SUPPLEMENTAL MATERIAL FOR: REAL-TIME MONITORING WITH RCA MODELS

LAJOS HORVÁTH AND LORENZO TRAPANI

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TABLE A.1. Empirical rejection frequencies under the null of no changepoint with covariates - Case $\beta_0 = 1$

			Weighted CUSUM			Standardized CUSUM		Weighted Page-CUSUM		
		ψ	0	0.25	0.45	0.5		0	0.25	0.45
						$c_{\alpha,0.5}$	$\widehat{c}_{\alpha,0.5}$			
m	m^*									
50	25		0.084	0.083	0.061	0.039	0.066	0.078	0.074	0.066
	50		0.103	0.107	0.095	0.057	0.083	0.103	0.092	0.084
	100		0.126	0.124	0.110	0.071	0.093	0.124	0.119	0.101
	200		0.139	0.132	0.121	0.074	0.106	0.136	0.143	0.125
100	50		0.083	0.077	0.063	0.036	0.056	0.075	0.070	0.060
	100		0.098	0.091	0.078	0.047	0.074	0.097	0.094	0.080
	200		0.103	0.102	0.090	0.061	0.072	0.103	0.111	0.092
	400		0.112	0.120	0.106	0.056	0.084	0.112	0.118	0.107
200	100		0.068	0.050	0.043	0.020	0.032	0.065	0.062	0.047
	200		0.098	0.087	0.082	0.041	0.070	0.092	0.098	0.092
	400		0.095	0.096	0.078	0.044	0.068	0.094	0.098	0.083
	800		0.109	0.114	0.091	0.056	0.073	0.103	0.106	0.100

A. FURTHER MONTE CARLO EVIDENCE AND GUIDELINES

A.1. **Further Monte Carlo evidence - the case of covariates.** We report empirical rejection frequencies in the presence of covariates in (5.1) in the STUR case; in the Extended Version,[1] we also report the stationary and explosive cases.

[1]Available at `https://drive.google.com/file/d/1KP1p7gomMKEt0bu_K2zoOvKOufnJpSWI/view`

Table A.2. Median delays and empirical rejection frequencies under alternatives - DGP with covariates

DGP		ψ	Weighted CUSUM 0	Weighted CUSUM 0.25	Weighted CUSUM 0.45	Standardized CUSUM 0.5 $c_{\alpha,0.5}$	Standardized CUSUM 0.5 $\widehat{c}_{\alpha,0.5}$	Weighted Page-CUSUM 0	Weighted Page-CUSUM 0.25	Weighted Page-CUSUM 0.45	OB
Case I (β_0=0.5)	m^*	100	52 (0.704)	45 (0.671)	34 (0.622)	36 (0.473)	31 (0.553)	55 (0.704)	45.5 (0.699)	31 (0.656)	
		200	80 (0.849)	61.5 (0.833)	47 (0.792)	52 (0.662)	46 (0.743)	81 (0.848)	59 (0.846)	43.5 (0.809)	54 (0.807)
		400	108.5 (0.915)	79 (0.911)	60 (0.880)	71.5 (0.759)	58 (0.834)	109 (0.909)	78 (0.912)	57 (0.881)	72.5 (0.765)
		800	135 (0.972)	94 (0.970)	75 (0.942)	94 (0.846)	81 (0.898)	136 (0.967)	95 (0.962)	70 (0.940)	104 (0.878)
Case II (β_0=1.05)	m^*	100	15 (1.000)	13 (1.000)	9 (1.000)	11 (1.000)	9 (1.000)	19 (1.000)	13 (1.000)	9 (1.000)	
		200	24 (1.000)	15 (1.000)	9 (1.000)	11 (1.000)	9 (1.000)	25 (1.000)	15 (1.000)	9 (1.000)	
		400	27 (1.000)	16 (1.000)	10 (1.000)	11 (1.000)	9 (1.000)	28 (1.000)	16 (1.000)	10 (1.000)	
		800	30 (1.000)	17 (1.000)	10 (1.000)	11 (1.000)	9 (1.000)	31 (1.000)	17 (1.000)	9 (1.000)	
Case III (β_0=1)	m^*	100	27 (0.997)	22 (0.996)	18 (0.995)	22 (0.992)	18 (0.994)	29 (0.998)	22 (0.997)	18 (0.997)	
		200	34 (1.000)	25 (1.000)	19 (1.000)	22 (1.000)	18 (1.000)	35 (1.000)	24 (1.000)	18 (1.000)	
		400	39 (1.000)	26 (1.000)	19 (1.000)	22 (1.000)	18 (1.000)	40 (1.000)	26 (1.000)	18 (1.000)	
		800	45 (1.000)	27 (1.000)	18 (1.000)	23 (1.000)	19 (1.000)	46 (1.000)	29 (1.000)	18 (1.000)	

For each DGP, we report the *mean* detection delay for only the cases where a changepoint is detected (thus leaving out the cases where no changepoint is detected). Numbers in round brackets represent the empirical rejection frequencies. We do not report median delays, unlike in Table 5.4. All medians are around zero.

A.2. **Further Monte Carlo evidence - power and delays in the presence of a smooth break.** In a separate set of experiments, we consider power and delays of our procedure in the presence of a smooth break. In particular, we consider a similar set-up as in (5.4) in the main paper, with the only change being in the deterministic part of the autoregressive parameter, viz.

$$y_i = \left(\beta_i + \epsilon_{i,1}\right) y_{i-1} + \epsilon_{i,2},$$

with

$$\beta_i = \begin{cases} \beta_0 + \Delta \left(i / \left\lfloor \frac{m^*}{4} \right\rfloor \right) I \left(1 \leq i - m \leq \left\lfloor \frac{m^*}{4} \right\rfloor \right) + \Delta I \left(i - m > \left\lfloor \frac{m^*}{4} \right\rfloor \right), & \text{‘fast transition’}, \\ \beta_0 + \Delta \left(i / \left\lfloor \frac{m^*}{2} \right\rfloor \right) I \left(1 \leq i - m \leq \left\lfloor \frac{m^*}{2} \right\rfloor \right) + \Delta I \left(i - m > \left\lfloor \frac{m^*}{2} \right\rfloor \right), & \text{‘medium transition’}, \\ \beta_0 + \Delta I \left(i / \left\lfloor \frac{m^*}{1.25} \right\rfloor \right) \left(1 \leq i - m \leq \left\lfloor \frac{m^*}{1.25} \right\rfloor \right) + \Delta I \left(i - m > \left\lfloor \frac{m^*}{1.25} \right\rfloor \right), & \text{‘slow transition’}. \end{cases}$$

We report results only for the case of no covariates; the case where $\mathbf{x}_i$ is present, from preliminary simulations, exhibits exactly the same patterns. As in the previous section, we report results only for the STUR case; results broadly exhibit the same patterns as in the case of an abrupt break, although tests based on standardization (i.e., using $\psi = 1/2$) now appear to perform slightly worse than tests using $\psi < 1/2$. Results for the stationary and explosive cases are reported in the Extended Version of the Supplement.[2]

TABLE A.3. Median delays and empirical rejection frequencies under a smooth change - STUR case

Case III (β_0=1)			ψ	Weighted CUSUM 0	Weighted CUSUM 0.25	Weighted CUSUM 0.45	Standardized CUSUM 0.5 $c_{\alpha,0.5}$	Standardized CUSUM 0.5 $\widehat{c}_{\alpha,0.5}$	Weighted Page-CUSUM 0	Weighted Page-CUSUM 0.25	Weighted Page-CUSUM 0.45
fast transition	m^*	100		42 (0.992)	40 (0.990)	40 (0.984)	46 (0.966)	42 (0.978)	43 (0.996)	38 (0.996)	38 (0.992)
		200		64 (1.000)	60 (1.000)	57 (1.000)	65 (0.996)	61 (1.000)	62 (1.000)	55 (1.000)	53.5 (1.000)
		400		102 (1.000)	94 (1.000)	95 (1.000)	107 (1.000)	99 (1.000)	99 (1.000)	90 (1.000)	89 (1.000)
		800		165 (1.000)	151 (1.000)	157 (1.000)	179 (1.000)	170 (1.000)	161 (1.000)	148 (1.000)	150 (1.000)
medium transition	m^*	100		56 (0.968)	55 (0.962)	56 (0.940)	63 (0.876)	60 (0.914)	56 (0.984)	53 (0.982)	55 (0.972)
		200		93 (0.998)	89 (1.000)	91 (1.000)	102 (0.990)	97 (0.994)	90 (1.000)	83.5 (1.000)	86 (1.000)
		400		154 (1.000)	147 (1.000)	154 (1.000)	177 (1.000)	161 (1.000)	149 (1.000)	141 (1.000)	145 (1.000)
		800		263 (1.000)	251 (1.000)	270 (1.000)	314 (1.000)	295 (1.000)	251.5 (1.000)	243 (1.000)	257 (1.000)
slow transition	m^*	100		71 (0.882)	70 (0.860)	74 (0.786)	79 (0.624)	76 (0.712)	70 (0.932)	69 (0.924)	72 (0.874)
		200		122 (0.992)	120 (0.990)	126 (0.970)	140 (0.894)	133 (0.946)	118 (0.996)	112 (0.996)	118 (0.992)
		400		212 (0.998)	208.5 (0.998)	222 (0.998)	258 (0.984)	239 (0.996)	202 (1.000)	194 (1.000)	208 (1.000)
		800		369 (1.000)	357 (1.000)	397 ()	477 (0.998)	443.5 (0.998)	350.5 (1.000)	346 (1.000)	381 ()

We report the median detection delay for only the cases where a changepoint is detected (thus leaving out the cases where no changepoint is detected). Numbers in round brackets represent the empirical rejection frequencies.

[2] https://drive.google.com/file/d/1KP1p7gomMKEtObu_K2zoOvKOufnJpSWI/view

A.3. **Further Monte Carlo evidence - power and delays in the presence of breaks of variable magnitude.** We consider - mainly in order to better gauge how the power of our procedure works - a further set of simulations for the STUR case, where we replicate the same exercise (with the same design) but with various sizes of the break Δ. In the Extended Version of the Supplement,[3] we also report results for the explosive case.

TABLE A.4. Median delays and empirical rejection frequencies for different break magnitudes - STUR case

Case III (β_0=1.0)			Weighted CUSUM			Standardized CUSUM		Weighted Page-CUSUM		
		ψ	0	0.25	0.45	0.5		0	0.25	0.45
						$c_{\alpha,0.5}$	$\widehat{c}_{\alpha,0.5}$			
$\Delta = 0.02$	m^*	100	65 (0.170)	52 (0.156)	37 (0.120)	47 (0.070)	38,5 (0.098)	60 (0.164)	50 (0.152)	38 (0.126)
		200	107 (0.298)	92.5 (0.278)	70 (0.240)	79 (0.140)	78 (0.190)	105.5 (0.290)	90 (0.306)	63 (0.252)
		400	189 (0.378)	160 (0.368)	119 (0.288)	134 (0.160)	124.5 (0.238)	186 (0.376)	150 (0.370)	121 (0.294)
		800	170 (0.416)	220 (0.428)	160.5 (0.314)	250 (0.192)	181 (0.236)	164 (0.404)	232 (0.396)	149 (0.306)
$\Delta = 0.04$	m^*	100	62 (0.580)	51 (0.540)	45 (0.440)	48 (0.260)	45 (0.372)	63 (0.570)	55.5 (0.554)	45.5 (0.456)
		200	94 (0.812)	81 (0.780)	71 (0.684)	83.5 (0.520)	71 (0.602)	97 (0.814)	76 (0.806)	64 (0.710)
		400	135 (0.934)	108 (0.914)	100.5 (0.852)	120.5 (0.658)	102 (0.770)	136 (0.924)	107 (0.914)	92 (0.852)
		800	157 (0.978)	114 (0.978)	105.5 (0.926)	163.5 (0.778)	131 (0.858)	159 (0.974)	123 (0.966)	107 (0.924)
$\Delta = 0.06$	m^*	100	48 (0.926)	42 (0.912)	39 (0.852)	46 (0.702)	40 (0.776)	50 (0.938)	44 (0.932)	39 (0.864)
		200	64 (0.990)	52 (0.980)	44 (0.976)	60 (0.922)	49 (0.966)	65 (0.992)	49 (0.992)	43 (0.988)
		400	78 (1.000)	57 (1.000)	48 (1.000)	67 (0.990)	53 (1.000)	80 (1.000)	57 (1.000)	45 (1.000)
		800	85 (1.000)	58 (1.000)	50 (1.000)	67 (0.998)	56 (1.000)	88 (1.000)	60 (1.000)	48.5 (1.000)
$\Delta = 0.08$	m^*	100	35.5 (0.990)	29 (0.982)	27 (0.976)	34 (0.938)	28 (0.960)	37 (0.992)	30 (0.992)	27 (0.986)
		200	44 (1.000)	33 (1.000)	27 (1.000)	33 (0.998)	28 (1.000)	46 (1.000)	32 (1.000)	25 (1.000)
		400	53 (1.000)	36 (1.000)	29 (1.000)	36 (1.000)	29 (1.000)	55 (1.000)	36 (1.000)	27 (1.000)
		800	58 (1.000)	38 (1.000)	29 (1.000)	37 (1.000)	31 (1.000)	60 (1.000)	40 (1.000)	29 (1.000)

We report the median detection delay for only the cases where a changepoint is detected (thus leaving out the cases where no changepoint is detected). Numbers in round brackets represent the empirical rejection frequencies.

[3] https://drive.google.com/file/d/1KP1p7gomMKEtObu_K2zoOvKOufnJpSWI/view

B. Further empirical evidence and complements to Section 6

B.1. **Complements to Section 6.** We complement the analysis and the findings in Section 6 in the main paper by briefly describing: (1) the test for the null of stationarity by Trapani (2021) which we use in Section 6.2; and (2) the 'linearity test' for the null that $Var\left(\epsilon_{i,1}\right)=0$ by Horváth and Trapani (2019).

The test by Trapani (2021) is based on computing

$$D_N=\frac{1}{N-p}\sum_{i=p+1}^{N}\frac{v_p}{v_p+y_i^2}, \tag{B.1}$$

where $v_p=p^{-1}\sum_{i=1}^{p}y_i^2$. Within this framework, the null and the alternative hypotheses are

$$\begin{cases} H_0: & y_i \text{ is strictly stationary} \\ H_A: & y_i \text{ is nonstationary} \end{cases}. \tag{B.2}$$

In order to use D_N, Trapani (2021) proposes a randomization thereof, based on defining

$$l_N=\exp\left(\exp\left(\left(\ln^2 N\right)\times D_N\right)-1\right),$$

and randomising l_N using an artificial sample of size R, and obtaining a statistic, $\Theta_{R,N}$, such that, as $\min\left(R,N\right)\to\infty$

$$\begin{array}{lll} \Theta_{R,N}\overset{D^*}{\to} & \chi_1^2 & \text{under } H_0, \\ R^{-1}\Theta_{R,N}\overset{P^*}{\to} & c_0>0 & \text{under } H_A, \end{array}$$

where P^* denotes the conditional probability with respect of the sample, and '$\overset{D^*}{\to}$' and '$\overset{P^*}{\to}$' denote conditional convergence in distribution and in probability according to P^*. In order to wash out dependence on the randomness, Trapani (2021) proposes running the test for $1\leq b\leq B$ iterations, each time defining a test statistic $\Theta_{R,N}^{(b)}$, and computing the *randomized confidence function*

$$Q_{R,N,B}(\alpha) = \frac{1}{B}\sum_{b=1}^{B} I\left[\Theta_{R,N}^{(b)} \leq c_\alpha\right], \tag{B.3}$$

where c_α is defined as $P\{\chi_1^2 \geq c_\alpha\} = \alpha$, for a given nominal level $\alpha \in (0,1)$. Hence, the decision rule in favor of H_0 is

$$Q_{R,N,B}(\alpha) \geq (1-\alpha) - \sqrt{\alpha(1-\alpha)}\sqrt{\frac{2\ln\ln B}{B}}. \tag{B.4}$$

As far as implementation is concerned, following the suggestions in Trapani (2021), we have set $p = \lfloor \ln N \rfloor$ and $R = B = N$, where N represents the sample size of the whole datasets. By way of robustness analysis, we have experimented with different values of R and B also, but results are unchanged. We note that the test by Trapani (2021) does not explicitly consider the possibility of changes in $Var(\epsilon_{i,1})$, but all the results in that paper can be shown to hold even in the case of shifts in the variance of $Var(\epsilon_{i,1})$ - essentially, this is due to the fact that the test checks whether the observations y_i explode or not, although, in the presence of heteroskedasticity, the null hypothesis can no longer be interpreted as 'strict stationarity', but merely as 'non-explosiveness'. We apply the test to the covariates used in the exercise in Section 6.2 in the main paper in Section B.3 hereafter.

The test by Horváth and Trapani (2019) follows a similar logic, and it is based on testing for

$$\begin{cases} H_0 : \text{Var}(\epsilon_{0,1}) = 0 \\ H_A : \text{Var}(\epsilon_{0,1}) > 0 \end{cases}. \tag{B.5}$$

In turn, this is based on estimating $Var(\epsilon_{i,1})$ using WLS, and subsequently use a similar randomization as above, thereafter constructing the randomized confidence function exactly as in (B.3). As Table B.1 shows, both Covid-19 hospitalizations and housing prices follow an RCA model, as opposed to a simple, linear AR model.

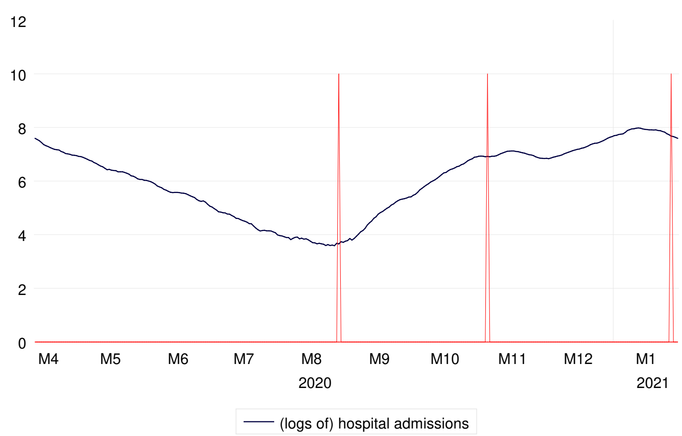
Table B.1. Tests for random autoregressive coefficients

Variable	Period	Sample size	test	NOTES
Covid-19 hospitalizations	Apr 11th, 2020 - Aug 15th, 2020	127	0.000 [reject;]	$\sqrt{Var(\epsilon_{0,1})} = 7.99 \times 10^{-3}$
	Aug 29th, 2020 - Oct 29th, 2020	62	0.000 [reject;]	$\sqrt{Var(\epsilon_{0,1})} = 2.71 \times 10^{-2}$
	Nov 5th, 2020 - Dec 31st, 2020	57	0.000 [reject;]	$\sqrt{Var(\epsilon_{0,1})} = 3.64 \times 10^{-2}$
House prices	Aug 20th, 2008 - Jan 14th, 2009	100	0.000 [reject;]	$\sqrt{Var(\epsilon_{0,1})} = 4.19 \times 10^{-2}$
	Mar 28th, 2008 - Jan 14th, 2009	200	0.000 [reject;]	$\sqrt{Var(\epsilon_{0,1})} = 3.99 \times 10^{-2}$

The main output in each entry of the table is the value taken by the randomized confidence function $Q_{R,N,B}(\alpha)$, testing for the null hypothesis of no randomness; the number in square brackets represent the threshold against which $Q_{R,N,B}(\alpha)$ is compared.

B.2. **Further empirical evidence on UK Covid-19 hospitalization data .** We report a graph of the logs of (one plus) the daily hospitalization data with the identified changepoints in Figure B.1.

Figure B.1. Daily Covid-19 hospitalizations - with changepoints - for England.



We have conducted, by way of comparison, an ex-post changepoint detection exercise, applying the techniques developed in Horváth and Trapani (2023) to the whole sample. We only report results obtained using Rényi statistics (corresponding to using a weighted version of the CUSUM process with weights $\kappa = 0.51$, 0.55, 0.65, 0.75, 0.85 and 1 - see Horváth and Trapani, 2023 for details).[4] As far as breakdates are concerned, we pick the ones corresponding to the 'majority vote' across κ, although discrepancies are, when present, in the region of few days ($2-5$ at most). We use binary segmentation, as also discussed in Horváth and Trapani (2023), to detect multiple breaks.

