Supplementary Material to "Testing for Coefficient Randomness in Local-to-Unity Autoregressions"

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Appendix A: Procedure to Determine α_1 Values for the Bonferroni-Wald Test

In this appendix, we describe the procedure to determine the α_1 values for the confidence interval for ρ_T given in Table 2. The procedure is based on simulating the asymptotic distribution of the Bonferroni-Wald test statistic with 5000 replications and calculating the frequency of H_0 being rejected. In each replication, we first generate $\{y_t\}_{t=1}^T$ by the mechanism

$$y_t = (1 + a/T)y_{t-1} + \varepsilon_t, \quad t = 1, 2, \ldots, T \tag{A.1}$$

with $y_0 = 0$, $T = 2000$, $a \in [-300, 10]$ and $\varepsilon_t \sim$ i.i.d $N(0, 1)$.

Note that ε_t used in (A.1) satisfies $\psi = \mathrm{Corr}(\varepsilon_t, \varepsilon_t^2 - \sigma_\varepsilon^2) = 0$. To calculate the false rejection frequencies for other ψ values, we artificially produce an environment where the Wald test statistic depends on $\psi \neq 0$. Noting that under the null,

$$z_t^2(\bar{\rho}_T) = 1 + (\varepsilon_t^2 - 1) + (\bar{a}/T - a/T)^2 y_{t-1}^2 - 2(\bar{a}/T - a/T)y_{t-1}\varepsilon_t,$$

and $\eta_t = \varepsilon_t^2 - 1$ (combined with ε_t) determines the value of ψ, we artificially replace $\eta_t = \varepsilon_t^2 - 1$ with $\eta_t^{rep} := \sqrt{1 - \psi^2}\eta_t + \psi\sqrt{2}\varepsilon_t$, obtaining

$$z_t^2(\bar{\rho}_T)^{rep} := 1 + \eta_t^{rep} + (\bar{a}/T - a/T)^2 y_{t-1}^2 - 2(\bar{a}/T - a/T)y_{t-1}\varepsilon_t,$$

in view of the distributional equivalence $W_\eta \stackrel{d}{=} \sqrt{1 - \psi^2}W_1 + \psi W_\varepsilon$ and the fact that $\mathbb{V}[\eta_t] = \sigma_\eta^2 = 2$ and $\mathrm{Corr}(\eta_t, \varepsilon_t) = 0$. In this replacement, the newly crafted variable η_t^{rep} takes over the role of $\eta_t = \varepsilon_t^2 - 1$, satisfying $\mathbb{E}[\eta_t^{rep}] = 0$, $\mathbb{V}[\eta_t^{rep}] = 2 = \mathbb{V}[\eta_t]$ and $\mathrm{Corr}(\eta_t^{rep}, \varepsilon_t) = \psi$ by construction. This replacement can be justified by the fact that under the null, the Wald test statistic asymptotically depends only on ψ (and a). It follows that the Bonferroni-Wald test statistic using $z_t^2(\bar{\rho}_T)^{rep}$ in place of $z_t^2(\bar{\rho}_T)$ asymptotically depends on ψ.

For each ψ, we calculate, under given α_1, the false rejection rates of the Bonferroni-Wald test following Algorithm 1 with a moving over a grid on $[-300, 10]$, and pick an α_1 value for the given ψ such that the false rejection rate is less than or equal to 0.05 for all a.

Appendix B: Proofs of Results in Section 2

In this appendix, we prove the theorems stated in Section 2. To do this, we need the following lemmas. The statements and proofs of Lemmas B.1-B.3 below are basically borrowed from those of Lemmas A.1, A.2 and 1 of Nishi and Kurozumi (2024).

Lemma B.1. *Under Assumption 1, with probability approaching one (w.p.a 1),* $(\rho_T + \omega_T v_t)$, $t = 1, \ldots, T$, *are all positive.*

Proof. This can be shown by noting that

$$\begin{aligned} P\left(\bigcup_{t=1}^{T}\{\rho_T + \omega_T v_t \leq 0\}\right) &\leq \sum_{t=1}^{T} P(\rho_T + \omega_T v_t \leq 0) \\ &\leq \sum_{t=1}^{T} P(|\omega_T v_t| \geq \rho_T) \\ &\leq \sum_{t=1}^{T} \frac{\omega_T^2}{\rho_T^2}\mathbb{E}[|v_t|^2] \\ &= \frac{|c|^2}{(1 + a/T)^2} T^{-1/2}\mathbb{E}[|v_t|^2] \to 0 \ (T \to \infty), \end{aligned}$$

where the third inequality holds by Markov's inequality, and the last convergence by the fact that $\mathbb{E}[|v_t|^2]$ is finite. □

Lemma B.2. *Under Assumption 1, we have*

$$\prod_{k=1}^{\lfloor Tr \rfloor}(\rho_T + \omega_T v_k) = e^{ar}\left(1 + cT^{-3/4}\sum_{k=1}^{\lfloor Tr \rfloor} v_k + O_p(T^{-1/2})\right),$$

uniformly in $r \in [0, 1]$.

Proof. By Lemma B.1, $\ln \prod_{k=1}^{\lfloor Tr \rfloor} \big(\rho_T + \omega_T v_k\big),\ 0 \le r \le 1$, exist w.p.a 1. Thus, we asymptotically have

$$\begin{aligned}
\prod_{k=1}^{\lfloor Tr \rfloor} (\rho_T + \omega_T v_k) &= \exp\bigg(\ln \prod_{k=1}^{\lfloor Tr \rfloor} (\rho_T + \omega_T v_k)\bigg) \\
&= \exp\bigg(\sum_{k=1}^{\lfloor Tr \rfloor} \ln(1 + a/T + c v_k/T^{3/4})\bigg) \\
&= \exp\bigg(\sum_{k=1}^{\lfloor Tr \rfloor} (a/T + c v_k/T^{3/4}) - \frac{1}{2}\sum_{k=1}^{\lfloor Tr \rfloor} \frac{(a/T + c v_k/T^{3/4})^2}{(1+\zeta_k)^2}\bigg), \qquad \text{(B.1)}
\end{aligned}$$

where we used a Taylor expansion for the third equality and $|\zeta_k| < |a/T + c v_k/T^{3/4}|$. As for the remainder term, we have for any $\epsilon > 0$,

$$\begin{aligned}
P\big(\max_{1\le k\le T} |\zeta_k| \ge \epsilon\big) &\le P\big(\max_{1\le k\le T} |a/T + c v_k/T^{3/4}| \ge \epsilon\big) \\
&\le \sum_{k=1}^{T} P(|c v_k/T^{3/4}| \ge \epsilon - |a/T|) \\
&\le c^2(\epsilon - |a/T|)^{-2} T^{-3/2} \sum_{k=1}^{T} \mathbb{E}[v_k^2] \quad \text{for } T \text{ sufficiently large} \\
&\to 0 \ \text{ as } T \to \infty,
\end{aligned}$$

since $\mathbb{E}[v_k^2] = 1$. Hence, $\max_{1\le k\le T} |\zeta_k|$ is $o_p(1)$. It follows that

$$\begin{aligned}
\sup_{0\le r\le 1} \bigg|\sum_{k=1}^{\lfloor Tr \rfloor} \frac{(a/T + c v_k/T^{3/4})^2}{(1+\zeta_k)^2}\bigg| &= \sum_{k=1}^{T} \frac{(a/T + c v_k/T^{3/4})^2}{(1+\zeta_k)^2} \\
&\le \frac{1}{\min_{1\le k\le T}(1+\zeta_k)^2} \sum_{k=1}^{T} \bigg(\frac{a^2}{T^2} + \frac{2ac}{T^{7/4}} \sum_{k=1}^{T} v_k + \frac{c^2}{T^{3/2}} \sum_{k=1}^{T} v_k^2\bigg) = O_p(T^{-1/2}),
\end{aligned}$$

