} \\ & - \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{p,t-2}(p)' g_t + \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{p,t-2}(p)' \bar{g}_{t,T-1}(p). \end{aligned}$$

For the first term in the decomposition, Lemma 21 gives

$$\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' g_t - E(w_{i,t-1}(p) g_{it}') \right\| = O_p\left(\frac{p}{\sqrt{NT}} + \frac{p \log T}{T} \right).$$

The second term is

$$\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'\bar{g}_{t,T-1}(p)\right\| \leq \frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\left\|w_{t-1}(p)\right\|\cdot\left\|\bar{g}_{t,T-1}(p)\right\|$$

$$= \quad O_p\left(\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\sqrt{Np}\sqrt{N\frac{1}{T-t}}\right) = O_p\left(\frac{\sqrt{p}}{\sqrt{T}}\right),$$

where the first equality follows from Lemma 20.

For the third term, we observe that

$$\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\bar{w}_{p,t-2}(p)'g_t\right\| \leq \frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\left\|g_t\right\|\cdot\left\|\bar{w}_{p,t-2}(p)\right\|$$

$$= \quad O_p\left(\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\sqrt{N}\sqrt{Np\frac{1}{t-p}}\right) = O_p\left(\frac{\sqrt{p}}{\sqrt{T}}\right),$$

where Lemma 20 provides the first equality.

Finally, again applying Lemma 20, we obtain

$$\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\bar{w}_{p,t-2}(p)'\bar{g}_{t,T-1}(p)\right\| \leq \frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\left\|\bar{w}_{p,t-2}(p)\right\|\cdot\left\|\bar{g}_{t,T-1}(p)\right\|$$

$$= \quad O_p\left(\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\sqrt{N\frac{1}{T-t}}\sqrt{Np\frac{1}{t-p}}\right) = O_p\left(\frac{\sqrt{p}}{\sqrt{T}}\right).$$

In sum, it holds that

$$\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}h_t(p)'g_t^* - E(w_{i,t-1}(p)g_{it}')\right\| = O_p\left(\frac{p}{\sqrt{NT}} + \frac{p\log T}{T} + \frac{\sqrt{p}}{\sqrt{T}}\right) = O_p\left(\frac{p}{\sqrt{T}}\right).$$

We now consider

$$\frac{1}{NT}\sum_{t=p+2}^{T-1}g_t^{*\prime}x_t^*(p) = \frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}g_t'w_{t-1}(p) - \frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}g_t'\bar{w}_{t,T-1}(p)$$

$$-\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\bar{g}_{t,T-1}(p)'w_{t-1}(p) + \frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\bar{g}_{t,T-2}(p)'\bar{w}_{t,T-1}(p).$$

For the first term in the decomposition, Lemma 21 gives

$$\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}g_t'w_{t-1}(p) - E(g_{it}w_{i,t-1}(p)')\right\| = O_p\left(\frac{p}{\sqrt{NT}} + \frac{p\log T}{T}\right).$$

The second term is

$$\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} g_t' \bar{w}_{t,T-1}(p) \right\| \leq \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \|g_t\| \cdot \|\bar{w}_{t,T-1}(p)\|$$

$$= O_p\left(\frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \sqrt{N} \sqrt{Np \frac{1}{T-t}} \right) = O_p\left(\frac{\sqrt{p}}{\sqrt{T}} \right),$$

where the first equality follows from Lemma 20.

For the third term, we observe that

$$\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{g}_{t,T-1}(p)' w_{t-1}(p) \right\| \leq \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \|w_{t-1}(p)\| \cdot \|\bar{g}_{t,T-1}(p)\|$$

$$= O_p\left(\frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \sqrt{Np} \sqrt{N \frac{1}{T-t}} \right) = O_p\left(\frac{\sqrt{p}}{\sqrt{T}} \right),$$

where Lemma 20 provides the first equality.

Finally, again applying Lemma 20, we obtain

$$\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{g}_{t,T-1}(p)' \bar{w}_{t,T-1}(p) \right\| \leq \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \|\bar{g}_{t,T-1}(p)\| \cdot \|\bar{w}_{t,T-1}(p)\|$$

$$= O_p\left(\frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \sqrt{Np \frac{1}{T-t}} \sqrt{N \frac{1}{T-t}} \right) = O_p\left(\frac{\sqrt{p}}{\sqrt{T}} \right).$$

In sum, it holds that

$$\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} g_t^{*\prime} x_t^*(p) - E(g_{it} w_{i,t-1}(p)') \right\| = O_p\left(\frac{p}{\sqrt{NT}} + \frac{p \log T}{T} + \frac{\sqrt{p}}{\sqrt{T}} \right) = O_p\left(\frac{p}{\sqrt{T}} \right).$$

Lastly, we consider

$$\begin{aligned} \frac{1}{NT} \sum_{t=p+2}^{T-1} g_t^{*\prime} g_t^* &= \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} g_t' g_t - \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} g_t' \bar{g}_{t,T-1} \\ &\quad - \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{g}_{t,T-1}' g_t + \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{g}_{t,T-1}' \bar{g}_{t,T-1}. \end{aligned}$$

For the first term in the decomposition, the CLT implies

$$\begin{aligned} &\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} g_t' g_t - E(g_{it} g_{it}') \right\| \\ \leq &\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} g_t' g_t - E(g_{it} g_{it}') \right\| + \left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{1}{T-t+1} g_t' g_t \right\|. \end{aligned}$$

The CLT implies

$$\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} g_t' g_t - E(g_{it} g_{it}') \right\| = O_p\left(\frac{1}{\sqrt{NT}}\right).$$

We also have

$$\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{1}{T-t+1} g_t' g_t \right\| \leq \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{1}{T-t+1} \left\| g_t \right\|^2 = O_p\left(\frac{\log T}{T}\right).$$

The second term is

$$\begin{aligned}
&\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} g_t' \bar{g}_{t,T-1} \right\| \leq \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \left\| g_t \right\| \cdot \left\| \bar{g}_{t,T-1} \right\| \\
= \;& O_p\left(\frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \sqrt{N} \sqrt{N \frac{1}{T-t}} \right) = O_p\left(\frac{1}{\sqrt{T}}\right),
\end{aligned}$$

where the first equality follows from Lemma 20.