TABLE B.2. Ex-post changepoint detection for Covid-19 daily hospitalization - England data.

Changepoint 1	Changepoint 2	Changepoint 3	Changepoint 4
Apr 10th, 2020 [1.009]	Aug 26th, 2020 [0.994]	Oct 29th, 2020 [1.010]	Jan 12th, 2021 [1.002]

The series ends at 30 January 2021. We use the logs of the original data (plus one, given that, in some days, hospitalizations are equal to zero): no further transformations are used.
All changepoints have been detected by all Rényi-type tests - no discrepancies were noted. Detected changepoints, and their estimated date, are presented in *chronological* order; breakdates have been estimated as the points in time where the majority of tests identifies a changepoint. Whilst details are available upon request, we note that breaks were detected with this order (from the first to be detected to the last one): break in August; break in April; break in January 2021; break in October.
For each changepoint, we report in square brackets, for reference, the left WLS estimates of β_0 - i.e., the value of β_0 *prior* to the breakdate.

Finally, in Table B.3 we report results using the test by Otto and Breitung (2023), as discussed in Section 5 in the main paper. Results are indeed quite similar, but note the (sometimes small, sometimes larger) increase in the detection delay.

B.3. **Further empirical evidence on housing data.** We report some preliminary information on our data. In Table B.4, we report the outcome of a standard unit root test on the three covariates used in our exercise.

[4]Using other weighing schemes give very similar results, available upon request.

TABLE B.3. Online changepoint detection for Covid-19 daily hospitalization - England data; using Otto and Breitung (2023) test.

Changepoint 1	Changepoint 2	Changepoint 3
Sep 8th, 2020	Nov 14th, 2020	Jan 29th, 2021
$\widehat{\beta} = 0.995$ [Apr 11th, 2020 - Aug 15th, 2020]	$\widehat{\beta} = 1.010$ [Sep 9th, 2020 - Oct 29th, 2020]	$\widehat{\beta} = 1.002$ [Nov 15th, 2020 - Dec 31st, 2020]

The series ends at 30 January 2021. We use the logs of the original data (plus one, given that, in some days, hospitalizations are equal to zero): no further transformations are used. For each changepoint, we report the sample on which estimation was performed in square brackets.

TABLE B.4. Unit root tests applied to covariates

Variable	Period	t-ADF	Trapani's (2019) test		Notes
			levels	first differences	
AAA	Mar 28th, 2008 - Oct 30th, 2009	-1.387 [0.863]	0.000 [reject; 0.9218]	0.942 [not reject; 0.9218]	Daily frequency; trend and intercept used in ADF
GS10	Mar 28th, 2008 - Oct 30th, 2009	-1.463 [0.840]	0.000 [reject; 0.9218]	0.940 [not reject; 0.9218]	Daily frequency; trend and intercept used in ADF
VXO	Mar 28th, 2008 - Oct 30th, 2009	-2.682 [0.244]	0.000 [reject; 0.9218]	0.965 [not reject; 0.9218]	Daily frequency; trend and intercept used in ADF
WEI	Jan 5th, 2008 - Dec 26th, 2009	3.070 [1.000]	0.000 [reject; 0.9123]	0.922 [not reject; 0.9123]	Weekly frequency; trend and intercept used in ADF

For each series, we have carried out a standard ADF test, choosing the number of lags in the Dickey-Fuller regression based on BIC. The numbers in square brackets are the p-values.
The test by Trapani (2021) for the null hypothesis of strict stationarity is applied to data in levels and in first differences. In both cases, the main output is the value taken by the randomized confidence function $Q_{R,N,B}(\alpha)$; the number in square brackets represent the threshold against which $Q_{R,N,B}(\alpha)$ is compared.

In Figure B.2, we plot housing prices in Los Angeles between March 28th, 2008, and October 30th, 2009.

In Table B.5, we report the findings when using the critical values $\widehat{c}_{\alpha,0.5}$ instead of $c_{\alpha,0.5}$.

Figure B.2. Logs of daily housing prices in Los Angeles - with estimated changepoint.

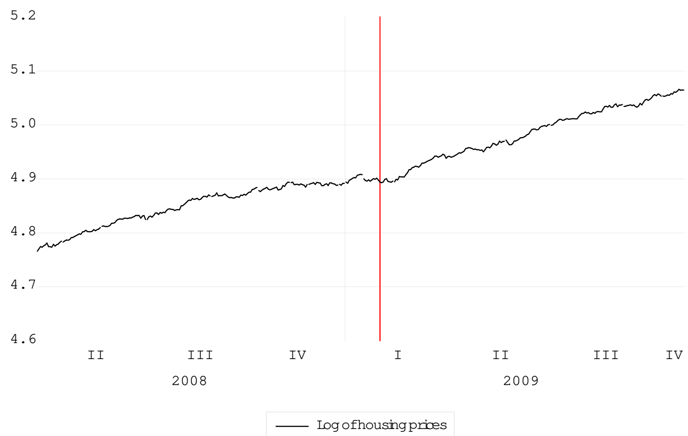


TABLE B.5. Online changepoint detection for Los Angeles daily housing prices.

Model: $y_i = (\beta_i + \epsilon_{i,1}) y_{i-1} + \epsilon_{i,2}$

m \ m^*	100	200
100	Jun 5th, 2009 [no changepoint found]	Jun 8th, 2009
200	Jun 5th, 2009 [no changepoint found]	Jun 8th, 2009

Model: $y_i = (\beta_i + \epsilon_{i,1}) y_{i-1} + \lambda_1 x_{1,i} + \lambda_2 x_{2,i} + \epsilon_{i,2}$

m \ m^*	100	200
100	Jun 2nd, 2009	Jun 2nd, 2009
200	May 29th, 2009	May 29th, 2009

Model: $y_i = (\beta_i + \epsilon_{i,1}) y_{i-1} + \lambda_1 x_{1,i} + \lambda_2 x_{2,i} + \lambda_3 x_{3,i} + \epsilon_{i,2}$

m \ m^*	100	200
100	Jun 2nd, 2009	Jun 2nd, 2009
200	May 29th, 2009	May 29th, 2009

Model: $y_i = (\beta_i + \epsilon_{i,1}) y_{i-1} + \lambda_1 x_{1,i} + \lambda_2 x_{2,i} + \lambda_3 x_{3,i} + \lambda_4 x_{4,i} + \epsilon_{i,2}$

m \ m^*	100	200
100	May 18th, 2009	May 18th, 2009
200	May 28th, 2009	May 28th, 2009

For each combination of m and m^*, we report the estimated breakdate. For all combinations of m and m^*, monitoring starts on January 15th, 2009. When $m = 100$, the training sample covers the period August 20th, 2008, till January 14th, 2009; when $m = 200$, the training sample covers the period March 28th, 2008, till January 14th, 2009. Similarly, when $m^* = 100$, the monitoring horizon stops at June 9th, 2009; when $m^* = 200$, the monitoring horizon stops at October 30th, 2009.

We have used the following notation for the regressors: $x_{1,i}$ denotes the 10 Year US Treasury Constant Maturity Rate, $x_{2,i}$ denotes the Moody's Seasoned Aaa Corporate Bond Yield, $x_{3,i}$ is the VXO volatility index, and $x_{4,i}$ is the WEI.

Horváth and Trapani (2023) find evidence of a changepoint on February 3rd, 2009, applying ex-post changepoint detection to the period January 5th, 1995 to October 23rd, 2012. The deterministic part of the autoregressive coefficient, β, is found to be equal to 0.99931 in the period before the changepoint, and 1.00007 afterwards.

C. Extensions

In the main paper, we present our results for the case where (1.1) does not have any deterministics. This allows for a simple presentation, where the presentation of the main point (i.e., the test statistics) is not overshadowed by the estimation problem.

Even when the model does not have a constant term, we normally use the LS estimators (OLS or WLS) with an intercept term fitted. Because in real unprocessed data, fitting LS estimators without an intercept term tends to induce bias. This is also the standard practice in the unit root literature and bubble testing literature: although the DGP is a random walk without a constant (or with a diminishing one which does not affect the asymptotics), the test statistics are still computed with an intercept term fitted.[5] As far as the RCA model is concerned, of course under nonstationarity the constant term is not identified, so adding it or not does not make a difference for inference (Aue and Horváth, 2011; Horváth and Trapani, 2023).

In this section, we study the extension of our methodology to the case of an RCA model with a constant and (for completeness) covariates, viz.

$$y_i = \mu + (\beta_i + \epsilon_{i,1})\, y_{i-1} + \boldsymbol{\lambda}_0^\intercal \mathbf{x}_i + \epsilon_{i,2}, \tag{C.1}$$

for $1 \le i \le N$, where we base our analysis on the maintained hypothesis that the intercept μ does not change. We focus on the weighted CUSUM process; for the sake of a concise discussion, we omit the analysis of the Page-CUSUM detectors, which can be conducted by adapting the arguments in the main paper and in this section.

As in the main paper, we assume that β_i is constant for an initial training period $1 \le i \le m$, and we test for the null hypothesis that

$$H_0 : \beta_0 = \beta_{m+1} = \beta_{m+2} = \ldots, \tag{C.2}$$

[5]We are grateful to an anonymous Referee for asking the question that led to the results in this section.

versus the same alternative hypothesis as in (3.13) We consider the following alternative, where the deterministic part of the autoregressive coefficient of (2.1) undergoes a change

$$y_i = \begin{cases} \mu + (\beta_0 + \epsilon_{i,1})\, y_{i-1} + \boldsymbol{\lambda}_0^\intercal \mathbf{x}_i + \epsilon_{i,2} & 1 \le i \le m + k^*, \\ \mu + (\beta_A + \epsilon_{i,1})\, y_{i-1} + \boldsymbol{\lambda}_0^\intercal \mathbf{x}_i + \epsilon_{i,2} & i > m + k^*, \end{cases} \tag{C.3}$$

where, as in the main paper, $\beta_0 \neq \beta_A$ and k^* is the time of change.

In this case, the WLS estimator using the data in the training sample is based on

$$\min_{\mu,\beta,\boldsymbol{\lambda}} \sum_{i=2}^{m} \frac{\left(y_i - (\mu + \beta y_{i-1} + \boldsymbol{\lambda}^\intercal \mathbf{x}_i)\right)^2}{1 + y_{i-1}^2}. \tag{C.4}$$

Letting, as before, $\mathbf{b} = (\beta, \boldsymbol{\lambda}^\intercal)^\intercal$, and defining the $(p+1) \times 1$ vector $\mathbf{w}_i = (y_{i-1}, \mathbf{x}_i^\intercal)^\intercal$, the solutions to (C.4) are

$$\begin{aligned} \widetilde{\mathbf{b}}_m &= \left[\sum_{i=2}^{m} \frac{\mathbf{w}_i \mathbf{w}_i^\intercal}{1 + y_{i-1}^2} - \left(\sum_{i=2}^{m} \frac{1}{1 + y_{i-1}^2} \right)^{-1} \left(\sum_{i=2}^{m} \frac{\mathbf{w}_i}{1 + y_{i-1}^2} \right) \left(\sum_{i=2}^{m} \frac{\mathbf{w}_i^\intercal}{1 + y_{i-1}^2} \right) \right]^{-1} \\ &\times \left[\sum_{i=2}^{m} \frac{\mathbf{w}_i y_i}{1 + y_{i-1}^2} - \left(\sum_{i=2}^{m} \frac{1}{1 + y_{i-1}^2} \right)^{-1} \left(\sum_{i=2}^{m} \frac{y_i}{1 + y_{i-1}^2} \right) \left(\sum_{i=2}^{m} \frac{\mathbf{w}_i}{1 + y_{i-1}^2} \right) \right], \end{aligned} \tag{C.5}$$

and

$$\widetilde{\mu}_m = \left[\sum_{i=2}^{m} \frac{1}{1 + y_{i-1}^2} \right]^{-1} \left[\sum_{i=2}^{m} \frac{y_i}{1 + y_{i-1}^2} - \widetilde{\mathbf{b}}_m^\intercal \sum_{i=2}^{m} \frac{\mathbf{w}_i}{1 + y_{i-1}^2} \right]. \tag{C.6}$$

Hence, our monitoring schemes can be based on the partial sums of the weighted residuals, using (similarly to (4.3)) the detector

$$\widetilde{Z}_m^X(k) = \left| \sum_{i=m+1}^{m+k} \frac{\left[y_i - \left(\widetilde{\mu}_m + \widetilde{\mathbf{b}}_m^\intercal \mathbf{w}_i \right) \right] y_{i-1}}{1 + y_{i-1}^2} \right|, \quad k \ge 1, \tag{C.7}$$

and the estimate

$$\widetilde{\mathfrak{s}}_m^2 = \frac{1}{m}\sum_{i=2}^{m} z_{\mu,i}^2,$$

with

$$z_{\mu,i} = \frac{\left(y_i - \left(\widetilde{\mu}_m + \widetilde{\beta}_m y_{i-1} + \widetilde{\boldsymbol{\lambda}}_m^{\intercal}\mathbf{x}_i\right)\right)}{1+y_{i-1}^2}\left(y_{i-1} - \frac{\sum_{i=2}^{m}\frac{y_{i-1}}{1+y_{i-1}^2}}{\sum_{i=2}^{m}\frac{1}{1+y_{i-1}^2}}\right).$$

We begin by noting that Horváth and Trapani (2023) show that, in the case with no covariates in (C.1), that

$$\widetilde{\mathfrak{s}}_m^2 = \mathfrak{s}^2 + O_P\left(m^{-\zeta}\right),$$

for some $\zeta > 0$. The same result can be shown by adapting their arguments, which we avoid for the sake of a concise discussion.

Hence, we define the stopping rules for the weighted CUSUM detectors as

$$\text{(C.8)} \qquad \widetilde{\tau}_{m,\psi}^{(x)} = \begin{cases} \inf\{k \geq 1 : \widetilde{Z}_m^X(k) \geq g_{m,\psi}^{(x)}(k)\}, \\ \infty, \text{ if } \widetilde{Z}_m^X(k) < g_{m,\psi}^{(x)}(k) \text{ for all } 1 \leq k < \infty, \end{cases}$$

and

$$\text{(C.9)} \qquad \widetilde{\tau}_{m,\psi}^{*(x)} = \begin{cases} \inf\{k \geq 1 : \widetilde{Z}_m^X(k) \geq g_{m,\psi}^{(x)}(k)\}, \\ m^*, \text{ if } \widetilde{Z}_m^X(k) < g_{m,\psi}^{(x)}(k) \text{ for all } 1 \leq k \leq m^*, \end{cases}$$

for an open-ended and a closed-ended monitoring procedure respectively, where $g_{m,\psi}^{(x)}(k)$ is defined in (4.5), using $\widetilde{\mathfrak{s}}_m^2$ instead of $\widehat{\mathfrak{s}}_m^2$.

Let $\overline{\mathbf{w}}_0 = \left(\overline{y}_{-1}, \mathbf{x}_0^{\intercal}\right)$. We modify Assumption 3.1 to ensure that the denominator of $\widetilde{\beta}_m$ is nonzero with probability 1 (see also Assumption 2 in Horváth and Trapani, 2019).

Assumption C.1. *If* $-\infty \leq E\log|\beta_0 + \epsilon_{0,1}| < 0$, *then it holds that* $P(\overline{\mathbf{w}}_0 = \mathbf{c}) < 1$ *for all vectors* $\mathbf{c}$.

Theorem C.1. *We assume that Assumptions of Theorem 3.1 are satisfied, with Assumption 3.1 replaced by Assumption C.1. Then, the same results as in Theorems 3.1-3.4, C.5, 3.7 and 3.8-3.9 hold for (C.8) and (C.9).*

D. Preliminary lemmas

We will use the following facts and notation: '$\overset{\mathcal{D}}{\to}$' denotes convergence in distribution; '$\overset{\mathcal{D}}{=}$' denotes equality in distribution; c_1, c_2, ... denote positive, finite constants which do not depend on sample sizes, and whose values may change from line to line.

Recall (2.1) under H_0

$$y_i = (\beta_0 + \epsilon_{i,1})\, y_{i-1} + \epsilon_{i,2}.$$

If $E \log |\beta_0 + \epsilon_{0,1}| < 0$, define the stationary solution

$$\overline{y}_i = \sum_{\ell=0}^{\infty} \left(\prod_{j=1}^{\ell} (\beta_0 + \epsilon_{i-j,1}) \right) \epsilon_{i-\ell,2}, \tag{C.1}$$

with the convention that $\prod_{\emptyset} = 0$. Finally, let

$$a_3 = E\left(\frac{\overline{y}_0^2}{1 + \overline{y}_0^2} \right). \tag{C.2}$$

Lemma C.1. *We assume that Assumption 2.1 is satisfied and that $E \log |\beta_0 + \epsilon_{0,1}| < 0$. Under H_0, it holds that there exist a $\kappa > 0$ and a $0 < c < 1$ such that*

$$E\, |y_i - \overline{y}_i|^{\kappa} = O\left(c^i\right),$$

as $i \to \infty$.

Proof. The proof is similar to the proof of Lemma A.2 in Horváth and Trapani (2016). Consider the expression

$$\overline{y}_i - y_i = y_0 \prod_{s=1}^{i} (\beta_0 + \epsilon_{s,1}) - \sum_{s=-\infty}^{0} \epsilon_{s,2} \prod_{z=s}^{i-1} (\beta_0 + \epsilon_{z+1,1}),$$

and note that, choosing $\kappa < 1$, Minkowski's inequality yields

$$E\, |\overline{y}_i - y_i|^{\kappa} \;\leq\; E\, |y_0|^{\kappa} \prod_{s=1}^{i} |\beta_0 + \epsilon_{s,1}|^{\kappa} + E \left| \sum_{s=-\infty}^{0} \epsilon_{s,2} \prod_{z=s}^{i-1} (\beta_0 + \epsilon_{z+1,1}) \right|^{\kappa}$$

$$\leq \quad E\left|y_0\right|^\kappa \prod_{s=1}^{i} E\left|\beta_0+\epsilon_{s,1}\right|^\kappa + \sum_{s=-\infty}^{0} E\left|\epsilon_{s,2}\right|^\kappa \prod_{z=s}^{i-1} E\left|\beta_0+\epsilon_{z+1,1}\right|^\kappa .$$

Aue et al. (2006) show (see the proof of their Lemma 2) that $E\left|\beta_0+\epsilon_{0,1}\right|^\kappa = c < 1$; hence the expression above becomes

$$E\left|\overline{y}_i - y_i\right|^\kappa \leq E\left|y_0\right|^\kappa c^i + E\left|\epsilon_{0,2}\right|^\kappa \sum_{s=-\infty}^{0} c^{i-s} \leq c_0 c^i,$$

whence the desired result. □

Lemma C.2. *We assume that Assumption 2.1 is satisfied. Under H_0, it holds that*
(i) if $E\log\left|\beta_0+\epsilon_{0,1}\right| < 0$, then

$$\text{(C.3)} \qquad \left|\left(\sum_{i=2}^{m}\frac{y_{i-1}^2}{1+y_{i-1}^2}\right)^{-1}\left(\sum_{i=2}^{m}\frac{\epsilon_{i,1}y_{i-1}^2}{1+y_{i-1}^2}\right) - \frac{1}{ma_3}\left(\sum_{i=2}^{m}\frac{\epsilon_{i,1}\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2}\right)\right| = O_P\left(\frac{1}{m}\right),$$

$$\text{(C.4)} \qquad \left|\left(\sum_{i=2}^{m}\frac{y_{i-1}^2}{1+y_{i-1}^2}\right)^{-1}\left(\sum_{i=2}^{m}\frac{\epsilon_{i,2}y_{i-1}}{1+y_{i-1}^2}\right) - \frac{1}{ma_3}\left(\sum_{i=2}^{m}\frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2}\right)\right| = O_P\left(\frac{1}{m}\right);$$

(ii) if either $E\log\left|\beta_0+\epsilon_{0,1}\right| = 0$ and Assumption 3.2 holds, or $E\log\left|\beta_0+\epsilon_{0,1}\right| > 0$ and Assumption 3.3 holds, then

$$\text{(C.5)} \qquad \left|\left(\sum_{i=2}^{m}\frac{y_{i-1}^2}{1+y_{i-1}^2}\right)^{-1}\left(\sum_{i=2}^{m}\frac{\epsilon_{i,1}y_{i-1}^2}{1+y_{i-1}^2}\right) - \frac{1}{m}\sum_{i=2}^{m}\epsilon_{i,1}\right| = O_P\left(m^{-1/2-\zeta}\right),$$

$$\text{(C.6)} \qquad \left|\left(\sum_{i=2}^{m}\frac{y_{i-1}^2}{1+y_{i-1}^2}\right)^{-1}\left(\sum_{i=2}^{m}\frac{\epsilon_{i,2}y_{i-1}}{1+y_{i-1}^2}\right)\right| = O_P\left(m^{-1/2-\zeta}\right),$$

for some $\zeta > 0$.

