The Supplementary Material to "Inference in median AR models with nonstationary and heavy-tailed heteroskedastic noises"

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I. Proofs of Theorems in Sections 2–4

This section provides the technical details for the results presented in Sections 2 to 4. Employing the standard transformation as detailed in Hamilton (1994) and by Assumption 2.1, for any $k \geq 0$, we derive the expansion:

$$y_{t,n} = \sum_{j=0}^{k} \pi_{j0}\varepsilon_{t-j,n} + \sum_{j=0}^{k} \pi_{j0} g\big(\frac{t-j}{n}\big) + \Pi'_{k+1} Y_{t-k-1,n}, \tag{S.1}$$

where Π_{k+1} is a p-dimensional vector with $\|\Pi_{k+1}\| = O(c_0^{k+1})$, and $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^p$.

Let $k_i = [n\lambda_i] \vee p$ for $i \in \{0, 1, \cdots, m+1\}$. For any $t \in \{k_{i-1}+1, \cdots, k_i\}$, by selecting $k = t - k_{i-1} - 1$ in equation (S.1), we obtain

$$y_{t,n} = \sum_{j=0}^{t-k_{i-1}-1} \pi_{j0}\varepsilon_{i,t-j}\big(\frac{t-j}{n}\big) + \sum_{j=0}^{t-k_{i-1}-1} \pi_{j0} g\big(\frac{t-j}{n}\big) + \Pi'_{t-k_{i-1}} Y_{k_{i-1},n}, \tag{S.2}$$

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where we utilize the equality from equation (2.7). Furthermore, we have

$$y_{t,n} = y_{i,t}(t/n) + \Delta_{t,n} + \Pi'_{t-k_{i-1}} Y_{k_{i-1},n}, \tag{S.3}$$

where $y_{i,t}(x)$ is defined in equation (2.6), and $\Delta_{t,n}$ is given by

$$\begin{aligned}\Delta_{t,n} = &\sum_{j=0}^{t-k_{i-1}-1} \pi_{j0}\Big[\varepsilon_{i,t-j}\big(\frac{t-j}{n}\big) - \varepsilon_{i,t-j}\big(\frac{t}{n}\big)\Big] - \sum_{j=t-k_{i-1}}^{\infty} \pi_{j0}\varepsilon_{i,t-j}\big(\frac{t}{n}\big) \\ &+ \sum_{j=0}^{t-k_{i-1}-1} \pi_{j0}\Big[g\big(\frac{t-j}{n}\big) - g\big(\frac{t}{n}\big)\Big] - \sum_{j=t-k_{i-1}}^{\infty} \pi_{j0} g\big(\frac{t}{n}\big).\end{aligned} \tag{S.4}$$

We now present two useful lemmas.

Lemma S.1. *Suppose that Assumptions 2.1–2.4 hold. Then, we have*

$$\sum_{t=k_{i-1}+1}^{k_i} |y_{t,n} - y_{i,t}(t/n)|^{\alpha_0} = O_p(n^{1-\alpha_0}),$$

as $n \to \infty$, *where* $k_i = [n\lambda_i]$ *for* $i \in \{1, \cdots, m+1\}$ *and* $k_0 = p$ *for* $i = 0$, *and the constant* α_0 *is given in Assumption 2.3.*

Proof of Lemma S.1. By the definition of $\Delta_{t,n}$ in (S.4) and using Assumptions 2.1–2.4, it is straightforward to prove that for any $t \in \{k_{i-1}+1, \cdots, k_i\}$, we have

$$\mathbb{E}[|\Delta_{t,n}|^{\alpha_0}] \le C\Big\{\sum_{j=0}^{t-k_{i-1}-1} c_0^{\alpha_0 j}(j/n)^{\alpha_0} + \big(\sum_{j=t-k_{i-1}}^{\infty} c_0^{\alpha_0 j}\big) \sup_{x\in[\lambda_{i-1},\lambda_i]} \big(\mathbb{E}[|\varepsilon_{i,0}(x)|^{\alpha_0}] + |g(x)|^{\alpha_0}\big)\Big\}, \tag{S.5}$$

where we utilize the fact that the distribution of $\varepsilon_{i,t}(x)$ is independent of t and the constant C depends solely on $g(x)$, $\phi_0(x)$, C_0, and $\mathbb{E}[|\eta_t|^{\alpha_0}]$. Given that $\alpha_0 \le 1$ and $c_0 \in (0,1)$, (S.5) indicates that

$$\mathbb{E}\Big(\sum_{t=k_{i-1}+1}^{k_i} |\Delta_{t,n}|^{\alpha_0}\Big) \le C'\Big\{\sum_{t=k_{i-1}+1}^{k_i}\big[\sum_{j=0}^{t-k_{i-1}-1} c_0^{\alpha_0 j}(j/n)^{\alpha_0} + \big(\sum_{j=t-k_{i-1}}^{\infty} c_0^{\alpha_0 j}\big)\big]\Big\} = O(n^{1-\alpha_0}),$$

where C' is a constant independent of n. Therefore, we have shown that

$$\sum_{t=k_{i-1}+1}^{k_i} |\Delta_{t,n}|^{\alpha_0} = O_p(n^{1-\alpha_0}). \tag{S.6}$$

On the other hand, for $Y_{k_i,n} = (y_{k_i,n}, y_{k_i-1,n}, \cdots, y_{k_i-p+1,n})'$, based on Assumptions 2.1–2.3 and (S.2), it is straightforward to deduce that

$$\sup_{n,i} \mathbb{E}\Big(\sum_{t=k_i-p+1}^{k_i} |y_{t,n} - \Pi'_{t-k_{i-1}} Y_{k_{i-1},n}|^{\alpha_0} \Big) < \infty. \tag{S.7}$$

Given that $\alpha_0 \in (0,1]$, the following fact holds:

$$\|Y_{k_i,n}\|^{\alpha_0} \leq \Big(\sum_{t=k_i-p+1}^{k_i} |y_{t,n} - \Pi'_{t-k_{i-1}} Y_{k_{i-1},n}|^{\alpha_0} \Big) + \Big(\sum_{t=k_i-p+1}^{k_i} \|\Pi_{t-k_{i-1}}\|^{\alpha_0} \Big) \|Y_{k_{i-1},n}\|^{\alpha_0}. \tag{S.8}$$

Thus, based on (S.7)–(S.8) and the fact that $\|Y_{p,n}\| = O_p(1)$, by using induction on i, we can directly show that

$$\|Y_{k_i,n}\| = O_p(1), \tag{S.9}$$

where $i = 0, 1, \cdots, m+1$. Finally, notice that (S.9) indicates that

$$\sum_{t=k_{i-1}+1}^{k_i} |\Pi'_{t-k_{i-1}} Y_{k_{i-1}}|^{\alpha_0} \leq \Big(\sum_{t=k_{i-1}+1}^{k_i} \|\Pi_{t-k_{i-1}}\|^{\alpha_0} \Big) \|Y_{k_{i-1},n}\|^{\alpha_0} = O_p(1). \tag{S.10}$$

Then, the conclusion is derived from (S.3), (S.6), and (S.10). □

Lemma S.2. *Suppose that $J(x_0, x_1, \cdots, x_p)$ is a bounded and α_0-Holder continuous function on $[0,1] \times \mathbb{R}^p$. Then, under Assumptions 2.1–2.4, we have*

$$\frac{1}{n} \sum_{t=p+1}^{n} J\Big(\frac{t}{n}, Y'_{t-1,n}\Big) \xrightarrow{p} \sum_{i=1}^{m+1} \int_{\lambda_{i-1}}^{\lambda_i} \mathbb{E}[J(s, Y'_{i,t-1}(s))] ds, \tag{S.11}$$

where $Y_{i,t-1}(s) = (y_{i,t-1}(s), \cdots, y_{i,t-p}(s))'$. Moreover, if Assumption 2.6 also holds, then we have

$$\frac{1}{n}\sum_{t=p+1}^{n} J\Big(\frac{t}{n}, Y'_{t-1,n}\Big) f_{t,n}(0) \xrightarrow{p} \sum_{i=1}^{m+1}\int_{\lambda_{i-1}}^{\lambda_i} \mathbb{E}[J(s, Y'_{i,t-1}(s)) f_{i,t}(0; s)]ds, \tag{S.12}$$

where $f_{t,n}(\cdot)$ is the conditional density of $\varepsilon_{t,n}$ given $\mathcal{F}_{t-1}$.

Proof of Lemma S.2. First, let us prove equation (S.11). Since $J(\cdot)$ is bounded, it suffices to show that, for any $i \in \{1, \cdots, m+1\}$,

$$\frac{1}{n}\sum_{t=k_{i-1}+p+1}^{k_i} J\Big(\frac{t}{n}, Y'_{t-1,n}\Big) \xrightarrow{p} \int_{\lambda_{i-1}}^{\lambda_i} \mathbb{E}[J(s, Y'_{i,t-1}(s))]ds,$$

where the lower bound $k_{i-1}+p+1$ is chosen to ensure that all the elements in $Y_{t-1,n}$ fall within the i-th segmentation, $\{y_{k_{i-1}+1}, \cdots, y_{k_i}\}$. Given the continuity of $J(\cdot)$ and the fact that $\alpha_0 \in (0, 1]$, we have

$$\begin{aligned}
&\Big|J\Big(\frac{t}{n}, Y'_{t-1,n}\Big) - J\Big(\frac{t}{n}, Y'_{i,t-1}(\frac{t}{n})\Big)\Big| \\
&\qquad \le C\sum_{j=1}^{p}\Big[|y_{t-j,n} - y_{i,t-j}(\frac{t-j}{n})|^{\alpha_0} + |y_{i,t-j}(\frac{t-j}{n}) - y_{i,t-j}(\frac{t}{n})|^{\alpha_0}\Big]
\end{aligned} \tag{S.13}$$

for some constant C. Based on Assumptions 2.1–2.4 and the definition of $y_{i,t}(x)$ given in equation (2.6), we can easily show that

$$\sum_{t=k_{i-1}+p+1}^{k_i}\sum_{j=1}^{p}|y_{i,t-j}(\frac{t-j}{n}) - y_{i,t-j}(\frac{t}{n})|^{\alpha_0} = O_p(n^{1-\alpha_0}). \tag{S.14}$$

Then, equations (S.13) through (S.14) and Lemma S.1 together imply that

$$\frac{1}{n}\sum_{t=k_{i-1}+p+1}^{k_i} J\Big(\frac{t}{n}, Y'_{t-1,n}\Big) = \frac{1}{n}\sum_{t=k_{i-1}+p+1}^{k_i} J\Big(\frac{t}{n}, Y'_{i,t-1}(\frac{t}{n})\Big) + O_p(n^{-\alpha_0}). \tag{S.15}$$

For any $\epsilon \in (0, \lambda_i - \lambda_{i-1})$, there exists a sequence $\{s_k\}_{k=0}^{l_\epsilon}$ such that

$$\lambda_{i-1} = s_0 < s_1 < \cdots < s_{l_\epsilon - 1} < s_{l_\epsilon} = \lambda_i \text{ and } s_k - s_{k-1} < \epsilon,$$

where l_ϵ only relies on ϵ. Let $\mathcal{S}_k = \{t : t/n \in (s_{k-1}, s_k]\}$. Then it follows that

$$\begin{aligned}
&\mathbb{E}\Bigg|\frac{1}{n}\sum_{t=k_{i-1}+p+1}^{k_i}\Big\{J\Big(\frac{t}{n}, Y'_{i,t-1}(\frac{t}{n})\Big) - \mathbb{E}\Big[J\Big(\frac{t}{n}, Y'_{i,t-1}(\frac{t}{n})\Big)\Big]\Big\} \\
&\qquad - \frac{1}{n}\sum_{k=1}^{l_\epsilon}\sum_{t\in\mathcal{S}_k}\Big\{J\Big(s_k, Y'_{i,t-1}(s_k)\Big) - \mathbb{E}\Big[J\Big(s_k, Y'_{i,t-1}(s_k)\Big)\Big]\Big\}\Bigg| \\
&\qquad \le \frac{2}{n}\sum_{k=1}^{l_\epsilon}\sum_{t\in\mathcal{S}_k}\mathbb{E}\Big[\Big|J\Big(\frac{t}{n}, Y'_{i,t-1}(\frac{t}{n})\Big) - J\Big(s_k, Y'_{i,t-1}(s_k)\Big)\Big|\Big] \\
&\qquad \le \frac{C'}{n}\sum_{k=1}^{l_\epsilon}\sum_{t\in\mathcal{S}_k}(s_k - t/n)^{\alpha_0} \le C'\epsilon^{\alpha_0},
\end{aligned} \tag{S.16}$$

where C' is a constant independent of ϵ. Meanwhile, based on the definition of $y_{i,t}(x)$ in (2.6), for a fixed s_k, $\{J(s_k, Y'_{i,t-1}(s_k))\}$ is a measurable function of the i.i.d. sequence $\{\eta_{t-1}, \eta_{t-2}, \cdots\}$. Consequently, $\{J(s_k, Y'_{i,t-1}(s_k))\}$ is both stationary and ergodic. Then we have

$$\frac{1}{n}\sum_{k=1}^{l_\epsilon}\sum_{t\in\mathcal{S}_k}\Big\{J\Big(s_k, Y'_{i,t-1}(s_k)\Big) - \mathbb{E}\Big[J\Big(s_k, Y'_{i,t-1}(s_k)\Big)\Big]\Big\} = o_p(1). \tag{S.17}$$

Thus, based on equations (S.16) through (S.17) and the arbitrariness of ϵ, we have actually shown that

$$\frac{1}{n}\sum_{t=k_{i-1}+p+1}^{k_i} J\Big(\frac{t}{n}, Y'_{i,t-1}(\frac{t}{n})\Big) = \frac{1}{n}\sum_{t=k_{i-1}+p+1}^{k_i}\mathbb{E}\Big[J\Big(\frac{t}{n}, Y'_{i,t-1}(\frac{t}{n})\Big)\Big] + o_p(1). \tag{S.18}$$

From the derivation of equation (S.16), it is evident that $\mathbb{E}[J(s, Y'_{i,t-1}(s))]$ is continuous in s. Then, it follows that

$$\frac{1}{n}\sum_{t=k_{i-1}+p+1}^{k_i}\mathbb{E}\Big[J\Big(\frac{t}{n}, Y'_{i,t-1}(\frac{t}{n})\Big)\Big] \longrightarrow \int_{\lambda_{i-1}}^{\lambda_i}\mathbb{E}[J(s, Y'_{i,t-1}(s))]ds. \tag{S.19}$$

By (S.15) and (S.18)–(S.19), the conclusion in (S.11) holds.