For the third term, we observe that

$$\begin{aligned}
&\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{g}_{t,T-1}' g_t \right\| \leq \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \left\| g_t \right\| \cdot \left\| \bar{g}_{t,T-1} \right\| \\
= \;& O_p\left(\frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \sqrt{N} \sqrt{N \frac{1}{T-t}} \right) = O_p\left(\frac{1}{\sqrt{T}}\right),
\end{aligned}$$

where Lemma 20 provides the first equality.

Finally, again applying Lemma 20, we obtain

$$\begin{aligned}
&\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{g}_{t,T-1}' \bar{g}_{t,T-1} \right\| \leq \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \left\| \bar{g}_{t,T-1} \right\|^2 \\
= \;& O_p\left(\frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} N \frac{1}{T-t} \right) = O_p\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}$$

In sum, it holds that

$$\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} g_t^{*\prime} g_t^* - E(g_{it} g_{it}') \right\| = O_p\left(\frac{1}{\sqrt{NT}} + \frac{\log T}{T} + \frac{1}{\sqrt{T}} \right) = O_p\left(\frac{p}{\sqrt{T}}\right).$$

□

Lemma 25. *Suppose that Assumptions 1 and 5 are satisfied. If $N \to \infty$, $T \to \infty$ and $p \to \infty$, then*

$$||D_1^x|| = O_p\left(\sqrt{p}\sum_{k=p+1}^{\infty}|\alpha_k|\right) = o_p(1).$$

Suppose that Assumptions 1 and 5 are satisfied. If $N \to \infty$, $T \to \infty$ and $p \to \infty$ with $p^2/T \to 0$, then

$$||\ell_p'(\hat{\Gamma}_p^{DFx})^{-1}D_1^x|| = O_p\left(\sum_{k=p+1}^{\infty}|\alpha_k|\right).$$

Proof. Similarly to the proof of Lemma 13, we have

$$E\left\|h_t(p)\right\|^2 = NE(tr(h_{it}(p)h_{it}(p)')) = N\sum_{k=1}^{p}E\left((h_{it,k})^2\right) = NpE((h_{it,1})^2) = O_p(Np),$$

so that $\|h_t(p)\| = O_p(\sqrt{Np})$ and

$$\left\|b_{t,p}^*\right\| = O_p\left(\sqrt{N}\sum_{k=p+1}^{\infty}|\alpha_k|\right)$$

by (2). Thus, it holds that

$$\begin{aligned}
\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}h_t(p)'b_{t,p}^*\right\| &\le \frac{1}{NT}\sum_{t=p+2}^{T-1}\left\|h_t(p)'b_{t,p}^*\right\| \le \frac{1}{NT}\sum_{t=p+2}^{T-1}\|h_t(p)\|\cdot\left\|b_{t,p}^*\right\| \\
&= \frac{1}{NT}\sum_{t=p+2}^{T-1}O_p\left(\sqrt{Np}\right)O_p\left(\sqrt{N}\sum_{k=p+1}^{\infty}|\alpha_k|\right) = O_p\left(\sqrt{p}\sum_{k=p+1}^{\infty}|\alpha_k|\right).
\end{aligned}$$

Next, we consider $\sum_{t=p+2}^{T-1}g_{it}^*b_{t,p}^*/(NT)$. Because

$$E(||g_t^*||^2) = N\sum_{k=1}^{k_x}E(g_{it,k}^2) = O(N),$$

it holds that

$$\begin{aligned}
\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}g_t^{*\prime}b_{t,p}^*\right\| &\le \frac{1}{NT}\sum_{t=p+2}^{T-1}\left\|g_t^{*\prime}b_{t,p}^*\right\| \le \frac{1}{NT}\sum_{t=p+2}^{T-1}\|g_t^*\|\cdot\left\|b_{t,p}^*\right\| \\
&= \frac{1}{NT}\sum_{t=p+2}^{T-1}O_p\left(\sqrt{N}\right)O_p\left(\sqrt{N}\sum_{k=p+1}^{\infty}|\alpha_k|\right) = O_p\left(\sum_{k=p+1}^{\infty}|\alpha_k|\right).
\end{aligned}$$

Summing up, we have $||D_1^x|| = O_p\left(\sqrt{p}\sum_{k=p+1}^{\infty}|\alpha_k|\right) = o_p(1)$

The proof concerning the order of $||\ell_p'(\hat{\Gamma}_p^D)^{-1}D_1^x||$ is analogous to that of the corresponding part of Lee, Okui and Shintani (2018, Lemma A.3) and is thus omitted.