Proof. We begin by showing (C.3)-(C.4). By Lemma C.1 and elementary algebra

$$\sum_{i=2}^{\infty}\left|\epsilon_{i,1}\right|\left|\frac{y_{i-1}^2}{1+y_{i-1}^2} - \frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2}\right| \leq \sum_{i=2}^{\infty}\left|\epsilon_{i,1}\right|\left|y_{i-1}^2 - \overline{y}_{i-1}^2\right| \leq \sum_{i=2}^{\infty}\left|\epsilon_{i,1}\right|\left|y_{i-1} - \overline{y}_{i-1}\right|\left(\left|y_{i-1}\right| + \left|\overline{y}_{i-1}\right|\right).$$

Since we can assume that $\kappa < 1$, it follows that

$$
\begin{aligned}
&E\left(\sum_{i=2}^{\infty}|\epsilon_{i,1}|\left|\frac{y_{i-1}^2}{1+y_{i-1}^2}-\frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2}\right|\right)^{\kappa/2} \\
\leq & \sum_{i=2}^{\infty}\left(E|\epsilon_{i,1}|^{\kappa/2}\right)E\left(\left|y_{i-1}-\overline{y}_{i-1}\right|^{\kappa/2}\left(|y_{i-1}|+\left|\overline{y}_{i-1}\right|\right)^{\kappa/2}\right) \\
\leq & \sum_{i=2}^{\infty}\left(E|\epsilon_{i,1}|^{\kappa/2}\right)\left(E\left|y_{i-1}-\overline{y}_{i-1}\right|^{\kappa}\right)^{1/2}\left(E|y_{i-1}|^{\kappa}+E\left|\overline{y}_{i-1}\right|^{\kappa}\right)^{1/2} \\
< & \infty,
\end{aligned}
$$

having used Assumption 2.1. Hence by Markov's inequality it follows that

$$
\sum_{i=2}^{\infty}|\epsilon_{i,1}|\left|\frac{y_{i-1}^2}{1+y_{i-1}^2}-\frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2}\right| = O_P(1), \tag{C.7}
$$

and by the same logic it also follows that

$$
\sum_{i=2}^{\infty}\left|\frac{y_{i-1}^2}{1+y_{i-1}^2}-\frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2}\right| = O_P(1).
$$

By Lemmas D.1-D.4 in Horváth and Trapani (2023), the sequence $\overline{y}_{i-1}^2/\left(1+\overline{y}_{i-1}^2\right)$ is a decomposable Bernoulli shift with all moments; hence, by Proposition 4.1 in Berkes et al. (2011) it follows that

$$
\left|\sum_{i=2}^{m}\frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2}-ma_3\right| = O_P\left(m^{1/2}\right),
$$

where a_3 is defined in (C.2), and similarly

$$
\left|\sum_{i=2}^{m}\frac{\epsilon_{i,1}\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2}\right| = O_P\left(m^{1/2}\right).
$$

Equation (C.3) now follows; (C.4) follows also from exactly the same logic.

We now turn to showing (C.5) and (C.6). When $E\log|\beta_0+\epsilon_{0,1}|=0$, Lemma A.4 in Horváth and Trapani (2016) implies that

$$P\left\{|y_i|\leq i^{\overline{\kappa}}\right\}\leq ci^{-\overline{\kappa}}; \tag{C.8}$$

when $E\log|\beta_0+\epsilon_{0,1}|>0$, Berkes et al. (2009) show that $|y_i|\to\infty$ a.s. exponentially fast, which implies (C.8). Hence by (C.8) it follows that

$$\sum_{i=2}^{m}E\left|\frac{y_{i-1}^2}{1+y_{i-1}^2}-1\right|=\sum_{i=2}^{m}E\left|\frac{1}{1+y_{i-1}^2}\right|I\left(|y_i|\leq i^{\overline{\kappa}}\right)+\sum_{i=2}^{m}E\left|\frac{1}{1+y_{i-1}^2}\right|I\left(|y_i|>i^{\overline{\kappa}}\right)=O\left(m^{1-\overline{\kappa}}\right).$$

By the independence between $\epsilon_{i,1}$ and y_{i-1} and by Assumption 2.1, it follows that

$$E\left(\sum_{i=2}^{m}\epsilon_{i,1}\left(\frac{y_{i-1}^2}{1+y_{i-1}^2}-1\right)\right)^2=\sum_{i=2}^{m}E\epsilon_{i,1}^2E\left(\frac{y_{i-1}^2}{1+y_{i-1}^2}-1\right)^2=O\left(m^{1-\overline{\kappa}}\right),$$

and similarly

$$E\left(\sum_{i=2}^{m}\frac{\epsilon_{i,2}y_{i-1}}{1+y_{i-1}^2}\right)^2=O\left(m^{1-\overline{\kappa}}\right).$$

The proof of (C.5) and (C.6) is now complete. □

Consider now the decomposition

$$Z_m(k)=\left|\left(\beta_0-\widehat{\beta}_m\right)\sum_{i=m+1}^{m+k}\frac{y_{i-1}^2}{1+y_{i-1}^2}+\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,1}y_{i-1}^2}{1+y_{i-1}^2}+\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,2}y_{i-1}}{1+y_{i-1}^2}\right|, \tag{C.9}$$

where $\widehat{\beta}_m$ is the WLS estimator computed using $\{y_1,...,y_m\}$.

In the next lemma, we obtain asymptotic representations for the terms in (C.9).

Lemma C.3. *We assume that Assumption 2.1 is satisfied. Under H_0:*

(i) if $E\log|\beta_0+\epsilon_{0,1}|<0$, then

$$\max_{1\leq k<\infty}k^{-1/2-\eta}\left|\sum_{i=m+1}^{m+k}\frac{y_{i-1}^2}{1+y_{i-1}^2}-ka_3\right|=O_P(1), \tag{C.10}$$

(C.11) $$\max_{1\le k<\infty}\left|\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,1}y_{i-1}^2}{1+y_{i-1}^2}-\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,1}\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2}\right|=O_P\left(1\right),$$

(C.12) $$\max_{1\le k<\infty}\left|\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,2}y_{i-1}}{1+y_{i-1}^2}-\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2}\right|=O_P\left(1\right),$$

for all $\eta>0$*;*

(ii) if either $E\log|\beta_0+\epsilon_{0,1}|=0$ *and Assumption 3.2 holds, or* $E\log|\beta_0+\epsilon_{0,1}|>0$ *and Assumption 3.3 holds, then*

(C.13) $$\max_{1\le k<\infty}k^{-1}\left|\sum_{i=m+1}^{m+k}\frac{y_{i-1}^2}{1+y_{i-1}^2}-k\right|=O_P\left(m^{-\zeta}\right),$$

(C.14) $$\max_{1\le k<\infty}k^{-1/2-\eta}\left|\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,1}y_{i-1}^2}{1+y_{i-1}^2}-\sum_{i=m+1}^{m+k}\epsilon_{i,1}\right|=O_P\left(m^{-\zeta}\right),$$

(C.15) $$\max_{1\le k<\infty}k^{-1/2-\eta}\left|\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,2}y_{i-1}}{1+y_{i-1}^2}\right|=O_P\left(m^{-\zeta}\right),$$

for some $\zeta>0$ *and for all* $\eta>0$*.*

Proof. We begin by considering the stationary case $E\log|\beta_0+\epsilon_{0,1}|<0$. On account of the proof of Lemma C.2, (C.10) follows if we show

$$\max_{1\le k<\infty}k^{-1/2-\eta}\left|\sum_{i=m+1}^{m+k}\frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2}-ka_3\right|=O_P\left(1\right).$$

This follows immediately, since $\overline{y}_{i-1}^2/\left(1+\overline{y}_{i-1}^2\right)$ is a decomposable Bernoulli shift with all moments, and the result is implied by the strong approximation in Aue et al. (2014). Further, (C.11) follows immediately from (C.7); (C.12) can be shown by the same logic. Consider now the case $E\log|\beta_0+\epsilon_{0,1}|\ge 0$. By (C.8)

$$E\sum_{i=m+1}^{\infty}\frac{1}{i}\frac{1}{1+y_{i-1}^2}=O\left(m^{-\widehat{\kappa}}\right),$$

for all $\widehat{\kappa} < \overline{\kappa}$. Hence, Abel's summation formula yields

$$\frac{1}{k}\sum_{i=m+1}^{m+k}\frac{1}{1+y_{i-1}^{2}}=\sum_{i=m+1}^{m+k}\frac{1}{i}\frac{1}{1+y_{i-1}^{2}}-\frac{1}{k}\sum_{i=m+1}^{m+k-1}\left((i+1)-i\right)\left(\sum_{j=m+1}^{i}\frac{1}{j}\frac{1}{1+y_{j-1}^{2}}\right).$$

Given that

$$\frac{1}{k}\sum_{i=m+1}^{m+k-1}E\left(\sum_{j=m+1}^{i}\frac{1}{j}\frac{1}{1+y_{j-1}^{2}}\right)\leq E\sum_{j=m+1}^{\infty}\frac{1}{j}\frac{1}{1+y_{j-1}^{2}}=O\left(m^{-\widehat{\kappa}}\right),$$

(C.13) follows from Markov's inequality. Letting $\mathcal{F}_i$ be the σ-field generated by $\{(\epsilon_{j,1},\epsilon_{j,2}), j\leq i\}$, note that

$$E\left(\frac{\epsilon_{i,1}}{1+y_{i-1}^{2}}\middle|\mathcal{F}_{i-1}\right)=0,$$

and therefore the sequence $\epsilon_{i,1}/\left(1+y_{i-1}^{2}\right)$ is a martingale difference sequence. Also, using the Burkholder's inequality (see e.g. Theorem 2.10 in Hall and Heyde, 2014) and (C.8)

$$\text{(C.16)}\quad E\left|\sum_{i=m+1}^{m+k}E\left[\left(\frac{\epsilon_{i,1}}{1+y_{i-1}^{2}}\right)^{2}\middle|\mathcal{F}_{i-1}\right]\right|^{\nu}$$

$$\leq c_1E\left(\sum_{i=m+1}^{m+k}E\left(\frac{1}{1+y_{i-1}^{2}}\right)^{2}\right)^{\nu/2}\leq c_2k^{\nu/2-1}\sum_{i=m+1}^{m+k}\left(E\left(\frac{1}{\left(1+y_{i-1}^{2}\right)^{\nu}}\right)\right)$$

$$\leq c_2k^{\nu/2-1}\sum_{i=m+1}^{m+k}\left(E\left(\frac{1}{\left(1+y_{i-1}^{2}\right)^{\nu}}I\left(|y_i|\leq i^{\overline{\kappa}}\right)\right)+E\left(\frac{1}{\left(1+y_{i-1}^{2}\right)^{\nu}}I\left(|y_i|>i^{\overline{\kappa}}\right)\right)\right)$$

$$\leq k^{\nu/2-1}\sum_{i=m+1}^{m+k}\left(c_3i^{-\overline{\kappa}}+c_4i^{-2\nu\overline{\kappa}}\right)\leq c_5\frac{k^{\nu/2}}{m^{\overline{\kappa}}}.$$

Similarly, we have

$$\text{(C.17)}\qquad \sum_{i=m+1}^{m+k}E\left|\frac{\epsilon_{i,1}}{1+y_{i-1}^{2}}\right|^{\nu}\leq c_6\sum_{i=m+1}^{m+k}E\left|\frac{1}{1+y_{i-1}^{2}}\right|^{\nu}\leq c_7\frac{k}{m^{\overline{\kappa}}}.$$

Using (C.16) and (C.17), and Rosenthal's maximal inequality for martingale difference sequences (see e.g. Theorem 2.12 in Hall and Heyde, 2014), it follows that

$$E \max_{1\le j\le k} \left| \sum_{i=m+1}^{m+j} \frac{\epsilon_{i,1}}{1+y_{i-1}^2} \right|^{\nu} \le c_8 \frac{k^{\nu/2}}{m^{\overline{\kappa}}}. \tag{C.18}$$

We now show (C.14) by noting that

$$\begin{aligned}
& P\left\{ \max_{1\le k<\infty} k^{-1/2-\eta} \left| \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,1}}{1+y_{i-1}^2} \right| > x \right\} \\
\le & \sum_{\ell=0}^{\infty} P\left\{ \max_{\exp(\ell)\le k\le \exp(\ell+1)} k^{-1/2-\eta} \left| \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,1}}{1+y_{i-1}^2} \right| > x \right\} \\
\le & \sum_{\ell=0}^{\infty} P\left\{ \max_{\exp(\ell)\le k\le \exp(\ell+1)} \left| \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,1}}{1+y_{i-1}^2} \right| > x \exp\left(\ell\left(\frac{1}{2}+\eta\right)\right) \right\} \\
\le & c_9 x^{-\nu} \sum_{\ell=0}^{\infty} \exp\left(-\nu\ell\left(\frac{1}{2}+\eta\right)\right) E \max_{1\le k\le \exp(\ell+1)} \left| \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,1}}{1+y_{i-1}^2} \right|^{\nu} \\
\le & c_{10} x^{-\nu} m^{-\overline{\kappa}} \sum_{\ell=0}^{\infty} \exp\left(-\nu\ell\left(\frac{1}{2}+\eta\right) + \frac{\nu}{2}(\ell+1)\right),
\end{aligned}$$

whence (C.14) follows immediately with $\zeta = \overline{\kappa}$. Equation (C.15) can be shown using the same logic. □

Lemma C.4. *We assume that Assumption 2.1 is satisfied. Under* H_0

(i) if $E \log|\beta_0 + \epsilon_{0,1}| < 0$*, then we can define two independent standard Wiener processes* $\{W_{m,1}(k), 1 \le k \le m\}$ *and* $\{W_{m,2}(k), 1 \le k < \infty\}$*, whose distribution does not depend on* m*, such that*

$$\begin{aligned}
& \max_{1\le k<m} k^{-1/2+\zeta} \left| \sum_{i=2}^{m} \left(\frac{\epsilon_{i,1}\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} + \frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2} \right) - \mathfrak{s}^{1/2} W_{m,1}(k) \right| \\
= & O_P(1),
\end{aligned} \tag{C.19}$$

(C.20)
$$\max_{1\le k<\infty} k^{-1/2+\zeta}\left|\sum_{i=m+1}^{m+k}\left(\frac{\epsilon_{i,1}\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2}+\frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2}\right)-\mathfrak{s}^{1/2}W_{m,2}(k)\right| =O_P(1),$$

for some $\zeta>0$*;*

(ii) if either $E\log|\beta_0+\epsilon_{0,1}|=0$ *and Assumption 3.2 holds, or* $E\log|\beta_0+\epsilon_{0,1}|>0$ *and Assumption 3.3 holds, then*

(C.21)
$$\max_{1\le k\le m} k^{-1/2+\zeta}\left|\sum_{i=1}^{k}\epsilon_{i,1}-\sigma_1 W_{m,1}(k)\right|=O_P(1),$$

(C.22)
$$\max_{1\le k<\infty} k^{-1/2+\zeta}\left|\sum_{i=m+1}^{m+k}\epsilon_{i,1}-\sigma_2 W_{m,2}(k)\right|=O_P(1),$$

for some $\zeta>0$.

Proof. Horváth and Trapani (2023) show that $\left(\epsilon_{i,1}\overline{y}_{i-1}^2+\epsilon_{i,2}\overline{y}_{i-1}\right)/\left(1+\overline{y}_{i-1}^2\right)$ is a decomposable Bernoulli shift under $-\infty\le E\log|\beta_0+\epsilon_{0,1}|<0$. Hence, the strong approximation shown in Aue et al. (2014) immediately yields (C.19) and (C.20). Equations (C.21) and (C.22) follow directly from Komlós et al. (1975) and Komlós et al. (1976). □

Let

(C.23)
$$\Gamma_m(k)=\begin{cases}\mathfrak{s}^{1/2}\left|W_{m,2}(k)-kW_{m,1}(m)\right| & \text{if } E\log|\beta_0+\epsilon_{0,1}|<0 \text{ holds},\\ \sigma_1\left|W_{m,2}(k)-kW_{m,1}(m)\right| & \text{if } E\log|\beta_0+\epsilon_{0,1}|\ge 0 \text{ holds}.\end{cases}$$

Lemma C.5. *Let* β_m *be a sequence such that, as* $m\to\infty$, $\gamma_m\to\infty$ *with* $\gamma_m=o(m)$*. Then, if assumptions of Theorem 3.4 are satisfied, it holds that under* H_0

$$\max_{\gamma_m\le k\le m^*}\frac{|Z_m(k)-\Gamma_m(k)|}{m^{1/2}\left(1+\dfrac{k}{m}\right)\left(\dfrac{k}{m+k}\right)^{1/2}}=O_P\left(\beta_m^{\zeta-1/2}\right),$$

for some $0<\zeta<1/2$.

Proof. Upon following the proof of (D.1), it can be shown that there exists a $0 < \zeta < 1/2$ such that

$$\max_{1 \le k \le m^*} \left(k^{\zeta} + \frac{k}{m} m^{\zeta} \right)^{-1} \left| Z_m(k) - \Gamma_m(k) \right| = O_P(1).$$

Hence we have

$$\begin{aligned}
&\max_{\gamma_m \le k \le m^*} \frac{\left| Z_m(k) - \Gamma_m(k) \right|}{m^{1/2} \left(1 + \frac{k}{m}\right) \left(\frac{k}{m+k}\right)^{1/2}} \\
=& O_P(1) \max_{\gamma_m \le k \le m^*} \frac{k^{\zeta} + \frac{k}{m} m^{\zeta}}{m^{1/2} \left(1 + \frac{k}{m}\right) \left(\frac{k}{m+k}\right)^{1/2}} \\
=& O_P\left(\gamma_m^{\zeta - 1/2}\right) + O_P\left(m^{\zeta - 1/2}\right) = O_P\left(\gamma_m^{\zeta - 1/2}\right).
\end{aligned}$$

□

Lemma C.6. *We assume that the conditions of Theorem 3.4 hold. Let*

$$\gamma_{m^*} = O\left(\exp\left(\log(m^*)^{1-\epsilon}\right)\right),$$

with $\epsilon > 0$ and arbitrarily small. Then, under H_0, on a suitably enlarged space, it is possible to construct two independent Wiener processes $\{W_{m,1}(k), 1 \le k \le T_m\}$ and $\{W_{m,2}(k), 1 \le k \le m\}$ whose distribution does not depend on m, such that

$$\max_{\gamma_{m^*} \le k \le m^*} \left| \frac{|Z_m(k)|}{g_{m,0.5}(k)} - \frac{\left| W_{m,2}(k) - \frac{k}{m} W_{m,1}(m) \right|}{g_{m,0.5}(k)} \right| = O_P\left(\exp\left(-c_0 \log(m^*)^{1-\epsilon}\right)\right),$$

for some $0 < c_0 < 1/2$.