For (S.12), notice that $f_{t,n}(0) = f_{i,t}(0; t/n) = f_\eta(0)/\sigma_i(t/n, \mathcal{F}_{t-1})$. Thus, by Assumption 2.6 and (S.13)-(S.14) and Lemma S.1, it is easy to get that

$$\frac{1}{n}\sum_{t=k_{i-1}+p+1}^{k_i} J\Big(\frac{t}{n}, Y'_{t-1,n}\Big) f_{t,n}(0) = \frac{1}{n}\sum_{t=k_{i-1}+p+1}^{k_i} J\Big(\frac{t}{n}, Y'_{i,t-1}(\frac{t}{n})\Big) f_{i,t}(0; \frac{t}{n}) + o_p(1). \tag{S.20}$$

In addition, by Assumption 2.6 and the dominated convergence theorem, it is not hard to prove that, for any $\epsilon > 0$, there exists $\delta \in (0, \epsilon)$, such that

$$\mathbb{E}[|f_{i,t}(0; x_1) - f_{i,t}(0; x_2)|] < \epsilon, \ \forall |x_1 - x_2| < \delta. \tag{S.21}$$

Based on equation (S.21), and utilizing the boundedness of $J(\cdot)$ and $f_{i,t}(\cdot)$, we can derive results analogous to those presented in equations (S.16) through (S.19). Consequently, equation (S.12) holds. □

Proof of Theorem 2.1. By the definition of $\hat{\theta}_n$ given in equation (2.4), it is evident that $\hat{u}_n = \sqrt{n}(\hat{\theta}_n - \theta_0)$ is the minimizer of the objective function $L_n(u)$, which is given by

$$L_n(u) = \sum_{t=p+1}^{n} \Big[w_{t,n}\Big(|\varepsilon_{t,n} - \frac{1}{\sqrt{n}} Z'_{t,n} u| - |\varepsilon_{t,n}|\Big)\Big]. \tag{S.22}$$

Utilizing the identity:

$$|x - y| - |x| = -y \times \mathrm{sgn}(x) + 2\int_0^y [I(x \le s) - I(x \le 0)] ds,$$

we can derive the decomposition of $L_n(u)$ as follows:

$$L_n(u) = -u' \sum_{t=p+1}^{n} \xi_{t,n} + 2\sum_{t=p+1}^{n} \mathbb{E}[\zeta_{t,n}(u)|\mathcal{F}_{t-1}] + 2\alpha_n(u), \tag{S.23}$$

where $\alpha_n(u) = \sum_{t=p+1}^n \{\zeta_{t,n}(u) - \mathbb{E}[\zeta_{t,n}(u)|\mathcal{F}_{t-1}]\}$ and

$$\xi_{t,n} = \frac{1}{\sqrt{n}} Z_{t,n} w_{t,n} \,\mathrm{sgn}(\varepsilon_{t,n}), \tag{S.24}$$

$$\zeta_{t,n}(u) = w_{t,n} \int_0^{\frac{1}{\sqrt{n}} u' Z_{t,n}} [I(\varepsilon_{t,n} \le s) - I(\varepsilon_{t,n} \le 0)] ds. \tag{S.25}$$

According to Assumption 2.5 and Assumption 2.7, we establish two key facts: (i) $\{u'\xi_{t,n}\}$ is a martingale difference sequence; (ii) $\max_t \{|u'\xi_{t,n}|\} < C/\sqrt{n}$ for some constant C. Furthermore, by Assumptions 2.1–2.4 and Assumption 2.7, Lemma S.2 indicates that

$$\begin{aligned}\sum_{t=p+1}^n \mathbb{E}[(u'\xi_{t,n})^2|\mathcal{F}_{t-1}] &= u' \times \Big(\frac{1}{n} \sum_{t=p+1}^n w_{t,n}^2 Z_{t,n} Z_{t,n}'\Big) \times u \\ &\xrightarrow{p} u'\Sigma_1 u. \end{aligned} \tag{S.26}$$

Then, applying the martingale central limit theorem as presented in Billingsley (1999), we obtain

$$T_n := \sum_{t=p+1}^n \xi_{t,n} \xrightarrow{d} \mathcal{N}(0, \Sigma_1). \tag{S.27}$$

On the other hand, based on Assumption 2.7 and Fubini's theorem, it is straightforward to show that

$$\begin{aligned}\sum_{t=p+1}^n \mathbb{E}[\zeta_{t,n}(u)|\mathcal{F}_{t-1}] = u' &\times \Big[\frac{1}{2n} \sum_{t=p+1}^n w_{t,n} Z_{t,n} Z_{t,n}' f_{t,n}(0)\Big] \times u \\ &+ \sum_{t=p+1}^n w_{t,n} \int_0^{\frac{1}{\sqrt{n}} u' Z_{t,n}} s[f_{t,n}(s_{t,n}) - f_{t,n}(0)] ds, \end{aligned} \tag{S.28}$$

where $|s_{t,n}| \le |u' Z_{t,n}|/\sqrt{n}$. Lemma S.2 further implies that

$$\frac{1}{n} \sum_{t=p+1}^n w_{t,n} Z_{t,n} Z_{t,n}' f_{t,n}(0) \xrightarrow{p} \Sigma_2. \tag{S.29}$$

For the second term on the right-hand side of equation (S.28), based on Assumptions 2.6 and 2.7, we find that

$$\Big|w_{t,n}\int_0^{\frac{1}{\sqrt{n}}u'Z_{t,n}} s[f_{t,n}(s_{t,n})-f_{t,n}(0)]ds\Big| \le \frac{C'}{n}\times \sup_{|y|\le\frac{1}{\sqrt{n}}\|u\|\|Z_{t,n}\|}|f_\eta(y/\sigma_0)-f_\eta(0)|, \tag{S.30}$$

where C' is a constant that depends solely on $w(\cdot)$, σ_0 and u.

Note that, by (S.2) and Assumptions 2.1–2.3, there exists $C''>0$ such that

$$\|Z_{t,n}\|\le C''[\tilde{Z}_t+\sum_{i=0}^m\|Y_{k_i,n}\|+1], \tag{S.31}$$

where $\tilde{Z}_t=\sum_{j=1}^p\{\sum_{k=0}^\infty(c_0^k\sum_{i=1}^{m+1}[\sup_x|\varepsilon_{i,t-j-k}(x)|])\}$ is stationary. Utilizing the result from equation (S.9), for any $\delta>0$, we can find some $M'>0$ such that $P(\sum_{i=0}^m\|Y_{k_i,n}\|+1>M')<\delta$. By (S.30)–(S.31), we further have

$$\begin{aligned}&\mathbb{E}\Big\{\sum_{t=p+1}^n\Big|w_{t,n}\int_0^{\frac{1}{\sqrt{n}}u'Z_{t,n}} s[f_{t,n}(s_{t,n})-f_{t,n}(0)]ds\Big|\Big\} \\ &\quad\le C'\delta\times\sup_{y\in\mathbb{R}}[|f_\eta(y)-f_\eta(0)|]+C'\times\mathbb{E}\{\sup_{|y|\le\frac{C''}{\sqrt{n}}\|u\|(\tilde{Z}_t+M')}|f_\eta(y/\sigma_0)-f_\eta(0)|\}.\end{aligned} \tag{S.32}$$

By Assumption 2.6 and the dominated convergence theorem, it is clear that

$$\mathbb{E}\{\sup_{|y|\le\frac{C''}{\sqrt{n}}\|u\|(\tilde{Z}_t+M')}|f_\eta(y/\sigma_0)-f_\eta(0)|\}\longrightarrow 0, \tag{S.33}$$

as $n\to\infty$. Therefore, based on equations (S.32) through (S.33) and the arbitrariness of δ, we have in fact demonstrated that

$$\mathbb{E}\Big\{\sum_{t=p+1}^n\Big|w_{t,n}\int_0^{\frac{1}{\sqrt{n}}u'Z_{t,n}} s[f_{t,n}(s_{t,n})-f_{t,n}(0)]ds\Big|\Big\}\longrightarrow 0. \tag{S.34}$$

Then, (S.28)–(S.29) and (S.34) imply that

$$\sum_{t=p+1}^{n} \mathbb{E}[\zeta_{t,n}(u)|\mathcal{F}_{t-1}] \xrightarrow{p} u'\Sigma_2 u/2. \tag{S.35}$$

Thus, based on equations (S.23), (S.27), and (S.35), along with Lemma S.3, for every fixed u, we find that

$$L_n(u) = -u'T_n + u'\Sigma_2 u + o_p(1). \tag{S.36}$$

Moreover, as demonstrated in equation (S.22), the objective function $L_n(u)$ is convex with respect to u. Consequently, by the convexity argument presented in Theorem 2 of Kato (2009), it follows that

$$\hat{u}_n = \sqrt{n}(\hat{\theta}_n - \theta_0) = 2^{-1}\Sigma_2^{-1} \times T_n + o_p(1) \xrightarrow{d} \mathcal{N}(0, \Sigma_2^{-1}\Sigma_1\Sigma_2^{-1}/4). \tag{S.37}$$

This completes the whole proof. □

Proof of Theorem 3.1. To facilitate understanding of the proof procedures, in this section, we denote the random vector X determined by $\{y_{t,n}\}$ as $X(\{y_{t,n}\})$, and that determined by both $\{b_t^*\}$ and $\{y_{t,n}\}$ as $X(\{b_t^*\}, \{y_{t,n}\})$. Then, for the theorem, we need to prove that

$$G_n(\{y_{t,n}\}) := \sup_{v\in\mathbb{R}^{p+r}} \left| P^*\left(\sqrt{n}[\hat{\theta}_n^*(\{b_t^*\}, \{y_{t,n}\}) - \hat{\theta}_n(\{y_{t,n}\})] \le v\right) - \Phi(v)\right| \xrightarrow{p} 0, \tag{S.38}$$

as $n \to \infty$, where P^* is the conditional probability given $\{y_{t,n}\}$, $\Phi(\cdot)$ is the distribution function of $\mathcal{N}(0, \Sigma_2^{-1}\Sigma_1\Sigma_2^{-1}/4)$, and the inequality for the vector v means coordinate-wise inequality.