for which we used $T^{-1/2}\sum_{k=1}^{T} v_k = O_p(1)$ by the CLT, and $T^{-1}\sum_{k=1}^{T} v_k^2 \xrightarrow{p} \mathbb{E}[v_k^2] = 1$ by the law of large numbers (LLN). Applying this result to (B.1) yields

$$\begin{aligned}\prod_{k=1}^{\lfloor Tr\rfloor}(\rho_T+\omega_T v_k) &= \exp\Big(\sum_{k=1}^{\lfloor Tr\rfloor}(a/T+cv_k/T^{3/4})\Big)\exp\Big(-\frac{1}{2}\sum_{k=1}^{\lfloor Tr\rfloor}\frac{(a/T+cv_k/T^{3/4})^2}{(1+\zeta_k)^2}\Big)\\ &= \exp\Big(\frac{\lfloor Tr\rfloor}{T}a\Big)\Big(1+cT^{-3/4}\sum_{k=1}^{\lfloor Tr\rfloor}v_k+O_p(T^{-1/2})\Big)\Big(1+O_p(T^{-1/2})\Big)\\ &= e^{ar}\Big(1+cT^{-3/4}\sum_{k=1}^{\lfloor Tr\rfloor}v_k+O_p(T^{-1/2})\Big),\end{aligned}$$

uniformly in $r\in[0,1]$ since

$$\sup_{0\le r\le 1}\Big|T^{-3/4}\sum_{k=1}^{\lfloor Tr\rfloor}v_k\Big| = T^{-1/4}\sup_{0\le r\le 1}\Big|T^{-1/2}\sum_{k=1}^{\lfloor Tr\rfloor}v_k\Big| = O_p(T^{-1/4})$$

by the FCLT, and $\exp(a\lfloor Tr\rfloor/T)=\exp[a(r+O(T^{-1}))]=e^{ar}(1+O(T^{-1}))$. □

Lemma B.3. *Consider model* (2) *under Assumption 1. Define the stochastic process Y_T on $[0,1]$ by $Y_T(r):=T^{-1/2}y_{\lfloor Tr\rfloor},\ 0\le r\le 1$. Then, $Y_T\Rightarrow\sigma_\varepsilon J_a$ in the Skorokhod space $D[0,1]$, where J_a solves $dJ_a(r)=aJ_a(r)dr+dW_\varepsilon(r)$.*

Proof. From (2), a backward substitution yields

$$y_t=\prod_{k=1}^{t}(\rho_T+\omega_T v_k)\sum_{s=1}^{t}\Big\{\prod_{k=1}^{s}(\rho_T+\omega_T v_k)\Big\}^{-1}\varepsilon_s+y_0\prod_{k=1}^{t}(\rho_T+\omega_T v_k).$$

By Lemma B.2 and $y_0=o_p(T^{1/2})$ by Assumption 1, we obtain

$$\begin{aligned}&Y_T(r)\\ &=T^{-1/2}y_{\lfloor Tr\rfloor}\\ &=e^{ar}\Big(1+cT^{-3/4}\sum_{k=1}^{\lfloor Tr\rfloor}v_k+O_p(T^{-1/2})\Big)\sum_{s=1}^{\lfloor Tr\rfloor}\exp\Big(-a\frac{s}{T}\Big)\Big(1-cT^{-3/4}\sum_{k=1}^{s}v_k+O_p(T^{-1/2})\Big)(T^{-1/2}\varepsilon_s)\\ &+T^{-1/2}y_0\Big(1+cT^{-3/4}\sum_{k=1}^{\lfloor Tr\rfloor}v_k+O_p(T^{-1/2})\Big)\\ &=e^{ar}\Big(T^{-1/2}\sum_{s=1}^{\lfloor Tr\rfloor}\exp\Big(-a\frac{s}{T}\Big)\varepsilon_s+P_{1,T}(r)+P_{2,T}(r)+P_{3,T}(r)\Big)+o_p(1),\end{aligned}$$

where

$$P_{1,T}(r) \coloneqq -cT^{-1/4} \sum_{s=1}^{\lfloor Tr \rfloor} \exp\Big(-a\frac{s}{T}\Big) \Big(T^{-1/2} \sum_{k=1}^{s} v_k + O_p(T^{-1/4})\Big) (T^{-1/2}\varepsilon_s),$$

$$P_{2,T}(r) \coloneqq cT^{-1/4} \Big(T^{-1/2} \sum_{k=1}^{\lfloor Tr \rfloor} v_k + O_p(T^{-1/4})\Big) \Big(T^{-1/2} \sum_{s=1}^{\lfloor Tr \rfloor} \exp\Big(-a\frac{s}{T}\Big)\varepsilon_s\Big),$$

and

$$P_{3,T}(r) \coloneqq -c^2 T^{-1/2} \Big(T^{-1/2} \sum_{k=1}^{\lfloor Tr \rfloor} v_k + O_p(T^{-1/4})\Big) \sum_{s=1}^{\lfloor Tr \rfloor} \exp\Big(-a\frac{s}{T}\Big) \Big(T^{-1/2} \sum_{k=1}^{s} v_k + O_p(T^{-1/4})\Big) (T^{-1/2}\varepsilon_s).$$

We show $P_{i,T} = o_p(1),\ i = 1,2,3$, thereby obtaining $Y_T(\cdot) \Rightarrow \sigma_\varepsilon \int_0^\cdot e^{a(\cdot - s)} dW_\varepsilon(s) = \sigma_\varepsilon J_a(\cdot)$. Now, for $P_{1,T}$, we have

$$\begin{aligned}
&\sum_{s=1}^{\lfloor Tr \rfloor} \exp\Big(-a\frac{s}{T}\Big) \Big(T^{-1/2} \sum_{k=1}^{s} v_k + O_p(T^{-1/4})\Big) (T^{-1/2}\varepsilon_s) \\
&\quad = T^{-1} \sum_{s=1}^{\lfloor Tr \rfloor} \exp\Big(-a\frac{s}{T}\Big) v_s \varepsilon_s + \exp(-a/T) \sum_{s=1}^{\lfloor Tr \rfloor} \exp\Big(-a\frac{s-1}{T}\Big) \Big(T^{-1/2} \sum_{k=1}^{s-1} v_k + O_p(T^{-1/4})\Big) (T^{-1/2}\varepsilon_s) \\
&\quad = T^{-1} \sum_{s=1}^{\lfloor Tr \rfloor} \exp\Big(-a\frac{s}{T}\Big) \mathbb{E}[\varepsilon_1 v_1] + \exp(-a/T) T^{-1} \sum_{s=1}^{\lfloor Tr \rfloor} \exp\Big(-a\frac{s-1}{T}\Big) (\varepsilon_s v_s - \mathbb{E}[\varepsilon_1 v_1]) \\
&\qquad + \exp(-a/T) \sum_{s=1}^{\lfloor Tr \rfloor} \exp\Big(-a\frac{s-1}{T}\Big) \Big(T^{-1/2} \sum_{k=1}^{s-1} v_k + O_p(T^{-1/4})\Big) (T^{-1/2}\varepsilon_s) \\
&\quad \Rightarrow \int_0^r e^{-as} ds \mathbb{E}[\varepsilon_1 v_1] + \sigma_\varepsilon \int_0^r e^{-as} W_v(s) dW_\varepsilon(s),
\end{aligned}$$

by Theorem 2.1 of Hansen (1992). This result, combined with $cT^{-1/4} \to 0$, gives $P_{1,T} = o_p(1)$. $P_{2,T}$ also vanishes because $cT^{-1/4} \to 0$ and

$$\Big(T^{-1/2} \sum_{k=1}^{\lfloor Tr \rfloor} v_k + O_p(T^{-1/4})\Big) \Big(T^{-1/2} \sum_{s=1}^{\lfloor Tr \rfloor} \exp\Big(-a\frac{s}{T}\Big)\varepsilon_s\Big) \Rightarrow \sigma_\varepsilon W_v(r) \int_0^r e^{-as} dW_\varepsilon(s).$$