$\square$

Lemma 26. *Suppose that Assumptions 1 and 5 are satisfied. If $N \to \infty$, $T \to \infty$ and $p \to \infty$ with $p/T \to 0$, then*

$$||D_2^x|| = O_p\left(\sqrt{\frac{p}{NT}}\right) = o_p(1).$$

Proof. We observe that

$$\begin{aligned}\frac{1}{NT}\sum_{t=p+2}^{T-1} h_t(p)'\epsilon_t^* &= \frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'\epsilon_t - \frac{1}{NT}\sum_{t=p+2}^{T-1}\sqrt{\frac{T-t}{T-t+1}}h_t(p)'\bar{\epsilon}_{t+1,T} \\ &\quad -\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\bar{w}_{p,t-2}(p)'\epsilon_t.\end{aligned}$$

As in the proof of Lemma 14, it holds that

$$\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'\epsilon_t\right\| = O_p\left(\sqrt{\frac{p}{NT}}\right).$$

Let $\dot{h}_t(p) = \sqrt{(T-t+1)/(T-t)}h_t(p) = w_{t-1}(p) - \bar{w}_{p,t-2}$. Then, the second term in the decomposition is

$$\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\dot{h}_t(p)'\bar{\epsilon}_{t+1,T} = \frac{1}{NT}\sum_{t=p+2}^{T-1}\sum_{m=t+1}^{T}\frac{1}{T-t+1}\dot{h}_t(p)'\epsilon_m.$$

Now, we have that

$$\begin{aligned}& E\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\sum_{m=t+1}^{T}\frac{1}{T-t+1}\dot{h}_t(p)'\epsilon_m\right\|^2 \\ =\;& \frac{1}{N^2T^2}\sum_{t=p+2}^{T-1}\sum_{t'=p+2}^{T-1}\frac{1}{T-t+1}\frac{1}{T-t'+1}\sum_{m=t+1}^{T}\sum_{m'=t'+1}^{T} tr\left(E\left(\dot{h}_t(p)'\epsilon_m\epsilon_{m'}'\dot{h}_{t'}(p)\right)\right) \\ =\;& \frac{\sigma^2}{N^2T^2}\sum_{t=p+2}^{T-1}\sum_{t'=p+2}^{T-1}\frac{T-\max(t,t')}{(T-t+1)(T-t'+1)} tr\left(E\left(\dot{h}_t(p)'\dot{h}_{t'}(p)\right)\right)\end{aligned}$$

given $E\left(\dot{h}_t(p)'\epsilon_m\epsilon_{m'}'\dot{h}_{t'}(p)\right) = 0$ if $m \neq m'$ and $E\left(\dot{h}_t(p)'\epsilon_m\epsilon_m'\dot{h}_{t'}(p)\right) = \sigma^2 E\left(\dot{h}_t(p)'\dot{h}_{t'}(p)\right)$.

We observe that

$$
\begin{aligned}
& \frac{\sigma^2}{N^2T^2}\sum_{t=p+2}^{T-1}\sum_{t'=p+2}^{T-1}\frac{T-\max(t,t')}{(T-t+1)(T-t'+1)}tr\left(E\left(\dot{h}_t(p)'\dot{h}_{t'}(p)\right)\right) \\
\leq\ & \frac{2\sigma^2}{N^2T^2}\sum_{t=p+2}^{T-1}\sum_{t'=t}^{T-1}\frac{T-t'}{(T-t+1)(T-t'+1)}tr\left(E\left(\dot{h}_t(p)'\dot{h}_{t'}(p)\right)\right) \\
=\ & \frac{2\sigma^2}{N^2T^2}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}tr\left(E\left(\dot{h}_t(p)'\tilde{\dot{h}}_{t,T-1}(p)\right)\right),
\end{aligned}
$$

where

$$
\tilde{\dot{h}}_{t,T-1}(p)=\frac{1}{T-t}\sum_{t'=t}^{T-1}\frac{T-t'}{T-t'+1}\dot{h}_{t'}(p).
$$

It holds that

$$
tr(E\left(\dot{h}_t(p)'\tilde{\dot{h}}_{t,T-1}(p)\right))\leq N(E(||\dot{h}_{it}(p)||^2))^{1/2}(E(||\tilde{\dot{h}}_{i,t,T-1}(p)||^2))^{1/2}=O\left(\frac{Np}{\sqrt{T-t}}\right),
$$

where $E||\tilde{\dot{h}}_{i,t,T-1}(p)||^2=O\left(Np/(T-t)\right)$ by the short memory assumption in Assumption 1. Thus, it holds that

$$
E\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\sum_{m=t+1}^{T}\frac{1}{T-t+1}\dot{h}_t(p)'\epsilon_m\right\|^2=O\left(\frac{1}{N^2T^2}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\frac{Np}{\sqrt{T-t}}\right)=O\left(\frac{p}{NT^{3/2}}\right).
$$

The Chebyshev inequality gives that

$$
\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\dot{h}_t(p)'\bar{\epsilon}_{t+1,T}\right\|=O_p\left(\frac{\sqrt{p}}{\sqrt{N}T^{3/4}}\right).
$$

For the third term, given $\bar{w}_{p,t-2}(p)'\epsilon_t$ is a martingale difference sequence, we observe that

$$
E\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\bar{w}_{p,t-2}(p)'\epsilon_t\right\|^2=\frac{1}{N^2T^2}\sum_{t=p+2}^{T-1}\left(\frac{T-t}{T-t+1}\right)^2\sigma^2tr(E(\bar{w}_{p,t-2}(p)'\bar{w}_{p,t-2}(p))).
$$

As $tr(E(\bar{w}_{p,t-2}(p)'\bar{w}_{p,t-2}(p)))=O(Np/(t-p))$ by Lemma 20 and $(T-t)^2/(T-t+1)^2<1$, we have

$$
E\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\bar{w}_{p,t-2}(p)'\epsilon_t\right\|^2=O\left(\frac{1}{N^2T^2}\sum_{t=p+2}^{T-1}\left(\frac{T-t}{T-t+1}\right)^2\frac{Np}{t-p}\right)=O\left(\frac{p\log T}{NT^2}\right).
$$

Therefore, the Chebyshev inequality gives

$$\left\| \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{p,t-2}(p)' \epsilon_t \right\| = O_p\left(\frac{\sqrt{p \log T}}{\sqrt{N}T} \right).$$

