SUPPLEMENTARY MATERIAL

Is Completeness Necessary? Estimation in Nonidentified Linear Models

Andrii Babii
UNC-Chapel Hill
and
Jean-Pierre Florens
Toulouse School of Economics

January 21, 2025

S.1 Generalized Inverse

In this section, we collect some facts about the generalized inverse operator from operator theory; see also Carrasco, Florens, and Renault (2007) for a comprehensive review of different aspects of the theory of ill-posed inverse models in econometrics. Let $\varphi \in \mathcal{E}$ be a structural parameter in a Hilbert space $\mathcal{E}$ and let $K : \mathcal{E} \to \mathcal{H}$ be a bounded linear operator mapping to a Hilbert space $\mathcal{H}$. Consider the functional equation

$$K\varphi = r.$$

If the operator K is not one-to-one, then the structural parameter φ is not point identified, and the identified set is a closed linear manifold described as $\Phi^{\mathrm{ID}} = \varphi + \mathcal{N}(K)$, where $\mathcal{N}(K) = \{\phi : K\phi = 0\}$ is the null space of K; see Figure S.1. The following result offers equivalent characterizations of the identified set; see Groetsch (1977), Theorem 3.1.1 for a formal proof.

Proposition S.1.1. *The identified set I_0 is characterized as a set of solutions to*

(i) the least-squares problem: $\min_{\phi \in \mathcal{E}} \|K\phi - r\|$;

(ii) the normal equations: $K^*K\phi = K^*r$, *where K^* is the adjoint operator of K.*

The generalized inverse is formally defined below.

Definition S.1.1. *The generalized inverse of the operator K is a unique linear operator $K^\dagger : \mathcal{R}(K) \oplus \mathcal{R}(K)^\perp \to \mathcal{E}$, defined by $K^\dagger r = \varphi_1$, where $\varphi_1 \in I_0$ is a unique solution to*

$$\min_{\phi \in I_0} \|\phi\|. \tag{S.1}$$

For nonidentified linear models, the generalized inverse maps r to the unique minimal norm element of I_0. It follows from equation (S.1) that φ_1 is a projection of 0 on the identified set. Therefore, φ_1 is the projection of the structural parameter φ on the orthogonal complement to the null space $\mathcal{N}(K)^\perp$, see Figure S.1, and we call φ_1 the best approximation to the structural parameter φ. The generalized inverse operator is typically a discontinuous map, as illustrated in the following proposition; see Groetsch (1977), pp.117-118 for more details.

Proposition S.1.2. *Suppose that the operator K is compact. Then the generalized inverse $K^\dagger$ is continuous if and only if $\mathcal{R}(K)$ is finite-dimensional.*

The following example illustrates this when K is an integral operator on spaces of square-integrable functions.

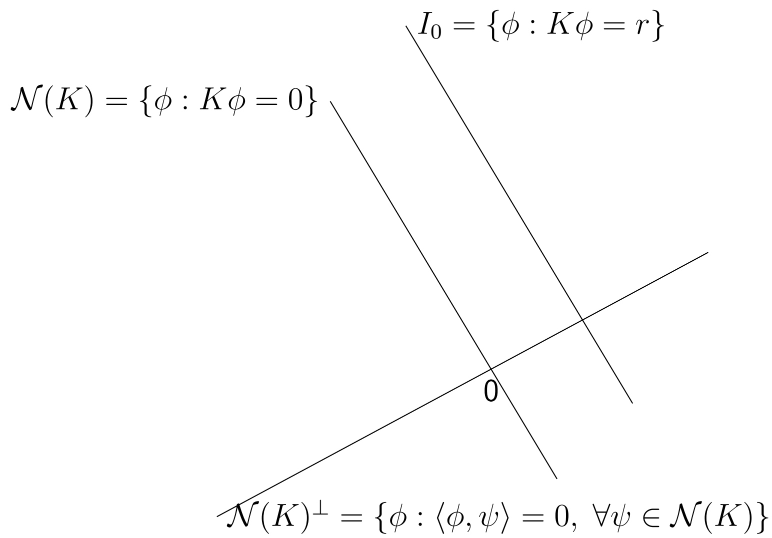

Figure S.1: Fundamental subspaces of $\mathcal{E}$.

Example S.1.1. *Suppose that K is an integral operator*

$$\begin{aligned} K : L_2 &\to L_2 \\ \phi &\mapsto \int \phi(z)k(z,w)\mathrm{d}z. \end{aligned}$$

Then K is compact whenever the kernel function k is square integrable. In this case, the generalized inverse is continuous if and only if k is a degenerate kernel function

$$k(z,w) = \sum_{j=1}^{m} \phi_j(z)\psi_j(w).$$

It is worth stressing that in the NPIV model, the kernel function k is typically a non-degenerate probability density function. Moreover, in econometric applications, r is usually estimated from the data, so that $K^\dagger \hat{r} \xrightarrow{p} K^\dagger r = \varphi_1$ may not hold even when $\hat{r} \xrightarrow{p} r$ due to the discontinuity of $K^\dagger$.[1] In other words, we are faced with an ill-posed inverse problem. Tikhonov regularization can be understood as a method that smooths out the discontinuities of the generalized inverse $(K^*K)^\dagger$.[2]