That $P_{3,T} = o_p(1)$ can be verified by a similar argument. □

Lemma B.4. *Consider model* (2) *under Assumptions 1 and 2. Then, we have* *(a)* $\hat{\sigma}^2_{\varepsilon,T}(\rho_T) \xrightarrow{p} \sigma^2_\varepsilon = \mathbb{E}[\varepsilon_t^2]$,

(b) $\hat{\sigma}^2_{\eta,T}(\rho_T) \xrightarrow{p} \sigma^2_\eta = \mathbb{E}[\eta_t^2]$,

(c) $\hat{\psi}_T(\rho_T) \xrightarrow{p} \psi = \mathbb{E}[\varepsilon_t \eta_t]/(\sigma_\varepsilon \sigma_\eta)$.

Proof. The proofs of parts (a) and (b) are the same as those of Lemmas 1(b) and (c) of Nishi and Kurozumi (2024).

(a) From $z_t(\rho_T) = y_t - \rho_T y_{t-1} = \omega_T v_t y_{t-1} + \varepsilon_t$, we have

$$\hat{\sigma}^2_{\varepsilon,T}(\rho_T) = T^{-1}\sum_{t=1}^T z_t^2(\rho_T)$$
$$= c^2 T^{-5/2}\sum_{t=1}^T y_{t-1}^2 v_t^2 + 2cT^{-7/4}\sum_{t=1}^T y_{t-1} v_t \varepsilon_t + T^{-1}\sum_{t=1}^T \varepsilon_t^2. \tag{B.2}$$

The first term is

$$T^{-5/2}\sum_{t=1}^T y_{t-1}^2 v_t^2 \le \sup_{0\le r\le 1}\left|T^{-1/2} y_{\lfloor Tr \rfloor}\right|^2 \cdot T^{-1/2} \cdot T^{-1}\sum_{t=1}^T v_t^2 = O_p(T^{-1/2}),$$

since $\sup_{0\le r\le 1}|y_{\lfloor Tr\rfloor}/\sqrt{T}| \Rightarrow \sup_{0\le r\le 1}|J_a(r)|$ by Lemma B.3 and the continuous mapping theorem (CMT). The second term is

$$\left|T^{-7/4}\sum_{t=1}^T y_{t-1} v_t \varepsilon_t\right| \le \sup_{0\le r\le 1}\left|T^{-1/2} y_{\lfloor Tr\rfloor}\right| T^{-1/4} \cdot T^{-1}\sum_{t=1}^T |\varepsilon_t v_t| = O_p(T^{-1/4}).$$

Substituting these results into (B.2), we have

$$\hat{\sigma}^2_{\varepsilon,T}(\rho_T) = T^{-1}\sum_{t=1}^T \varepsilon_t^2 + o_p(1) \xrightarrow{p} \sigma^2_\varepsilon.$$

(b) We write $\hat{\sigma}^2_{\eta,T}(\rho_T)$ as

$$\hat{\sigma}^2_{\eta,T}(\rho_T) = T^{-1}\sum_{t=1}^T \left\{ z_t^2(\rho_T) - \hat{\sigma}^2_{\varepsilon,T}(\rho_T)\right\}^2 = T^{-1}\sum_{t=1}^T z_t^4(\rho_T) - \hat{\sigma}^4_{\varepsilon,T}(\rho_T). \tag{B.3}$$

The first term is

$$T^{-1}\sum_{t=1}^T z_t^4(\rho_T) = c^4 T^{-4}\sum_{t=1}^T y_{t-1}^4 v_t^4 + 4c^3 T^{-13/4}\sum_{t=1}^T y_{t-1}^3 \varepsilon_t v_t^3 + 6c^2 T^{-5/2}\sum_{t=1}^T y_{t-1}^2 \varepsilon_t^2 v_t^2$$
$$+ 4cT^{-7/4}\sum_{t=1}^T y_{t-1}\varepsilon_t^3 v_t + T^{-1}\sum_{t=1}^T \varepsilon_t^4,$$

for which we have

$$\left|T^{-4}\sum_{t=1}^{T} y_{t-1}^4 v_t^4\right| \le \sup_{0\le r\le 1}\left|T^{-1/2}y_{\lfloor Tr\rfloor}\right|^4 T^{-1}\cdot T^{-1}\sum_{t=1}^{T} v_t^4 = O_p(T^{-1}),$$

$$\left|T^{-13/4}\sum_{t=1}^{T} y_{t-1}^3 \varepsilon_t v_t^3\right| \le \sup_{0\le r\le 1}\left|T^{-1/2}y_{\lfloor Tr\rfloor}\right|^3 T^{-3/4}\cdot T^{-1}\sum_{t=1}^{T} |\varepsilon_t v_t^3| = O_p(T^{-3/4}),$$

$$\left|T^{-5/2}\sum_{t=1}^{T} y_{t-1}^2 \varepsilon_t^2 v_t^2\right| \le \sup_{0\le r\le 1}\left|T^{-1/2}y_{\lfloor Tr\rfloor}\right|^2 T^{-1/2}\cdot T^{-1}\sum_{t=1}^{T} |\varepsilon_t^2 v_t^2| = O_p(T^{-1/2}),$$

and

$$\left|T^{-7/4}\sum_{t=1}^{T} y_{t-1} \varepsilon_t^3 v_t\right| \le \sup_{0\le r\le 1}\left|T^{-1/2}y_{\lfloor Tr\rfloor}\right| T^{-1/4}\cdot T^{-1}\sum_{t=1}^{T} |\varepsilon_t^3 v_t| = O_p(T^{-1/4}).$$

Thus, the first term of (B.3) is

$$T^{-1}\sum_{t=1}^{T} z_t^4(\rho_T) = T^{-1}\sum_{t=1}^{T}\varepsilon_t^4 + o_p(1) \xrightarrow{p} \mathbb{E}[\varepsilon_t^4].$$

Hence

$$\hat{\sigma}_{\eta,T}^2(\rho_T) \xrightarrow{p} \mathbb{E}[\varepsilon_t^4] - \sigma_\varepsilon^4 = \sigma_\eta^2.$$

(c) It suffices to show

$$T^{-1}\sum_{t=1}^{T} z_t(\rho_T)\Big\{z_t^2(\rho_T) - \hat{\sigma}_{\varepsilon,T}^2(\rho_T)\Big\} \xrightarrow{p} \mathbb{E}[\varepsilon_t^3].$$