Notice that $\hat{u}_n^*(\{b_t^*\}, \{y_{t,n}\}) = \sqrt{n}[\hat{\theta}_n^*(\{b_t^*\}, \{y_{t,n}\}) - \theta_0]$ is the minimizer of the following objective function:

$$L_n^*(u; \{b_t^*\}, \{y_{t,n}\}) = -u' \sum_{t=p+1}^{n} [(b_t^* - 1)\xi_{t,n}(\{y_{t,n}\})] + 2 \sum_{t=p+1}^{n} [(b_t^* - 1)\zeta_{t,n}(u; \{y_{t,n}\})] + L_n(u; \{y_{t,n}\}), \tag{S.39}$$

where $L_n(u; \{y_{t,n}\})$, $\xi_{t,n}(\{y_{t,n}\})$ and $\zeta_{t,n}(u; \{y_{t,n}\})$ are defined in (S.23)–(S.25). For any fixed $u \in \mathbb{R}^{p+r}$, employing Fubini's theorem and Assumptions 2.6–2.7, it is straightforward to demonstrate that

$$\sum_{t=p+1}^{n} \mathbb{E}[\zeta_{t,n}^2(u; \{y_{t,n}\})] = O(n^{-1/2}). \tag{S.40}$$

Let $(\Omega, \mathcal{A}, P)$ be the underlying probability space for the sequence $\{y_{t,n}\}$. According to (S.26)–(S.27), (S.35), Lemma S.3 and (S.40), and by applying Dudley's almost sure representation theorem (see Theorem 9.4 in Pollard (1990) or Theorem 1.10.4 in van der Vaart and Wellner (1996)) in the product space $\mathbb{R}^\infty$, we can demonstrate the existence of an extended probability space $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{P})$ and a sequence of measurable and perfect mappings $\phi_n : \bar{\Omega} \to \Omega$ such that:

(i): $P(A) = \bar{P}(\phi_n^{-1}(A))$ for any $A \in \mathcal{A}$, namely, $P = \bar{P} \circ \phi_n^{-1}$;

(ii): There exist a measurable set $\bar{A}_0 \in \bar{\mathcal{A}}$ with $\bar{P}(\bar{A}_0) = 1$ and a measurable mapping $\bar{T}_0 : \bar{\Omega} \to \mathbb{R}^{p+r}$ with $\bar{P} \circ \bar{T}_0^{-1} \sim \mathcal{N}(0, \Sigma_2^{-1}\Sigma_1\Sigma_2^{-1}/4)$ such that for any $\bar{\omega} \in \bar{A}_0$, the following convergence results hold simultaneously:

$$\sum_{t=p+1}^{n} \xi_{t,n}(\{y_{t,n}(\phi_n(\bar{\omega}))\}) \to \bar{T}_0(\bar{\omega}); \sum_{t=p+1}^{n} \xi_{t,n}(\{y_{t,n}(\phi_n(\bar{\omega}))\})\xi'_{t,n}(\{y_{t,n}(\phi_n(\bar{\omega}))\}) \to \Sigma_1;$$
$$\sum_{t=p+1}^{n} \zeta_{t,n}(u;\{y_{t,n}(\phi_n(\bar{\omega}))\}) \to u'\Sigma_2 u/2 \text{ and } \sum_{t=p+1}^{n} [\zeta_{t,n}(u;\{y_{t,n}(\phi_n(\bar{\omega}))\})]^2 \to 0, \ \forall u \in \mathbb{Q}^{p+r}. \tag{S.41}$$

According to Assumption 3.1 and invoking Lindeberg's central limit theorem, for any fixed $\bar{\omega}$, we establish the following convergences:

$$\sum_{t=p+1}^{n} [(b_t^* - 1)\xi_{t,n}(\{y_{t,n}(\phi_n(\bar{\omega}))\})] \xrightarrow{d^*} \mathcal{N}(0, \Sigma_1); \tag{S.42}$$
$$\sum_{t=p+1}^{n} [(b_t^* - 1)\zeta_{t,n}(u;\{y_{t,n}(\phi_n(\bar{\omega}))\})] \xrightarrow{p^*} 0, \forall u \in \mathbb{Q}^{p+r}. \tag{S.43}$$

It is important to note that $\mathbb{Q}^{p+r}$ constitutes a countable dense subset within $\mathbb{R}^{p+r}$. Subsequently, utilizing equations (S.41)–(S.43), in conjunction with Lemma 3 and Theorem 2 from Kato (2009), for any fixed $\bar{\omega} \in \bar{A}_0$, we deduce

$$\sqrt{n}[\hat{\theta}_n(\{y_{t,n}(\phi_n(\bar{\omega}))\}) - \theta_0] = 2^{-1}\Sigma_2^{-1} \sum_{t=p+1}^{n} \xi_{t,n}(\{y_{t,n}(\phi_n(\bar{\omega}))\}) + o(1), \tag{S.44}$$
$$\sqrt{n}[\hat{\theta}_n^*(\{b_t^*\}, \{y_{t,n}(\phi_n(\bar{\omega}))\}) - \theta_0] = 2^{-1}\Sigma_2^{-1} \sum_{t=p+1}^{n} b_t^*\xi_{t,n}(\{y_{t,n}(\phi_n(\bar{\omega}))\}) + o_p^*(1). \tag{S.45}$$

From equations (S.42), (S.44), and (S.45), we infer that

$$\sqrt{n}[\hat{\theta}_n^*(\{b_t^*\}, \{y_{t,n}(\phi_n(\bar{\omega}))\}) - \hat{\theta}_n(\{y_{t,n}(\phi_n(\bar{\omega}))\})] \xrightarrow{d^*} \mathcal{N}(0, \Sigma_2^{-1}\Sigma_1\Sigma_2^{-1}/4). \tag{S.46}$$

Given that the distribution function $\Phi(\cdot)$ is continuous, by the multivariate version of Pólya's theorem, equation (S.46) is equivalent to

$$G_n(\{y_{t,n}(\phi_n(\bar{\omega}))\}) \longrightarrow 0, \ \forall \bar{\omega} \in \bar{A}_0. \tag{S.47}$$

Note that the distribution of $G_n(\{y_{t,n}(\phi_n(\cdot))\})$ is identical to that of $G_n(\{y_{t,n}(\cdot)\})$, as a consequence of $P = \bar{P} \circ \phi_n^{-1}$. Hence, the conclusion holds from (S.47). $\square$

Proof of Theorem 4.1. By Theorem 2.1 and Lemma S.5, it is evident that

$$\sup_{z\in\mathcal{Z}} |\tilde{S}_n(\hat{\theta}_n, z) - \tilde{S}_n(\theta_0, z) + 2\tilde{V}'(z)\sqrt{n}(\hat{\theta}_n - \theta_0)| = o_p(1). \tag{S.48}$$

By (S.27), Lemma S.4, and the continuity of $\tilde{V}(z)$, in the space $l^\infty(\mathcal{Z})$, we have

$$\tilde{S}_n(\theta_0, z) - \tilde{V}'(z)\Sigma_2^{-1}T_n \text{ is asymptotically tight,} \tag{S.49}$$

where T_n is defined in (S.27). Furthermore, by applying the martingale central limit theorem and Lemma S.2, for any $a_1, a_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathcal{Z}$, it can be demonstrated that

$$\begin{aligned} a_1[\tilde{S}_n(\theta_0, z_1) - \tilde{V}'(z_1)\Sigma_2^{-1}T_n] + a_2[\tilde{S}_n(\theta_0, z_2) - \tilde{V}'(z_2)\Sigma_2^{-1}T_n] \\ \xrightarrow{d} \mathcal{N}(0, a_1^2\Lambda_0(z_1, z_1) + 2a_1a_2\Lambda_0(z_1, z_2) + a_2^2\Lambda_0(z_2, z_2)). \end{aligned} \tag{S.50}$$

Consequently, (S.49) and (S.50) imply that

$$\tilde{S}_n(\theta_0, z) - \tilde{V}'(z)\Sigma_2^{-1}T_n \rightsquigarrow \tilde{S}_0(z). \tag{S.51}$$

Thus, the theorem follows from equations (S.37), (S.48), and (S.51), along with the application of the continuous mapping theorem. $\square$

Proof of Theorem 4.2. Initially, we recall several notations, as delineated below, which were previously defined in the proofs associated with Lemma S.5:

$$\mathcal{C}_n = \{\theta : \sqrt{n}\|\theta - \theta_0\| < C\},$$

$$\tilde{R}_{t,n}(\theta, z) = \tilde{r}_{t,n}(\theta, z) - \tilde{r}_{t,n}(\theta_0, z) = 2\tilde{w}_{t,n} \exp(z'\tilde{Y}_{t-1,n})[I(\varepsilon_{t,n} > Z'_{t,n}(\theta - \theta_0)) - I(\varepsilon_{t,n} > 0)],$$

$$\tilde{R}^{\pm}_{t,n}(\theta, z, \delta) = 2\tilde{w}_{t,n} \exp(z'\tilde{Y}_{t-1,n})[I(\varepsilon_{t,n} > Z'_{t,n}(\theta - \theta_0) \mp \frac{C\delta}{\sqrt{n}}\|Z_{t,n}\|) - I(\varepsilon_{t,n} > 0)],$$

where $\theta \in \mathbb{R}^{r+p}$, $z \in \mathcal{Z}$, and $C, \delta > 0$. In addition, for any $z_1, z_2 \in \mathcal{Z}$, we define

$$\Lambda_n(z_1, z_2) = n^{-1} \sum_{t=p+1}^{n} \Big\{[\tilde{r}_{t,n}(\theta_0, z_1) - \tilde{V}'(z_1)\Sigma_2^{-1} Z_{t,n} w_{t,n} \operatorname{sgn}(\varepsilon_{t,n})] \times [\tilde{r}_{t,n}(\theta_0, z_2) - \tilde{V}'(z_2)\Sigma_2^{-1} Z_{t,n} w_{t,n} \operatorname{sgn}(\varepsilon_{t,n})]\Big\}.$$

Then, based on Lemma S.2 and Assumption 2.7, and by leveraging the boundedness and the Lipschitz continuity of $\exp(z'\tilde{Y}_{t-1,n})$, it is easy to check that

$$\sup_{z_1, z_2 \in \mathcal{Z}} |\Lambda_n(z_1, z_2) - \Lambda_0(z_1, z_2)| = o_p(1), \tag{S.52}$$

where $\Lambda_0(z_1, z_2)$ is defined in Theorem 4.1.

Step 1: Assume that $C \in \mathbb{N}$ and $\delta \in \mathbb{Q} \cap (0, 1)$. Then, there exists a sequence of open balls $\{B_{C\delta}(u_i)\}_{i=1}^{K_\delta}$ such that $B_C(0) \subset \cup_{i=1}^{K_\delta} B_{C\delta}(u_i)$, where $u_i \in B_C(0)$ and $K_\delta \leq \left(\frac{3}{\delta}\right)^{r+p}$. Let $\theta_{in} = \theta_0 + u_i/\sqrt{n}$, then we have $\mathcal{C}_n \subset \cup_{i=1}^{K_\delta} \mathcal{C}_{in}$ with $\mathcal{C}_{in} = B_{\frac{C\delta}{\sqrt{n}}}(\theta_{in})$. For $\tilde{S}^*_n(\theta, z)$, we have

$$\tilde{S}^*_n(\theta, z) - \tilde{S}^*_n(\theta_0, z) = \frac{1}{\sqrt{n}} \sum_{t=p+1}^{n} \mathbb{E}[\tilde{R}_{t,n}(\theta, z)] + \frac{1}{\sqrt{n}} \sum_{t=p+1}^{n} (b^*_t - 1)\mathbb{E}[\tilde{R}_{t,n}(\theta, z)] + \frac{1}{\sqrt{n}} \sum_{t=p+1}^{n} b^*_t\{\tilde{R}_{t,n}(\theta, z) - \mathbb{E}[\tilde{R}_{t,n}(\theta, z)]\}. \tag{S.53}$$

Observe that when $\theta \in \mathcal{C}_n$ and $z \in \mathcal{Z}$, $n^{-1/2}\sum_{t=p+1}^{n}\mathbb{E}[\tilde{R}_{t,n}(\theta, z)|\mathcal{F}_{t-1}]$ is uniformly bounded. Subsequently, based on Lemma S.5 (ii) and applying the dominated convergence theorem, we conclude that

$$\sup_{\theta\in\mathcal{C}_n, z\in\mathcal{Z}}\Big|\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\mathbb{E}[\tilde{R}_{t,n}(\theta, z)] + 2\tilde{V}'(z)\sqrt{n}(\theta-\theta_0)\Big| = o(1). \tag{S.54}$$

Furthermore, by a simple calculation, we can show that there exists a constant $C' > 0$ such that for any $\theta_1, \theta_2 \in \mathcal{C}_n$ and $z_1, z_2 \in \mathcal{Z}$, the following inequality holds:

$$\begin{aligned}&\Big|\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}(b_t^*-1)\{\mathbb{E}[\tilde{R}_{t,n}(\theta_1, z_1)] - \mathbb{E}[\tilde{R}_{t,n}(\theta_2, z_2)]\}\Big| \\ &\le \frac{C'}{\sqrt{n}}\sum_{t=p+1}^{n}\Big(|b_t^*-1| \times [\|\theta_1-\theta_2\| + n^{-1/2}\|z_1-z_2\|]\Big).\end{aligned} \tag{S.55}$$

It can be verified that

$$\sup_{\theta\in\mathcal{C}_n, z\in\mathcal{Z}}\frac{1}{n}\sum_{t=p+1}^{n}\{\mathbb{E}[\tilde{R}_{t,n}(\theta, z)]\}^2 = o(1). \tag{S.56}$$

Consequently, based on equations (S.55) and (S.56), Assumption 3.1, and the independence of $\{b_t^*\}$, we conclude that

$$\sup_{\theta\in\mathcal{C}_n, z\in\mathcal{Z}}\Big|\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}(b_t^*-1)\mathbb{E}[\tilde{R}_{t,n}(\theta, z)]\Big| \xrightarrow{p^*} 0. \tag{S.57}$$