Next, we consider $\sum_{t=p+2}^{T-1} g_t^{*\prime} \epsilon_t^* / (NT)$. Its expectation is zero because of the strict exogeneity of g_{it}. Its variance is

$$\begin{aligned} & E\left(\left(\frac{1}{NT} \sum_{t=p+2}^{T-1} g_t^{*\prime} \epsilon_t^* \right) \left(\frac{1}{NT} \sum_{t=p+2}^{T-1} g_t^{*\prime} \epsilon_t^* \right)' \right) \\ =& \sigma^2 \frac{1}{N^2 T^2} \sum_{t=p+2}^{T-1} E(g_t^{*\prime} g_t^*) = \sigma^2 \frac{1}{NT^2} \sum_{t=p+2}^{T-1} E(g_{it}^* g_{it}^{*\prime}) = O\left(\frac{1}{NT} \right). \end{aligned}$$

Thus, we have

$$\frac{1}{NT} \sum_{t=p+2}^{T-1} g_t^{*\prime} \epsilon_t^* = O_p\left(\frac{1}{\sqrt{NT}} \right).$$

To sum up, we have

$$||D_2^x|| = O_p\left(\sqrt{\frac{p}{NT}} \right) + O_p\left(\frac{\sqrt{p}}{\sqrt{N} T^{3/4}} \right) + O\left(\frac{\sqrt{p \log T}}{\sqrt{N} T} \right) + O_p\left(\frac{1}{\sqrt{NT}} \right) = O_p\left(\sqrt{\frac{p}{NT}} \right).$$

□

Lemma 27. *Suppose that Assumptions 1 and 5 are satisfied. If $N \to \infty$, $T \to \infty$ and $p \to \infty$ with $p^3/(NT) \to 0$ and $p^2/T \to 0$, then*

$$\sqrt{NT} \ell_p' (\Gamma_p^x)^{-1} D_2^x / v_p^x \to_d N(0, 1).$$

Proof. We observe that

$$\frac{1}{NT} \sum_{t=p+2}^{T-1} h_t(p)' \epsilon_t^* = \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} w_{t-1}(p)' \epsilon_t + D_{22},$$

where

$$D_{22} = -\frac{1}{NT} \sum_{t=p+2}^{T-1} \sqrt{\frac{T-t}{T-t+1}} h_t(p)' \bar{\epsilon}_{t+1,T} - \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \bar{w}_{p,t-2}(p)' \epsilon_t.$$

The proof of Lemma 26 shows that

$$\|D_{22}\| = O_p\left(\frac{\sqrt{p}}{\sqrt{N} T^{3/4}} \right) + O\left(\frac{\sqrt{p \log T}}{\sqrt{N} T} \right) = O_p\left(\frac{\sqrt{p}}{\sqrt{N} T^{3/4}} \right)$$

so that $\sqrt{NT}||D_{22}|| = o_p(1)$ if $p^2/T \to 0$.

Next, consider

$$\frac{1}{NT}\sum_{t=p+2}^{T-1} g_t^{*\prime}\epsilon_t^* = \frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}g_t'\epsilon_t + D_{22}^x,$$

where

$$D_{22}^x = -\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{1}{T-t+1}g_t'\epsilon_t - \frac{1}{NT}\sum_{t=p+2}^{T-1}\sqrt{\frac{T-t}{T-t+1}}g_t^{*\prime}\bar{\epsilon}_{t+1,T}.$$

We consider the first term of D_{22}^x. Its mean is zero. Also, we have

$$\begin{aligned}
&E\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{1}{T-t+1}g_t'\epsilon_t\right\|^2 \\
=&\frac{1}{N^2}T^2\sum_{t=p+2}^{T-1}\sum_{t'=p+2}^{T-1}\frac{1}{T-t+1}\frac{1}{T-t'+1}tr(E(g_t'\epsilon_t\epsilon_t'g_t)) \\
=&\frac{\sigma^2}{NT^2}\sum_{t=p+2}^{T-1}\frac{1}{(T-t+1)^2}tr(E(g_{it}g_{it}')) = O\left(\frac{1}{NT^2}\right).
\end{aligned}$$

Thus, this term is of order $O_p(1/\sqrt{NT^2})$.

The second term of D_{22}^x is

$$\frac{1}{NT}\sum_{t=p+2}^{T-1}\sqrt{\frac{T-t}{T-t+1}}g_t^{*\prime}\bar{\epsilon}_{t+1,T} = \frac{1}{NT}\sum_{t=p+2}^{T-1}\sum_{m=t+1}^{T}\sqrt{\frac{1}{(T-t)(T-t+1)}}g_t^{*\prime}\epsilon_m.$$

Its mean is zero. We also have

$$\begin{aligned}
&E\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\sum_{m=t+1}^{T}\sqrt{\frac{1}{(T-t)(T-t+1)}}g_t^{*\prime}\epsilon_m\right\|^2 \\
=&\frac{1}{N^2T^2}\sum_{t=p+2}^{T-1}\sum_{t'=p+2}^{T-1}\sqrt{\frac{1}{(T-t)(T-t+1)}}\sqrt{\frac{1}{(T-t')(T-t'+1)}}\sum_{m=t+1}^{T}\sum_{m'=t+1}^{T}tr(E(g_t^{*\prime}\epsilon_m\epsilon_m'g_{t'}^*)) \\
=&\frac{\sigma^2}{NT^2}\sum_{t=p+2}^{T-1}\sum_{t'=p+2}^{T-1}(T-\max(t,t'))\sqrt{\frac{1}{(T-t)(T-t+1)}}\sqrt{\frac{1}{(T-t')(T-t'+1)}}tr(E(g_{it}^*g_{it'}^{*\prime})) \\
\leq&\frac{\sigma^2}{NT^2}\sum_{t'=p+2}^{T-1}\sqrt{\frac{1}{(T-t')(T-t'+1)}}\sum_{t=p+2}^{T-1}\sqrt{\frac{T-t}{T-t+1}}|tr(E(g_{it}^*g_{it'}^{*\prime}))| \\
\leq&\frac{\sigma^2}{NT^2}\sum_{t'=p+2}^{T-1}\sqrt{\frac{1}{(T-t')(T-t'+1)}}M = O\left(\frac{1}{NT^2}\right).
\end{aligned}$$

The second inequality follows because of the short-memory property of g_{it}^*. Thus, the second term is of order $O_p(1/\sqrt{NT^2})$.