Proof. The proof follows from the same arguments as Lemma C.5, of which this lemma is a special case. □

We now report some preliminary results to prove the main results in Section 4. We begin by studying the WLS estimator $\widehat{\beta}_m$. Define

$$\mathbf{Q}_m = \begin{bmatrix} y_1 & \mathbf{x}_2^\intercal \\ y_2 & \mathbf{x}_3^\intercal \\ . & . \\ y_{m-1} & \mathbf{x}_m^\intercal \end{bmatrix},$$

and the diagonal matrix

$$\mathbf{W}_m = \text{diag}\left\{\frac{1}{1+y_1^2}, \frac{1}{1+y_2^2}, ..., \frac{1}{1+y_{m-1}^2}\right\}.$$

Then it holds that

$$\widehat{\mathbf{b}}_m = \left(\mathbf{Q}_m^\intercal \mathbf{W}_m \mathbf{Q}_m\right)^{-1} \mathbf{Q}_m^\intercal \mathbf{W}_m \mathbf{Y}_m,$$

where $\mathbf{Y}_m = (y_2, y_3, ..., y_m)^\intercal$. Using the recursion defined in (4.1), we obtain

$$\widehat{\mathbf{b}}_m - \mathbf{b}_0 = \left(\mathbf{Q}_m^\intercal \mathbf{W}_m \mathbf{Q}_m\right)^{-1} \mathbf{Q}_m^\intercal \mathbf{W}_m \mathbf{E}_m,$$

having defined $\mathbf{b}_0 = (\beta_0, \boldsymbol{\lambda}_0^\intercal)^\intercal$ and

$$\mathbf{E}_m = \left(\epsilon_{1,2} y_1 + \epsilon_{2,2}, \epsilon_{1,3} y_2 + \epsilon_{2,3}, ..., \epsilon_{1,m} y_{m-1} + \epsilon_{2,m}\right)^\intercal.$$

Based on the above, under the condition for stationarity $E \log |\beta_0 + \epsilon_{0,1}| < 0$, it can be verified that

$$\overline{y}_i = (\beta_0 + \epsilon_{i,1})\, \overline{y}_{i-1} + \boldsymbol{\lambda}_0^\intercal \mathbf{x}_i + \epsilon_{i,2}, \quad -\infty < i < \infty,$$

has a unique stationary, causal solution. Consider the variables

$$\text{(C.24)} \qquad \mathbf{z}_i = \frac{1}{\left(1+\overline{y}_{i-1}^2\right)^{1/2}} \left(\overline{y}_{i-1}, \mathbf{x}_i^\intercal\right)^\intercal,$$

(C.25) $$\boldsymbol{\eta}_i = \left(\frac{\left(\epsilon_{i,1}\overline{y}_{i-1} + \epsilon_{i,2}\right)\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2}, \frac{\left(\epsilon_{i,1}\overline{y}_{i-1} + \epsilon_{i,2}\right)\mathbf{x}_i^{\intercal}}{1+\overline{y}_{i-1}^2} \right)^{\intercal},$$

and define

(C.26) $$\mathbf{Q} = E\left(\mathbf{z}_1\mathbf{z}_1^{\intercal}\right),$$

(C.27) $$\mathbf{C} = E\left(\boldsymbol{\eta}_0\boldsymbol{\eta}_0^{\intercal}\right),$$

and

(C.28) $$\mathbf{a} = E\left(\frac{\left(\overline{y}_1, \mathbf{x}_2^{\intercal}\right)^{\intercal}\overline{y}_1}{1+\overline{y}_1^2} \right).$$

Lemma C.7. *We assume that $E\log|\beta_0 + \epsilon_{0,1}| < 0$, and that Assumptions 2.1, 4.1 and 4.2 are satisfied. Then it holds that*

$$\widehat{\mathbf{b}}_m - \mathbf{b}_0 = \frac{1}{m}\mathbf{Q}^{-1}\sum_{i=2}^{m}\boldsymbol{\eta}_i + o_P\left(m^{-1/2-\zeta}\right),$$

for some $\zeta > 0$, where $\boldsymbol{\eta}_i$ and $\mathbf{Q}$ are defined in (C.25) and (C.26).

Proof. We begin by writing explicitly the solution of (4.1) as

(C.29) $$\overline{y}_i = \sum_{\ell=0}^{\infty}\left(\prod_{j=1}^{\ell}\left(\beta_0 + \epsilon_{i-j,1}\right)\right)\left(\mathbf{x}_{i-\ell}^{\intercal}\boldsymbol{\lambda}_0 + \epsilon_{i-\ell,2}\right),$$

with the convention that $\prod_{\emptyset} = 0$. With minor modifications of the arguments in Aue et al. (2006), it can be shown that there exist a $\kappa > 0$ and a constant $0 < c < 1$ such that

(C.30) $$E\left|y_i - \overline{y}_i\right|^{\kappa} = O\left(c^i\right),$$

as $i \to \infty$. Using (C.30) it can be shown that

$$\left\|\mathbf{Q}_m^{\intercal}\mathbf{W}_m\mathbf{Q}_m - \overline{\mathbf{Q}}_m\right\| = O_P\left(1\right),$$

where

$$\overline{\mathbf{Q}}_m = \sum_{i=2}^{m} \overline{\mathbf{z}}_i \overline{\mathbf{z}}_i^\intercal,$$

and $\overline{\mathbf{z}}_i$ is constructed in the same way as $\mathbf{z}_i$ defined in (C.24), replacing y_i with $\overline{y}_i$. We now show that $\overline{y}_i$ defined in (C.29) is a decomposable Bernoulli shift. Indeed, let

$$\overline{y}_{i,k} = \sum_{\ell=0}^{k} \left(\prod_{j=1}^{\ell} (\beta_0 + \epsilon_{i-j,1}) \right) \left(\mathbf{x}_{i-\ell}^\intercal \boldsymbol{\lambda}_0 + \epsilon_{i-\ell,2} \right) + \sum_{\ell=k+1}^{\infty} \left(\prod_{j=1}^{\ell} \left(\beta_0 + \epsilon_{i-j,1}^*\right) \right) \left(\mathbf{x}_{i-\ell,k}^\intercal \boldsymbol{\lambda}_0 + \epsilon_{i-\ell,2}^* \right),$$

where for $\ell \geq 0$

$$\mathbf{x}_{i-\ell,k} = \begin{cases} \mathbf{g}\left(\eta_{i-\ell}, \eta_{i-\ell-1}, ..., \eta_k, \eta_{k-1}^*, \eta_{k-2}^*, ...\right) & \text{if } \ell < i-k, \\ \mathbf{g}\left(\eta_{i-\ell}^*, \eta_{i-\ell-1}^*, ...\right) & \text{if } \ell \geq i-k, \end{cases}$$

the η_j^*s are independent copies of η_0, $\left(\epsilon_{j,1}^*, \epsilon_{j,2}^*\right)$ are independent copies of $(\epsilon_{0,1}, \epsilon_{0,2})$, and the sequences $\left\{\eta_j, -\infty < j < \infty\right\}$, $\left\{\eta_j^*, -\infty < j < \infty\right\}$, $\{(\epsilon_{j,1}, \epsilon_{j,2}), -\infty < j < \infty\}$ and $\left\{\left(\epsilon_{j,1}^*, \epsilon_{j,2}^*\right),\right.$ $\left.-\infty < j < \infty\right\}$ are independent. Defining $u_{i-\ell} = \mathbf{x}_{i-\ell}^\intercal \boldsymbol{\lambda}_0 + \epsilon_{i-\ell,2}$, and recalling Assumptions 4.1 and 4.2, by the same arguments as in Aue et al. (2006) we can show that, under $E \log |\beta_0 + \epsilon_{0,1}| < 0$, there exists a $\kappa > 0$ such that

$$E \left|\beta_0 + \epsilon_{0,1}\right|^\kappa < 1.$$

Using $\kappa < 1$, we have

$$\begin{aligned} & E \left| \sum_{\ell=k+1}^{\infty} \left(\prod_{j=1}^{\ell} (\beta_0 + \epsilon_{i-j,1}) \right) \left(\mathbf{x}_{i-\ell}^\intercal \boldsymbol{\lambda}_0 + \epsilon_{i-\ell,2} \right) \right|^{\kappa/2} \\ & \leq \sum_{\ell=k+1}^{\infty} E \left(\left| \prod_{j=1}^{\ell} (\beta_0 + \epsilon_{i-j,1}) \right|^{\kappa/2} \left| \mathbf{x}_{i-\ell}^\intercal \boldsymbol{\lambda}_0 + \epsilon_{i-\ell,2} \right|^{\kappa/2} \right) \\ & \leq \sum_{\ell=k+1}^{\infty} \left(E \left| \prod_{j=1}^{\ell} (\beta_0 + \epsilon_{i-j,1}) \right|^{\kappa} \right)^{1/2} \left(E \left| \mathbf{x}_{i-\ell}^\intercal \boldsymbol{\lambda}_0 + \epsilon_{i-\ell,2} \right|^{\kappa} \right)^{1/2} \end{aligned}$$

$$= \sum_{\ell=k+1}^{\infty} \left(E\left|\beta_0 + \epsilon_{0,1}\right|^{\kappa}\right)^{\ell/2} \left(E\left|\mathbf{x}_0^{\intercal}\boldsymbol{\lambda}_0 + \epsilon_{0,2}\right|^{\kappa}\right)^{1/2}$$
$$\leq c_1 \sum_{\ell=k+1}^{\infty} \rho^{\ell/2} \leq c_2 c_3^k,$$

where $\rho = E\left|\beta_0 + \epsilon_{0,1}\right|^{\kappa}$, and $0 < c_3 < 1$, having used Assumptions 4.1 and 4.2. By the same token, we have

$$E\left|\sum_{\ell=k+1}^{\infty}\left(\prod_{j=1}^{\ell}\left(\beta_0 + \epsilon_{i-j,1}^*\right)\right)\left(\mathbf{x}_{i-\ell,k}^{\intercal}\boldsymbol{\lambda}_0 + \epsilon_{i-\ell,2}^*\right)\right|^{\kappa/2}$$
$$\leq \sum_{\ell=k+1}^{\infty} E\left(\left|\prod_{j=1}^{\ell}\left(\beta_0 + \epsilon_{i-j,1}^*\right)\right|^{\kappa/2}\left|\mathbf{x}_{i-\ell,k}^{\intercal}\boldsymbol{\lambda}_0 + \epsilon_{i-\ell,2}^*\right|^{\kappa/2}\right)$$
$$\leq \sum_{\ell=k+1}^{\infty}\left(E\left|\prod_{j=1}^{\ell}\left(\beta_0 + \epsilon_{i-j,1}^*\right)\right|^{\kappa}\right)^{1/2}\left(E\left|\mathbf{x}_{i-\ell,k}^{\intercal}\boldsymbol{\lambda}_0 + \epsilon_{i-\ell,2}^*\right|^{\kappa}\right)^{1/2}$$
$$= \sum_{\ell=k+1}^{\infty} \left(E\left|\beta_0 + \epsilon_{0,1}\right|^{\kappa}\right)^{\ell/2} \left(E\left|\mathbf{x}_0^{\intercal}\boldsymbol{\lambda}_0 + \epsilon_{0,2}\right|^{\kappa}\right)^{1/2} \leq c_4 c_5^k,$$

with $0 < c_5 < 1$. Hence it holds that

$$E\left|\overline{y}_i - \overline{y}_{i,k}\right|^{\kappa} \leq c_6 c_7^{\kappa}, \tag{C.31}$$

for some $c_6 > 0$ and $0 < c_7 < 1$, which shows that $\overline{y}_i$ is a decomposable Bernoulli shift. This immediately yields, by the approximations developed in Aue et al. (2014), that

$$\left\|\frac{1}{m}\overline{\mathbf{Q}}_m - \mathbf{Q}\right\| = O_P\left(m^{-1/2}\right). \tag{C.32}$$

We now turn to studying the numerator of $\widehat{\beta}_m - \beta_0$. We begin with some preliminary results. Using (C.31), it follows that, for all $\kappa_3 > 0$ and for $c_7 < c_9 < 1$

(C.33) $$E\left|\frac{\overline{y}_i^2}{1+\overline{y}_i^2} - \frac{\overline{y}_{i,k}^2}{1+\overline{y}_{i,k}^2}\right|$$

$$\leq E \left| \frac{|\overline{y}_i| + |\overline{y}_{i,k}|}{(1+\overline{y}_i^2)\left(1+\overline{y}_{i,k}^2\right)} \left|\overline{y}_i - \overline{y}_{i,k}\right| \right|^{\kappa_3}$$

$$\leq E \left(c_8 \left|\overline{y}_i - \overline{y}_{i,k}\right| I \left(\left|\overline{y}_i - \overline{y}_{i,k}\right| \leq c_9^k\right) + c_{10} I \left(\left|\overline{y}_i - \overline{y}_{i,k}\right| \geq c_9^k\right)\right)^{\kappa_3}$$

$$\leq c_{11} c_9^{k\kappa_3} + c_{12} P \left(\left|\overline{y}_i - \overline{y}_{i,k}\right| \geq c_9^k\right) \leq c_{11} c_9^{k\kappa_3} + c_{12} c_9^{-k} E \left|\overline{y}_i - \overline{y}_{i,k}\right|$$

$$\leq c_{13} c_{14}^k,$$

for some $0 < c_9 < 1$. Using the same arguments, for all $\kappa_4 < \kappa_1$

$$E \left\| \frac{\mathbf{x}_i \overline{y}_{i-1}}{1+\overline{y}_{i-1}^2} - \frac{\mathbf{x}_{i,k} \overline{y}_{i-1,k}}{1+\overline{y}_{i-1,k}^2} \right\|^{\kappa_4} \leq c_{15} c_{16}^k, \tag{C.34}$$

and for all $\kappa_5 < \kappa_2/2$

$$E \left\| \frac{\mathbf{x}_i \mathbf{x}_i^\intercal}{1+\overline{y}_{i-1}^2} - \frac{\mathbf{x}_{i,k} \mathbf{x}_{i,k}^\intercal}{1+\overline{y}_{i-1,k}^2} \right\|^{\kappa_5} \leq c_{17} c_{18}^k, \tag{C.35}$$

where $0 < c_{16}, c_{18} < 1$. This entails that all the sequences studied above are decomposable Bernoulli shifts. Thus

$$\left\| \mathbf{Q}_m^\intercal \mathbf{W}_m \mathbf{E}_m - \overline{\mathbf{E}}_m \right\| = O_P(1),$$

where

$$\overline{\mathbf{E}}_m = \sum_{i=2}^m \boldsymbol{\eta}_i,$$

and $\boldsymbol{\eta}_i$ is defined in (C.25).Repeating the arguments above, it can also be shown that, for all $\kappa_6 < \kappa_1$

$$E \left\| \boldsymbol{\eta}_i - \boldsymbol{\eta}_{i,k} \right\|^{\kappa_6} \leq c_{19} c_{20}^k,$$

with $\boldsymbol{\eta}_{i,k}$ defined as the other coupling constructions above and $0 < c_{20} < 1$. The final result now follows from the strong approximation derived in Aue et al. (2014). □

Lemma C.8. *We assume that* $E\log|\beta_0+\epsilon_{0,1}|<0$*, and that Assumptions 2.1, 4.1 and 4.2 are satisfied. Then it holds that*

$$\max_{1\leq k<\infty} k^{-1/2-\eta}\left\|\sum_{i=m+1}^{m+k}\left(y_{i-1},\mathbf{x}_i^{\intercal}\right)\frac{y_{i-1}}{1+y_{i-1}^2}-k\mathbf{a}\right\|=O_P\left(1\right), \tag{C.36}$$

for all $\eta>0$*, and*

$$\max_{1\leq k<\infty}\left|\sum_{i=m+1}^{m+k}\frac{\left(\epsilon_{i,1}y_{i-1}+\epsilon_{i,2}\right)y_{i-1}}{1+y_{i-1}^2}-\sum_{i=m+1}^{m+k}\frac{\left(\epsilon_{i,1}\overline{y}_{i-1}+\epsilon_{i,2}\right)\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2}\right|=O_P\left(1\right). \tag{C.37}$$

Proof. The proof is based on the results derived in the proof of Lemma C.7. Equation (C.36) follows from exactly the same arguments as the proof of (C.10), and (C.37) follows from the proof of (C.11)-(C.12). □

Lemma C.9. *We assume that* $E\log|\beta_0+\epsilon_{0,1}|<0$*, and that Assumptions 2.1, 4.1 and 4.2 are satisfied. Then, on a suitably enlarged probability space, it is possible to define two independent Wiener processes* $\{W_{m,1}(k),1\leq k\leq m\}$ *and* $\{W_{m,2}(k),1\leq k<\infty\}$*, whose distributions do not depend on* m*, such that*

$$\left|\sum_{i=2}^{m}\mathbf{a}^{\intercal}\mathbf{Q}\boldsymbol{\eta}_i-\mathfrak{s}_{x,1}W_{m,1}\left(m\right)\right|=O_P\left(m^{1/2-\zeta}\right),$$

and

$$\max_{1\leq k<\infty}\frac{1}{k^{1/2-\zeta}}\left|\sum_{i=m+1}^{m+k}\frac{\left(\epsilon_{i,1}\overline{y}_{i-1}+\epsilon_{i,2}\right)\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2}-\mathfrak{s}_{x,2}W_{m,2}\left(k\right)\right|=O_P\left(1\right),$$

for some $0<\zeta<1/2$*, where* $\mathfrak{s}_{x,1}$ *and* $\mathfrak{s}_{x,2}$ *are defined in (4.7).*

Proof. In the proof of Lemma C.7 we have shown that $\boldsymbol{\eta}_i$ is a decomposable Bernoulli shift; the desired result follows from Aue et al. (2014). □

Lemma C.10. *We assume that* $E \log |\beta_0 + \epsilon_{0,1}| > 0$*, and that Assumptions 2.1, 4.1-4.3 are satisfied. Then it holds that*

$$\left\| \widehat{\mathbf{b}}_m - \mathbf{b}_0 \right\| = O_P\left(m^{-1/2}\right).$$

Proof. Recall that

$$\widehat{\mathbf{b}}_m - \mathbf{b}_0 = \left(\mathbf{Q}_m^\intercal \mathbf{W}_m \mathbf{Q}_m\right)^{-1} \mathbf{Q}_m^\intercal \mathbf{W}_m \mathbf{E}_m.$$

Repeating the proof of Theorem 3.1 in Berkes et al. (2009), it follows that $|y_i| \to \infty$ a.s. exponentially fast; this immediately yields

$$\left\| \mathbf{Q}_m^\intercal \mathbf{W}_m \mathbf{Q}_m - \mathbf{B}_m \right\| = O_P(1),$$

where $\mathbf{B}_m$ is a $(p+1) \times (p+1)$ symmetric matrix with elements $\{B_{i,j}, 1 \leq i, j \leq p+1\}$ defined as

$$\begin{aligned} B_{1,1} &= m, \\ (B_{1,2}, ..., B_{1,p+1}) &= \sum_{i=2}^{\infty} \frac{\mathbf{x}_i^\intercal y_{i-1}}{1 + y_{i-1}^2}, \\ \{B_{i,j}, 2 \leq i, j \leq p+1\} &= \sum_{i=2}^{\infty} \frac{\mathbf{x}_i \mathbf{x}_i^\intercal}{1 + y_{i-1}^2}. \end{aligned}$$

Since we already know from the above that $\|\mathbf{E}_m\| = O_P\left(m^{1/2}\right)$, the result follows immediately. □

E. Proofs

Proof of Theorem 3.1. Recall (C.23). We begin by showing that

$$\max_{1\le k<\infty} \frac{|Z_m(k) - \Gamma_m(k)|}{m^{1/2}\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{\psi}} = o_P(1). \tag{D.1}$$

We begin by considering the case $E\log|\beta_0 + \epsilon_{0,1}| < 0$; we write

$$\begin{aligned}
&\left(\beta_0 - \widehat{\beta}_m\right)\sum_{i=m+1}^{m+k}\frac{y_{i-1}^2}{1+y_{i-1}^2} + \frac{k}{m}\sum_{i=2}^{m}\left(\frac{\epsilon_{i,1}\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} + \frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2}\right)\\
=&\left(\beta_0 - \widehat{\beta}_m\right)\left(\sum_{i=m+1}^{m+k}\frac{y_{i-1}^2}{1+y_{i-1}^2} - ka_3\right)\\
&- ka_3\left(\left(\sum_{i=2}^{m}\frac{y_{i-1}^2}{1+y_{i-1}^2}\right)^{-1} - \frac{1}{ma_3}\right)\left(\sum_{i=2}^{m}\left(\frac{\epsilon_{i,1}y_{i-1}^2}{1+y_{i-1}^2} + \frac{\epsilon_{i,2}y_{i-1}}{1+y_{i-1}^2}\right)\right)\\
&+ \frac{k}{m}\sum_{i=2}^{m}\left(\epsilon_{i,1}\left(\frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} - \frac{y_{i-1}^2}{1+y_{i-1}^2}\right) + \epsilon_{i,2}\left(\frac{\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2} - \frac{y_{i-1}}{1+y_{i-1}^2}\right)\right)\\
=&R_{m,1}(k) + R_{m,2}(k) + R_{m,3}(k).
\end{aligned}$$