Analogous to the approach used in equations (S.115) and (S.118), it can be derived that

$$\begin{aligned}\sup_{\theta\in\mathcal{C}_n, z\in\mathcal{Z}}\Big[\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}b_t^*\{\tilde{R}_{t,n}(\theta, z) - \mathbb{E}[\tilde{R}_{t,n}(\theta, z)]\}\Big] &\le \frac{C''\delta}{n}\sum_{t=p+1}^{n}b_t^* \\ &+ \max_{1\le i\le K_\delta}\sup_{z\in\mathcal{Z}}\Big|\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\Big[(b_t^*-1)\{\tilde{R}_{t,n}^{+}(\theta_{in}, z, \delta) - \mathbb{E}[\tilde{R}_{t,n}^{+}(\theta_{in}, z, \delta)]\}\Big]\Big| \\ &+ \max_{1\le i\le K_\delta}\sup_{z\in\mathcal{Z}}\Big|\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\{\tilde{R}_{t,n}^{+}(\theta_{in}, z, \delta) - \mathbb{E}[\tilde{R}_{t,n}^{+}(\theta_{in}, z, \delta)]\}\Big|.\end{aligned} \tag{S.58}$$

Step 2: Observe that the set $\{(C,\delta) : C \in \mathbb{N}, \delta \in \mathbb{Q} \cap (0,1)\}$ is countable. Following a similar approach to the proofs for Theorem 3.1, based on equations (S.26)–(S.27), (S.35), Lemma S.3, (S.40), (S.52), and Lemmas S.5–S.6, and by applying Dudley's almost sure representation theorem again, on an extended probability space $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{P})$, there exist a sequence of measurable and perfect mappings $\phi_n : \bar{\Omega} \to \Omega$ and a set $\bar{A}_0$ with $\bar{P}(\bar{A}_0) = 1$, such that for any $\bar{\omega} \in \bar{A}_0$, the convergence results of (S.41) and the following convergences occur simultaneously:

$$\sup_{\theta \in \mathcal{C}_n, z \in \mathcal{Z}} \left| \frac{1}{\sqrt{n}} \sum_{t=p+1}^{n} \{\tilde{R}_{t,n}(\theta, z; \{y_{t,n}(\phi_n(\bar{\omega}))\}) + 2\tilde{V}(z)\sqrt{n}(\theta - \theta_0)\} \right| \to 0, \tag{S.59}$$

$$\max_{1 \le i \le K_\delta} \sup_{z \in \mathcal{Z}} \left| \frac{1}{\sqrt{n}} \sum_{t=p+1}^{n} \{\tilde{R}^{\pm}_{t,n}(\theta_{in}, z, \delta; \{y_{t,n}(\phi_n(\bar{\omega}))\}) - \mathbb{E}[\tilde{R}^{\pm}_{t,n}(\theta_{in}, z, \delta)]\} \right| \to 0, \tag{S.60}$$

$$\max_{1 \le i \le K_\delta} \sup_{z \in \mathcal{Z}} \left[\frac{1}{n} \sum_{t=p+1}^{n} \{\tilde{R}^{\pm}_{t,n}(\theta_{in}, z, \delta; \{y_{t,n}(\phi_n(\bar{\omega}))\}) - \mathbb{E}[\tilde{R}^{\pm}_{t,n}(\theta_{in}, z, \delta)]\}^2 \right] \to 0, \tag{S.61}$$

$$\sup_{z_1, z_2 \in \mathcal{Z}} |\Lambda_n(z_1, z_2; \{y_{t,n}(\phi_n(\bar{\omega}))\}) - \Lambda_0(z_1, z_2)| \to 0, \tag{S.62}$$

where $C \in \mathbb{N}$ and $\delta \in \mathbb{Q} \cap (0,1)$.

Notice that $\{(b_t^* - 1)\{\tilde{R}^{+}_{t,n}(\theta_{in}, z, \delta; \{y_{t,n}(\phi_n(\bar{\omega}))\}) - \mathbb{E}[\tilde{R}^{+}_{t,n}(\theta_{in}, z, \delta)]\}\}$ consists of independent random variables for a given $\bar{\omega}$. Subsequently, based on equation (S.61) and following the tightness proofs for $\tilde{S}_n(\theta_0, z)$ as outlined in Lemma S.4, for any $C \in \mathbb{N}$ and $\delta \in \mathbb{Q} \cap (0,1)$, we have

$$\max_{1 \le i \le K_\delta} \sup_{z \in \mathcal{Z}} \left| \frac{1}{\sqrt{n}} \sum_{t=p+1}^{n} \left[(b_t^* - 1)\{\tilde{R}^{+}_{t,n}(\theta_{in}, z, \delta; \{y_{t,n}(\phi_n(\bar{\omega}))\}) - \mathbb{E}[\tilde{R}^{+}_{t,n}(\theta_{in}, z, \delta)]\} \right] \right| \xrightarrow{p^*} 0. \tag{S.63}$$

Consequently, based on equations (S.58), (S.60), (S.63), and the arbitrariness of δ, we can further demonstrate that, for any $\epsilon > 0$,

$$P^*\Big(\sup_{\theta\in\mathcal{C}_n,z\in\mathcal{Z}} \frac{1}{\sqrt{n}}\sum_{t=p+1}^{n} b_t^*\{\tilde{R}_{t,n}(\theta,z;\{y_{t,n}(\phi_n(\bar{\omega}))\}) - \mathbb{E}[\tilde{R}_{t,n}(\theta,z)]\} > \epsilon\Big) \to 0.$$

By a similar argument, we can also establish that

$$P^*\Big(\inf_{\theta\in\mathcal{C}_n,z\in\mathcal{Z}} \frac{1}{\sqrt{n}}\sum_{t=p+1}^{n} b_t^*\{\tilde{R}_{t,n}(\theta,z;\{y_{t,n}(\phi_n(\bar{\omega}))\}) - \mathbb{E}[\tilde{R}_{t,n}(\theta,z)]\} < -\epsilon\Big) \to 0.$$

Therefore, it follows that

$$\sup_{\theta\in\mathcal{C}_n,z\in\mathcal{Z}} \Big|\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n} b_t^*\{\tilde{R}_{t,n}(\theta,z;\{y_{t,n}(\phi_n(\bar{\omega}))\}) - \mathbb{E}[\tilde{R}_{t,n}(\theta,z)]\}\Big| \xrightarrow{p^*} 0. \tag{S.64}$$

Accordingly, by equations (S.53)-(S.54), (S.57), (S.59), and (S.64), and applying the conclusions in (S.44) and (S.45), it is straightforward to demonstrate that

$$\begin{aligned}
&\tilde{S}_n^*(\hat{\theta}_n^*, z; \{b_t^*\}, \{y_{t,n}(\phi_n(\bar{\omega}))\}) - \tilde{S}_n(\hat{\theta}_n, z; \{y_{t,n}(\phi_n(\bar{\omega}))\}) \qquad \text{(S.65)}\\
&\quad = \Big\{\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n} (b_t^*-1)\tilde{r}_{t,n}(\theta_0, z; \{y_{t,n}(\phi_n(\bar{\omega}))\})\\
&\qquad\qquad - \tilde{V}'(z)\Sigma_2^{-1}\sum_{t=p+1}^{n}(b_t^*-1)\xi_{t,n}(\{y_{t,n}(\phi_n(\bar{\omega}))\})\Big\} + o_p^*(1).
\end{aligned}$$

Ultimately, by equation (S.62) and applying the tightness proofs in Lemma S.4, we establish that

$$\begin{aligned}
&\Big\{\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n} (b_t^*-1)\tilde{r}_{t,n}(\theta_0, z; \{y_{t,n}(\phi_n(\bar{\omega}))\})\\
&\qquad - \tilde{V}'(z)\Sigma_2^{-1}\sum_{t=p+1}^{n}(b_t^*-1)\xi_{t,n}(\{y_{t,n}(\phi_n(\bar{\omega}))\})\Big\} \rightsquigarrow \tilde{S}_0(z). \qquad \text{(S.66)}
\end{aligned}$$

Thus, we conclude that on an extended probability space,

$$\tilde{T}_n^*(\{b_t^*\}, \{y_{t,n}(\phi_n(\bar{\omega}))\}) \xrightarrow{d^*} \tilde{T}_0, \forall\bar{\omega}\in\bar{A}_0. \tag{S.67}$$

This completes the whole proof, as a consequence of $P = \bar{P}\circ\phi_n^{-1}$. $\square$

Proof of Theorem 4.3. Upon carefully examining the proofs of Lemmas S.1–S.2, and utilizing the fact that $\sup_{t,n}\mathbb{E}|a_{t,n}|^{\alpha_0} < \infty$ as stated in Assumption 2.3, it follows that under the same conditions as those in Lemmas S.1-S.2, we can deduce that

$$\sum_{t=k_{i-1}+1}^{k_i} |y_{t,n} - y_{i,t}(t/n)|^{\alpha_0} = O_p(n^{1-\alpha_0/2}), \tag{S.68}$$

$$\frac{1}{n}\sum_{t=p+1}^{n} J\Big(\frac{t}{n}, Y'_{t-1,n}\Big) \xrightarrow{p} \sum_{i=1}^{m+1}\int_{\lambda_{i-1}}^{\lambda_i} \mathbb{E}[J(s, Y'_{i,t-1}(s))]ds, \tag{S.69}$$

$$\frac{1}{n}\sum_{t=p+1}^{n} J\Big(\frac{t}{n}, Y'_{t-1,n}\Big)f_{t,n}(0) \xrightarrow{p} \sum_{i=1}^{m+1}\int_{\lambda_{i-1}}^{\lambda_i} \mathbb{E}[J(s, Y'_{i,t-1}(s))f_{i,t}(0;s)]ds. \tag{S.70}$$

Step 1: Let $\breve{\varepsilon}_{t,n} = \varepsilon_{t,n} + a_{t,n}/\sqrt{n}$. Then, $\hat{u}_n = \sqrt{n}(\hat{\theta}_n - \theta_0)$ is the minimizer of the objective function $\check{L}_n(u)$, defined as

$$\check{L}_n(u) = -u'\check{T}_n + 2\sum_{t=p+1}^{n}\mathbb{E}[\check{\zeta}_{t,n}(u)|\mathcal{F}_{t-1}] + 2\check{\alpha}_n(u), \tag{S.71}$$

where $\check{T}_n = \sum_{t=p+1}^{n}\check{\xi}_{t,n}$ and $\check{\alpha}_n(u) = \sum_{t=p+1}^{n}\{\check{\zeta}_{t,n}(u) - \mathbb{E}[\check{\zeta}_{t,n}(u)|\mathcal{F}_{t-1}]\}$, and

$$\check{\xi}_{t,n} = \frac{1}{\sqrt{n}}Z_{t,n}w_{t,n}\,\mathrm{sgn}(\breve{\varepsilon}_{t,n}), \tag{S.72}$$

$$\check{\zeta}_{t,n}(u) = w_{t,n}\int_0^{\frac{1}{\sqrt{n}}u'Z_{t,n}} [I(\breve{\varepsilon}_{t,n} \le s) - I(\breve{\varepsilon}_{t,n} \le 0)]ds. \tag{S.73}$$

For $\check{T}_n$, we have the following decomposition:

$$\check{T}_n = \sum_{t=p+1}^{n}\check{\xi}_{t,n,1} + \sum_{t=p+1}^{n}\check{\xi}_{t,n,2} \equiv \check{T}_{n,1} + \check{T}_{n,2}, \tag{S.74}$$

where $\check{\xi}_{t,n,1} = \frac{1}{\sqrt{n}}Z_{t,n}w_{t,n}\,\mathrm{sgn}(\varepsilon_{t,n})$ and $\check{\xi}_{t,n,2} = \frac{1}{\sqrt{n}}Z_{t,n}w_{t,n}[\mathrm{sgn}(\breve{\varepsilon}_{t,n}) - \mathrm{sgn}(\varepsilon_{t,n})]$. By Assumptions 2.5–2.7 and using the martingale central limit theorem and equation (S.69), it is straightforward to prove that

$$\check{T}_{n,1} \xrightarrow{d} \mathcal{N}(0, \Sigma_1). \tag{S.75}$$

On the other hand, by Assumption 2.6-2.7 and $\sup_{t,n} \mathbb{E}|a_{t,n}|^{\alpha_0} < \infty$, we can deduce that

$$\sum_{t=p+1}^{n} [\check{\xi}_{t,n,2} - \mathbb{E}(\check{\xi}_{t,n,2}|\mathcal{F}_{t-1})] = o_p(1). \tag{S.76}$$

Meanwhile, by $a_{t,n} = M\sigma_{t,n}$ and $f_{t,n}(0) = f_\eta(0)/\sigma_{t,n}$ and using (S.69), we have

$$\sum_{t=p+1}^{n} \mathbb{E}(\check{\xi}_{t,n,2}|\mathcal{F}_{t-1}) = \frac{2}{n}\sum_{t=p+1}^{n} w_{t,n}Z_{t,n}a_{t,n}f_{t,n}(0) + o_p(1). \tag{S.77}$$

$$\frac{2}{n}\sum_{t=p+1}^{n} w_{t,n}Z_{t,n}a_{t,n}f_{t,n}(0) = \frac{2Mf_\eta(0)}{n}\sum_{t=p+1}^{n} w_{t,n}Z_{t,n} \xrightarrow{p} \delta_a. \tag{S.78}$$