A simple calculation gives

$$\begin{aligned}
&T^{-1}\sum_{t=1}^{T} z_t(\rho_T)\{z_t^2(\rho_T) - \hat{\sigma}_{\varepsilon,T}^2(\rho_T)\} \\
&= T^{-1}\sum_{t=1}^{T}(cT^{-3/4}y_{t-1}v_t + \varepsilon_t)\{c^2T^{-3/2}y_{t-1}^2 v_t^2 + 2cT^{-3/4}y_{t-1}\varepsilon_t v_t + \varepsilon_t^2 - \hat{\sigma}_{\varepsilon,T}^2(\rho_T)\} \\
&= T^{-1}\sum_{t=1}^{T}\varepsilon_t^3 + A_T,
\end{aligned}$$

where

$$\begin{aligned}
A_T \coloneqq\ & c^3T^{-13/4}\sum_{t=1}^{T} y_{t-1}^3 v_t^3 + 3c^2T^{-5/2}\sum_{t=1}^{T} y_{t-1}^2\varepsilon_t v_t^2 + 3cT^{-7/4}\sum_{t=1}^{T} y_{t-1}\varepsilon_t^2 v_t \\
&- \hat{\sigma}_{\varepsilon,T}^2(\rho_T)\times\Big\{cT^{-7/4}\sum_{t=1}^{T} y_{t-1}v_t + T^{-1}\sum_{t=1}^{T}\varepsilon_t\Big\}.
\end{aligned}$$

The first term of A_T satisfies

$$\left| c^3 T^{-13/4} \sum_{t=1}^{T} y_{t-1}^3 v_t^3 \right| \le |c|^3 \sup_{0\le r\le 1} |y_{\lfloor Tr \rfloor}/\sqrt{T}|^3 \times T^{-7/4} \sum_{t=1}^{T} |v_t|^3 = O_p(T^{-3/4}).$$

Similarly, we can prove that the other terms of A_T is $O_p(T^{-1/4})$. Thus

$$T^{-1} \sum_{t=1}^{T} z_t(\rho_T)\{z_t^2(\rho_T) - \hat{\sigma}_{\varepsilon,T}^2(\rho_T)\} = T^{-1} \sum_{t=1}^{T} \varepsilon_t^3 + o_p(1) \xrightarrow{p} \mathbb{E}[\varepsilon_t^3].$$

□

For later reference, we give several results on the weak convergence of components of test statistics.

Lemma B.5. *Consider model* (2) *under Assumptions 1 and 2. Then, we have*

(a)

$$T^{-3/2} \sum_{t=1}^{T} x_{2,t-1} z_t^2(\rho_T) \Rightarrow \sigma_\eta \sigma_\varepsilon^2 \int_0^1 \tilde{J}_{a,2}(r) dW_\eta(r) + c^2 \sigma_\varepsilon^4 \int_0^1 (\tilde{J}_{a,2})^2(r) dr + 2c\sigma_\varepsilon^4 q \int_0^1 \tilde{J}_{a,1}(r)\tilde{J}_{a,2}(r) dr,$$

(b)

$$T^{-1} \sum_{t=1}^{T} x_{1,t-1} z_t^2(\rho_T) \Rightarrow \sigma_\eta \sigma_\varepsilon \int_0^1 \tilde{J}_{a,1}(r) dW_\eta(r) + c^2 \sigma_\varepsilon^3 \int_0^1 \tilde{J}_{a,1}(r)\tilde{J}_{a,2}(r) dr + 2c\sigma_\varepsilon^3 q \int_0^1 (\tilde{J}_{a,1})^2(r) dr,$$

(c) $\hat{\sigma}^2_{\widetilde{\xi^*}}(\rho_T) \xrightarrow{p} \sigma_\eta^2$,

(d) $\hat{\sigma}^2_{\widetilde{\xi^{**}}}(\rho_T) \xrightarrow{p} \sigma_\eta^2$,

where $\hat{\sigma}^2_{\widetilde{\xi^*}}(\rho_T)$ *and* $\hat{\sigma}^2_{\widetilde{\xi^{**}}}(\rho_T)$ *are the OLS variance estimators of* (9) *and* (14), *respectively.*

Proof. For part (a), a straightforward calculation gives

$$T^{-3/2} \sum_{t=1}^{T} x_{2,t-1} z_t^2(\rho_T) = B_{1,T} + B_{2,T} + B_{3,T},$$

where

$$B_{1,T} := T^{-3/2}\sum_{t=1}^{T}\Big(y_{t-1}^2 - T^{-1}\sum_{t=1}^{T} y_{t-1}^2\Big)(\varepsilon_t^2-\sigma_\varepsilon^2),$$
$$B_{2,T} := c^2T^{-3}\sum_{t=1}^{T}\Big(y_{t-1}^2 - T^{-1}\sum_{t=1}^{T} y_{t-1}^2\Big)y_{t-1}^2 v_t^2,$$

and

$$B_{3,T} := 2cT^{-9/4}\sum_{t=1}^{T}\Big(y_{t-1}^2 - T^{-1}\sum_{t=1}^{T} y_{t-1}^2\Big)y_{t-1}\varepsilon_t v_t.$$

It is straightforward to show $B_{1,T} \Rightarrow \sigma_\eta\sigma_\varepsilon^2\int_0^1 \tilde{J}_{a,2}(r)dW_\eta(r)$, using Lemma B.3 and Theorem 2.1 of Hansen (1992). As for $B_{2,T}$, we have

$$\begin{aligned}B_{2,T} &= c^2T^{-3}\sum_{t=1}^{T}\Big(y_{t-1}^2 - T^{-1}\sum_{t=1}^{T} y_{t-1}^2\Big)^2 + c^2T^{-3}\sum_{t=1}^{T}\Big(y_{t-1}^2 - T^{-1}\sum_{t=1}^{T} y_{t-1}^2\Big)y_{t-1}^2(v_t^2-1)\\ &= c^2\int_0^1\Big(Y_T^2(r)-\int_0^1 Y_T^2(s)ds\Big)^2dr + c^2T^{-1/2}\int_0^1\Big(Y_T^2(r)-\int_0^1 Y_T^2(s)ds\Big)Y_T^2(r)dW_{v^2-1,T}\\ &\Rightarrow c^2\sigma_\varepsilon^4\int_0^1(\tilde{J}_{a,2})^2(r)dr,\end{aligned}$$

where the last convergence follows from Lemma B.3, the fact that $\int_0^1(Y_T^2(r)-\int_0^1 Y_T^2(s)ds)Y_T^2(r)dW_{v^2-1,T} = O_p(1)$ by Theorem 2.1 of Hansen (1992), and the CMT. By a similar argument, we obtain

$$\begin{aligned}B_{3,T} &= 2c\sigma_\varepsilon qT^{-5/2}\sum_{t=1}^{T}\Big(y_{t-1}^2 - T^{-1}\sum_{t=1}^{T} y_{t-1}^2\Big)\Big(y_{t-1} - T^{-1}\sum_{t=1}^{T} y_{t-1}\Big)\\ &\quad + 2cT^{-9/4}\sum_{t=1}^{T}\Big(y_{t-1}^2 - T^{-1}\sum_{t=1}^{T} y_{t-1}^2\Big)y_{t-1}(\varepsilon_t v_t-\sigma_{\varepsilon v})\\ &\Rightarrow 2c\sigma_\varepsilon^4 q\int_0^1 \tilde{J}_{a,2}(r)\tilde{J}_{a,1}(r)dr.\end{aligned}$$

Therefore, we arrive at

$$T^{-3/2}\sum_{t=1}^{T} x_{2,t-1}z_t^2(\rho_T) \Rightarrow \sigma_\eta\sigma_\varepsilon^2\int_0^1\tilde{J}_{a,2}(r)dW_\eta(r) + c^2\sigma_\varepsilon^4\int_0^1(\tilde{J}_{a,2})^2(r)dr + 2c\sigma_\varepsilon^4 q\int_0^1\tilde{J}_{a,2}(r)\tilde{J}_{a,1}(r)dr,$$

as desired.