Using Lemmas C.2 and C.3 with $\eta < 1/2$

$$\begin{aligned}
&\max_{1\le k<\infty}\frac{|R_{m,1}(k)|}{m^{1/2}\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{\psi}}\\
=&O_P\left(m^{-1}\right)\max_{1\le k\le M}\frac{k^{1/2+\eta}}{\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{\psi}}\\
&+ O_P\left(m^{-1}\right)\max_{M<k<\infty}\frac{k^{1/2+\eta}}{\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{\psi}}\\
=&O_P(1)\left(m^{-1/2}\max_{1\le k\le M}k^{\eta} + \max_{M<k<\infty}k^{\eta-1/2}\right) = o_P(1).
\end{aligned}$$

Using the same logic, it can also be shown that

$$\max_{1\leq k<\infty}\frac{\left|R_{m,2}\left(k\right)\right|}{m^{1/2}\left(1+\dfrac{k}{m}\right)\left(\dfrac{k}{m+k}\right)^{\psi}}=o_{P}\left(1\right),$$

$$\max_{1\leq k<\infty}\frac{\left|R_{m,3}\left(k\right)\right|}{m^{1/2}\left(1+\dfrac{k}{m}\right)\left(\dfrac{k}{m+k}\right)^{\psi}}=o_{P}\left(1\right).$$

Using Lemma C.4, it follows that

$$\max_{1\leq k<\infty}\frac{\left|\dfrac{k}{m}\sum_{i=2}^{m}\left(\dfrac{\epsilon_{i,1}\overline{y}_{i-1}^{2}}{1+\overline{y}_{i-1}^{2}}+\dfrac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}_{i-1}^{2}}\right)-\dfrac{k}{m}\left(a_{1}\sigma_{1}^{2}+a_{2}\sigma_{2}^{2}\right)^{1/2}W_{m,1}\left(m\right)\right|}{m^{1/2}\left(1+\dfrac{k}{m}\right)\left(\dfrac{k}{m+k}\right)^{\psi}}\tag{D.2}$$

$$=O_{P}\left(1\right)\max_{1\leq k<\infty}\frac{k}{m}\frac{m^{1/2-\zeta}}{m^{1/2}\left(1+\dfrac{k}{m}\right)\left(\dfrac{k}{m+k}\right)^{\psi}}$$

$$=O_{P}\left(m^{-\zeta}\right)\max_{1\leq k<\infty}\left(\frac{k}{m+k}\right)^{1-\psi}=o_{P}\left(1\right).$$

By the same token, we can show that

$$\max_{1\leq k<\infty}\frac{\left|\sum_{i=m+1}^{m+k}\left(\epsilon_{i,1}\left(\dfrac{\overline{y}_{i-1}^{2}}{1+\overline{y}_{i-1}^{2}}-\dfrac{y_{i-1}^{2}}{1+y_{i-1}^{2}}\right)\right)\right|}{m^{1/2}\left(1+\dfrac{k}{m}\right)\left(\dfrac{k}{m+k}\right)^{\psi}}\tag{D.3}$$

$$=O_{P}\left(1\right)\max_{1\leq k<\infty}\frac{1}{m^{1/2}\left(1+\dfrac{k}{m}\right)\left(\dfrac{k}{m+k}\right)^{\psi}}$$

$$=O_{P}\left(m^{-1/2}\right)\max_{1\leq k<\infty}\frac{m}{m+k}\left(\frac{k}{m+k}\right)^{-\psi}=O_{P}\left(m^{-1/2+\psi}\right)=o_{P}\left(1\right),$$

and

(D.4)
$$\max_{1\le k<\infty} \frac{\left| \sum_{i=m+1}^{m+k} \left(\epsilon_{i,2} \left(\frac{\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2} - \frac{y_{i-1}}{1+y_{i-1}^2} \right) \right) \right|}{m^{1/2} \left(1+\frac{k}{m}\right) \left(\frac{k}{m+k}\right)^{\psi}}$$

$$= O_P(1) \max_{1\le k<\infty} \frac{1}{m^{1/2} \left(1+\frac{k}{m}\right) \left(\frac{k}{m+k}\right)^{\psi}}$$

$$= O_P\left(m^{-1/2}\right) \max_{1\le k<\infty} \frac{m}{m+k} \left(\frac{k}{m+k}\right)^{-\psi} = O_P\left(m^{-1/2+\psi}\right) = o_P(1),$$

having used Lemma C.3. Similarly, by Lemma C.4

(D.5)
$$\max_{1\le k<\infty} \frac{\left| \sum_{i=m+1}^{m+k} \left(\frac{\epsilon_{i,1}\overline{y}_{i-1}^2 + \epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2} \right) - \left(a_1\sigma_1^2 + a_2\sigma_2^2\right)^{1/2} W_{m,2}(k) \right|}{m^{1/2} \left(1+\frac{k}{m}\right) \left(\frac{k}{m+k}\right)^{\psi}}$$

$$= O_P(1) \max_{1\le k<\infty} \frac{k^{1/2-\zeta}}{m^{1/2} \left(1+\frac{k}{m}\right) \left(\frac{k}{m+k}\right)^{\psi}} = o_P(1).$$

Putting all together, (D.1) has been shown for the stationary case. Considering now the case $E\log|\beta_0+\epsilon_{0,1}| \ge 0$, write

$$\left(\beta_0 - \widehat{\beta}_m\right) \sum_{i=m+1}^{m+k} \frac{y_{i-1}^2}{1+y_{i-1}^2} + \frac{k}{m} \sum_{i=2}^{m} \epsilon_{i,1}$$

$$= \left(\beta_0 - \widehat{\beta}_m\right) \left(\sum_{i=m+1}^{m+k} \frac{y_{i-1}^2}{1+y_{i-1}^2} - k \right)$$

$$- k \left(\left(\sum_{i=2}^{m} \frac{y_{i-1}^2}{1+y_{i-1}^2} \right)^{-1} - \frac{1}{m} \right) \left(\sum_{i=2}^{m} \left(\frac{\epsilon_{i,1}y_{i-1}^2}{1+y_{i-1}^2} + \frac{\epsilon_{i,2}y_{i-1}}{1+y_{i-1}^2} \right) \right)$$

$$- \frac{k}{m} \left(\sum_{i=2}^{m} \frac{\epsilon_{i,1}}{1+y_{i-1}^2} + \sum_{i=2}^{m} \frac{\epsilon_{i,2}y_{i-1}}{1+y_{i-1}^2} \right)$$

$$=R_{m,4}(k)+R_{m,5}(k)+R_{m,6}(k).$$

The arguments are similar to the above. Indeed, using Lemmas C.2-C.4

(D.6)
$$\begin{aligned}&\max_{1\le k<\infty}\frac{|R_{m,4}(k)|}{m^{1/2}\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{\psi}}\\ =&O_P\left(m^{-1/2}\right)\max_{1\le k<\infty}\frac{km^{-\zeta}}{m^{1/2}\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{\psi}}\\ =&O_P\left(m^{-\zeta}\right)=o_P(1).\end{aligned}$$

Similarly it can be shown that

(D.7)
$$\max_{1\le k<\infty}\frac{|R_{m,5}(k)|}{m^{1/2}\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{\psi}}=O_P\left(m^{-\zeta}\right)=o_P(1),$$

(D.8)
$$\max_{1\le k<\infty}\frac{|R_{m,6}(k)|}{m^{1/2}\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{\psi}}=O_P\left(m^{-\zeta+\eta}\right)=o_P(1),$$

having chosen $\eta<\zeta$ in the last equation. Lemma C.4 yields

(D.9)
$$\max_{1\le k<\infty}\frac{\left|\frac{k}{m}\sum_{i=2}^{m}\epsilon_{i,1}-\sigma_1\frac{k}{m}W_{m,1}(m)\right|}{m^{1/2}\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{\psi}}=O_P\left(m^{\zeta-1/2}\right)=o_P(1),$$

and

(D.10)
$$\max_{1\le k<\infty}\frac{\left|\sum_{i=m+1}^{m+k}\epsilon_{i,1}-\sigma_1W_{m,2}(k)\right|}{m^{1/2}\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{\psi}}=O_P\left(m^{\zeta-1/2}\right)=o_P(1).$$

similarly, by Lemma C.3

(D.11)
$$\max_{1\le k<\infty}\frac{\left|\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,1}}{1+y_{i-1}^2}\right|+\left|\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,2}y_{i-1}}{1+y_{i-1}^2}\right|}{m^{1/2}\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{\psi}}$$
$$=O_P\left(m^{-\zeta}\right)\max_{1\le k<\infty}\frac{k^{1/2+\eta}}{m^{1/2}\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{\psi}}$$
$$=O_P\left(m^{1/2-\zeta}\right)\max_{1\le k<\infty}\frac{k^{1/2+\eta-\psi}}{(m+k)^{1/2+\eta-\psi+1/2-\eta}}=O_P\left(m^{\eta-\zeta}\right)=o_P\left(1\right),$$

having chosen $\eta<\zeta$. Putting all together, (D.1) now follows for the case $E\log|\beta_0+\epsilon_{0,1}|\geq 0$.

Let now

(D.12)
$$\Gamma\left(t\right)=\begin{cases}\mathfrak{s}^{1/2}\left|W_2\left(t\right)-tW_1\left(1\right)\right| & \text{if } E\log|\beta_0+\epsilon_{0,1}|<0 \text{ holds}\\ \sigma_1\left|W_2\left(t\right)-tW_1\left(1\right)\right| & \text{if } E\log|\beta_0+\epsilon_{0,1}|\geq 0 \text{ holds}\end{cases},$$

where $\{W_1(k), 1\le k\le m\}$ and $\{W_2(k), 1\le k<\infty\}$ are two independent standard Wiener processes. Using the fact that the distribution of $W_{m,1}(\cdot)$ and $W_{m,2}(\cdot)$ does not depend on m, and exploiting the scale transformation and the continuity of the Wiener process

$$\max_{1\le k<\infty}\frac{\Gamma_m\left(k\right)}{m^{1/2}\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{\psi}}$$
$$\stackrel{\mathcal{D}}{=}\max_{1\le k<\infty}\frac{\Gamma\left(\frac{k}{m}\right)}{\left(1+\frac{k}{m}\right)\left(\frac{k/m}{1+k/m}\right)^{\psi}}$$
$$\stackrel{\mathcal{D}}{=}\sup_{0<t<\infty}\frac{\Gamma\left(t\right)}{\left(1+t\right)\left(\frac{t}{1+t}\right)^{\psi}}.$$

Computing the covariance kernel, it can be easily verified that

$$\left\{ \frac{W_2(t) - tW_1(1)}{1+t}, t \geq 0 \right\} \stackrel{\mathcal{D}}{=} \left\{ W\left(\frac{t}{1+t} \right), t \geq 0 \right\}, \tag{D.13}$$

where $W(\cdot)$ denotes a standard Wiener. Hence

$$\max_{1 \leq k < \infty} \frac{\Gamma_m(k)}{m^{1/2}\left(1 + \frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{\psi}} \stackrel{\mathcal{D}}{=} \sup_{0<t<\infty} \frac{\mathfrak{s}\left| W\left(\frac{t}{1+t}\right)\right|}{\left(\frac{t}{1+t}\right)^{\psi}},$$

whence Theorem 3.1 follows. □

Proof of Theorem 3.2. Using exactly the same arguments as in the proof of Theorem 3.1, it follows that

$$\max_{1 \leq k \leq m^*} \frac{|Z_m(k) - \Gamma_m(k)|}{m^{1/2}\left(1 + \frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{\psi}} = o_P(1). \tag{D.14}$$

Further, it is immediate to see that

$$\begin{aligned} &\max_{1 \leq k \leq m^*} \frac{\Gamma_m(k)}{m^{1/2}\left(1 + \frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{\psi}} \\ \stackrel{\mathcal{D}}{=} &\max_{1 \leq k \leq m^*} \frac{\Gamma\left(\frac{k}{m}\right)}{\left(1 + \frac{k}{m}\right)\left(\frac{k/m}{1+k/m}\right)^{\psi}} \\ \stackrel{\mathcal{D}}{=} &\sup_{0<t<m_0} \frac{\Gamma(t)}{(1+t)\left(\frac{t}{1+t}\right)^{\psi}}. \end{aligned}$$

Using (D.13), it finally follows

$$\sup_{0<t<m_0} \frac{\Gamma(t)}{(1+t)\left(\frac{t}{1+t}\right)^{\psi}} \stackrel{\mathcal{D}}{=} \sup_{0<t<m_0} \frac{\mathfrak{s}\left|W\left(\frac{t}{1+t}\right)\right|}{\left(\frac{t}{1+t}\right)^{\psi}},$$

whence the desired result. □

Proof of Theorem 3.4. We begin by noting that, repeating *verbatim* the proof of (D.1) with $\psi = 1/2$, it can be shown that

$$\max_{1\le k\le m^*} \frac{|Z_m(k) - \Gamma_m(k)|}{m^{1/2}\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{1/2}} = O_P(1). \tag{D.15}$$

Recall also that the distribution of $\Gamma_m(k)$ does not depend on m, so that

$$\left\{m^{-1/2}\Gamma_m(mt), t\ge 0\right\} \stackrel{\mathcal{D}}{=} \{\Gamma(t), t\ge 0\},$$

where $\Gamma(t)$ is defined in (D.12); further recall that, by (D.13)

$$\left\{\frac{1}{\mathfrak{s}}\frac{\Gamma(t)}{1+t}, t\ge 0\right\} \stackrel{\mathcal{D}}{=} \left\{\left|W\left(\frac{t}{1+t}\right)\right|, t\ge 0\right\}.$$

We now report some results concerning $\Gamma_m(k)$. We begin by studying

$$\frac{1}{\mathfrak{s}} \max_{1\le k\le \log m} \frac{\Gamma_m(k)}{m^{1/2}\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{1/2}} \tag{D.16}$$

$$\stackrel{\mathcal{D}}{=} \frac{1}{\mathfrak{s}} \max_{1/m\le t\le(\log m)/m} \frac{\Gamma(t)}{(1+t)\left(\frac{t}{1+t}\right)^{1/2}} \stackrel{\mathcal{D}}{=} \max_{1/m\le t\le(\log m)/m} \frac{\left|W\left(\frac{t}{1+t}\right)\right|}{\left(\frac{t}{1+t}\right)^{1/2}}$$

$$\stackrel{\mathcal{D}}{=} \max_{1/(m+1)\le u\le(\log m)/(m+\log m)} \frac{|W(u)|}{u^{1/2}} \stackrel{\mathcal{D}}{=} \max_{1\le v\le \log m} \frac{|W(v)|}{v^{1/2}}$$

$$=O_P\left(\sqrt{\log\log\log m}\right),$$

where we have repeatedly used the scale transformation of the Wiener process, and the Law of the Iterated Logarithm in the last passage. Similarly

$$\frac{1}{\mathfrak{z}}\max_{m/\log m\le k\le m^*}\frac{\Gamma_m(k)}{m^{1/2}\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{1/2}} \tag{D.17}$$

$$\overset{\mathcal{D}}{=}\frac{1}{\mathfrak{z}}\max_{1/\log m\le t\le m^*/\log m}\frac{\Gamma(t)}{(1+t)\left(\frac{t}{1+t}\right)^{1/2}}\overset{\mathcal{D}}{=}\max_{1/\log m\le t\le m^*/\log m}\frac{\left|W\left(\frac{t}{1+t}\right)\right|}{\left(\frac{t}{1+t}\right)^{1/2}}$$

$$\overset{\mathcal{D}}{=}\max_{1/(\log m+1)\le u\le m^*/(m+m^*)}\frac{|W(u)|}{u^{1/2}}\overset{\mathcal{D}}{=}\max_{1\le v\le m^*(\log m+1)/(m+m^*)}\frac{|W(v)|}{v^{1/2}}$$

$$=O_P\left(\sqrt{\log\log\frac{m^*(\log m+1)}{m+m^*}}\right)=O_P\left(\sqrt{\log\log\log m}\right).$$

Finally, by the same token, note that

$$\frac{1}{\mathfrak{z}}\max_{\log m\le k\le m/\log m}\frac{\Gamma_m(k)}{m^{1/2}\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{1/2}}$$

$$\overset{\mathcal{D}}{=}\frac{1}{\mathfrak{z}}\max_{\log m\le k\le m/\log m}\frac{\Gamma_m\left(\frac{k}{m}\right)}{\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{1/2}}$$

$$\overset{\mathcal{D}}{=}\frac{1}{\mathfrak{z}}\max_{(\log m)/m\le t\le 1/\log m}\frac{\Gamma(t)}{(1+t)\left(\frac{t}{1+t}\right)^{1/2}}\overset{\mathcal{D}}{=}\max_{(\log m)/m\le t\le 1/\log m}\frac{\left|W\left(\frac{t}{1+t}\right)\right|}{\left(\frac{t}{1+t}\right)^{1/2}}$$

$$\overset{\mathcal{D}}{=}\max_{(\log m)/(m+\log m)\le t\le 1/(\log m+1)}\frac{|W(u)|}{u^{1/2}}\overset{\mathcal{D}}{=}\max_{1\le v\le(m+\log m)/((1+\log m)\log m)}\frac{|W(v)|}{v^{1/2}},$$

and using the Law of the Iterated Logarithm it follows that

$$\text{(D.18)} \qquad \lim_{m\to\infty} \frac{1}{\sqrt{2\log\log\dfrac{m+\log m}{(1+\log m)\log m}}} \max_{1\le v\le (m+\log m)/((1+\log m)\log m)} \frac{|W(v)|}{v^{1/2}} = 1 \quad \text{a.s.}$$

Given that, as $m \to \infty$

$$\frac{\log\log\dfrac{m+\log m}{(1+\log m)\log m}}{\log\log m} = 1,$$

it follows that the term in (D.18) dominates, and therefore, putting all together

$$\lim_{m\to\infty} P\left\{ \max_{1\le k\le m^*} \frac{Z_m(k)}{m^{1/2}\left(1+\dfrac{k}{m}\right)\left(\dfrac{k}{m+k}\right)^{1/2}} = \max_{1\le v\le (m+\log m)/((1+\log m)\log m)} \frac{|W(v)|}{v^{1/2}} \right\} = 1.$$

Using the Darling-Erdős theorem (Darling and Erdős, 1956), it follows that

$$\lim_{m\to\infty} P\left\{ \gamma\left(\log\frac{m+\log m}{(1+\log m)\log m}\right) \max_{1\le v\le (m+\log m)/((1+\log m)\log m)} \frac{|W(v)|}{v^{1/2}} \right.$$
$$\left. \le x + \delta\left(\log\frac{m+\log m}{(1+\log m)\log m}\right)\right\} = \exp\left(-\exp\left(-x\right)\right).$$