Thus, equations (S.74)–(S.78) indicate that

$$\check{T}_n \xrightarrow{d} \mathcal{N}(\delta_a, \Sigma_1). \tag{S.79}$$

Furthermore, by Assumptions 2.6–2.7 and equation (S.70) and using the same proof process for equation (S.35) and Lemma S.3, it is straightforward to show that

$$\sum_{t=p+1}^{n} \mathbb{E}[\check{\zeta}_{t,n}(u)|\mathcal{F}_{t-1}] = u'\Sigma_2 u/2 + o_p(1), \quad \check{\alpha}_n(u) = o_p(1). \tag{S.80}$$

Then, equations (S.71), (S.79), and (S.80) imply that, under the local alternative, equation (S.37) is modified to

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = 2^{-1}\Sigma_2^{-1} \times \check{T}_n + o_p(1) \xrightarrow{d} \mathcal{N}(2^{-1}\Sigma_2^{-1}\delta_a, \Sigma_2^{-1}\Sigma_1\Sigma_2^{-1}/4). \tag{S.81}$$

Step 2: Now, we investigate the limiting distribution of $\tilde{S}_n(\hat{\theta}_n, z)$ under the local alternative hypothesis. In this scenario, observe that $\varepsilon_{t,n}(\theta) = \check{\varepsilon}_{t,n} - Z'_{t,n}(\theta - \theta_0)$. Consequently, we have

$$\tilde{S}_n(\theta_0, z) = \frac{1}{\sqrt{n}}\sum_{t=p+1}^{n} \tilde{w}_{t,n}\,\mathrm{sgn}(\check{\varepsilon}_{t,n})\exp(z'\tilde{Y}_{t-1,n}). \tag{S.82}$$

For $\tilde{S}_n(\theta_0, z)$, we substitute $\varepsilon_{t,n}$ with $\check{\varepsilon}_{t,n}$ in the proof of Lemma S.4 and note that the following inequality holds:

$$
\begin{aligned}
\mathbb{E}\Big|\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}[\tilde{w}_{t,n}\operatorname{sgn}(\check{\varepsilon}_{t,n})(\prod_{i=1}^{l}\tilde{y}_{t-j_i,n})]\Big| &\le \mathbb{E}\Big|\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}[\tilde{w}_{t,n}\operatorname{sgn}(\varepsilon_{t,n})(\prod_{i=1}^{l}\tilde{y}_{t-j_i,n})]\Big| \\
&+ \mathbb{E}\Big|\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}[\tilde{w}_{t,n}[\operatorname{sgn}(\check{\varepsilon}_{t,n}) - \operatorname{sgn}(\varepsilon_{t,n}) - \mathbb{E}\{[\operatorname{sgn}(\check{\varepsilon}_{t,n}) - \operatorname{sgn}(\varepsilon_{t,n})]|\mathcal{F}_{t-1}\}](\prod_{i=1}^{l}\tilde{y}_{t-j_i,n})]\Big| \\
&+ \mathbb{E}\Big|\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}[\tilde{w}_{t,n}[\mathbb{E}\{[\operatorname{sgn}(\check{\varepsilon}_{t,n}) - \operatorname{sgn}(\varepsilon_{t,n})]|\mathcal{F}_{t-1}\}](\prod_{i=1}^{l}\tilde{y}_{t-j_i,n})]\Big|. \quad \text{(S.83)}
\end{aligned}
$$

Subsequently, by the moment inequality for the sum of martingale difference sequences and using $\check{\varepsilon}_{t,n} = \varepsilon_{t,n} + a_{t,n}/\sqrt{n}$, there exists a constant C_3 such that

$$
\mathbb{E}\Big|\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}[\tilde{w}_{t,n}\operatorname{sgn}(\check{\varepsilon}_{t,n})(\prod_{i=1}^{l}\tilde{y}_{t-j_i,n})]\Big| \le C_3^l. \quad \text{(S.84)}
$$

Hence, the asymptotic tightness is derived from equations (S.84) and (S.108)-(S.112). Concurrently, it is straightforward to demonstrate that for any fixed $z \in \mathcal{Z}$,

$$
\begin{aligned}
\tilde{S}_n(\theta_0, z) &= \frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\tilde{w}_{t,n}\operatorname{sgn}(\varepsilon_{t,n})\exp(z'\tilde{Y}_{t-1,n}) \quad \text{(S.85)} \\
&+ \frac{2Mf_\eta(0)}{n}\sum_{t=p+1}^{n}\tilde{w}_{t,n}\exp(z'\tilde{Y}_{t-1,n}) + o_p(1).
\end{aligned}
$$

On the other hand, it can be directly shown that Lemma S.5 remains valid under the alternative hypothesis. Therefore, by equations (S.82), (S.85), and Lemma S.5, we can deduce that

$$
\begin{aligned}
\tilde{S}_n(\hat{\theta}_n, z) &= \frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}[\tilde{w}_{t,n}\exp(z'\tilde{Y}_{t-1,n}) - \tilde{V}'(z)\Sigma_2^{-1}Z_{t,n}w_{t,n}]\operatorname{sgn}(\varepsilon_{t,n}) \quad \text{(S.86)} \\
&+ \frac{2Mf_\eta(0)}{n}\sum_{t=p+1}^{n}[\tilde{w}_{t,n}\exp(z'\tilde{Y}_{t-1,n}) - \tilde{V}'(z)\Sigma_2^{-1}Z_{t,n}w_{t,n}] + o_p(1).
\end{aligned}
$$

Consequently, it follows that

$$\tilde{S}_n(\hat{\theta}_n, z) \rightsquigarrow \tilde{S}_0(z) + MD_a(z). \tag{S.87}$$

This completes the proof for part (i) of Theorem 4.3.

Step 3: Now, we concentrate on the bootstrap procedures. By following the same proof procedures used for Theorem 3.1 and Theorem 4.2, it is straightforward to demonstrate that

$$\sqrt{n}(\hat{\theta}_n^* - \hat{\theta}_n) = 2^{-1}\Sigma_2^{-1} \times \frac{1}{\sqrt{n}} \sum_{t=p+1}^{n} (b_t^* - 1) Z_{t,n} w_{t,n} \operatorname{sgn}(\check{\varepsilon}_{t,n}) + o_p^*(1). \tag{S.88}$$

Observe that the center δ_a in equation (S.81) has been omitted because the i.i.d. sequence $\{b_t^* - 1\}$ has a zero center. Moreover, we can deduce that

$$\begin{aligned} &\tilde{S}_n^*(\hat{\theta}_n^*, z) - \tilde{S}_n(\hat{\theta}_n, z) \\ &= \frac{1}{\sqrt{n}} \sum_{t=p+1}^{n} \Big\{ (b_t^* - 1)[\tilde{w}_{t,n} \exp(z'\tilde{Y}_{t-1,n}) - \tilde{V}'(z)\Sigma_2^{-1} Z_{t,n} w_{t,n}] \operatorname{sgn}(\check{\varepsilon}_{t,n}) \Big\} + o_p^*(1). \end{aligned} \tag{S.89}$$

Therefore, conclusion (ii) in Theorem 4.3 follows from equation (S.89). □

II. Some Technical Lemmas

In this section, we give some technical lemmas.

Lemma S.3. *Suppose that Assumptions 2.1–2.7 hold. Then we have*

$$\alpha_n(u) = \sum_{t=p+1}^{n} \{\zeta_{t,n}(u) - \mathbb{E}[\zeta_{t,n}(u)|\mathcal{F}_{t-1}]\} \xrightarrow{p} 0,$$

as $n \to \infty$, where $\zeta_{t,n}(u)$ is defined in (S.25).

Proof of Lemma S.3. Note that $\zeta_{t,n}(u)$ can be rewritten as follows.

$$\zeta_{t,n}(u) = w_{t,n} u_n' Z_{t,n} M_{t,n}(u_n), \tag{S.90}$$

where $u_n = n^{-1/2}u$, $M_{t,n}(u) = \int_0^1 X_{t,n}(u' Z_{t,n} s) ds$ and $X_{t,n}(x) = I(\varepsilon_{t,n} \le x) - I(\varepsilon_{t,n} \le 0)$. By the Cauchy-Schwarz inequality, we obtain the following result:

$$|\alpha_n(u)| \le \|u_n\| \times \Big\| \sum_{t=p+1}^{n} \Big(w_{t,n} Z_{t,n} \big\{ M_{t,n}(u_n) - \mathbb{E}[M_{t,n}(u_n)|\mathcal{F}_{t-1}] \big\} \Big) \Big\|. \tag{S.91}$$

Therefore, to establish the conclusion, it suffices to demonstrate that, for any $j \le r+p$,

$$\sum_{t=p+1}^{n} \Big(w_{t,n} Z_{t,n}^{(j)} \big\{ M_{t,n}(u_n) - \mathbb{E}[M_{t,n}(u_n)|\mathcal{F}_{t-1}] \big\} \Big) = o_p(\sqrt{n} + n|u_n|), \tag{S.92}$$

where $Z_{t,n}^{(j)}$ denotes the j-th component of the vector $Z_{t,n}$.

Let $m_{t,n} = w_{t,n} Z_{t,n}^{(j)}$ and $\tilde{f}_{t,n}(u) = m_{t,n} M_{t,n}(u)$, and further define $D_n(u)$ as

$$D_n(u) = \frac{1}{\sqrt{n}} \sum_{t=p+1}^{n} \{ \tilde{f}_{t,n}(u) - \mathbb{E}[\tilde{f}_{t,n}(u)|\mathcal{F}_{t-1}] \}. \tag{S.93}$$

Since $u_n \to 0$, we only need to prove that, for any $\eta \in (0,1)$,

$$\sup_{\|u\|<\eta} \frac{|D_n(u)|}{1 + \sqrt{n}\|u\|} = o_p(1), \text{ as } n \to \infty. \tag{S.94}$$

For (S.94), we can assume $m_{t,n} \ge 0$ (if not, we can utilize the equality $m_{t,n} = \max\{m_{t,n}, 0\} - \max\{-m_{t,n}, 0\}$ to adjust it), and we divide the proof into three steps.

Step 1: Based on Assumptions 2.6–2.7, there exists a constant C_1 such that $2\sup_t (w_{t,n}\|Z_{t,n}\|^2) \sup_{y,t} f_{t,n}(y) < C_1$. Then, for any $\epsilon \in (0, C_1)$ and $\delta \in (0, \eta]$, using the basic result for packing number (as seen in Problem 2.1.6 of van der Vaart and Wellner (1996)), we can find a sequence of open balls $\{B_{\frac{\epsilon\delta}{C_1}}(u_i)\}_{i=1}^{K_\epsilon}$ that cover the open ball $B_\delta(0)$, where $u_i \in B_\delta(0)$ and $K_\epsilon \leq \left(\frac{3C_1}{\epsilon}\right)^{r+p}$. Therefore, there exists a partition $\{U_i(\delta)\}_{i=1}^{K_\epsilon}$ of $B_\delta(0)$ such that $U_i(\delta) \subset B_{\frac{\epsilon\delta}{C_1}}(u_i)$.

Thus, for any $u \in U_i(\delta)$, utilizing the monotonicity of $X_{t,n}(x)$ and the fact that $m_{t,n} \geq 0$, we have the following uniform result:

$$\tilde{f}_{t,n,i}^{-} \leq \tilde{f}_{t,n}(u) \leq \tilde{f}_{t,n,i}^{+}, \tag{S.95}$$

where $\tilde{f}_{t,n,i}^{\pm} = m_{t,n} \int_0^1 X_{t,n}(u_i' Z_{t,n} s \pm \frac{\epsilon\delta}{C_1}\|Z_{t,n}\|) ds$. Moreover, applying Fubini's theorem along with Assumptions 2.6 and 2.7, and Assumption 2.2, we derive that

$$\mathbb{E}[(\tilde{f}_{t,n,i}^{+} - \tilde{f}_{t,n,i}^{-})|\mathcal{F}_{t-1}] < \epsilon\delta. \tag{S.96}$$

Step 2: Let $\delta_k = \eta/2^k$ for $k = 0, 1, \cdots$, and define $B(k) = B_{\delta_k}(0)$ and $A(k) = B(k)/B(k+1)$. Based on the results from Step 1, for a fixed k, there exists a partition $\{U_i(\delta_k)\}_{i=1}^{K_\epsilon}$ of $B(k)$. Then, by (S.95)-(S.96), for any $u \in U_i(\delta_k)$, we have

$$\begin{aligned} D_n(u) &\leq \frac{1}{\sqrt{n}} \sum_{t=p+1}^{n} [\tilde{f}_{t,n,i}^{+} - \mathbb{E}(\tilde{f}_{t,n,i}^{+}|\mathcal{F}_{t-1})] + \sqrt{n}\epsilon\delta_k \\ &\equiv D_n^{+}(u_i) + \sqrt{n}\epsilon\delta_k. \end{aligned} \tag{S.97}$$

Since $A(k) = B(k)/B(k+1)$, (S.97) implies that

$$\begin{aligned} P\Big(\sup_{u\in A(k)} \frac{D_n(u)}{1+\sqrt{n}\|u\|} > 6\epsilon\Big) &\leq P\Big(\sup_{u\in A(k)} D_n(u) > 3\sqrt{n}\epsilon\delta_k\Big) \\ &\leq P\Big(\max_{1\leq i\leq K_\epsilon} D_n^{+}(u_i) > \sqrt{n}\epsilon\delta_k\Big) \\ &\leq K_\epsilon \times \frac{\max_i \mathbb{E}[D_n^{+}(u_i)]^2}{n\epsilon^2\delta_k^2}. \end{aligned} \tag{S.98}$$