Part (b) can be proven in a similar fashion. Write $T^{-1}\sum_{t=1}^{T} x_{1,t-1}z_t^2(\rho_T)$ as

$$T^{-1}\sum_{t=1}^{T} x_{1,t-1}z_t^2(\rho_T) = C_{1,T} + C_{2,T} + C_{3,T},$$

where $C_{1,T} := T^{-1}\sum_{t=1}^{T} x_{1,t-1}(\varepsilon_t^2 - \sigma_\varepsilon^2)$, $C_{2,T} := c^2T^{-\frac{5}{2}}\sum_{t=1}^{T} x_{1,t-1}y_{t-1}^2v_t^2$, and $C_{3,T} := 2cT^{-\frac{7}{4}}\sum_{t=1}^{T} x_{1,t-1}y_{t-1}\varepsilon_t v_t$. Then, it is straightforward to show

$$C_{1,T} \Rightarrow \sigma_\eta\sigma_\varepsilon \int_0^1 \tilde{J}_{a,1}(r)dW_\eta(r),$$
$$C_{2,T} \Rightarrow c^2\sigma_\varepsilon^3 \int_0^1 \tilde{J}_{a,1}(r)\tilde{J}_{a,2}(r)dr,$$

and

$$C_{3,T} \Rightarrow 2c\sigma_\varepsilon^3 q \int_0^1 (\tilde{J}_{a,1})^2(r)dr.$$

Combining the above results completes the proof of part (b).

To prove part (c), we define $M := I_T - X(X'X)^{-1}X$ and write $\hat{\sigma}^2_{\widetilde{\xi^*}}(\rho_T)$ as

$$\begin{aligned}
\hat{\sigma}^2_{\widetilde{\xi^*}}(\rho_T) &= T^{-1}\widetilde{\Xi^*}'M\widetilde{\Xi^*} \\
&= T^{-1}\sum_{t=1}^{T}(\widetilde{\xi_t^*})^2 - T^{-1}\begin{pmatrix}\sum_{t=1}^{T} x_{1,t-1}\xi_t^* & \sum_{t=1}^{T} x_{2,t-1}\xi_t^*\end{pmatrix} \\
&\times \begin{pmatrix}\sum_{t=1}^{T}(x_{1,t-1})^2 & \sum_{t=1}^{T} x_{1,t-1}x_{2,t-1} \\ \sum_{t=1}^{T} x_{2,t-1}x_{1,t-1} & \sum_{t=1}^{T}(x_{2,t-1})^2\end{pmatrix}^{-1}\begin{pmatrix}\sum_{t=1}^{T} x_{1,t-1}\xi_t^* \\ \sum_{t=1}^{T} x_{2,t-1}\xi_t^*\end{pmatrix}. \qquad \text{(B.4)}
\end{aligned}$$

The first term of (B.4) becomes

$$\begin{aligned}
T^{-1}\sum_{t=1}^{T}(\widetilde{\xi_t^*})^2 &= T^{-1}\sum_{t=1}^{T}(\xi_t^*)^2 - \left(T^{-1}\sum_{t=1}^{T}\xi_t^*\right)^2 \\
&= T^{-1}\sum_{t=1}^{T}\{c^2T^{-3/2}y_{t-1}^2(v_t^2-1) + 2cT^{-3/4}y_{t-1}(\varepsilon_t v_t - \sigma_{\varepsilon v}) + (\varepsilon_t^2 - \sigma_\varepsilon^2)\}^2 \\
&- \left[T^{-1}\sum_{t=1}^{T}\{c^2T^{-3/2}y_{t-1}^2(v_t^2-1) + 2cT^{-3/4}y_{t-1}(\varepsilon_t v_t - \sigma_{\varepsilon v}) + (\varepsilon_t^2 - \sigma_\varepsilon^2)\}\right]^2 \\
&= T^{-1}\sum_{t=1}^{T}(\varepsilon_t^2 - \sigma_\varepsilon^2)^2 + o_p(1) \\
&\xrightarrow{p} \sigma_\eta^2.
\end{aligned}$$

The second term of (B.4) satisfies

$$
\begin{aligned}
&T^{-1}\left(\sum_{t=1}^{T} x_{1,t-1}\xi_t^* \quad \sum_{t=1}^{T} x_{2,t-1}\xi_t^*\right)\begin{pmatrix}\sum_{t=1}^{T}(x_{1,t-1})^2 & \sum_{t=1}^{T} x_{1,t-1}x_{2,t-1}\\ \sum_{t=1}^{T} x_{2,t-1}x_{1,t-1} & \sum_{t=1}^{T}(x_{2,t-1})^2\end{pmatrix}^{-1}\begin{pmatrix}\sum_{t=1}^{T} x_{1,t-1}\xi_t^*\\ \sum_{t=1}^{T} x_{2,t-1}\xi_t^*\end{pmatrix}\\
&= T^{-1}\left(T^{-1}\sum_{t=1}^{T} x_{1,t-1}(\varepsilon_t^2-\sigma_\varepsilon^2)+o_p(1) \quad T^{-3/2}\sum_{t=1}^{T} x_{2,t-1}(\varepsilon_t^2-\sigma_\varepsilon^2)+o_p(1)\right)\\
&\times\begin{pmatrix}T^{-2}\sum_{t=1}^{T}(x_{1,t-1})^2 & T^{-5/2}\sum_{t=1}^{T} x_{1,t-1}x_{2,t-1}\\ T^{-5/2}\sum_{t=1}^{T} x_{2,t-1}x_{1,t-1} & T^{-3}\sum_{t=1}^{T}(x_{2,t-1})^2\end{pmatrix}^{-1}\begin{pmatrix}T^{-1}\sum_{t=1}^{T} x_{1,t-1}(\varepsilon_t^2-\sigma_\varepsilon^2)+o_p(1)\\ T^{-3/2}\sum_{t=1}^{T} x_{2,t-1}(\varepsilon_t^2-\sigma_\varepsilon^2)+o_p(1)\end{pmatrix}\\
&= O_p(T^{-1}). \qquad \text{(B.5)}
\end{aligned}
$$

Hence, we obtain $\hat{\sigma}^2_{\widetilde{\xi^*}}(\rho_T) \xrightarrow{p} \sigma_\eta^2$, as desired.

To prove part (d), let $\widetilde{Z_1} := (\widetilde{z_1}(\rho_T), \widetilde{z_2}(\rho_T), \ldots, \widetilde{z_T}(\rho_T))'$. Then, $\widetilde{Z_2^*}(\rho_T)$ is expressed as

$$
\widetilde{Z_2^*}(\rho_T) = \frac{1}{\sqrt{1-\hat{\psi}_T^2(\rho_T)}}\left\{\tilde{Z}_2(\rho_T) - \frac{\hat{\sigma}_{\eta,T}(\rho_T)\hat{\psi}_T(\rho_T)}{\hat{\sigma}_{\varepsilon,T}(\rho_T)}\widetilde{Z_1}(\rho_T)\right\},
$$