To sum up, we have $||D_{22}^x|| = o_p(\sqrt{NT})$. It therefore follows that

$$\left\| \sqrt{NT}\ell_p'(\Gamma_p^x)^{-1} \begin{pmatrix} D_{22} \\ D_{22}^x \end{pmatrix} \right\| \leq \|\ell_p\|_1 \cdot \left\|(\Gamma_p^2)^{-1}\right\|_1 \cdot \left\| \sqrt{NT} \begin{pmatrix} D_{22} \\ D_{22}^x \end{pmatrix} \right\| = o_p(1),$$

by the assumption that $\|\ell_p\|_1 = O(1)$ and Assumption 1.

Lemma 23 gives

$$\sqrt{NT}\ell_p'(\Gamma_p^x)^{-1} \frac{1}{NT} \sum_{t=p+2}^{T-1} \frac{T-t}{T-t+1} \begin{pmatrix} w_{t-1}(p)' \\ g_t' \end{pmatrix} \epsilon_t / v_p^x \to_d N(0,1).$$

□

8 Lag selection

8.1 Proof of Theorem 10

Proof. The proof is similar to that of Theorem 8 in Lee, Okui, and Shintani (2018). First, we note that $\hat{p} < p_{\min}$ if and only if $|t_p(\hat{\alpha}(p))| < z_{0.5\alpha}$ for any $p \in \{p_{\min}, \ldots, p_{\max}\}$. We thus need to prove that

$$P\left(\max_{p_{\min} \leq p \leq p_{\max}} |t_p(\hat{\alpha}(p))| > z_{0.5\alpha} \right) \to 1.$$

Let

$$\tilde{t}_p(\hat{\alpha}(p)) = e_p' \left(\Gamma_p\right)^{-1} \frac{1}{\sqrt{NT}} \sum_{t=p+2}^{T-1} w_{t-1}(p) \epsilon_t / v_p.$$

We shall prove that

$$\max_{p_{\min} \leq p \leq p_{\max}} \left| t_p(\hat{\alpha}(p)) - \tilde{t}_p(\hat{\alpha}(p)) \right| = o_p(1). \tag{6}$$

Note that

$$\begin{aligned} & t_p(\hat{\alpha}(p)) - \tilde{t}_p(\hat{\alpha}(p)) \\ =& e_p' \left(\Gamma_p\right)^{-1} \frac{1}{\sqrt{NT}} \sum_{t=p+2}^{T-1} \frac{1}{T-t+1} w_{t-1}(p) \epsilon_t / v_p \\ &+ \sqrt{NT} e_p' (\hat{\Gamma}_p^{DF})^{-1} D_1 / v_p + \sqrt{NT} e_p' ((\hat{\Gamma}_p^{DF})^{-1} - \Gamma_p^{-1}) D_2 / v_p + \sqrt{NT} e_p' \Gamma_p^{-1} D_{22} / v_p \\ &+ \sqrt{NT} \alpha_p / v_p + e_p' \left(\Gamma_p\right)^{-1} \frac{1}{\sqrt{NT}} \sum_{t=p+2}^{T-1} w_{t-1}(p) \epsilon_t \left(v_p^{-1} - \hat{v}_p^{-1}\right). \end{aligned}$$

We first consider

$$
\begin{aligned}
&E\left(\max_{p_{\min}\le p\le p_{\max}}\left(e_p'\left(\Gamma_p\right)^{-1}\frac{1}{\sqrt{NT}}\sum_{t=p+2}^{T-1}\frac{1}{T-t+1}w_{t-1}(p)\epsilon_t/v_p\right)^2\right)\\
&\le\left(\sum_{p=p_{\min}}^{p_{\max}}\left(e_p'\left(\Gamma_p\right)^{-1}\frac{1}{\sqrt{NT}}\sum_{t=p+2}^{T-1}\frac{1}{T-t+1}w_{t-1}(p)\epsilon_t/v_p\right)^2\right)\\
&=\sum_{p=p_{\min}}^{p_{\max}}\frac{1}{NT}\sum_{t=p+2}^{T}\frac{1}{T-t+1}=O\left(\frac{p_{\max}\log T}{NT}\right).
\end{aligned}
$$

Thus, the Chebyshev inequality implies that

$$
\max_{p_{\min}\le p\le p_{\max}}e_p'\left(\Gamma_p\right)^{-1}\frac{1}{\sqrt{NT}}\sum_{t=p+2}^{T-1}\frac{1}{T-t+1}w_{t-1}(p)\epsilon_t/v_p=O_p\left(\sqrt{\frac{p_{\max}\log T}{NT}}\right)=o_p(1).
$$

Next, considering the argument in the proof of Lemma 16, it holds that

$$
\begin{aligned}
&\left\|\hat{\Gamma}_p^{DF}-\Gamma_p\right\|\\
&\le\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'w_{t-1}(p)-\Gamma_p\right\|+\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'\bar{w}_{t,T-1}(p)\right\|\\
&\quad+\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\bar{w}_{p,t-2}(p)'w_{t-1}(p)\right\|+\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\bar{w}_{p,t-2}(p)'\bar{w}_{t,T-1}(p)\right\|\\
&\le\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p)'w_{t-1}(p)-\Gamma_p\right\|+\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\left\|w_{t-1}(p)\right\|\cdot\left\|\bar{w}_{t,T-1}(p)\right\|\\
&\quad+\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\left\|w_{t-1}(p)\right\|\cdot\left\|\bar{w}_{p,t-2}(p)\right\|+\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\left\|\bar{w}_{p,t-2}(p)\right\|\cdot\left\|\bar{w}_{t,T-1}(p)\right\|.
\end{aligned}
$$