The desired result follows upon noting that, by elementary arguments

$$\left|\gamma\left(\frac{m+\log m}{(1+\log m)\log m}\right) - \gamma(m)\right| \to 0,$$
$$\left|\delta\left(\frac{m+\log m}{(1+\log m)\log m}\right) - \delta(m)\right| \to 0,$$

as $m \to \infty$. □

Proof of Theorem 3.3. We start with the proof of (3.4), considering the case $E\log|\beta_0 + \epsilon_{0,1}| < 0$. As in the proof of Theorem 3.1, it suffices to show that

$$\max_{1\le k\le m^*} \frac{Z_m^*(k)}{\overline{g}_{m,\psi}(k)} \xrightarrow{\mathcal{D}} \frac{1}{\overline{c}_{\alpha,\psi}} \sup_{0\le t\le 1} \frac{|W(t)|}{t^{\psi}},$$

where

$$Z_m^*(k) = \left| \left(\sum_{i=2}^{m} \frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} \right)^{-1} \left(\sum_{i=2}^{m} \frac{\epsilon_{i,1}\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} + \sum_{i=2}^{m} \frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2} \right) \left(\sum_{i=m+1}^{m+k} \frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} \right) \right.$$
$$\left. + \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,1}\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} + \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2} \right|,$$

where recall that $\overline{y}_i$ is the stationary solution of (2.1). Using the proofs of Lemmas C.2 and C.3, it follows by routine calculations that

$$\left| \left(\sum_{i=2}^{m} \frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} \right)^{-1} - \frac{1}{ma_3} \right| = O_P(m),$$

and

$$\left| \sum_{i=2}^{m} \frac{\epsilon_{i,1}\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} + \sum_{i=2}^{m} \frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2} \right| = O_P\left(m^{1/2}\right),$$

where a_3 is defined in (C.2). The approximations in Aue et al. (2014) yield

$$(m^*)^{-1/2+\psi} \max_{1\le k\le m^*} \frac{1}{k^\psi} \left| \sum_{i=m+1}^{m+k} \left(\frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} - a_3 \right) \right| \stackrel{\mathcal{D}}{\to} a_4 \sup_{0\le t\le 1} \frac{|W(t)|}{t^\psi},$$

where

$$a_4 = \sum_{h=-\infty}^{\infty} E\left[\left(\frac{\overline{y}_0^2}{1+\overline{y}_0^2} - a_3 \right) \left(\frac{\overline{y}_h^2}{1+\overline{y}_h^2} - a_3 \right) \right].$$

Thus we conclude

$$(m^*)^{-1/2+\psi} \left(\sum_{i=2}^{m} \frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} \right)^{-1} \left| \sum_{i=2}^{m} \frac{\epsilon_{i,1}\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} + \sum_{i=2}^{m} \frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2} \right| \max_{1\le k\le m^*} \frac{1}{k^\psi} \sum_{i=m+1}^{m+k} \frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2}$$
$$= O_P\left(m^{-1/2}\right) (m^*)^{-1/2+\psi} \max_{1\le k\le m^*} \frac{1}{k^\psi} \sum_{i=m+1}^{m+k} \frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2}$$
$$= O_P\left(m^{-1/2}\right) (m^*)^{-1/2+\psi} k^{1-\psi}$$
$$+ O_P\left(m^{-1/2}\right) (m^*)^{-1/2+\psi} \max_{1\le k\le m^*} \frac{1}{k^\psi} \sum_{i=m+1}^{m+k} \left(\frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} - a_3 \right)$$

$$=O_P\left(\left(\frac{m^*}{m}\right)^{1/2}+m^{-1/2}\right)=o_P\left(1\right).$$

Using Aue et al. (2014), we can define two independent Wiener processes $\left\{W_{m,1}\left(x\right),x\geq 0\right\}$ and $\left\{W_{m,2}\left(x\right),x\geq 0\right\}$ such that

$$\max_{1\leq k\leq m^*}\frac{1}{\left(m^*\right)^{1/2-\psi}k^{\psi}}\left|\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,1}\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2}+\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2}-\left(a_1^{1/2}\sigma_1W_{m,1}\left(k\right)+a_2^{1/2}\sigma_2W_{m,2}\left(k\right)\right)\right|$$

$$=o_P\left(1\right).$$

By the scale transformation and the continuity of the Wiener process, it follows that

$$\max_{1\leq k\leq m^*}\frac{1}{\left(m^*\right)^{1/2-\psi}k^{\psi}}\left|a_1^{1/2}\sigma_1W_{m,1}\left(k\right)+a_2^{1/2}\sigma_2W_{m,2}\left(k\right)\right|$$

$$\stackrel{\mathcal{D}}{=}\max_{1/m^*\leq k/m^*\leq 1}\left(\frac{k}{m^*}\right)^{-\psi}\left|a_1^{1/2}\sigma_1W_{m,1}\left(\frac{k}{m^*}\right)+a_2^{1/2}\sigma_2W_{m,2}\left(\frac{k}{m^*}\right)\right|$$

$$\stackrel{\mathcal{D}}{\rightarrow}\sup_{0\leq u\leq 1}\frac{1}{t^{\psi}}\left|a_1^{1/2}\sigma_1W_1\left(t\right)+a_2^{1/2}\sigma_2W_2\left(t\right)\right|,$$

where W_1 and W_2 are independent Wiener processes. Since

$$\left\{a_1^{1/2}\sigma_1W_1\left(t\right)+a_2^{1/2}\sigma_2W_2\left(t\right),t\geq 0\right\}\stackrel{\mathcal{D}}{=}\left\{\mathscr{s}W\left(t\right),t\geq 0\right\},$$

where $\left\{W\left(t\right),t\geq 0\right\}$ is a Wiener process, the result follows.

Next we consider the case $\psi=1/2$, and we establish a Darling-Erdős limiting law for $Z_m^*\left(k\right)$. Firstly note that

$$\max_{1\leq k\leq m^*}\frac{1}{k^{1/2}}\left(\sum_{i=2}^{m}\frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2}\right)^{-1}\left(\sum_{i=2}^{m}\frac{\epsilon_{i,1}\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2}+\sum_{i=2}^{m}\frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2}\right)\sum_{i=m+1}^{m+k}\frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2}$$

$$=O_P\left(\left(\frac{m^*}{m}\right)^{1/2}\right),$$

$$
\frac{1}{(\log\log m^*)^{1/2}} \max_{1\le k\le m^*} \frac{1}{k^{1/2}} \left| \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,1}\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} + \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2} \right| = O_P(1),
$$

and

$$
\frac{1}{(\log\log m^*)^{1/2}} \max_{\log m^*\le k\le m^*/\log m^*} \frac{1}{k^{1/2}} \left| \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,1}\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} + \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2} \right| \xrightarrow{\mathcal{P}} c > 0.
$$

The above entails that we need to consider the maximum of $Z_m^*(k)$ over $\log m^* \le k \le m^*/\log m^*$. We have

$$
\begin{aligned}
&\max_{\log m^*\le k\le m^*/\log m^*} \frac{1}{k^{1/2}} \left(\sum_{i=2}^{m} \frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} \right)^{-1} \left(\sum_{i=2}^{m} \frac{\epsilon_{i,1}\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} + \sum_{i=2}^{m} \frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2} \right) \sum_{i=m+1}^{m+k} \frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} \\
=&O_P\left(\left(\frac{m^*}{m}\right)^{1/2} (\log m^*)^{-1/2} \right).
\end{aligned}
$$

By the Law of the Iterated Logarithm, it follows that

$$
\max_{1\le k\le \log m^*} \frac{1}{k^{1/2}} \left| \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,1}\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} + \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2} \right| = O_P\left((\log\log\log m^*)^{1/2} \right).
$$

Hence we need to show that, for all $-\infty < x < \infty$

$$
\begin{aligned}
&\lim_{m\to\infty} P\left\{ \gamma(\log m^*) \frac{1}{\mathfrak{s}} \max_{1\le k\le m^*} \frac{1}{k^{1/2}} \left| \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,1}\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} + \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}_{i-1}^2} \right| \le x + \delta(\log m^*) \right\} \\
&= \exp(-\exp(-x)),
\end{aligned}
$$

which follows from the same logic as the proof of Theorem 3.4.

We now turn to the case $E\log|\beta_0+\epsilon_{0,1}| \ge 0$. Following the proof of Lemma C.2, it can be shown that

$$
\text{(D.19)} \qquad \left| \left(\sum_{i=2}^{m} \frac{y_{i-1}^2}{1+y_{i-1}^2} \right)^{-1} - \frac{1}{m} \right| = O_P\left(m^{-3/2}\right),
$$

$$\text{(D.20)} \qquad \left| \left(\sum_{i=2}^{m} \frac{\epsilon_{i,1} y_{i-1}^2}{1+y_{i-1}^2} + \sum_{i=2}^{m} \frac{\epsilon_{i,2} y_{i-1}}{1+y_{i-1}^2} \right) - \sum_{i=2}^{m} \epsilon_{i,1} \right| = O_P(1),$$

$$\text{(D.21)} \qquad \max_{1\le k<\infty} \left| \sum_{i=m+1}^{m+k} \frac{y_{i-1}^2}{1+y_{i-1}^2} - k \right| = O_P(c^m),$$

and

$$\text{(D.22)} \qquad \max_{1\le k<\infty} \left| \left(\sum_{i=m+1}^{m+k} \frac{\epsilon_{i,1} y_{i-1}^2}{1+y_{i-1}^2} + \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,2} y_{i-1}}{1+y_{i-1}^2} \right) - \sum_{i=m+1}^{m+k} \epsilon_{i,1} \right| = O_P(c^m),$$

with some $0 < c < 1$. Thus we get

$$\begin{aligned}
&\left(\sum_{i=2}^{m} \frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} \right)^{-1} \left| \sum_{i=2}^{m} \frac{\epsilon_{i,1} \overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} + \sum_{i=2}^{m} \frac{\epsilon_{i,2} \overline{y}_{i-1}}{1+\overline{y}_{i-1}^2} \right| (m^*)^{-1/2+\psi} \max_{1\le k\le m^*} \frac{1}{k^{\psi}} \sum_{i=m+1}^{m+k} \frac{\overline{y}_{i-1}^2}{1+\overline{y}_{i-1}^2} \\
=&O_P\left(m^{-1/2}\right) (m^*)^{-1/2+\psi} \max_{1\le k\le m^*} \frac{1}{k^{\psi}} (k + c^m) \\
=&O_P\left(\left(\frac{m^*}{m} \right)^{1/2} \right) = o_P(1).
\end{aligned}$$

Our assumptions entail that

$$(m^*)^{-1/2+\psi} \max_{1\le k\le m^*} \frac{1}{k^{\psi}} \left| \sum_{i=m+1}^{m+k} \epsilon_{i,1} \right| \xrightarrow{\mathcal{D}} \sigma_1 \sup_{0\le t\le 1} \frac{|W(t)|}{t^{\psi}},$$

where $\{W(t), t \ge 0\}$ is a Wiener process; this completes the proof of (3.4). Considering the case $\psi = 1/2$, we can follow the same arguments as above, using (D.19)-(D.22). We obtain

$$\left| \max_{1\le k<m^*} \frac{|Z_m^*(k)|}{k^{1/2}} - \max_{\log m^* \le k < m^*/\log m^*} \frac{1}{k^{1/2}} \left| \sum_{i=m+1}^{m+k} \epsilon_{i,1} \right| \right| = o_P\left((\log\log m^*)^{-1/2} \right).$$

Hence we need to prove only that

$$\lim_{m\to\infty} P\left\{ \gamma(\log m^*) \frac{1}{\sigma_1} \max_{\log m^* \le k < m^*/\log m^*} \frac{1}{k^{1/2}} \left| \sum_{i=m+1}^{m+k} \epsilon_{i,1} \right| \le x + \delta(\log m^*) \right\} = \exp(-\exp(-x)),$$

which is shown in Csörgő and Horváth (1997). □

Proof of Theorem 3.5. Since the distributions of $W_{m,1}(\cdot)$ and $W_{m,2}(\cdot)$ do not depend on m, Corollary C.6 entails that, for all $-\infty < x < \infty$

$$P\left\{\max_{h_{m^*}\leq k\leq m^*}\frac{Z_m(k)}{g_{m,0.5}(k)}\geq x\right\} = P\left\{\max_{h_{m^*}\leq k\leq m^*}\frac{\left|W_1(k)-\frac{k}{m}W_2(m)\right|}{m^{1/2}\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{1/2}}\geq x\right\}+o(1),$$

as $m^*\to\infty$. In turn, this entails that we can construct critical values $\widehat{c}_{\alpha,0.5}$ based on

$$P\left\{\max_{h_{m^*}\leq k\leq m^*}\frac{\left|W_1(k)-\frac{k}{m}W_2(m)\right|}{m^{1/2}\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{1/2}}\geq \widehat{c}_{\alpha,0.5}\right\}=\alpha.$$

Using the scale transformation for Wiener processes

$$\begin{aligned}
&\max_{h_{m^*}\leq k\leq m^*}\frac{\left|W_1(k)-\frac{k}{m}W_2(m)\right|}{m^{1/2}\left(1+\frac{k}{m}\right)\left(\frac{k}{m+k}\right)^{1/2}}\\
\stackrel{\mathcal{D}}{=}&\max_{h_{m^*}/m^*\leq k/m^*\leq 1}\frac{\left(1+\frac{k}{m}\right)\left|W\left(\frac{k/m}{1+k/m}\right)\right|}{\left(1+\frac{k}{m}\right)\left(\frac{k/m}{1+k/m}\right)^{1/2}}\\
\stackrel{\mathcal{D}}{=}&\max_{h_{m^*}/m^*\leq\tau\leq 1}\frac{\left|W\left(\frac{\tau}{1+\tau}\right)\right|}{\left(\frac{\tau}{1+\tau}\right)^{1/2}}\\
\stackrel{\mathcal{D}}{=}&\max_{h_{m^*}/(m^*+h_{m^*})\leq u\leq 1/2}\frac{|W(u)|}{u^{1/2}}\stackrel{\mathcal{D}}{=}\max_{1\leq v\leq\phi_m}\frac{|W(v)|}{v^{1/2}}\\
\stackrel{\mathcal{D}}{=}&\max_{1\leq v\leq\exp(\log\phi_m)}\frac{|W(v)|}{v^{1/2}},
\end{aligned}$$

where $W(\cdot)$ is a standard Wiener process and $\phi_m = (m^* + h_{m^*}) / (2h_{m^*})$. Using equation (18) in Vostrikova (1981), we have

$$P\left\{\max_{1\leq v\leq \exp(\log\phi_m)} \frac{|W(v)|}{v^{1/2}} \geq \widehat{c}_{\alpha,0.5}\right\} \simeq \frac{\widehat{c}_{\alpha,0.5}\exp\left(-\frac{1}{2}\widehat{c}^2_{\alpha,0.5}\right)}{(2\pi)^{1/2}}\left(\log\phi_m + \frac{4-\log\phi_m}{\widehat{c}^2_{\alpha,0.5}} + O\left(\widehat{c}^{-4}_{\alpha,0.5}\right)\right),$$

as $\widehat{c}_{\alpha,0.5}\to\infty$. This proves the claim. □

Proof of Theorem 3.6. The proof follows, with minor modifications, the proofs of the results above, so we only outline its main passages. We begin with part *(i)*, and consider the case $E\log|\beta_0+\epsilon_{0,1}|<0$. Lemma C.2 entails that we need to prove only that

$$\begin{aligned}(\text{D.23})\qquad &\max_{1\leq k<\infty}\frac{1}{g_{m,\psi}(k)}\max_{1\leq \ell\leq k}\left|\sum_{i=m+\ell}^{m+k}\left(\frac{\epsilon_{i,1}\overline{y}^2_{i-1}}{1+\overline{y}^2_{i-1}}+\frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}^2_{i-1}}\right)\right.\\ &\left.-\frac{k-\ell}{m}\sum_{i=2}^{m}\left(\frac{\epsilon_{i,1}\overline{y}^2_{i-1}}{1+\overline{y}^2_{i-1}}+\frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}^2_{i-1}}\right)\right|\\ &\stackrel{\mathcal{D}}{\to}\sup_{0<x<\infty}\frac{\sup_{0\leq t\leq x}|(W_2(x)-W_2(t))-(x-t)W_1(1)|}{(1+x)(x/(1+x))^{\psi}},\end{aligned}$$

where recall that $\overline{y}_i$ is the stationary solution of (2.1). The approximations in Lemma C.4 imply

$$\begin{aligned}&\max_{1\leq k<\infty}\frac{1}{g_{m,\psi}(k)}\max_{1\leq \ell\leq k}\left|\sum_{i=m+\ell}^{m+k}\left(\frac{\epsilon_{i,1}\overline{y}^2_{i-1}}{1+\overline{y}^2_{i-1}}+\frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}^2_{i-1}}\right)-\frac{k-\ell}{m}\sum_{i=2}^{m}\left(\frac{\epsilon_{i,1}\overline{y}^2_{i-1}}{1+\overline{y}^2_{i-1}}+\frac{\epsilon_{i,2}\overline{y}_{i-1}}{1+\overline{y}^2_{i-1}}\right)\right.\\ &\left.-\mathfrak{s}\left(W_{m,2}(k)-W_{m,2}(\ell)-\frac{k-\ell}{m}W_{m,1}(m)\right)\right|\\ =&o_P(1).\end{aligned}$$

By the scale transformation and the continuity of the Wiener process we have

$$\begin{aligned}&\max_{1\leq k<\infty}\left(\frac{k+m}{k}\right)^{\psi}\frac{m}{m+k}\max_{1\leq \ell\leq k}\left|\left(W_{m,2}(k)-W_{m,2}(\ell)-\frac{k-\ell}{m}W_{m,1}(m)\right)\right|\\ &\stackrel{\mathcal{D}}{\to}\sup_{0<x<\infty}\left(\frac{1+x}{x}\right)^{\psi}\frac{1}{1+x}\sup_{0\leq t\leq x}|(W_2(x)-W_2(t))-(x-t)W_1(1)|,\end{aligned}$$

whence the desired result.

Next we consider the nonstationary case. Applying Lemma C.3*(ii)*, we can replace (D.23) with the proof of

$$\max_{1\leq k<\infty}\frac{1}{g_{m,\psi}(k)}\max_{1\leq\ell\leq k}\left|\sum_{i=m+\ell}^{m+k}\frac{\epsilon_{i,1}y_{i-1}^2}{1+y_{i-1}^2}-\frac{k-\ell}{m}\sum_{i=2}^{m}\frac{\epsilon_{i,1}y_{i-1}^2}{1+y_{i-1}^2}\right|$$
$$\stackrel{\mathcal{D}}{\to}\sup_{0<x<\infty}\left(\frac{1+x}{x}\right)^{\psi}\frac{1}{1+x}\sup_{0\leq t\leq x}\left|\left(W_2(x)-W_2(t)\right)-(x-t)W_1(1)\right|,$$

which follows from the approximations in Lemma C.4*(ii)*.

We conclude our proof by showing that

$$P\left\{\sup_{0<x<\infty}\left(\frac{1+x}{x}\right)^{\psi}\frac{1}{1+x}\sup_{0\leq t\leq x}\left|\left(W_2(x)-W_2(t)\right)-(x-t)W_1(1)\right|<\infty\right\}=1. \tag{D.24}$$

When $0<x\leq 1$, using Theorem 2.1 in Garsia et al. (1970),[6] it follows that there exists a random variable ζ such that $E|\zeta|^p<\infty$ for all $p\geq 1$ and

$$\left(\frac{1+x}{x}\right)^{\psi}\frac{1}{1+x}\sup_{0\leq t\leq x}\left|\left(W_2(x)-W_2(t)\right)\right|$$
$$\leq|\zeta|\left(\frac{1+x}{x}\right)^{\psi}\frac{1}{1+x}\sup_{0\leq t\leq x}\left|\left((x-t)\log\left(\frac{1}{x-t}\right)\right)^{1/2}\right|$$

a.s., whence

(D.25)

$$\left(\frac{1+x}{x}\right)^{\psi}\frac{1}{1+x}\sup_{0\leq t\leq x}\left|\left(W_2(x)-W_2(t)\right)-(x-t)W_1(1)\right|$$
$$\leq|\zeta|\left(\frac{1+x}{x}\right)^{\psi}\frac{1}{1+x}\sup_{0\leq t\leq x}\left|\left((x-t)\log\left(\frac{1}{x-t}\right)\right)^{1/2}\right|+W_1(1)\left(\frac{1+x}{x}\right)^{\psi}\frac{1}{1+x}\sup_{0\leq t\leq x}|x-t|.$$

[6] See also Lemma 4.1 in Csörgő and Horváth, 1993.