which yields

$$
\begin{aligned}
\hat{\sigma}^2_{\widetilde{\xi^{**}}} &= T^{-1}\widetilde{Z_2^*}(\rho_T)'M\widetilde{Z_2^*}(\rho_T)\\
&= \frac{1}{1-\hat{\psi}_T^2(\rho_T)}\left\{D_{1,T} - 2\frac{\hat{\sigma}_{\eta,T}(\rho_T)\hat{\psi}_T(\rho_T)}{\hat{\sigma}_{\varepsilon,T}(\rho_T)}D_{2,T} + \frac{\hat{\sigma}^2_{\eta,T}(\rho_T)\hat{\psi}_T^2(\rho_T)}{\hat{\sigma}^2_{\varepsilon,T}(\rho_T)}D_{3,T}\right\}, \qquad \text{(B.6)}
\end{aligned}
$$

where $D_{1,T} := T^{-1}\tilde{Z}_2(\rho_T)'M\tilde{Z}_2(\rho_T)$, $D_{2,T} := T^{-1}\tilde{Z}_2(\rho_T)'M\widetilde{Z_1}(\rho_T)$, and $D_{3,T} := T^{-1}\widetilde{Z_1}(\rho_T)'M\widetilde{Z_1}(\rho_T)$. Since $D_{1,T}$ is $\hat{\sigma}^2_{\widetilde{\xi^*}}(\rho_T)$, we have already proven in part (c) that

$$
D_{1,T} \xrightarrow{p} \sigma_\eta^2. \qquad \text{(B.7)}
$$

In view of equation (9), $D_{2,T}$ becomes

$$
\begin{aligned}
D_{2,T} &= T^{-1}\widetilde{\Xi^*}(\rho_T)'M\widetilde{Z_1}(\rho_T)\\
&= T^{-1}\sum_{t=1}^{T}\xi_t^*\widetilde{z}_t(\rho_T) - T^{-1}\left(\sum_{t=1}^{T} x_{1,t-1}\xi_t^* \quad \sum_{t=1}^{T} x_{2,t-1}\xi_t^*\right)\\
&\quad\times\begin{pmatrix}\sum_{t=1}^{T}(x_{1,t-1})^2 & \sum_{t=1}^{T} x_{1,t-1}x_{2,t-1}\\ \sum_{t=1}^{T} x_{2,t-1}x_{1,t-1} & \sum_{t=1}^{T}(x_{2,t-1})^2\end{pmatrix}^{-1}\begin{pmatrix}\sum_{t=1}^{T} x_{1,t-1}z_t(\rho_T)\\ \sum_{t=1}^{T} x_{2,t-1}z_t(\rho_T)\end{pmatrix}.
\end{aligned}
$$

For the first term of $D_{2,T}$, we have

$$\begin{aligned}T^{-1}\sum_{t=1}^{T}\xi_t^*\widetilde{z}_t(\rho_T) &= T^{-1}\sum_{t=1}^{T}\{c^2T^{-3/2}y_{t-1}^2(v_t^2-1)+2cT^{-3/4}y_{t-1}(\varepsilon_t v_t-\sigma_{\varepsilon v})+(\varepsilon_t^2-\sigma_\varepsilon^2)\}\\ &\quad\times\left(z_t(\rho_T)-T^{-1}\sum_{t=1}^{T}z_t(\rho_T)\right)\\ &= T^{-1}\sum_{t=1}^{T}(\varepsilon_t^2-\sigma_\varepsilon^2)\varepsilon_t+o_p(1)\xrightarrow{p}\mathbb{E}[\varepsilon_t^3].\end{aligned}$$

We can also show that the second term of $D_{2,T}$ is $O_p(T^{-1})$ in the same way as we did in (B.5). Thus, we get

$$D_{2,T}\xrightarrow{p}\mathbb{E}[\varepsilon_t^3]. \tag{B.8}$$

Lastly, $D_{3,T}$ becomes

$$\begin{aligned}D_{3,T} &= T^{-1}\sum_{t=1}^{T}\widetilde{z}_t^2(\rho_T)-T^{-1}\left(\sum_{t=1}^{T}x_{1,t-1}z_t(\rho_T)\quad\sum_{t=1}^{T}x_{2,t-1}z_t(\rho_T)\right)'\\ &\quad\times\begin{pmatrix}\sum_{t=1}^{T}(x_{1,t-1})^2 & \sum_{t=1}^{T}x_{1,t-1}x_{2,t-1}\\ \sum_{t=1}^{T}x_{2,t-1}x_{1,t-1} & \sum_{t=1}^{T}(x_{2,t-1})^2\end{pmatrix}^{-1}\begin{pmatrix}\sum_{t=1}^{T}x_{1,t-1}z_t(\rho_T)\\ \sum_{t=1}^{T}x_{2,t-1}z_t(\rho_T)\end{pmatrix}\\ &= \hat{\sigma}_{\varepsilon,T}^2(\rho_T)-\left(T^{-1}\sum_{t=1}^{T}z_t(\rho_T)\right)^2+O_p(T^{-1})\xrightarrow{p}\sigma_\varepsilon^2.\end{aligned} \tag{B.9}$$

Substituting (B.7), (B.8), and (B.9) into (B.6) and applying Lemma B.4, we deduce

$$\begin{aligned}\hat{\sigma}_{\widetilde{\xi^{**}}}^2 &\xrightarrow{p}\frac{1}{1-\psi^2}\left\{\sigma_\eta^2-2\frac{\sigma_\eta\psi}{\sigma_\varepsilon}\mathbb{E}[\varepsilon_t^3]+\frac{\sigma_\eta^2\psi^2}{\sigma_\varepsilon^2}\sigma_\varepsilon^2\right\}\\ &= \frac{1}{1-\psi^2}(\sigma_\eta^2-2\sigma_\eta^2\psi^2+\sigma_\eta^2\psi^2)=\sigma_\eta^2.\end{aligned}$$

□

Proof of Theorem 1. First, note that

$$\mathrm{LN}_T(\rho_T)=\frac{T^{-3/2}\sum_{t=1}^{T}x_{2,t-1}z_t^2(\rho_T)}{\hat{\sigma}_{\eta,T}(\rho_T)\left\{T^{-3}\sum_{t=1}^{T}(x_{2,t-1})^2\right\}^{1/2}}.$$

Then, using Lemmas B.4 and B.5 and the CMT, we deduce

$$\begin{aligned}
\mathrm{LN}_T(\rho_T) \Rightarrow & \frac{\sigma_\eta \sigma_\varepsilon^2 \int_0^1 \tilde{J}_{a,2}(r) dW_\eta(r) + c^2 \sigma_\varepsilon^4 \int_0^1 (\tilde{J}_{a,2})^2(r) dr + 2c\sigma_\varepsilon^4 q \int_0^1 \tilde{J}_{a,1}(r)\tilde{J}_{a,2}(r) dr}{\sigma_\eta \Big\{ \sigma_\varepsilon^4 \int_0^1 (\tilde{J}_{a,2})^2(r) dr \Big\}^{1/2}} \\
= & \frac{\int_0^1 \tilde{J}_{a,2}(r) dW_\eta(r)}{\{ \int_0^1 \big(\tilde{J}_{a,2}\big)^2(r) dr \}^{1/2}} + \frac{\sigma_\varepsilon^2}{\sigma_\eta} \Bigg[\frac{c^2 \int_0^1 \big(\tilde{J}_{a,2}\big)^2(r) dr + 2cq \int_0^1 \tilde{J}_{a,2}(r)\tilde{J}_{a,1}(r) dr}{\{ \int_0^1 \big(\tilde{J}_{a,2}\big)^2(r) dr \}^{1/2}} \Bigg].
\end{aligned}$$