We consider the following term:

$$
\bar{w}_{p,t-2}(p)=\frac{1}{t-p-1}(w_p(p)+\cdots+w_{t-2}(p)).
$$

Lemma A.15 of Dzemski and Okui (2024) implies that

$$
P\left(\max_{1\le i\le N}\max_{0\le p'\le p_{\max}}|w_{p-p',i}+\cdots+w_{t-2-p',i}|\ge C\sqrt{t-p-1}\log(Np_{\max})\right)\to 0.
$$

Thus,

$$
\max_{p_{\min}\le p\le p_{\max}}\left\|\bar{w}_{p,t-2}(p)\right\|=O_p\left(\sqrt{\frac{Np_{\max}}{t-p_{\max}-1}}\log(Np_{\max})\right).
$$

Therefore, we have

$$
\begin{aligned}
\max_{p_{\min}\le p\le p_{\max}} \left\|\hat{\Gamma}_p^{DF}-\Gamma_p\right\| \le & \left\|\frac{1}{NT}\sum_{t=p_{\min}+2}^{T-1}\frac{T-t}{T-t+1}w_{t-1}(p_{\max})'w_{t-1}(p_{\max})-\Gamma_{p_{\max}}\right\| \\
&+\left\|\frac{1}{NT}\sum_{t=p_{\min}+2}^{p_{\max}}\frac{T-t}{T-t+1}w_{t-1}(p_{\max})'w_{t-1}(p_{\max})\right\| \\
&+\frac{1}{NT}\sum_{t=p_{\min}+2}^{T-1}\frac{T-t}{T-t+1}\left\|w_{t-1}(p_{\max})\right\|\cdot\left\|\bar{w}_{t,T-1}(p_{\max})\right\| \\
&+\frac{1}{NT}\sum_{t=p_{\min}+2}^{T-1}\frac{T-t}{T-t+1}\left\|w_{t-1}(p_{\max})\right\|\cdot\left\|\bar{w}_{p,t-2}(p)\right\| \\
&+\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\left\|\bar{w}_{p,t-2}(p)\right\|\cdot\left\|\bar{w}_{t,T-1}(p_{\max})\right\| \\
&=O_p\left(\frac{p_{\max}}{\sqrt{NT}}+\frac{p_{\max}\log T}{T}\right)+O_p\left(\frac{p_{\max}^3}{T}\right) \\
&+O_p\left(\frac{1}{NT}\sum_{t=p_{\min}+2}^{T-1}\frac{T-t}{T-t+1}\sqrt{Np_{\max}}\sqrt{Np_{\max}\frac{1}{T-t}}\right) \\
&+O_p\left(\frac{1}{NT}\sum_{t=p_{\min}+2}^{T-1}\frac{T-t}{T-t+1}\sqrt{Np_{\max}}\sqrt{\frac{Np_{\max}}{t-p_{\max}-1}}\log(Np_{\max})\right) \\
&+O_p\left(\frac{1}{NT}\sum_{t=p_{\min}+2}^{T-1}\frac{T-t}{T-t+1}\sqrt{Np_{\max}\frac{1}{T-t}}\sqrt{\frac{Np_{\max}}{t-p_{\max}-1}}\log(Np_{\max})\right) \\
=&O_p\left(\frac{p_{\max}}{\sqrt{T}}\log(Np_{\max})\right)=o_p(1).
\end{aligned}
$$

Similarly, it holds that

$$
\max_{p_{\min}\le p\le p_{\max}}\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}h_t(p)'h_t(p)-\Gamma_p\right\|=o_p(1).
$$

It thus holds that

$$
\max_{p_{\min}\le p\le p_{\max}}\left|\hat{v}_p-v_p\right|=o_p(1)
$$

and

$$
\max_{p_{\min}\le p\le p_{\max}}\left|\hat{v}_p^{-1}-v_p^{-1}\right|=o_p(1).
$$

Next, we consider D_1. Following the argument made in the proof of Lemma 17, it holds that

$$\max_{p_{\min}\le p\le p_{\max}}\left\|\sqrt{NT}D_1\right\| = O_p\left(\sqrt{NTp_{\max}}\sum_{k=p_{\min}+1}^{\infty}|\alpha_k|\right) = o_p(1).$$

We now examine D_{22}. The order is examined by using the fact that

$$E\left(\max_{p_{\min}\le p\le p_{\max}}||D_{22}||^2\right) \le E\left(\sum_{p=p_{\min}}^{p_{\max}}||D_{22}||^2\right) = O\left(\sum_{p=p_{\min}}^{p_{\max}}E(||D_{22}||^2)\right)$$

so that

$$\max_{p_{\min}\le p\le p_{\max}}||D_{22}|| = O_p\left(\sqrt{\sum_{p=p_{\min}}^{p_{\max}}E(||D_{22}||^2)}\right).$$

Note that

$$D_{22} = -\frac{1}{NT}\sum_{t=p+2}^{T-1}\sqrt{\frac{T-t}{T-t+1}}h_t(p)'\bar{\epsilon}_{t+1,T} - \frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\bar{w}_{p,t-2}(p)'\epsilon_t.$$

Let $\dot{h}_t(p) = \sqrt{(T-t+1)/(T-t)}h_t(p) = w_{t-1}(p) - \bar{w}_{p,t-2}$. Then, the second term in the decomposition is

$$\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\dot{h}_t(p)'\bar{\epsilon}_{t+1,T} = \frac{1}{NT}\sum_{t=p+2}^{T-1}\sum_{m=t+1}^{T}\frac{1}{T-t+1}\dot{h}_t(p)'\epsilon_m.$$