S.2 Degenerate U-statistics in Hilbert Spaces

S.2.1 Wiener-Itô Integral

This section reviews key results on the asymptotic distribution of degenerate U-statistics in Hilbert spaces. Let $(\mathcal{X}, \Sigma, \mu)$ be a measure space, and let H be a separable Hilbert space. We denote by $L_2(\mathcal{X}^m, H)$ the space of functions $f : \mathcal{X}^m \to H$ satisfying $\mathbb{E}\|f(X_1,\ldots,X_m)\|^2 < \infty$. A stochastic process $\{\mathbb{W}(A), A \in \Sigma_\mu\}$, indexed by the σ-field $\Sigma_\mu = \{A \in \Sigma : \mu(A) < \infty\}$ is called a *Gaussian random measure* if:

1. For all $A \in \Sigma_\mu$,
$$\mathbb{W}(A) \sim N(0, \mu(A)).$$
2. For any collection of disjoint sets $(A_k)_{k=1}^K$ in Σ_μ, the random variables $\mathbb{W}(A_k), k = 1,\ldots,K$ are independent, and
$$\mathbb{W}\left(\bigcup_{k=1}^K A_k\right) = \sum_{k=1}^K \mathbb{W}(A_k).$$

[1] In practice, the situation is even more complex because the operator K is also estimated from the data.

[2] By Proposition S.1.1, solving $K\varphi = r$ is equivalent to solving $K^*K\varphi = K^*r$. The latter is more attractive to work with because the spectral theory of self-adjoint operators in Hilbert spaces applies to K^*K.

Now, let $(A_k)_{k=1}^K$ be pairwise disjoint sets in Σ_μ, and consider the set S_m of simple functions $f \in L_2(\mathcal{X}^m, H)$ of the form

$$f(x_1, \dots, x_m) = \sum_{i_1,\dots,i_m=1}^{K} c_{i_1,\dots,i_m} \mathbb{1}_{A_{i_1}}(x_1) \times \cdots \times \mathbb{1}_{A_{i_m}}(x_m),$$

where $c_{i_1,\dots,i_m} = 0$ if any two indices $i_1, \dots, i_m$ are equal, i.e., f vanishes on the diagonal. For a Gaussian random measure $\mathbb{W}$ corresponding to P, we define the random operator $J_m : S_m \to H$ by

$$J_m(f) = \sum_{i_1,\dots,i_m=1}^{K} c_{i_1,\dots,i_m} \mathbb{W}(A_{i_1}) \dots \mathbb{W}(A_{i_m}).$$

The operator J_m has three notable properties:

1. Linearity;
2. $\mathbb{E} J_m(f) = 0$;
3. Isometry: $\mathbb{E}\langle J_m(f), J_m(g)\rangle_H = \langle f, g\rangle_{L_2(\mathcal{X}^m, H)}$.

Since S_m is dense in $L_2(\mathcal{X}^m, H)$, J_m can be extended to a continuous linear isometry on $L_2(\mathcal{X}^m, H)$, known as the Wiener-Itô integral.

Example S.2.1. *Let $(B_t)_{t\geq 0}$ denote a real-valued Brownian motion. For any interval $(t, s] \subset [0, \infty)$, define $\mathbb{W}((t, s]) = B_s - B_t$, which is a Gaussian random measure (with μ as the Lebesgue measure). The Wiener-Itô integral $J : L_2([0, \infty), \mathrm{d}t) \to \mathbf{R}$ is then given by $J(f) = \int f(t)\mathrm{d}B_t$.*

S.2.2 Central Limit Theorem

Consider a probability space $(\mathcal{X}, \Sigma, P)$, where $\mathcal{X}$ is a separable metric space and Σ is a Borel σ-algebra. Let $(X_i)_{i=1}^n$ be i.i.d. random variables taking values in $(\mathcal{X}, \Sigma, P)$. Define a symmetric function $h : \mathcal{X} \times \mathcal{X} \to H$, where H is a separable Hilbert space. The H-valued U-statistic of degree 2 is given by

$$\mathbf{U}_n = \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} h(X_i, X_j).$$

The U-statistic is called degenerate if $\mathbb{E}h(x_1, X_2) = 0$. The following theorem provides the limiting distribution of degenerate H-valued U-statistics; see Korolyuk and Borovskich (1994), Theorem 4.10.2 for a detailed proof.

Theorem S.2.1. *Suppose that* $\mathbf{U}_n$ *is a degenerate* U*-statistic such that* $\mathbb{E}h(X_1, X_2) = 0$ *and* $\mathbb{E}\|h(X_1, X_2)\|^2 < \infty$*. Then*

$$n\mathbf{U}_n \xrightarrow{d} J(h),$$

where $J(h) = \iint_{\mathcal{X}\times\mathcal{X}} h(x_1, x_2)\mathbb{W}(\mathrm{d}x_1)\mathbb{W}(\mathrm{d}x_2)$ *is a stochastic Wiener-Itô integral, and* $\mathbb{W}$ *is a Gaussian random measure on* H*.*

References

Carrasco, M., J.-P. Florens, and E. Renault (2007): "Linear inverse problems in structural econometrics estimation based on spectral decomposition and regularization," *Handbook of Econometrics*, 6B, 5633–5751.

Groetsch, C. W. (1977): *Generalized inverses of linear operators*. Chapman & Hall Pure and Applied Mathematics.

Korolyuk, V. S., and Y. V. Borovskich (1994): *Theory of U-statistics*. Springer Science & Business Media.