□

Proof of Theorem 2. First, by Lemma B.5(c) and the CMT, the denominator of $t_{\hat{\omega}_T^2}(\rho_T)$ divided by $T^{3/2}$ becomes

$$\begin{aligned}
\hat{\sigma}_{\tilde{\xi}^*}(\rho_T) T^{-3/2} (X_2' M_1 X_2)^{1/2} &= \hat{\sigma}_{\tilde{\xi}^*}(\rho_T) \{ T^{-3} (M_1 X_2)'(M_1 X_2) \}^{1/2} \\
&= \hat{\sigma}_{\tilde{\xi}^*}(\rho_T) \Big\{ T^{-3} \sum_{t=1}^T \Big(x_{2,t-1} - \frac{\sum_{t=1}^T x_{1,t-1} x_{2,t-1}}{\sum_{t=1}^T (x_{1,t-1})^2} x_{1,t-1} \Big)^2 \Big\}^{1/2} \\
&= \hat{\sigma}_{\tilde{\xi}^*}(\rho_T) \Big\{ \int_0^1 \Big(\tilde{Y}_{2,T}(r) - \frac{\int_0^1 \tilde{Y}_{1,T}(s) \tilde{Y}_{2,T}(s) ds}{\int_0^1 (\tilde{Y}_{1,T})^2(s) ds} \tilde{Y}_{1,T}(r) \Big)^2 \Big\}^{1/2} \\
&\Rightarrow \sigma_\eta \sigma_\varepsilon^2 \Big[\int_0^1 Q_a^2(r) dr \Big]^{1/2},
\end{aligned}$$

where $\tilde{Y}_{1,T}(r) \coloneqq Y_T(r) - \int_0^1 Y_T(s) ds$ and $\tilde{Y}_{2,T}(r) \coloneqq Y_T^2(r) - \int_0^1 Y_T^2(s) ds$. Next, applying Lemma B.5, the numerator of $t_{\hat{\omega}_T^2}(\rho_T)$ divided by $T^{3/2}$ is seen to satisfy

$$\begin{aligned}
T^{-3/2} X_2' M_1 \tilde{Z}_2(\rho_T) = & T^{-3/2} \sum_{t=1}^T x_{2,t-1} z_t^2(\rho_T) - \frac{T^{-5/2} \sum_{t=1}^T x_{1,t-1} x_{2,t-1} T^{-1} \sum_{t=1}^T x_{1,t-1} z_t^2(\rho_T)}{T^{-2} \sum_{t=1}^T (x_{1,t-1})^2} \\
& \Rightarrow \sigma_\eta \sigma_\varepsilon^2 \int_0^1 \tilde{J}_{a,2}(r) dW_\eta(r) + c^2 \sigma_\varepsilon^4 \int_0^1 (\tilde{J}_{a,2})^2(r) dr + 2c\sigma_\varepsilon^4 q \int_0^1 \tilde{J}_{a,1}(r) \tilde{J}_{a,2}(r) dr \\
& - \sigma_\varepsilon^3 \int_0^1 \tilde{J}_{a,1}(r) \tilde{J}_{a,2}(r) dr \\
& \times \frac{\sigma_\eta \sigma_\varepsilon \int_0^1 \tilde{J}_{a,1}(r) dW_\eta(r) + c^2 \sigma_\varepsilon^3 \int_0^1 \tilde{J}_{a,1}(r) \tilde{J}_{a,2}(r) dr + 2c\sigma_\varepsilon^3 q \int_0^1 (\tilde{J}_{a,1})^2(r) dr}{\sigma_\varepsilon^2 \int_0^1 (\tilde{J}_{a,1})^2(r) dr} \\
& = \sigma_\eta \sigma_\varepsilon^2 \int_0^1 Q_a(r) dW_\eta(r) + c^2 \sigma_\varepsilon^4 \int_0^1 Q_a^2(r) dr.
\end{aligned}$$

Combining the above results gives

$$t_{\hat{\omega}_T^2}(\rho_T) \Rightarrow \frac{\int_0^1 Q_a(r)dW_\eta(r)}{\left[\int_0^1 Q_a^2(r)dr\right]^{1/2}} + \frac{c^2\sigma_\varepsilon^2}{\sigma_\eta}\left[\int_0^1 Q_a^2(r)dr\right]^{1/2}.$$

To derive the asymptotic distribution of $W_T(\rho_T)$, note that

$$\begin{aligned} W_T(\rho_T) &= \hat{\sigma}_{\tilde{\xi}^*}^{-2}(X'\tilde{Z}_2(\rho_T))'(X'X)^{-1}(X'\tilde{Z}_2(\rho_T)) \\ &= \hat{\sigma}_{\tilde{\xi}^*}^{-2}\begin{pmatrix} T^{-1}\sum_{t=1}^T x_{1,t-1}z_t^2(\rho_T) \\ T^{-3/2}\sum_{t=1}^T x_{2,t-1}z_t^2(\rho_T)\end{pmatrix}'\begin{pmatrix} T^{-2}\sum_{t=1}^T(x_{1,t-1})^2 & T^{-5/2}\sum_{t=1}^T x_{1,t-1}x_{2,t-1} \\ T^{-5/2}\sum_{t=1}^T x_{2,t-1}x_{1,t-1} & T^{-3}\sum_{t=1}^T(x_{2,t-1})^2\end{pmatrix}^{-1} \\ &\qquad \times \begin{pmatrix} T^{-1}\sum_{t=1}^T x_{1,t-1}z_t^2(\rho_T) \\ T^{-3/2}\sum_{t=1}^T x_{2,t-1}z_t^2(\rho_T)\end{pmatrix}. \end{aligned}$$

Then, applying Lemma B.5 and the CMT, we get

$$\begin{aligned} W_T(\rho_T) \Rightarrow &\left\{ \begin{pmatrix} \int_0^1 \tilde{J}_{a,1}(r)dW_\eta(r) \\ \int_0^1 \tilde{J}_{a,2}(r)dW_\eta(r)\end{pmatrix} + \frac{\sigma_\varepsilon^2}{\sigma_\eta}\begin{pmatrix} c^2\int_0^1 \tilde{J}_{a,1}(r)\tilde{J}_{a,2}(r)dr + 2cq\int_0^1\left(\tilde{J}_{a,1}\right)^2(r)dr \\ c^2\int_0^1\left(\tilde{J}_{a,2}\right)^2(r)dr + 2cq\int_0^1 \tilde{J}_{a,1}(r)\tilde{J}_{a,2}(r)dr\end{pmatrix}\right\}' \\ &\times \begin{pmatrix} \int_0^1\left(\tilde{J}_{a,1}\right)^2(r)dr & \int_0^1 \tilde{J}_{a,1}(r)\tilde{J}_{a,2}(r)dr \\ \int_0^1 \tilde{J}_{a,2}(r)\tilde{J}_{a,1}(r)dr & \int_0^1\left(\tilde{J}_{a,2}\right)^2(r)dr\end{pmatrix}^{-1} \\ &\times \left\{ \begin{pmatrix} \int_0^1 \tilde{J}_{a,1}(r)dW_\eta(r) \\ \int_0^1 \tilde{J}_{a,2}(r)dW_\eta(r)\end{pmatrix} + \frac{\sigma_\varepsilon^2}{\sigma_\eta}\begin{pmatrix} c^2\int_0^1 \tilde{J}_{a,1}(r)\tilde{J}_{a,2}(r)dr + 2cq\int_0^1\left(\tilde{J}_{a,1}\right)^2(r)dr \\ c^2\int_0^1\left(\tilde{J}_{a,2}\right)^2(r)dr + 2cq\int_0^1 \tilde{J}_{a,1}(r)\tilde{J}_{a,2}(r)dr\end{pmatrix}\right\}, \end{aligned}$$

completing the proof. □

To prove Theorem 3, we use the following lemma.