Now, we have that

$$\begin{aligned}
& E\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\sum_{m=t+1}^{T}\frac{1}{T-t+1}\dot{h}_t(p)'\epsilon_m\right\|^2 \\
= & \frac{1}{N^2T^2}\sum_{t=p+2}^{T-1}\sum_{t'=p+2}^{T-1}\frac{1}{T-t+1}\frac{1}{T-t'+1}\sum_{m=t+1}^{T}\sum_{m'=t'+1}^{T}tr\left(E\left(\dot{h}_t(p)'\epsilon_m\epsilon_{m'}'\dot{h}_{t'}(p)\right)\right) \\
= & \frac{\sigma^2}{N^2T^2}\sum_{t=p+2}^{T-1}\sum_{t'=p+2}^{T-1}\frac{T-\max(t,t')}{(T-t+1)(T-t'+1)}tr\left(E\left(\dot{h}_t(p)'\dot{h}_{t'}(p)\right)\right)
\end{aligned}$$

given $E\left(\dot{h}_t(p)'\epsilon_m\epsilon_{m'}'\dot{h}_{t'}(p)\right) = 0$ if $m \ne m'$ and $E\left(\dot{h}_t(p)'\epsilon_m\epsilon_m'\dot{h}_{t'}(p)\right) = \sigma^2 E\left(\dot{h}_t(p)'\dot{h}_{t'}(p)\right)$. We observe that

$$\begin{aligned}
& \frac{\sigma^2}{N^2T^2}\sum_{t=p+2}^{T-1}\sum_{t'=p+2}^{T-1}\frac{T-\max(t,t')}{(T-t+1)(T-t'+1)}tr\left(E\left(\dot{h}_t(p)'\dot{h}_{t'}(p)\right)\right) \\
\le & \frac{2\sigma^2}{N^2T^2}\sum_{t=p+2}^{T-1}\sum_{t'=t}^{T-1}\frac{T-t'}{(T-t+1)(T-t'+1)}tr\left(E\left(\dot{h}_t(p)'\dot{h}_{t'}(p)\right)\right) \\
= & \frac{2\sigma^2}{N^2T^2}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}tr\left(E\left(\dot{h}_t(p)'\tilde{\dot{h}}_{t,T-1}(p)\right)\right),
\end{aligned}$$

where

$$\tilde{\dot{h}}_{t,T-1}(p) = \frac{1}{T-t}\sum_{t'=t}^{T-1}\frac{T-t'}{T-t'+1}\dot{h}_{t'}(p).$$

It holds that

$$tr(E\left(\dot{h}_t(p)'\tilde{\dot{h}}_{t,T-1}(p)\right)) \leq N(E(||\dot{h}_{it}(p)||^2))^{1/2}(E(||\tilde{\dot{h}}_{i,t,T-1}(p)||^2))^{1/2} = O\left(\frac{Np}{\sqrt{T-t}}\right),$$

where $E||\tilde{\dot{h}}_{i,t,T-1}(p)||^2 = O\left(Np/(T-t)\right)$ by the short memory assumption in Assumption 1. Thus, it holds that

$$E\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\sum_{m=t+1}^{T}\frac{1}{T-t+1}\dot{h}_t(p)'\epsilon_m\right\|^2 = O\left(\frac{1}{N^2T^2}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\frac{Np}{\sqrt{T-t}}\right) = O\left(\frac{p}{NT^{3/2}}\right).$$

For the third term, given $\bar{w}_{p,t-2}(p)'\epsilon_t$ is a martingale difference sequence, we observe that

$$E\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\bar{w}_{p,t-2}(p)'\epsilon_t\right\|^2 = \frac{1}{N^2T^2}\sum_{t=p+2}^{T-1}\left(\frac{T-t}{T-t+1}\right)^2\sigma^2 tr(E(\bar{w}_{p,t-2}(p)'\bar{w}_{p,t-2}(p))).$$

As $tr(E(\bar{w}_{p,t-2}(p)'\bar{w}_{p,t-2}(p))) = O(Np/(t-p))$ by Lemma 8 and $(T-t)^2/(T-t+1)^2 < 1$, we have

$$E\left\|\frac{1}{NT}\sum_{t=p+2}^{T-1}\frac{T-t}{T-t+1}\bar{w}_{p,t-2}(p)'\epsilon_t\right\|^2 = O\left(\frac{1}{N^2T^2}\sum_{t=p+2}^{T-1}\left(\frac{T-t}{T-t+1}\right)^2\frac{Np}{t-p}\right) = O\left(\frac{p\log T}{NT^2}\right).$$

Therefore, it holds that

$$\begin{aligned}\max_{p_{\min}\leq p\leq p_{\max}}\left\|\sqrt{NT}D_{22}\right\| =& O_p\left(\sqrt{\sum_{p=p_{\min}}^{p_{\max}}\left(\frac{p}{NT^{3/2}}+\frac{p\log T}{NT^2}\right)}\right)\\ =& O_p\left(\frac{p_{\max}}{N^{1/2}T^{3/4}}+\frac{p_{\max}\sqrt{\log T}}{N^{1/2}T}\right) = o_p(1).\end{aligned}$$

Lastly,

$$\max_{p_{\min}\leq p\leq p_{\max}}|\sqrt{NT}e_p'\alpha(p)| = \max_{p_{\min}\leq p\leq p_{\max}}|\sqrt{NT}\alpha_p| \leq \sqrt{NT}\sum_{k=p_{\min}+1}^{p_{\max}}|\alpha_k| \to 0$$

by the assumption. Therefore, we have (6).