Note that the sequence $\{\tilde{f}^+_{t,n,i} - \mathbb{E}(\tilde{f}^+_{t,n,i}|\mathcal{F}_{t-1})\}$ is a martingale difference sequence. Therefore,

$$\begin{aligned}\mathbb{E}[D^+_n(u_i)]^2 &\leq \frac{1}{n}\sum_{t=p+1}^{n}\mathbb{E}(\tilde{f}^+_{t,n,i})^2\\ &\leq \frac{2}{n}\sum_{t=p+1}^{n}\mathbb{E}[w^2_{t,n}\|Z_{t,n}\|^3\sup_{y\in\mathbb{R}} f_{t,n}(y)]\times\delta_k \equiv \pi_n(\delta_k).\end{aligned}\tag{S.99}$$

Based on equations (S.98) through (S.99), we obtain that

$$P\Big(\sup_{u\in A(k)}\frac{D_n(u)}{1+\sqrt{n}\|u\|} > 6\epsilon\Big) \leq \frac{K_\epsilon \pi_n(\delta_k)}{n\epsilon^2\delta_k^2}.\tag{S.100}$$

By a similar argument as presented in equation (S.100), we can derive the same bound for the lower tail. Consequently, it follows that

$$P\Big(\sup_{u\in A(k)}\frac{|D_n(u)|}{1+\sqrt{n}\|u\|} > 6\epsilon\Big) \leq \frac{2K_\epsilon \pi_n(\delta_k)}{n\epsilon^2\delta_k^2}.\tag{S.101}$$

In addition, Assumptions 2.6-2.7 indicate that

$$\sup_n \pi_n(\delta_k) \longrightarrow 0, \text{ as } k\to\infty.\tag{S.102}$$

Step 3: For any $\epsilon' > 0$, based on equation (S.102), we can select some $k_{\epsilon'}$ such that

$$\sup_n \frac{2K_\epsilon\pi_n(\delta_k)}{\epsilon^2\eta^2} < \epsilon',\ \forall k > k_{\epsilon'}.\tag{S.103}$$

Meanwhile, let $k_n > k_{\epsilon'}$ be the unique integer satisfying $n^{-1/2} \leq 2^{-k_n} < 2n^{-1/2}$.

Note that $B_\eta(0) = B(k_n+1)\cup B^c(k_n+1)$, where $B^c(k_n+1) = \cup_{k=0}^{k_n}A(k)$. Then, based on equation (S.101), the definition of $\delta_k = \eta/2^k$, and the inequality $n^{-1/2} \leq$

2^{-k_n}, we have

$$P\Big(\sup_{u\in B^c(k_n+1)}\frac{|D_n(u)|}{1+\sqrt{n}\|u\|}>6\epsilon\Big)\le\sum_{k=0}^{k_n}P\Big(\sup_{u\in A(k)}\frac{|D_n(u)|}{1+\sqrt{n}\|u\|}>6\epsilon\Big)$$
$$\le\sum_{k=0}^{k_{\epsilon'}}\frac{2^{2k+1}K_\epsilon\pi_n(\delta_k)}{n\epsilon^2\eta^2}+\sum_{k=k_{\epsilon'}+1}^{k_n}\frac{\epsilon' 4^k}{n}$$
$$\le O(n^{-1})+4\epsilon'. \tag{S.104}$$

On the other hand, given that $2^{-k_n}<2n^{-1/2}$, $1+\sqrt{n}\|u\|\ge1$, and $\eta\in(0,1)$, a similar procedure to that used for equations (S.100) through (S.101) implies that

$$P\Big(\sup_{u\in B(k_n+1)}\frac{|D_n(u)|}{1+\sqrt{n}\|u\|}>2\epsilon\Big)\le\frac{2K_\epsilon\pi_n(\delta_{k_n+1})}{\epsilon^2}\longrightarrow0, \tag{S.105}$$

as $n\to\infty$.

Then, based on equations (S.104) through (S.105) and the arbitrariness of ϵ', we have demonstrated that

$$P\Big(\sup_{u\in B_\eta(0)}\frac{|D_n(u)|}{1+\sqrt{n}\|u\|}>6\epsilon\Big)\longrightarrow0,\quad\text{as } n\to\infty. \tag{S.106}$$

This completes the whole proof. □

Lemma S.4. *Suppose that all the conditions in Theorem 2.1 hold. Then we have*

$$\tilde{S}_n(\theta_0,z)\rightsquigarrow\tilde{S}_1(z),$$

as $n\to\infty$, where $\tilde{S}_1(z)$ is a centered Gaussian process with covariance function $\Lambda_1(z_1,z_2)=\sum_{i=1}^{m+1}\int_{\lambda_{i-1}}^{\lambda_i}\mathbb{E}\{\tilde{w}^2(x,Y'_{i,t-1}(x))\exp((z_1+z_2)'\tilde{Y}_{i,t-1}(x))\}dx$.

Proof of Lemma S.4. We begin by demonstrating the asymptotic tightness of $\tilde{S}_n(\theta_0,z)$. Specifically, it is essential to establish that for any $\epsilon,\eta>0$, there exists a $\delta>0$ such that:

$$\limsup_{n\to\infty} P(\sup_{\|z_1-z_2\|<\delta} |\tilde{S}_n(\theta_0, z_1) - \tilde{S}_n(\theta_0, z_2)| > \epsilon) < \eta. \tag{S.107}$$

For any $z_1, z_2 \in \mathcal{Z}$ with $\|z_1 - z_2\| < 1$, given the boundedness of $\mathcal{Z}$ and $\tilde{Y}_{t-1,n}$, and by applying the Taylor expansion of the exponential function, we derive the following result:

$$\begin{aligned}
\tilde{S}_n(\theta_0, z_1) - \tilde{S}_n(\theta_0, z_2) &= \frac{1}{\sqrt{n}} \sum_{t=p+1}^{n} \Big\{ \tilde{w}_{t,n} \operatorname{sgn}(\varepsilon_{t,n})[\exp(z_1'\tilde{Y}_{t-1,n}) - \exp(z_2'\tilde{Y}_{t-1,n})] \Big\} \\
&= \frac{1}{\sqrt{n}} \sum_{l=1}^{\infty} \sum_{t=p+1}^{n} \Big\{ \tilde{w}_{t,n} \operatorname{sgn}(\varepsilon_{t,n})[(z_1'\tilde{Y}_{t-1,n})^l - (z_2'\tilde{Y}_{t-1,n})^l]/l! \Big\} \\
&\equiv \sum_{l=1}^{\infty} \tilde{S}_{n,l}(\theta_0, z_1, z_2).
\end{aligned} \tag{S.108}$$

Furthermore, let $(j_1, j_2, \cdots, j_l)$ be a combination of $j_i \in \{1, \cdots, \tilde{p}\}$. Then,

$$\tilde{S}_{n,l}(\theta_0, z_1, z_2) = \sum_{(j_1,\cdots,j_l)} \Big\{ \Big(\frac{1}{\sqrt{n}} \sum_{t=p+1}^{n} [\tilde{w}_{t,n} \operatorname{sgn}(\varepsilon_{t,n})(\prod_{i=1}^{l} \tilde{y}_{t-j_i,n})] \Big) \times [\prod_{i=1}^{l} z_{1,j_i} - \prod_{i=1}^{l} z_{2,j_i}]/l! \Big\}, \tag{S.109}$$

where z_{k,j_i} is the j_i-th component of z_k. Additionally, through a straightforward calculation, it is evident that

$$\Big| \prod_{i=1}^{l} z_{1,j_i} - \prod_{i=1}^{l} z_{2,j_i} \Big| \le \|z_1 - z_2\| \times (1 + \|z_1\| + \|z_2\|)^l. \tag{S.110}$$

Hence, for any $\delta \in (0, 1)$, equations (S.109)–(S.110) imply that

$$\begin{aligned}
\mathbb{E}[\sup_{\|z_1-z_2\|<\delta} |\tilde{S}_{n,l}(\theta_0, z_1, z_2)|] &\le \delta C_1^l/l! \times \sum_{(j_1,\cdots,j_l)} \mathbb{E}\Big| \frac{1}{\sqrt{n}} \sum_{t=p+1}^{n} [\tilde{w}_{t,n} \operatorname{sgn}(\varepsilon_{t,n})(\prod_{i=1}^{l} \tilde{y}_{t-j_i,n})] \Big| \\
&\le \delta C_1^l C_2^l \tilde{p}^l/l!,
\end{aligned} \tag{S.111}$$

where C_1 depends solely on the domain of $\mathcal{Z}$, and C_2 depends on $\tilde{w}(\cdot)$ and $\phi(\cdot)$. Consequently, by equations (S.108) and (S.111), we have demonstrated that:

$$\mathbb{E}\Big\{\sup_{\|z_1-z_2\|<\delta}|\tilde{S}_n(\theta_0,z_1)-\tilde{S}_n(\theta_0,z_2)|\Big\} \leq \delta \times \exp(C_1C_2\tilde{p}). \tag{S.112}$$

The conclusion in (S.107) is thus derived from the arbitrariness of δ and (S.112).

To establish the finite-dimensional convergence of $\tilde{S}_n(\theta_0,z)$, without loss of generality, we focus on the two-dimensional case. Let $a_1,a_2\in\mathbb{R}$ and $z_1,z_2\in\mathcal{Z}$. We have

$$a_1\tilde{S}_n(\theta_0,z_1)+a_2\tilde{S}_n(\theta_0,z_2)=\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\Big\{\tilde{w}_{t,n}\,\mathrm{sgn}(\varepsilon_{t,n})[a_1\exp(z_1'\tilde{Y}_{t-1,n})+a_2\exp(z_2'\tilde{Y}_{t-1,n})]\Big\}.$$

Observe that the summands form a martingale difference sequence (m.d.s.). By the martingale central limit theorem and following similar proof procedures as for equation (S.27), it follows that:

$$a_1\tilde{S}_n(\theta_0,z_1)+a_2\tilde{S}_n(\theta_0,z_2)\xrightarrow{d}\mathcal{N}(0,a_1^2\Lambda_1(z_1,z_1)+2a_1a_2\Lambda_1(z_1,z_2)+a_2^2\Lambda_1(z_2,z_2)). \tag{S.113}$$

Thus, the lemma is established from equations (S.107) and (S.113). □

Lemma S.5. *Suppose that all the conditions in Theorem 2.1 hold, and denote* $\mathcal{C}_n=\{\theta:\sqrt{n}\|\theta-\theta_0\|<C\}$ *for some constant* $C>0$*. Then it follows that*

(i). $\sup_{\theta\in\mathcal{C}_n,z\in\mathcal{Z}}\left|\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\{\tilde{R}_{t,n}(\theta,z)-\mathbb{E}[\tilde{R}_{t,n}(\theta,z)|\mathcal{F}_{t-1}]\}\right|=o_p(1)$,

(ii). $\sup_{\theta\in\mathcal{C}_n,z\in\mathcal{Z}}\left|\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\{\mathbb{E}[\tilde{R}_{t,n}(\theta,z)|\mathcal{F}_{t-1}]\}+2\tilde{V}'(z)\sqrt{n}(\theta-\theta_0)\right|=o_p(1)$,

where $\tilde{R}_{t,n}(\theta,z)=\tilde{r}_{t,n}(\theta,z)-\tilde{r}_{t,n}(\theta_0,z)$ *with* $\tilde{r}_{t,n}(\theta,z)$ *given in (4.3).*

Proof of Lemma S.5. (i) According to the definition of $\tilde{r}_{t,n}(\theta, z)$ in (4.3), we can deduce that

$$\tilde{R}_{t,n}(\theta, z) = 2\tilde{w}_{t,n} \exp(z'\tilde{Y}_{t-1,n})[I(\varepsilon_{t,n} > Z'_{t,n}(\theta - \theta_0)) - I(\varepsilon_{t,n} > 0)]. \tag{S.114}$$

To establish result (i), we use the standard argument similar to that for Lemma S.3. For any $\delta \in (0, 1)$, there exists a sequence of open balls $\{B_{C\delta}(u_i)\}_{i=1}^{K_\delta}$ such that $B_C(0) \subset \cup_{i=1}^{K_\delta} B_{C\delta}(u_i)$, where $u_i \in B_C(0)$ and $K_\delta \leq \left(\frac{3}{\delta}\right)^{r+p}$. Furthermore, let $\theta_{in} = \theta_0 + u_i/\sqrt{n}$, then we have $\mathcal{C}_n \subset \cup_{i=1}^{K_\delta} \mathcal{C}_{in}$ with $\mathcal{C}_{in} = B_{\frac{C\delta}{\sqrt{n}}}(\theta_{in})$.