It is now easy to see that

$$|\zeta|\left(\frac{1+x}{x}\right)^{\psi}\frac{1}{1+x}\sup_{0\leq t\leq x}\left|\left((x-t)\log\left(\frac{1}{x-t}\right)\right)^{1/2}\right|=O_{P}(1).$$

As far as the second term in (D.25) is concerned, it immediately follows that

$$|W_{1}(1)|\left(\frac{1+x}{x}\right)^{\psi}\frac{1}{1+x}\sup_{0\leq t\leq x}|x-t|=O_{P}(1)\left(\frac{1+x}{x}\right)^{\psi-1}=O_{P}(1).$$

Also, by the Law of the Iterated Logarithm, it follows that

$$\begin{aligned}&\sup_{1\leq x<\infty}\left(\frac{1+x}{x}\right)^{\psi}\frac{1}{1+x}\sup_{0\leq t\leq x}\left|\left(W_{2}(x)-W_{2}(t)\right)-(x-t)W_{1}(1)\right|\\ \leq&\sup_{1\leq x<\infty}\frac{1}{x}2\sup_{0\leq t\leq x}|W_{2}(x)|+|W_{1}(1)|<\infty,\end{aligned}$$

a.s., whence (D.24) follows.

Finally, as far as parts *(ii)* and *(iii)* of the theorem are concerned, the proofs are based on the observation that

$$\max_{1\leq k\leq m^{*}}\frac{1}{g_{m,\psi}^{*}(k)}\max_{1\leq\ell\leq k}\frac{k-\ell}{m}\left|\sum_{i=2}^{m}\frac{\epsilon_{i,1}y_{i-1}^{2}}{1+y_{i-1}^{2}}+\frac{\epsilon_{i,2}y_{i-1}}{1+y_{i-1}^{2}}\right|=o_{P}(1),$$

and the same using $\overline{g}_{m,\psi}(k)$; hence, the arguments above can be repeated with minor modifications. □

Proof of Corollary 3.1. The result is shown e.g. in the proof of Theorem 3.4 in Horváth and Trapani (2016), or Corollary 3.1 in Horváth and Trapani (2023). □

In order to prove the next set of results, we will make use of the following decomposition, valid under (3.13). Let the recursion under the alternative be defined as

$$y_{(2),i}=\left(\beta_{A}+\epsilon_{i,1}\right)y_{(2),i-1}+\epsilon_{i,2},$$

for $i > 0$, with initial value $y_{(2),0}$. Then we have

(D.26)

$$Z_m(k) = \left| \sum_{i=m+1}^{m+k^*} \frac{\epsilon_{i,1} y_{i-1}^2 + \epsilon_{i,2} y_{i-1}}{1 + y_{i-1}^2} - \left(\widehat{\beta}_m - \beta_0\right) \left(\sum_{i=m+1}^{m+k^*} \frac{y_{i-1}^2}{1+y_{i-1}^2} + \sum_{i=m+k^*+1}^{m+k} \frac{y_{(2),i-1}^2}{1+y_{(2),i-1}^2} \right) \right.$$
$$+ (\beta_A - \beta_0) \left(\sum_{i=m+1}^{m+k} \frac{y_{(2),i-1}^2}{1+y_{(2),i-1}^2} - \sum_{i=m+1}^{m+k^*} \frac{y_{(2),i-1}^2}{1+y_{(2),i-1}^2} \right)$$
$$\left. + \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,1} y_{(2),i-1}^2 + \epsilon_{i,2} y_{(2),i-1}}{1+y_{(2),i-1}^2} - \sum_{i=m+1}^{m+k^*} \frac{\epsilon_{i,1} y_{(2),i-1}^2 + \epsilon_{i,2} y_{(2),i-1}}{1+y_{(2),i-1}^2} \right|$$
$$= |I + II + III + IV + V|.$$

Proof of Theorem 3.7. Based on the results above, the theorem follows from standard arguments - see e.g. Theorem 4.1 in Horváth and Trapani (2023). □

Proof of Theorem 3.8. The proof is inspired by Aue and Horváth (2004), but it differs substantially in some arguments. Let

$$N(m;x) = N = \left(\frac{\sigma_{(2)} c_{\alpha,\psi} m^{1/2-\psi}}{\Delta_m} - \left(\sigma_{(2)} x \frac{c_{\alpha,\psi}^{1/2-\psi} m^{(1-2\psi)^2}}{\Delta_m^{3/2-\psi}} \right)^{1/(1-\psi)} \right)^{1/(1-\psi)}, \tag{D.27}$$

and note the following facts

$$\lim_{m\to\infty} \frac{N}{m} = 0, \tag{D.28}$$

$$\lim_{m\to\infty} \sqrt{N} |\Delta_m| = \infty, \tag{D.29}$$

$$\lim_{m\to\infty} \frac{k^*}{m} = 0, \tag{D.30}$$

$$\lim_{m\to\infty} \frac{k^*}{N} = 0, \tag{D.31}$$

$$\lim_{m\to\infty} \frac{1}{\sigma_{(2)}} \left(\frac{N}{m}\right)^{\psi-1/2} \left(c_{\alpha,\psi} - \frac{N\Delta_m}{m^{1/2}(N/m)^{\psi}} \right) = x, \tag{D.32}$$

and, as $m \to \infty$

$$N \approx \left(\frac{\sigma_{(2)} c_{\alpha,\psi} m^{1/2-\psi}}{\Delta_m} \right)^{1/(1-\psi)}. \tag{D.33}$$

Recall (D.26). We begin by showing that

$$\left(\frac{N}{m} \right)^{\psi-1/2} \left(\max_{1 \le k \le k^*} \frac{|Z_m(k)|}{m^{1/2}(1+k/m)(k/(k+m))^{\psi}} - \frac{N\Delta_m}{m^{1/2}(N/m)^{\psi}} \right) \xrightarrow{\mathcal{P}} -\infty. \tag{D.34}$$

We begin by noting that, by (D.33)

$$\lim_{m \to \infty} \frac{N\Delta_m}{m^{1/2}(N/m)^{\psi}} = c_{\alpha,\psi} > 0. \tag{D.35}$$

Further, since for $k \le k^*$

$$Z_m(k) = \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,1} y_{i-1}^2 + \epsilon_{i,2} y_{i-1}}{1+y_{i-1}^2} - \left(\widehat{\beta}_m - \beta_0 \right) \sum_{i=m+1}^{m+k} \frac{y_{i-1}^2}{1+y_{i-1}^2},$$

it holds that

$$\begin{aligned}
&\max_{1 \le k \le k^*} \frac{|Z_m(k)|}{m^{1/2}(1+k/m)(k/(k+m))^{\psi}} \\
&\le \max_{1 \le k \le k^*} \frac{\left| \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,1} y_{i-1}^2 + \epsilon_{i,2} y_{i-1}}{1+y_{i-1}^2} \right|}{m^{1/2}(1+k/m)(k/(k+m))^{\psi}} \\
&\quad + \max_{1 \le k \le k^*} \frac{\left| \left(\widehat{\beta}_m - \beta_0 \right) \sum_{i=m+1}^{m+k} \frac{y_{i-1}^2}{1+y_{i-1}^2} \right|}{m^{1/2}(1+k/m)(k/(k+m))^{\psi}}.
\end{aligned}$$

Recalling that $\widehat{\beta}_m - \beta_0 = O_P\left(m^{-1/2}\right)$, irrespective of whether y_i is stationary or not, and using (C.10) and (C.13), it follows that, for all values of $E \log |\beta_0 + \epsilon_{i,1}|$

$$\begin{aligned}
&\max_{1 \le k \le k^*} \frac{\left| \left(\widehat{\beta}_m - \beta_0\right) \sum_{i=m+1}^{m+k} \frac{y_{i-1}^2}{1+y_{i-1}^2} \right|}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{\psi}} \\
=& O_P\left(1\right) m^{-1/2} \max_{1 \le k \le k^*} \frac{k}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{\psi}} \\
=& O_P\left(\left(\frac{k^*}{m}\right)^{1-\psi}\right) = o_P\left(1\right),
\end{aligned}$$

by (D.30). Also

$$\begin{aligned}
&\max_{1 \le k \le k^*} \frac{\left| \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,1} y_{i-1}^2 + \epsilon_{i,2} y_{i-1}}{1+y_{i-1}^2} \right|}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{\psi}} \\
\le& \max_{1 \le k \le k^*} \frac{\left| \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,1} y_{i-1}^2 + \epsilon_{i,2} y_{i-1}}{1+y_{i-1}^2} - W_m\left(k\right) \right|}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{\psi}} \\
&+ \max_{1 \le k \le k^*} \frac{\left|W_m\left(k\right)\right|}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{\psi}},
\end{aligned}$$

where $W_m\left(k\right)$ is a Wiener process. Using Lemma C.4, it holds that, for all values of $E \log |\beta_0 + \epsilon_{i,1}|$

$$\begin{aligned}
&\max_{1 \le k \le k^*} \frac{\left| \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,1} y_{i-1}^2 + \epsilon_{i,2} y_{i-1}}{1+y_{i-1}^2} - W_m\left(k\right) \right|}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{\psi}} \\
=& O_P\left(1\right) \max_{1 \le k \le k^*} \frac{k^{\zeta}}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{\psi}} \\
=& O_P\left(\left(\frac{k^*}{m}\right)^{1/2-\psi}\right) = o_P\left(1\right),
\end{aligned}$$

for some $0 < \zeta < 1/2$; further, by the Law of the Iterated Logarithm for Wiener processes (see e.g. Breiman, 1968)

$$\begin{aligned}
&\max_{1 \le k \le k^*} \frac{|W_m(k)|}{m^{1/2}(1+k/m)(k/(k+m))^{\psi}} \\
=&O_P(1) \max_{1 \le k \le k^*} \frac{k^{1/2}(\log\log k)^{1/2}}{m^{1/2}(1+k/m)(k/(k+m))^{\psi}} \\
=&(\log\log k^*)^{1/2} O_P\left(\left(\frac{k^*}{m}\right)^{1/2-\psi}\right) = o_P(1).
\end{aligned}$$

Putting all together, it follows that

$$\max_{1 \le k \le k^*} \frac{|Z_m(k)|}{m^{1/2}(1+k/m)(k/(k+m))^{\psi}} = o_P(1),$$

whence, using (D.35), (D.34) follows. We now turn to showing that

$$\left(\frac{N}{m}\right)^{\psi-1/2} \max_{k^* \le k \le N} \frac{\left|Z_m(k) - \left(\sigma_{(2)} W'_m(k) + a_{(2)} k \Delta_m\right)\right|}{m^{1/2}(1+k/m)(k/(k+m))^{\psi}} = o_P(1), \tag{D.36}$$

where $\{W'_m(k), k \ge 1\}$ is a standard Wiener and

$$a_{(2)} = \begin{cases} E\dfrac{\overline{y}^2_{(2),0}}{1+\overline{y}^2_{(2),0}} & \text{if } E\log|\beta_A + \epsilon_{i,1}| < 0 \\ 1 & \text{if } E\log|\beta_A + \epsilon_{i,1}| \ge 0 \end{cases}. \tag{D.37}$$

Indeed, considering I in (D.26), it holds that

$$\begin{aligned}
&\left|\widehat{\beta}_m - \beta_0\right| \left(\frac{N}{m}\right)^{\psi-1/2} \max_{k^* \le k \le N} \frac{\sum_{i=m+1}^{m+k^*} \frac{y^2_{i-1}}{1+y^2_{i-1}} + \sum_{i=m+k^*+1}^{m+k} \frac{y^2_{i-1}}{1+y^2_{i-1}}}{m^{1/2}(1+k/m)(k/(k+m))^{\psi}} \\
=&O_P(1)\, m^{-1/2} \left(\frac{N}{m}\right)^{\psi-1/2} \max_{k^* \le k \le N} \frac{c_0 k^* + c_1(k-k^*)}{m^{1/2}(1+k/m)(k/(k+m))^{\psi}} \\
=&O_P(1) \frac{N^{\psi}}{m^{1/2}} = o_P(1),
\end{aligned}$$

by (D.28). Also, note that, using (D.28)-(D.32)

$$\begin{aligned}
&\left(\frac{N}{m}\right)^{\psi-1/2}\max_{k^*\leq k\leq N}\frac{k^*\Delta_m}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{\psi}}\\
=&O\left(1\right)\Delta_m N^{\psi-1/2}\left(k^*\right)^{1-\psi}=o\left(1\right),
\end{aligned}$$

as shown in equation (3.13) in Aue and Horváth (2004). Therefore it is not hard to see that

$$\left(\frac{N}{m}\right)^{\psi-1/2}\Delta_m\max_{k^*\leq k\leq N}\frac{\displaystyle\sum_{i=m+1}^{m+k}\frac{y_{(2),i-1}^2}{1+y_{(2),i-1}^2}-\sum_{i=m+k^*+1}^{m+k}\frac{y_{(2),i-1}^2}{1+y_{(2),i-1}^2}}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{\psi}}=o_P\left(1\right);$$

also, using Lemma C.3 it follows that, irrespective of $E\log\left|\beta_A+\epsilon_{i,1}\right|$

$$\left(\frac{N}{m}\right)^{\psi-1/2}\Delta_m\left(\max_{k^*\leq k\leq N}\frac{\left|\displaystyle\sum_{i=m+1}^{m+k}\frac{y_{(2),i-1}^2}{1+y_{(2),i-1}^2}-ka_{(2)}\right|}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{\psi}}-\frac{Na_{(2)}}{m^{1/2}\left(k/m\right)^{\psi}}\right)\xrightarrow{\mathcal{P}}-\infty.$$

Similarly, noting that

$$\sum_{i=m+1}^{m+k^*}\frac{\epsilon_{i,1}y_{i-1}^2+\epsilon_{i,2}y_{i-1}}{1+y_{i-1}^2}=O_P\left(\sqrt{k^*}\right),$$

for all values of $E\log\left|\beta_A+\epsilon_{i,1}\right|$, it follows that

$$\begin{aligned}
&\left(\frac{N}{m}\right)^{\psi-1/2}\max_{k^*\leq k\leq N}\frac{\left|\displaystyle\sum_{i=m+1}^{m+k^*}\frac{\epsilon_{i,1}y_{i-1}^2+\epsilon_{i,2}y_{i-1}}{1+y_{i-1}^2}\right|}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{\psi}}\\
=&O_P\left(1\right)\left(\frac{N}{m}\right)^{\psi-1/2}\max_{k^*\leq k\leq N}\frac{\sqrt{k^*}}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{\psi}}\\
=&O_P\left(\left(\frac{k^*}{N}\right)^{1/2-\psi}\right)=o_P\left(1\right),
\end{aligned}$$

and by the same token it can be shown that

$$\begin{aligned}&\left(\frac{N}{m}\right)^{\psi-1/2}\max_{k^*\leq k\leq N}\frac{\left|\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,1}y^2_{(2),i-1}+\epsilon_{i,2}y_{(2),i-1}}{1+y^2_{(2),i-1}}-\sum_{i=m+k^*+1}^{m+k}\frac{\epsilon_{i,1}y^2_{(2),i-1}+\epsilon_{i,2}y_{(2),i-1}}{1+y^2_{(2),i-1}}\right|}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{\psi}}\\ =\ & o_P\left(1\right),\end{aligned}$$

and also

$$\begin{aligned}&\left(\frac{N}{m}\right)^{\psi-1/2}\max_{k^*\leq k\leq N}\frac{\left|\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,1}y^2_{(2),i-1}+\epsilon_{i,2}y_{(2),i-1}}{1+y^2_{(2),i-1}}-\sigma_{(2)}W'_m\left(k\right)\right|}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{\psi}}\\ =&O_P\left(1\right)\left(\frac{N}{m}\right)^{\psi-1/2}\max_{k^*\leq k\leq N}\frac{k^{\zeta}}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{\psi}}\\ =&O_P\left(1\right)N^{\psi-1/2}\max_{k^*\leq k\leq N}k^{\zeta-\psi}=o_P\left(1\right),\end{aligned}$$

for some $0<\zeta<1/2$. Putting all together, (D.36) finally follows. From hereon, the proof of Theorem 1.1 in Aue and Horváth (2004) can be followed *verbatim*, yielding the final result. □

Proof of Theorem 3.9. The proof builds on the results derived in Aue et al. (2008) and on the proof of Theorem 3.8, and we report only the main passages. Let

$$N'\left(m;x\right)=N'=\sigma^2_{(2)}\left(\frac{c_{\alpha,0.5}-x}{\Delta_m}\right)^2, \tag{D.38}$$

and note that (see equation (4.2) in Aue et al., 2008)

$$\lim_{m\rightarrow\infty}N'\left(\frac{2\sigma^2_{(2)}\log\log m}{\Delta^2_m}\right)^{-1}=1, \tag{D.39}$$

and, using (3.21)

$$\lim_{m\rightarrow\infty}\frac{k^*}{m}=0. \tag{D.40}$$

We begin by showing that

$$\max_{1\le k\le k^*}\frac{|Z_m(k)|}{m^{1/2}(1+k/m)(k/(k+m))^{1/2}}-\sqrt{N'}\Delta_m\overset{\mathcal{P}}{\to}-\infty. \tag{D.41}$$

We begin by noting that, by (D.39)

$$\lim_{m\to\infty}\frac{\sqrt{N'}\Delta_m}{\sqrt{\log\log m}}=1. \tag{D.42}$$

Recalling that, for $k\le k^*$

$$Z_m(k)=\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,1}y_{i-1}^2+\epsilon_{i,2}y_{i-1}}{1+y_{i-1}^2}-\left(\widehat{\beta}_m-\beta_0\right)\sum_{i=m+1}^{m+k}\frac{y_{i-1}^2}{1+y_{i-1}^2},$$

it holds that

$$\begin{aligned}&\max_{1\le k\le k^*}\frac{|Z_m(k)|}{m^{1/2}(1+k/m)(k/(k+m))^{1/2}}\\ \le&\max_{1\le k\le k^*}\frac{\left|\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,1}y_{i-1}^2+\epsilon_{i,2}y_{i-1}}{1+y_{i-1}^2}\right|}{m^{1/2}(1+k/m)(k/(k+m))^{1/2}}\\ &+\max_{1\le k\le k^*}\frac{\left|\left(\widehat{\beta}_m-\beta_0\right)\sum_{i=m+1}^{m+k}\frac{y_{i-1}^2}{1+y_{i-1}^2}\right|}{m^{1/2}(1+k/m)(k/(k+m))^{1/2}}.\end{aligned}$$