We now show that the distribution of $\max_{p_{\min}\leq p\leq p_{\max}}\tilde{t}_p(\hat{\alpha}(p))$ can be approximated by the maximum of Gaussian random variables. Let y_p, $p = p_{\min},\dots p_{\max}$ be a random sequence whose joint distribution is multivariate Gaussian and whose covariance structure is the same as that of $\tilde{t}_p(\hat{\alpha}(p))$. Noting that $e_p'\Gamma_p^{-1}w_{t-1}(p)\epsilon_t/v_p$ is a martingale difference sequence and the long-run variance is 1, Theorem 1 of Chang, Chen, and Wu (2024) gives that

$$\sup_{a\in\mathbb{R}}\left|P\left(\max_{p_{\min}\le p\le p_{\max}}\tilde{t}_p(\hat{\alpha}(p))<a\right)-P\left(\max_{p_{\min}\le p\le p_{\max}}y_p<a\right)\right|=o(1). \tag{7}$$

We provide several observations about $a(p)=e_p'\left(\Gamma_p\right)^{-1}w_{t-1}(p)\epsilon_t/v_p$. It is a martingale difference sequence. The covariance structure of $a(p)$ is as follows. By the definition of v_p, $Var(a(p))=1$. We then examine the covariances. For $p_1<p_2$, we have

$$\begin{aligned}E(a(p_1)a(p_2))=&\sigma^2 e_{p_1}'\left(\Gamma_{p_1}\right)^{-1}\left[I_{p_1},0_{p_1\times(p_2-p_1)}\right]\Gamma_{p_2}\left(\Gamma_{p_2}\right)^{-1}e_{p_2}\\=&\sigma^2 e_{p_1}'\left[\left(\Gamma_{p_1}\right)^{-1},0_{p_1\times(p_2-p_1)}\right]e_{p_2}=0,\end{aligned}$$

where I_{p_1} is the p_1 dimensional identity matrix and $0_{p_1\times(p_2-p_1)}$ is the $p_1\times(p_2-p_1)$ matrix of zeros. We thus have

$$E(\tilde{t}_{p_1}(\hat{\alpha}(p_1))\tilde{t}_{p_2}(\hat{\alpha}(p_2)))=\frac{N(T-p_2-2)}{NT}\sigma^2 e_{p_1}'\left[\left(\Gamma_{p_1}\right)^{-1},0_{p_1\times(p_2-p_1)}\right]e_{p_2}=0.$$

Because $a(p)$, $p=p_{\min},\dots,p_{\max}$ is an uncorrelated sequence as shown above, we have

$$P\left(\max_{p_{\min}\le p\le p_{\max}}y_p>z_{0.5\alpha}\right)=1-(1-0.5\alpha)^{p_{\max}-p_{\min}}\to 1,$$

and

$$P\left(\max_{p_{\min}\le p\le p_{\max}}|y_p|>z_{0.5\alpha}\right)\to 1. \tag{8}$$

We obtain the desired result by combining (6), (7) and (8).

□

9 The order of Γ_Δ^{-1}

As an example, we consider the case in which w_{it} is i.i.d. across individuals and over time so that $\gamma_0=\sigma^2$, $\gamma_j=0$ for $j>0$. The i.i.d. assumption is satisfied when all AR coefficients (α_k's) are zero.

For simplicity, we assume that $\gamma_0=\sigma^2=1$. In this case, we have

$$\Gamma_\Delta=\begin{pmatrix}-1 & 1 & 0 & \dots & 0 & 0\\ 0 & -1 & 1 & \dots & 0 & 0\\ \dots & \dots & & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & -1 & 1\\ 0 & 0 & 0 & \dots & 0 & -1\end{pmatrix}.$$

We first derive λ_p. The inverse of Γ_Δ is

$$\Gamma_\Delta^{-1} = -\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 1 & \dots & 1 & 1 \\ \dots & \dots & & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

It therefore follows that $\Gamma_\Delta^{-1\prime}\Gamma_\Delta^{-1}$ is a matrix whose (a, b)-th element is $\min(a, b)$. This implies that the maximum eigenvalue is p because for $l \in \mathbb{R}^p$, we have

$$l'\Gamma_\Delta^{-1\prime}\Gamma_\Delta^{-1}l = \sum_{a=1}^{p}\sum_{b=1}^{p}\min(a,b)l_a l_b \leq p\sum_{a=1}^{p}\sum_{b=1}^{p}|l_a| \cdot |l_b| \leq p||l||^2,$$

where l_a and l_b be the a-th and b-th elements of l, and the upper bound under the condition that $||l|| = 1$ is achieved by $l = (0, \dots, 0, 1)$. This means that $\lambda_p = \sqrt{p}$.

Next, we derive the bounds of $v_{AH,p}^2$. We have

$$\Gamma_\Delta^{\prime -1} + \Gamma_\Delta^{-1} = -\begin{pmatrix} 2 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2 & 1 & \dots & 1 & 1 \\ \dots & \dots & & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 2 & 1 \\ 1 & 1 & 1 & \dots & 1 & 2 \end{pmatrix}.$$

The eigenvalues of this matrix are $-(p + 1)$ (single eigenvalue) and -1 (the degree of multiplicity is $p - 1$). These eigenvalues can be verified directly. Define $A_\Gamma = \Gamma_\Delta^{\prime -1} + \Gamma_\Delta^{-1}$. For $v = (1, \dots, 1)'$, we find that $A_\Gamma v = -(p + 1)v$, establishing $-(p + 1)$ as an eigenvalue. Consider the vectors $v_1 = (1, -1, 0, \dots, 0)', v_2 = (1, 1, -2, 0, \dots, 0)', \dots, v_{p-1} = (1, 1, \dots, 1, -2)'$. For each $j = 1, \dots, p - 1$, $A_\Gamma v_j = -v_j$, demonstrating that -1 is an eigenvalue with multiplicity $p - 1$. Hence, the definition of ℓ_p implies that

$$M_1\sigma^2 \leq v_{AH,p}^2 \leq pM_2\sigma^2$$

asymptotically when $(T - p) \to \infty$.

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