Notice that, for any $\theta \in \mathcal{C}_{in}$, by the monotone property of the indicator function and the positivity of $\tilde{w}_{t,n}$ and $\exp(z'\tilde{Y}_{t-1,n})$, we uniformly have

$$\tilde{R}^{-}_{t,n}(\theta_{in}, z, \delta) \leq \tilde{R}_{t,n}(\theta, z) \leq \tilde{R}^{+}_{t,n}(\theta_{in}, z, \delta), \tag{S.115}$$

where $\tilde{R}^{\pm}_{t,n}(\theta_{in}, z, \delta)$ are given by

$$\tilde{R}^{\pm}_{t,n}(\theta_{in}, z, \delta) = 2\tilde{w}_{t,n} \exp(z'\tilde{Y}_{t-1,n})[I(\varepsilon_{t,n} > Z'_{t,n}(\theta_{in} - \theta_0) \mp \frac{C\delta}{\sqrt{n}}\|Z_{t,n}\|) - I(\varepsilon_{t,n} > 0)]. \tag{S.116}$$

Given that $\tilde{w}(\cdot)$ satisfies Assumption 2.7 and, in conjunction with Assumption 2.6, considering the boundedness of $\mathcal{Z}$ and $\phi(\cdot)$, we can deduce the following inequality:

$$\mathbb{E}\{[\tilde{R}^{+}_{t,n}(\theta_{in}, z, \delta) - \tilde{R}^{-}_{t,n}(\theta_{in}, z, \delta)]|\mathcal{F}_{t-1}\} \leq \frac{C'\delta}{\sqrt{n}}, \tag{S.117}$$

where C' denotes a constant that is independent of δ, θ_{in}, z, and n.

Subsequently, based on equations (S.115) and (S.117), we can straightforwardly derive that

$$\sup_{\theta\in\mathcal{C}_n, z\in\mathcal{Z}} \frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\{\tilde{R}_{t,n}(\theta,z)-\mathbb{E}[\tilde{R}_{t,n}(\theta,z)|\mathcal{F}_{t-1}]\}$$
$$\leq \max_{1\leq i\leq K_\delta}\sup_{z\in\mathcal{Z}} \frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\{\tilde{R}^{+}_{t,n}(\theta_{in},z,\delta)-\mathbb{E}[\tilde{R}^{+}_{t,n}(\theta_{in},z,\delta)|\mathcal{F}_{t-1}]\}+C'\delta. \quad \text{(S.118)}$$

Furthermore, we have

$$\mathbb{E}\Big[\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\{\tilde{R}^{+}_{t,n}(\theta_{in},z,\delta)-\mathbb{E}[\tilde{R}^{+}_{t,n}(\theta_{in},z,\delta)|\mathcal{F}_{t-1}]\}\Big]^2$$
$$\leq \frac{1}{n}\sum_{t=p+1}^{n}\mathbb{E}[\tilde{R}^{+}_{t,n}(\theta_{in},z,\delta)]^2 \leq \frac{C''(\delta+1)}{\sqrt{n}}, \quad \text{(S.119)}$$

where C'' is a constant independent of δ, θ_{in}, z, and n. By following the proof strategy of (S.107), it is easy to show that

$$\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\{\tilde{R}^{+}_{t,n}(\theta_{in},z,\delta)-\mathbb{E}[\tilde{R}^{+}_{t,n}(\theta_{in},z,\delta)|\mathcal{F}_{t-1}]\} \text{ is asymptotically tight.} \quad \text{(S.120)}$$

Given that K_δ is finite, inequalities (S.119) and (S.120) imply that

$$\max_{1\leq i\leq K_\delta}\sup_{z\in\mathcal{Z}}\Big|\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\{\tilde{R}^{+}_{t,n}(\theta_{in},z,\delta)-\mathbb{E}[\tilde{R}^{+}_{t,n}(\theta_{in},z,\delta)|\mathcal{F}_{t-1}]\}\Big| = o_p(1). \quad \text{(S.121)}$$

Based on (S.118) and (S.121) and the arbitrariness of δ, we conclude that, for any $\epsilon > 0$,

$$P\Big(\sup_{\theta\in\mathcal{C}_n, z\in\mathcal{Z}} \frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\{\tilde{R}_{t,n}(\theta,z)-\mathbb{E}[\tilde{R}_{t,n}(\theta,z)|\mathcal{F}_{t-1}]\} > \epsilon\Big) \to 0,$$

as $n\to\infty$. By a similar argument, we can also establish that

$$P\Big(\inf_{\theta\in\mathcal{C}_n, z\in\mathcal{Z}} \frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\{\tilde{R}_{t,n}(\theta,z)-\mathbb{E}[\tilde{R}_{t,n}(\theta,z)|\mathcal{F}_{t-1}]\} < -\epsilon\Big) \to 0.$$

This completes the proof for the conclusion (i).

(ii) Given $\varepsilon_{t,n} = \eta_t \sigma_{t,n}$ and Assumption 2.6, it follows that

$$\frac{1}{\sqrt{n}} \sum_{t=p+1}^{n} \{\mathbb{E}[\tilde{R}_{t,n}(\theta, z)|\mathcal{F}_{t-1}]\} = -\frac{2}{n} \sum_{t=p+1}^{n} \left[\tilde{w}_{t,n} \exp(z'\tilde{Y}_{t-1,n}) Z'_{t,n} f_\eta(0)/\sigma_{t,n}\right] \times \sqrt{n}(\theta - \theta_0)$$
$$+ \frac{2}{n} \sum_{t=p+1}^{n} \left[\tilde{w}_{t,n} \exp(z'\tilde{Y}_{t-1,n}) Z'_{t,n}[f_\eta(0) - f_\eta(s_t)]/\sigma_{t,n}\right] \times \sqrt{n}(\theta - \theta_0), \quad \text{(S.122)}$$

where $|s_t| \le C\sigma_0^{-1}\|Z_{t,n}\|/\sqrt{n}$ for all t. By Lemma S.2 and the Lipschitz continuity of $\exp(z'\tilde{Y}_{t-1,n})$, we can demonstrate that

$$\sup_{z\in\mathcal{Z}} \left\| \frac{1}{n} \sum_{t=p+1}^{n} \tilde{w}_{t,n} \exp(z'\tilde{Y}_{t-1,n}) Z'_{t,n} f_\eta(0)/\sigma_{t,n} - \tilde{V}'(z) \right\| = o_p(1), \quad \text{(S.123)}$$
$$\sup_{z\in\mathcal{Z}} \left\| \frac{1}{n} \sum_{t=p+1}^{n} \tilde{w}_{t,n} \exp(z'\tilde{Y}_{t-1,n}) Z'_{t,n}[f_\eta(0) - f_\eta(s_t)]/\sigma_{t,n} \right\| = o_p(1). \quad \text{(S.124)}$$

Then the conclusion (ii) follows from the fact that $\sup_{\theta\in\mathcal{C}_n} \sqrt{n}\|\theta - \theta_0\| < C$. □

Lemma S.6. *Suppose that all the conditions in Theorem 2.1 hold. Then, for any $C, \delta > 0$, it follows that*

(i). $\sup_{z\in\mathcal{Z}} \left| \frac{1}{\sqrt{n}} \sum_{t=p+1}^{n} \{\tilde{R}^{\pm}_{t,n}(\theta_n, z, \delta) - \mathbb{E}[\tilde{R}^{\pm}_{t,n}(\theta_n, z, \delta)]\} \right| = o_p(1),$

(ii). $\sup_{z\in\mathcal{Z}} \left[\frac{1}{n} \sum_{t=p+1}^{n} \{\tilde{R}^{\pm}_{t,n}(\theta_n, z, \delta) - \mathbb{E}[\tilde{R}^{\pm}_{t,n}(\theta_n, z, \delta)]\}^2 \right] = o_p(1),$

where $\theta_n = \theta_0 + u/\sqrt{n}$ for some $u \in \mathbb{R}^{r+p}$ satisfying $\|u\| < C$, and $\tilde{R}^{\pm}_{t,n}(\theta_n, z, \delta)$ are defined in equation (S.116).

Proof of Lemma S.6. Since the proof procedures are very similar, we present only the details for the conclusions pertaining to the sequence $\{\tilde{R}^{+}_{t,n}(\theta_n, z, \delta)\}$. Firstly, by examining the proofs of equations (S.119) and (S.121), it follows that the analogous results are obtained when θ_{in} is replaced by θ_n, specifically,

$$\sup_{z\in\mathcal{Z}}\left\{\frac{1}{n}\sum_{t=p+1}^{n}\mathbb{E}[\tilde{R}_{t,n}^{+}(\theta_n,z,\delta)]^2\right\}=o(1), \tag{S.125}$$

$$\sup_{z\in\mathcal{Z}}\left|\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\{\tilde{R}_{t,n}^{+}(\theta_n,z,\delta)-\mathbb{E}[\tilde{R}_{t,n}^{+}(\theta_n,z,\delta)|\mathcal{F}_{t-1}]\}\right|=o_p(1). \tag{S.126}$$

By equation (S.126), for result (i), it suffices to demonstrate that

$$\sup_{z\in\mathcal{Z}}\left|\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\{\mathbb{E}[\tilde{R}_{t,n}^{+}(\theta_n,z,\delta)|\mathcal{F}_{t-1}]-\mathbb{E}[\tilde{R}_{t,n}^{+}(\theta_n,z,\delta)]\}\right|=o_p(1). \tag{S.127}$$

By the definition of $\tilde{R}_{t,n}^{+}(\theta_n,z,\delta)$, we obtain

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\mathbb{E}[\tilde{R}_{t,n}^{+}(\theta_n,z,\delta)|\mathcal{F}_{t-1}]&=-\frac{2f_\eta(0)}{n}\sum_{t=p+1}^{n}\left\{\tilde{w}_{t,n}\exp(z'\tilde{Y}_{t-1,n})[Z_{t,n}'u-\|Z_{t,n}\|C\delta]/\sigma_{t,n}\right\}\\&+\frac{2}{n}\sum_{t=p+1}^{n}\left\{\tilde{w}_{t,n}\exp(z'\tilde{Y}_{t-1,n})(Z_{t,n}'u-\|Z_{t,n}\|C\delta)[f_\eta(0)-f_\eta(s_t)]/\sigma_{t,n}\right\},\end{aligned} \tag{S.128}$$

where $|s_t|\leq n^{-1/2}C(1+\delta)\|Z_{t,n}\|/\sigma_0$. Subsequently, by Lemma S.2 and the Lipschitz continuity of $\exp(z'\tilde{Y}_{t-1,n})$, we can further show that

$$\sup_{z\in\mathcal{Z}}\left|\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\mathbb{E}[\tilde{R}_{t,n}^{+}(\theta_n,z,\delta)|\mathcal{F}_{t-1}]+2\tilde{V}(z)u-2\tilde{v}(z)C\delta\right|=o_p(1), \tag{S.129}$$

where $\tilde{v}(z)$ is given by

$$\tilde{v}(z)=\sum_{i=1}^{m+1}\int_{\lambda_{i-1}}^{\lambda_i}\mathbb{E}[\tilde{w}(x,Y_{i,t-1}'(x))\exp(z'\tilde{Y}_{i,t-1}(x))\|Z_{i,t}(x)\|f_{i,t}(0;x)]dx.$$

From equation (S.128), it is evident that $n^{-1/2}\sum_{t=p+1}^{n}\mathbb{E}[\tilde{R}_{t,n}^{+}(\theta_n,z,\delta)|\mathcal{F}_{t-1}]$ is uniformly bounded for all n and z. Therefore, by equation (S.129) and applying the dominated convergence theorem, we conclude that

$$\sup_{z\in\mathcal{Z}}\left|\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\mathbb{E}[\tilde{R}_{t,n}^{+}(\theta_n,z,\delta)]+2\tilde{V}(z)u-2\tilde{v}(z)C\delta\right|=o(1). \tag{S.130}$$

Then, the conclusion (S.127) follows from (S.129) and (S.130).

For the result (ii), based on the definition of $\tilde{R}^{+}_{t,n}(\theta_n, z, \delta)$ and Assumption 2.7, and by leveraging the boundedness and the Lipschitz continuity of $\exp(z'\tilde{Y}_{t-1,n})$, it is straightforward to demonstrate that there exists a constant C_1 such that for any $z_1, z_2 \in \mathcal{Z}$, the following inequality holds:

$$\sup_n \left[\frac{1}{n}\sum_{t=p+1}^{n} |\{\tilde{R}_{t,n}(\theta_n, z_1, \delta)\}^2 - \{\tilde{R}_{t,n}(\theta_n, z_2, \delta)\}^2|\right] \le C_1\|z_1 - z_2\|. \tag{S.131}$$

In addition, (S.125) implies that, for any $z \in \mathcal{Z}$, $n^{-1}\sum_{t=p+1}^{n}\{\tilde{R}_{t,n}(\theta_n, z, \delta)\}^2 = o_p(1)$. Then, by (S.131), we have

$$\sup_{z\in\mathcal{Z}} \left[\frac{1}{n}\sum_{t=p+1}^{n} \{\tilde{R}_{t,n}(\theta_n, z, \delta)\}^2\right] = o_p(1). \tag{S.132}$$

Thus, the conclusion holds from (S.125) and (S.132). □

III. Proofs for Theorem 7.1

In this section, we present details for Theorem 7.1 in Section 7.