Using the fact that $\widehat{\beta}_m-\beta_0=O_P\left(m^{-1/2}\right)$, irrespective of whether y_i is stationary or not, and (C.10) and (C.13), it follows that, for all values of $E\log|\beta_0+\epsilon_{i,1}|$

$$\begin{aligned}&\max_{1\le k\le k^*}\frac{\left|\left(\widehat{\beta}_m-\beta_0\right)\sum_{i=m+1}^{m+k}\frac{y_{i-1}^2}{1+y_{i-1}^2}\right|}{m^{1/2}(1+k/m)(k/(k+m))^{1/2}}\\ =&O_P(1)\,m^{-1/2}\max_{1\le k\le k^*}\frac{k}{m^{1/2}(k/m)^{1/2}}=O_P\left(\left(\frac{k^*}{m}\right)^{1/2}\right)=o_P(1),\end{aligned}$$

by (D.30). Also

$$
\begin{aligned}
&\max_{1\le k\le k^*}\frac{\left|\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,1}y_{i-1}^2+\epsilon_{i,2}y_{i-1}}{1+y_{i-1}^2}\right|}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{1/2}}\\
\le &\max_{1\le k\le k^*}\frac{\left|\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,1}y_{i-1}^2+\epsilon_{i,2}y_{i-1}}{1+y_{i-1}^2}-W_m\left(k\right)\right|}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{1/2}}\\
&+\max_{1\le k\le k^*}\frac{\left|W_m\left(k\right)\right|}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{1/2}},
\end{aligned}
$$

where $W_m\left(k\right)$ is a Wiener process. Using Lemma C.4

$$
\begin{aligned}
&\max_{1\le k\le k^*}\frac{\left|\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,1}y_{i-1}^2+\epsilon_{i,2}y_{i-1}}{1+y_{i-1}^2}-W_m\left(k\right)\right|}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{1/2}}\\
=&O_P\left(1\right)\max_{1\le k\le k^*}\frac{k^{\zeta}}{m^{1/2}\left(k/m\right)^{1/2}}=O_P\left(1\right),
\end{aligned}
$$

for some $0<\zeta<1/2$ and all values of $E\log\left|\beta_0+\epsilon_{i,1}\right|$. Further, using the Law of the Iterated Logarithm for Wiener processes (see e.g. Breiman, 1968), it follows that

$$
\begin{aligned}
&\max_{1\le k\le k^*}\frac{\left|W_m\left(k\right)\right|}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{1/2}}\\
=&O_P\left(1\right)\max_{1\le k\le k^*}\frac{k^{1/2}\left(\log\log k\right)^{1/2}}{m^{1/2}\left(k/m\right)^{1/2}}=O_P\left(\left(\log\log k^*\right)^{1/2}\right)=O_P\left(\left(\log\log\log m\right)^{1/2}\right),
\end{aligned}
$$

having used (3.21). Putting all together, it follows that

$$
\max_{1\le k\le k^*}\frac{\left|Z_m\left(k\right)\right|}{m^{1/2}\left(1+k/m\right)\left(k/\left(k+m\right)\right)^{1/2}}=O_P\left(\left(\log\log\log m\right)^{1/2}\right),
$$

whence, using (D.42), (D.41) follows. We now turn to showing that

$$\text{(D.43)} \qquad \max_{k^* \le k \le N'} \frac{\left| Z_m(k) - \left(\sigma_{(2)} W_m'(k) + a_{(2)} k \Delta_m \right) \right|}{m^{1/2} (1 + k/m) (k/(k+m))^{1/2}} = o_P\left((\log \log m)^{1/2} \right),$$

where $\{W_m'(k), k \ge 1\}$ is a standard Wiener and $a_{(2)}$ is defined in (D.37). As in the proof of Theorem 3.8, it follows that, for all values of $E \log |\beta_0 + \epsilon_{i,1}|$

$$\begin{aligned} & \left| \widehat{\beta}_m - \beta_0 \right| \max_{k^* \le k \le N'} \frac{\sum_{i=m+1}^{m+k^*} \frac{y_{i-1}^2}{1+y_{i-1}^2} + \sum_{i=m+k^*+1}^{m+k} \frac{y_{i-1}^2}{1+y_{i-1}^2}}{m^{1/2} (1 + k/m) (k/(k+m))^{1/2}} \\ =& O_P(1) m^{-1/2} \max_{k^* \le k \le N'} \frac{c_0 k^* + c_1 (k - k^*)}{m^{1/2} (1 + k/m) (k/(k+m))^{1/2}} = O_P\left(\left(\frac{N'}{m} \right)^{1/2} \right) = o_P(1). \end{aligned}$$

Moreover

$$\begin{aligned} \text{(D.44)} \qquad & \Delta_m \max_{k^* \le k \le N'} \frac{\sum_{i=m+1}^{m+k} \frac{y_{(2),i-1}^2}{1+y_{(2),i-1}^2} - \sum_{i=m+k^*+1}^{m+k} \frac{y_{i-1}^2}{1+y_{i-1}^2}}{m^{1/2} (1 + k/m) (k/(k+m))^{1/2}} \\ =& O_P(1) \max_{k^* \le k \le N'} \frac{k^* \Delta_m}{m^{1/2} (1 + k/m) (k/(k+m))^{1/2}} \\ =& O(1) \Delta_m (k^*)^{1/2} = o\left((\log \log m)^{1/4} \right), \end{aligned}$$

and

$$\text{(D.45)} \qquad \lim_{m \to \infty} (N')^{-1/2} \max_{k^* \le k \le N'} \frac{\sum_{i=m+1}^{m+k} \frac{y_{(2),i-1}^2}{1+y_{(2),i-1}^2}}{m^{1/2} (1 + k/m) (k/(k+m))^{1/2}} = a_{(2)} > 0.$$

Recalling that

$$\sum_{i=m+1}^{m+k^*} \frac{\epsilon_{i,1} y_{i-1}^2 + \epsilon_{i,2} y_{i-1}}{1 + y_{i-1}^2} = O_P\left(\sqrt{k^*} \right),$$

for all values of $E \log |\beta_A + \epsilon_{i,1}|$, it follows that

$$\text{(D.46)} \qquad \max_{k^* \leq k \leq N'} \frac{\left| \sum_{i=m+1}^{m+k^*} \frac{\epsilon_{i,1} y_{i-1}^2 + \epsilon_{i,2} y_{i-1}}{1+y_{i-1}^2} \right|}{m^{1/2} (1+k/m) \left(k/(k+m)\right)^{1/2}} = O_P(1) \max_{k^* \leq k \leq N} \frac{\sqrt{k^*}}{k^{1/2}} = O_P(1);$$

furthermore we have

(D.47)

$$\max_{k^* \leq k \leq N'} \frac{\left| \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,1} y_{(2),i-1}^2 + \epsilon_{i,2} y_{(2),i-1}}{1+y_{(2),i-1}^2} - \sigma_{(2)} W_m'(k) \right|}{m^{1/2} (1+k/m) \left(k/(k+m)\right)^{1/2}} = O_P(1) \max_{k^* \leq k \leq N} k^{\zeta - 1/2} = O_P(1).$$

Recalling that $k^* = o(N')$ and using the Law of the Iterated Logarithm, it follows from (D.47) that

$$\text{(D.48)} \qquad \lim_{m \to \infty} \frac{1}{\sqrt{2 \log \log N'}} \max_{k^* \leq k \leq N'} \frac{\left| \sum_{i=m+1}^{m+k} \frac{\epsilon_{i,1} y_{(2),i-1}^2 + \epsilon_{i,2} y_{(2),i-1}}{1+y_{(2),i-1}^2} \right|}{m^{1/2} (1+k/m) \left(k/(k+m)\right)^{1/2}} \xrightarrow{\mathcal{P}} c_1 > 0.$$

Putting together (D.44)-(D.48), it follows that

$$\lim_{m \to \infty} P \left\{ \max_{k^* \leq k \leq N'} \frac{|Z_m(k)|}{m^{1/2} (1+k/m) \left(k/(k+m)\right)^{1/2}} = \max_{k^* \leq k \leq N'} \frac{\left|\sigma_{(2)} W_m'(k) + a_{(2)} k\right|}{m^{1/2} (1+k/m) \left(k/(k+m)\right)^{1/2}} \right\} = 1.$$

From hereon, the proof of Theorem 2.2 in Aue et al. (2008) can be followed *verbatim*, yielding the final result. □

Proof of Theorem 4.1. We begin by decomposing the detector $Z_m^X(k)$ as

$$\text{(D.49)} \qquad Z_m^X(k) = \left| \left(\mathbf{b}_0 - \widehat{\mathbf{b}}_m \right)^\intercal \sum_{i=m+1}^{m+k} \left(y_{i-1}, \mathbf{x}_i^\intercal\right) \frac{y_{i-1}}{1+y_{i-1}^2} + \sum_{i=m+1}^{m+k} \frac{\left(\epsilon_{i,1} y_{i-1} + \epsilon_{i,2}\right) y_{i-1}}{1+y_{i-1}^2} \right|.$$

Henceforth, the proof is very similar to the proof of Theorem 3.1, and we only report the main passages where the two proofs differ.

Consider first the case $E\log|\beta_0+\epsilon_{0,1}|<0$; using Lemmas C.7-C.9 instead of Lemmas C.2-C.4, it can be shown along the same lines as in the proof of Theorem 3.1 that

$$\max_{1\leq k<\infty}\frac{Z_m^X(k)}{m^{1/2}\left(1+\frac{k}{\mathfrak{s}_x^2 m}\right)\left(\frac{k}{k+\mathfrak{s}_x^2 m}\right)^{\psi}}\xrightarrow{\mathcal{D}}\sup_{0<t<\infty}\frac{\left|-\mathfrak{s}_{x,1}tW_1(1)+\mathfrak{s}_{x,2}W_2(t)\right|}{\left(1+\frac{t}{\mathfrak{s}_x^2}\right)\left(\frac{t}{t+\mathfrak{s}_x^2}\right)^{\psi}},$$

where $\{W_1(t),t\geq 0\}$ and $\{W_2(t),t\geq 0\}$ are two independent standard Wiener processes. It is not hard to see that

$$\sup_{0<t<\infty}\frac{\left|-\mathfrak{s}_{x,1}tW_1(1)+\mathfrak{s}_{x,2}W_2(t)\right|}{\left(1+\frac{t}{\mathfrak{s}_x^2}\right)\left(\frac{t}{t+\mathfrak{s}_x^2}\right)^{\psi}}\overset{\mathcal{D}}{=}\sup_{0<u<\infty}\frac{\left|-\mathfrak{s}_{x,1}\mathfrak{s}_x^2 uW_1(1)+\mathfrak{s}_{x,2}W_2(\mathfrak{s}_x^2 u)\right|}{(1+u)\left(\frac{u}{1+u}\right)^{\psi}},$$

and that, for $u\leq v$

$$\begin{aligned}&E\left(\left(-\mathfrak{s}_{x,1}\mathfrak{s}_x^2 uW_1(1)+\mathfrak{s}_{x,2}W_2\left(\mathfrak{s}_x^2 u\right)\right)\left(-\mathfrak{s}_{x,1}\mathfrak{s}_x^2 vW_1(1)+\mathfrak{s}_{x,2}W_2\left(\mathfrak{s}_x^2 v\right)\right)\right)\\=&\mathfrak{s}_{x,1}^2\left(\mathfrak{s}_x^2\right)^2 uv+\mathfrak{s}_{x,2}^2\mathfrak{s}_x^2 u=\frac{\mathfrak{s}_{x,2}^4}{\mathfrak{s}_{x,1}^2}(uv+u).\end{aligned}$$

Hence it follows that

$$\left\{\frac{-\mathfrak{s}_{x,1}\mathfrak{s}_x^2 uW_1(1)+\mathfrak{s}_{x,2}W_2(\mathfrak{s}_x^2 u)}{1+u},u\geq 0\right\}\overset{\mathcal{D}}{=}\left\{\frac{\mathfrak{s}_{x,2}^2}{\mathfrak{s}_{x,1}}W(u),u\geq 0\right\},$$

where $\{W(u),u\geq 0\}$ is an independent standard Wiener process. This completes the proof when $E\log|\beta_0+\epsilon_{0,1}|<0$.

When $E\log|\beta_0+\epsilon_{0,1}|>0$, consider (D.49) again. Seeing as

$$\left\|\sum_{i=m+1}^{\infty}\frac{\mathbf{x}_i y_{i-1}}{1+y_{i-1}^2}\right\|=O_P(1), \tag{D.50}$$

it is easy to see that

$$
\begin{aligned}
&\left|\left(\mathbf{b}_0-\widehat{\mathbf{b}}_m\right)^{\intercal}\left(\sum_{i=m+1}^{m+k}\left(y_{i-1},\mathbf{x}_i^{\intercal}\right)\frac{y_{i-1}}{1+y_{i-1}^2}-\sum_{i=m+1}^{m+k}\left(y_{i-1},\mathbf{0}_p^{\intercal}\right)\frac{y_{i-1}}{1+y_{i-1}^2}\right)\right| \\
\leq&\left\|\mathbf{b}_0-\widehat{\mathbf{b}}_m\right\|\left\|\sum_{i=m+1}^{m+k}\left(y_{i-1},\mathbf{x}_i^{\intercal}\right)\frac{y_{i-1}}{1+y_{i-1}^2}-\sum_{i=m+1}^{m+k}\left(y_{i-1},\mathbf{0}_p^{\intercal}\right)\frac{y_{i-1}}{1+y_{i-1}^2}\right\| \\
=&O_P\left(m^{-1/2}\right),
\end{aligned}
$$

where $\mathbf{0}_p$ is a p-dimensional vector of zeros. Similarly, using (D.50) and standard algebra, it follows that

$$
\left\|\left(\mathbf{b}_0-\widehat{\mathbf{b}}_m\right)^{\intercal}-\left(\left(\sum_{i=2}^{m}\frac{y_{i-1}^2}{1+y_{i-1}^2}\right)^{-1}\left(\sum_{i=2}^{m}\frac{\epsilon_{i,1}y_{i-1}^2}{1+y_{i-1}^2}\right),\mathbf{0}_p^{\intercal}\right)\right\|=O_P\left(m^{-1/2}\right),
$$

and by (C.13)

$$
\max_{1\leq k<\infty}\frac{1}{k}\left|\left(\sum_{i=2}^{m}\frac{y_{i-1}^2}{1+y_{i-1}^2}\right)^{-1}\left(\sum_{i=2}^{m}\frac{\epsilon_{i,1}y_{i-1}^2}{1+y_{i-1}^2}\right)\left(\sum_{i=m+1}^{m+k}\frac{y_{i-1}^2}{1+y_{i-1}^2}-k\right)\right|=O_P\left(m^{-1/2-\zeta}\right),
$$

for some $\zeta>0$, and similarly to (C.5)-(C.6)

$$
\left|\left(\sum_{i=2}^{m}\frac{y_{i-1}^2}{1+y_{i-1}^2}\right)^{-1}\left(\sum_{i=2}^{m}\frac{\epsilon_{i,1}y_{i-1}^2}{1+y_{i-1}^2}\right)-\frac{1}{m}\sum_{i=2}^{m}\epsilon_{i,1}\right|=O_P\left(m^{1/2-\zeta}\right).
$$

Thus, putting all together

$$
\max_{1\leq k<\infty}\frac{1}{k}\left|\left(\mathbf{b}_0-\widehat{\mathbf{b}}_m\right)^{\intercal}\sum_{i=m+1}^{m+k}\left(y_{i-1},\mathbf{x}_i^{\intercal}\right)\frac{y_{i-1}}{1+y_{i-1}^2}-\frac{k}{m}\sum_{i=2}^{m}\epsilon_{i,1}\right|=O_P\left(m^{-1/2-\zeta}\right).
$$

Finally, similarly to (C.14) it can be shown

$$
\max_{1\leq k<\infty}k^{-1/2-\eta}\left|\sum_{i=m+1}^{m+k}\frac{\epsilon_{i,1}y_{i-1}^2}{1+y_{i-1}^2}-\sum_{i=m+1}^{m+k}\epsilon_{i,1}\right|=O_P\left(m^{-\zeta}\right),
$$

for some $\zeta > 0$ and for all $\eta > 0$. Putting all together and recalling that, when $E \log |\beta_0 + \epsilon_{0,1}| > 0$, $\mathfrak{s}_x^2 = 1$, it follows that

$$\max_{1 \leq k < \infty} \frac{\left| Z_m^X(k) - \left| \sum_{i=m+1}^{m+k} \epsilon_{i,1} - \frac{k}{m} \sum_{i=2}^{m} \epsilon_{i,1} \right| \right|}{m^{1/2} \left(1 + \frac{k}{m}\right) \left(\frac{k}{k+m}\right)^{\psi}} = o_P(1).$$

Hereafter, the final result follows by making appeal to the proof of Theorem 3.1. □

Proof of Theorem 4.2. The proof follows immediately from the approximations derived above, and from the proof of Theorem 3.4. □

Proof of Theorem 4.10. The proof follows by the same arguments as the proof of Theorem 3.6, and we therefore omit it to save space. □

Proof of Theorem 4.4. The proof follows from the same arguments as in the case with no covariates. □

Proof of Theorem C.1. The proof is essentially the same as the proofs above, so we only report some arguments thereof. We begin with the stationary case $E \log |\beta_0 + \epsilon_{0,1}| < 0$. In such a case, the proof follows from marginally adapting the above, upon defining $\widetilde{\mathbf{x}}_i = (1, \mathbf{x}_i^\intercal)^\intercal$ and subsequently redefining $\mathbf{z}_i$, $\boldsymbol{\eta}_i$, $\mathbf{Q}$, $\mathbf{C}$ and $\mathbf{a}$ defined in (C.24)-(C.28) as

$$\widetilde{\mathbf{z}}_i = \frac{1}{\left(1 + \overline{y}_{i-1}^2\right)^{1/2}} \left(\overline{y}_{i-1}, \widetilde{\mathbf{x}}_i^\intercal\right)^\intercal,$$

$$\widetilde{\boldsymbol{\eta}}_i = \left(\frac{\left(\epsilon_{i,1}\overline{y}_{i-1} + \epsilon_{i,2}\right) \overline{y}_{i-1}}{1 + \overline{y}_{i-1}^2}, \frac{\left(\epsilon_{i,1}\overline{y}_{i-1} + \epsilon_{i,2}\right) \widetilde{\mathbf{x}}_i^\intercal}{1 + \overline{y}_{i-1}^2} \right)^\intercal,$$

$$\widetilde{\mathbf{Q}} = E\left(\widetilde{\mathbf{z}}_1 \widetilde{\mathbf{z}}_1^\intercal\right),$$

$$\widetilde{\mathbf{C}} = E\left(\widetilde{\boldsymbol{\eta}}_0 \widetilde{\boldsymbol{\eta}}_0^\intercal\right),$$

$$\widetilde{\mathbf{a}} = E\left(\frac{\left(\overline{y}_1, \widetilde{\mathbf{x}}_2^\intercal\right)^\intercal \overline{y}_1}{1 + \overline{y}_1^2} \right).$$

We note that the denominator of $\widetilde{\mathbf{b}}_m$ can be approximated by

$$E\frac{\overline{\mathbf{w}}_0\overline{\mathbf{w}}_0^\intercal}{1+\overline{y}_{-1}^2} - \left(E\frac{1}{1+\overline{y}_{-1}^2}\right)^{-1}\left(E\frac{\overline{\mathbf{w}}_0}{1+\overline{y}_{-1}^2}\right)\left(E\frac{\overline{\mathbf{w}}_0}{1+\overline{y}_{-1}^2}\right)^\intercal,$$

where $\overline{\mathbf{w}}_i = \left(\overline{y}_{i-1}, \mathbf{x}_i^\intercal\right)^\intercal$. In order to ensure that this is nonzero, note that, for all nontrivial vectors $\mathbf{b}$, it holds that

$$\left(E\frac{\mathbf{b}^\intercal\overline{\mathbf{w}}_0}{1+\overline{y}_{-1}^2}\right)^2 \leq \left(E\frac{1}{1+\overline{y}_{-1}^2}\right)\left(E\frac{\mathbf{b}^\intercal\overline{\mathbf{w}}_0\overline{\mathbf{w}}_0^\intercal\mathbf{b}}{1+\overline{y}_{-1}^2}\right), \tag{D.51}$$

by the Cauchy-Schwartz inequality, whence also

$$\left(E\frac{\mathbf{b}^\intercal\overline{\mathbf{w}}_0\overline{\mathbf{w}}_0^\intercal\mathbf{b}}{1+\overline{y}_{-1}^2}\right) \geq \left(E\frac{1}{1+\overline{y}_{-1}^2}\right)^{-1}\mathbf{b}^\intercal\left(E\frac{\overline{\mathbf{w}}_0}{1+\overline{y}_{-1}^2}\right)\left(E\frac{\overline{\mathbf{w}}_0^\intercal}{1+\overline{y}_{-1}^2}\right)\mathbf{b}.$$

The equality in (D.51) holds if and only if

$$P\left(\frac{\mathbf{b}^\intercal\overline{\mathbf{w}}_0\overline{\mathbf{w}}_0^\intercal\mathbf{b}}{1+\overline{y}_{-1}^2} = c_0\frac{1}{1+\overline{y}_{-1}^2}\right) = 1,$$

for some constant c_0. In turn, this holds if and only if

$$P\left(\mathbf{b}^\intercal\overline{\mathbf{w}}_0\overline{\mathbf{w}}_0^\intercal\mathbf{b} = c_0\right) = 1,$$

which requires $P\left(\overline{\mathbf{w}}_0 = \mathbf{c}_1\right) = 1$, for some vector of constants $\mathbf{c}_1$. But this is ruled out by Assumption C.1.

When $E\log|\beta_0+\epsilon_{0,1}| > 0$, the fact that $|y_i|$ grows exponentially readily entails that

$$\left(\sum_{i=2}^m\frac{1}{1+y_{i-1}^2}\right)^{-1}\left(\sum_{i=2}^m\frac{\overline{\mathbf{w}}_i}{1+y_{i-1}^2}\right)\left(\sum_{i=2}^m\frac{\overline{\mathbf{w}}_i^\intercal}{1+y_{i-1}^2}\right) = O_P(1),$$

whence the result in Lemma C.10 holds also in this case *mutatis mutandis*. The rest of the proof then follows from the same passages as above. □

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