Proof of Theorem 7.1. Analogous to the approach for Theorem 2.1, observe that $\hat{u}_n(\tau) = \sqrt{n}(\hat{\theta}_n(\tau) - \theta_0)$ is the minimizer of the objective function $L_n(u, \tau)$, defined as:

$$\begin{aligned} L_n(u,\tau) &= \sum_{t=p+1}^{[n\tau]} w_{t,n}\left[|\varepsilon_{t,n} - \frac{1}{\sqrt{n}}Z'_{t,n}u| - |\varepsilon_{t,n}|\right], \\ &= -u'T_n(\tau) + 2\sum_{t=p+1}^{[n\tau]} \mathbb{E}[\zeta_{t,n}(u)|\mathcal{F}_{t-1}] + 2\alpha_n(u,\tau), \end{aligned} \tag{S.133}$$

where $T_n(\tau) = \sum_{t=p+1}^{[n\tau]} \xi_{t,n}$ and $\alpha_n(u,\tau) = \sum_{t=p+1}^{[n\tau]} \{\zeta_{t,n}(u) - \mathbb{E}[\zeta_{t,n}(u)|\mathcal{F}_{t-1}]\}$. We first establish the following three facts:

$$T_n(\tau) \rightsquigarrow T_0(\tau), \tag{S.134}$$

$$\sup_{\tau\in\mathcal{T}} \Big| \sum_{t=p+1}^{[n\tau]} \mathbb{E}[\zeta_{t,n}(u)|\mathcal{F}_{t-1}] - u'\Sigma_2(\tau)u/2 \Big| = o_p(1), \tag{S.135}$$

$$\sup_{\tau\in\mathcal{T}} |\alpha(u,\tau)| = o_p(1). \tag{S.136}$$

For (S.134), we first show the asymptotic tightness of $T_n(\tau)$, that is, for any $\epsilon, \eta > 0$, there exists $\delta > 0$ such that

$$\limsup_{n\to\infty} P\big(\sup_{|\tau_1-\tau_2|<\delta} \|T_n(\tau_1) - T_n(\tau_2)\| > \epsilon\big) < \eta. \tag{S.137}$$

Because $\{\xi_{t,n}\}$ is an m.d.s., by the Burkholder inequality and Assumption 2.7, for any $q > 2$ and $k_1, k_2 \in \mathbb{Z}$, we have

$$\mathbb{E}\Big(\|\sum_{t=k_1}^{k_2} \xi_{t,n}\|^q\Big) \leq C_1\Big(\frac{k_2 - k_1 + 1}{n}\Big)^{q/2}, \tag{S.138}$$

where C_1 is a constant only relying on $w(\cdot)$ and q. Furthermore, by the maximum inequality in Proposition 1 in Wu (2007) and (S.138), for any $k_1, d \in \mathbb{Z}$, we have

$$\begin{aligned}\Big[\mathbb{E}\Big(\max_{k\leq 2^d} \|\sum_{t=k_1+1}^{k_1+k} \xi_{t,n}\|^q\Big)\Big]^{1/q} &\leq \sum_{r=0}^{d} \Big[\sum_{m=1}^{2^{d-r}} \mathbb{E}\Big(\|\sum_{t=k_1+1+2^r(m-1)}^{k_1+2^r m} \xi_{t,n}\|^q\Big)\Big]^{1/q} \\ &\leq C_2 2^{d/2}/\sqrt{n}. \end{aligned}\tag{S.139}$$

Therefore, by the duality argument, (S.139) indicates that for $\tau_0 \in (0,1)$ and $\delta > 0$,

$$\mathbb{E}\Big(\sup_{|\tau|<\delta} \|T_n(\tau_0+\tau) - T_n(\tau_0)\|^q\Big) \leq (2C_2)^q(\delta + 1/n)^{q/2}. \tag{S.140}$$

Since $\mathcal{T}$ is a subset of $(0,1]$, for any $\delta \in (0,1)$, we can always divide $\mathcal{T}$ into the union $\cup_{i=1}^{k_\delta}[\delta_{i-1}, \delta_i]$ with $\delta_i - \delta_{i-1} \in (0, \delta)$ and $k_\delta < 3/\delta$, then by (S.140), it is obvious that

$$\limsup_{n\to\infty} P(\sup_{|\tau_1-\tau_2|<\delta} \|T_n(\tau_1) - T_n(\tau_2)\| > \epsilon) \leq C_3\delta^{q/2-1}/\epsilon^q. \tag{S.141}$$

By the fact that $q > 2$ and the arbitrariness of δ, (S.137) holds.

On the other hand, we need prove the finite-dimensional convergence of $T_n(\tau)$. Without loss of generality, we focus only on the two-dimensional case. Let $\beta_1, \beta_2 \in \mathbb{R}^{r+p}$ and $\tau_1, \tau_2 \in (0,1)$ with $\tau_1 < \tau_2$, then by similar proof procedures for (S.27), it is not hard to show that

$$\beta_1' T_n(\tau_1) + \beta_2' T_n(\tau_2) \xrightarrow{d} \mathcal{N}(0, \beta_1'\Sigma_1(\tau_1)\beta_1 + 2\beta_1'\Sigma_1(\tau_1)\beta_2 + \beta_2'\Sigma_1(\tau_2)\beta_2). \tag{S.142}$$

As a result, (S.134) holds from (S.137) and (S.142).

For (S.135), by checking the proof process for (S.28) and using (S.34), it is obvious that

$$\begin{aligned}
&\sup_{\tau\in\mathcal{T}} \Big| \sum_{t=p+1}^{[n\tau]} \mathbb{E}[\zeta_{t,n}(u)|\mathcal{F}_{t-1}] - u'\big[\frac{1}{2n}\sum_{t=p+1}^{[n\tau]} w_{t,n} Z_{t,n} Z_{t,n}' f_{t,n}(0)\big] u \Big| \\
&\leq \sum_{t=p+1}^{n} \Big| w_{t,n} \int_0^{\frac{1}{\sqrt{n}} u' Z_{t,n}} s[f_{t,n}(s_{t,n}) - f_{t,n}(0)] ds \Big| = o_p(1).
\end{aligned} \tag{S.143}$$

Notice that Assumptions 2.6-2.7 imply that there exists some deterministic constant C_4 such that for any $\tau_1 < \tau_2$

$$\Big| \frac{1}{n} \sum_{t=[n\tau_1]+1}^{[n\tau_2]} w_{t,n} Z_{t,n} Z_{t,n}' f_{t,n}(0) \Big| \leq C_4(\tau_2 - \tau_1 + 1/n). \tag{S.144}$$

Thus, combine the following point-wise convergence

$$\frac{1}{n} \sum_{t=p+1}^{[n\tau]} w_{t,n} Z_{t,n} Z_{t,n}' f_{t,n}(0) = \Sigma_2(\tau) + o_p(1), \tag{S.145}$$

the conclusion (S.135) holds.

For (S.136), we only need to show the functional version of (S.93) holds, namely,

$$\sup_{\|u\|<\eta} \sup_{\tau} \frac{|D_n(u,\tau)|}{1+\sqrt{n}\|u\|} = o_p(1), \tag{S.146}$$

as $n \to \infty$, where $D_n(u,\tau)$ is defined as follows.

$$D_n(u,\tau) = \frac{1}{\sqrt{n}} \sum_{t=p+1}^{[n\tau]} \{\tilde{f}_{t,n}(u) - \mathbb{E}[\tilde{f}_{t,n}(u)|\mathcal{F}_{t-1}]\}. \tag{S.147}$$

Note that for a fixed u, by the Doob's maximum inequality for the martingale difference sequence and repeating the procedures as Step 2 in Lemma S.3, we can easily show that the counterpart of (S.101):

$$P\Big(\sup_{u\in A(k)} \sup_{\tau} \frac{|D_n(u,\tau)|}{1+\sqrt{n}\|u\|} > 6\epsilon\Big) \le \frac{2K_\epsilon \pi_n(\delta_k)}{n\epsilon^2\delta_k^2}, \tag{S.148}$$

where $\pi_n(\delta_k)$ is defined in (S.99). Then the conclusion can be easily derived following the same process in Step 3 for Lemma S.3.

Finally, since $\Sigma_2(\tau)$ is invertible for any $\tau > 0$, then by the continuity of $\Sigma_2(\tau)$ and the compactness of $\mathcal{T}$, the maximum eigenvalue of $\Sigma_2(\tau)$ is bounded from above and the minimum eigenvalue of $\Sigma_2(\tau)$ is bounded away from 0. Meanwhile, notice that $L_n(u,\tau)$ is convex in u for each τ and bounded in τ for each u, then by Theorem 2 in Kato (2009) and (S.134)-(S.136), the theorem holds. This completes the proof. □

IV. Additional Simulation Studies

In this section, we extend our analysis with additional simulation results. We maintain the same parameters and configurations as detailed in Section 5, with the exception that the distribution of the bootstrap weight b_t^* is altered to follow a standard exponential distribution (i.e., Exp(1)). This modification allows us to explore the sensitivity of our simulation outcomes to changes in the weight distribution, providing further insights into the robustness of our methodology. The outcomes of these simulations are encapsulated in Tables S.1 to S.2. A comparison reveals that the performance of the bootstrap tests in Sections 3-4 is strikingly similar to those presented in Tables 7 to 10, suggesting that the bootstrap methodology employed within this study exhibits a considerable degree of stability regardless of the specific distributional characteristics of the weight b_t^*.

Table S.1: The size and power of the bootstrap test for the AR(2) coefficient in model (5.7), where b_t^* is altered to follow a standard exponential distribution

			$\eta_t \sim \mathcal{N}(0,1)$			$\eta_t \sim t_3$		
$g_1(x) \sim$	λ_1	n	$\kappa = 0.0$	$\kappa = 0.2$	$\kappa = 0.4$	$\kappa = 0.0$	$\kappa = 0.2$	$\kappa = 0.4$
(5.3)	0.25	400	0.051	0.665	0.995	0.047	0.820	1.000
		800	0.043	0.914	1.000	0.043	0.992	1.000
	0.50	400	0.047	0.688	0.997	0.043	0.850	1.000
		800	0.045	0.944	1.000	0.057	0.991	1.000
	0.75	400	0.053	0.734	0.999	0.040	0.879	1.000
		800	0.058	0.942	1.000	0.046	0.997	1.000
(5.4)	0.25	400	0.049	0.666	0.999	0.043	0.823	1.000
		800	0.050	0.943	1.000	0.053	0.984	1.000
	0.50	400	0.054	0.699	0.993	0.041	0.872	1.000
		800	0.050	0.928	1.000	0.043	0.998	1.000
	0.75	400	0.052	0.726	0.999	0.050	0.915	1.000
		800	0.053	0.954	1.000	0.042	0.986	1.000
(5.5)	0.25	400	0.046	0.670	0.997	0.037	0.850	1.000
		800	0.053	0.921	1.000	0.056	0.984	1.000
	0.50	400	0.041	0.697	0.998	0.053	0.881	1.000
		800	0.059	0.947	1.000	0.048	0.991	1.000
	0.75	400	0.042	0.700	0.996	0.050	0.895	1.000
		800	0.058	0.951	1.000	0.045	0.994	1.000

Table S.2: The size and power of $\tilde{T}_n$ in model (5.8), where b_t^* is altered to follow a standard exponential distribution

			$g_1(x) \sim$ (5.3)			$g_1(x) \sim$ (5.4)			$g_1(x) \sim$ (5.5)		
$\eta_t \sim$	λ_1	n	$\kappa = 0.0$	$\kappa = 0.2$	$\kappa = 0.4$	$\kappa = 0.0$	$\kappa = 0.2$	$\kappa = 0.4$	$\kappa = 0.0$	$\kappa = 0.2$	$\kappa = 0.4$
$\mathcal{N}(0,1)$	0.25	400	0.031	0.379	0.893	0.040	0.700	0.986	0.042	0.362	0.923
		800	0.048	0.677	0.997	0.037	0.958	1.000	0.051	0.681	0.997
	0.50	400	0.044	0.340	0.857	0.041	0.745	0.996	0.038	0.398	0.916
		800	0.039	0.591	0.996	0.067	0.963	1.000	0.045	0.662	0.999
	0.75	400	0.046	0.333	0.865	0.051	0.812	1.000	0.031	0.434	0.935
		800	0.038	0.585	0.994	0.057	0.988	1.000	0.047	0.701	0.996
t_3	0.25	400	0.061	0.332	0.856	0.052	0.690	0.999	0.029	0.308	0.821
		800	0.040	0.577	0.993	0.061	0.932	1.000	0.052	0.575	0.980
	0.50	400	0.027	0.304	0.841	0.040	0.674	0.993	0.042	0.316	0.801
		800	0.044	0.562	0.983	0.045	0.930	1.000	0.057	0.578	0.995
	0.75	400	0.035	0.309	0.844	0.038	0.760	0.999	0.038	0.341	0.795
		800	0.050	0.582	0.992	0.048	0.945	1.000	0.046	0.557	0.994

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