Lemma B.6. *Consider model* (2) *under Assumptions 1 and 2. Then, we have*

(a)

$$\begin{aligned} T^{-3/2}\sum_{t=1}^T x_{2,t-1}z_t^{2*}(\rho_T) \Rightarrow &\sigma_\eta\sigma_\varepsilon^2\int_0^1 \tilde{J}_{a,2}(r)dW_1(r) \\ &+ (1-\psi^2)^{-1/2}\Big\{c^2\sigma_\varepsilon^4\int_0^1(\tilde{J}_{a,2})^2(r)dr + 2c\sigma_\varepsilon^4 q\int_0^1 \tilde{J}_{a,1}(r)\tilde{J}_{a,2}(r)dr\Big\}, \end{aligned}$$

(b)

$$T^{-1}\sum_{t=1}^{T} x_{1,t-1}z_t^{2*}(\rho_T) \Rightarrow \sigma_\eta\sigma_\varepsilon \int_0^1 \tilde{J}_{a,1}(r)dW_1(r) + (1-\psi^2)^{-1/2}\Big\{c^2\sigma_\varepsilon^3\int_0^1 \tilde{J}_{a,1}(r)\tilde{J}_{a,2}(r)dr + 2c\sigma_\varepsilon^3 q\int_0^1 (\tilde{J}_{a,1})^2(r)dr\Big\}.$$

Proof. To prove part (a), note that

$$T^{-3/2}\sum_{t=1}^{T} x_{2,t-1}z_t^{2*}(\rho_T) = \frac{1}{\sqrt{1-\hat{\psi}_T^2(\rho_T)}}\Big\{E_{1,T} - \frac{\hat{\sigma}_{\eta,T}(\rho_T)\hat{\psi}_T(\rho_T)}{\hat{\sigma}_{\varepsilon,T}(\rho_T)}E_{2,T}\Big\},$$

where $E_{1,T} := T^{-3/2}\sum_{t=1}^{T} x_{2,t-1}z_t^2(\rho_T)$ and $E_{2,T} := T^{-3/2}\sum_{t=1}^{T} x_{2,t-1}z_t(\rho_T)$. By Lemma B.5(a), $E_{1,T}$ satisfies

$$E_{1,T} \Rightarrow \sigma_\eta\sigma_\varepsilon^2\int_0^1 \tilde{J}_{a,2}(r)dW_\eta(r) + c^2\sigma_\varepsilon^4\int_0^1 (\tilde{J}_{a,2})^2(r)dr + 2c\sigma_\varepsilon^4 q\int_0^1 \tilde{J}_{a,1}(r)\tilde{J}_{a,2}(r)dr.$$

As for $E_{2,T}$, a straightforward calculation yields

$$\begin{aligned} E_{2,T} &= T^{-3/2}\sum_{t=1}^{T} x_{2,t-1}(cT^{-3/4}y_{t-1}v_t + \varepsilon_t) \\ &= T^{-3/2}\sum_{t=1}^{T} x_{2,t-1}\varepsilon_t + O_p(T^{-1/4}) \Rightarrow \sigma_\varepsilon^3\int_0^1 \tilde{J}_{a,2}(r)dW_\varepsilon(r). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} T^{-3/2}\sum_{t=1}^{T} x_{2,t-1}z_t^{2*}(\rho_T) &\Rightarrow \frac{1}{\sqrt{1-\psi^2}}\Big\{\sigma_\eta\sigma_\varepsilon^2\int_0^1 \tilde{J}_{a,2}(r)dW_\eta(r) + c^2\sigma_\varepsilon^4\int_0^1 (\tilde{J}_{a,2})^2(r)dr \\ &\qquad + 2c\sigma_\varepsilon^4 q\int_0^1 \tilde{J}_{a,1}(r)\tilde{J}_{a,2}(r)dr - \frac{\sigma_\eta\psi}{\sigma_\varepsilon}\sigma_\varepsilon^3\int_0^1 \tilde{J}_{a,2}(r)dW_\varepsilon(r)\Big\} \\ &\overset{d}{=} \sigma_\eta\sigma_\varepsilon^2\int_0^1 \tilde{J}_{a,2}(r)dW_1(r) \\ &\qquad + \frac{1}{\sqrt{1-\psi^2}}\Big(c^2\sigma_\varepsilon^4\int_0^1 (\tilde{J}_{a,2})^2(r)dr + 2c\sigma_\varepsilon^4 q\int_0^1 \tilde{J}_{a,1}(r)\tilde{J}_{a,2}(r)dr\Big), \end{aligned}$$

in view of (3). This proves part (a). The proof of part (b) is similar, and thus is omitted. □

Proof of Theorem 3. The proof for $\mathrm{LN}_T^*(\rho_T)$ is essentially the same as that of Theorem 1 except that we consider $\sum_{t=1}^T x_{2,t-1} z_t^{2*}(\rho_T)$ in the numerator of the test statistic. Dividing both the numerator and denominator by $T^{3/2}$ and applying Lemma B.6(a) leads to the desired result. The proof for the augmented tests goes along the same lines as those of Theorem 2 if we replace $z_t^2(\rho_T)$ with $z_t^{2*}(\rho_T)$ and apply Lemmas B.5(d) and B.6. □

Appendix C: Proofs of Results in Section 3

In this appendix, we prove the asymptotic results mentioned in Section 3: namely, the asymptotic distribution of $\hat{\rho}_T$ and the consistency of $\hat{\sigma}_{\varepsilon,T}^2(\hat{\rho}_T)$, $\hat{\sigma}_{\eta,T}^2(\hat{\rho}_T)$ and $\hat{\psi}_T(\hat{\rho}_T)$.

Lemma C.1. *Consider model* (2) *under Assumptions 1 and 2. Then, we have*

$$T(\hat{\rho}_T - \rho_T) \Rightarrow \frac{\int_0^1 J_a(r) W_\varepsilon(r)}{\int_0^1 J_a^2(r) dr}.$$

Proof. From the definition of $\hat{\rho}_T$, we have

$$\begin{aligned} T(\hat{\rho}_T - \rho_T) &= \frac{T^{-1}\sum_{t=1}^T y_{t-1}(cT^{-3/4} y_{t-1} v_t + \varepsilon_t)}{T^{-2}\sum_{t=1}^T y_{t-1}^2} \\ &= \frac{cT^{-1/4}\int_0^1 Y_T^2(r) dW_{v,T}(r) + \sigma_\varepsilon \int_0^1 Y_T(r) dW_{\varepsilon,T}(r)}{\int_0^1 Y_T^2(r) dr} \Rightarrow \frac{\int_0^1 J_a(r) dW_\varepsilon(r)}{\int_0^1 J_a^2(r)}. \end{aligned}$$

□

Lemma C.2. *Consider model* (2) *under Assumptions 1 and 2. Then, we have*

(a) $\hat{\sigma}_{\varepsilon,T}^2(\hat{\rho}_T) \xrightarrow{p} \sigma_\varepsilon^2$,

(b) $\hat{\sigma}_{\eta,T}^2(\hat{\rho}_T) \xrightarrow{p} \sigma_\eta^2$,

(c) $\hat{\psi}_T(\hat{\rho}_T) \xrightarrow{p} \psi$.

Proof. For part (a), $\hat{\sigma}^2_{\varepsilon,T}(\hat{\rho}_T)$ satisfies

$$\begin{aligned}\hat{\sigma}^2_{\varepsilon,T}(\hat{\rho}_T) &= T^{-1}\sum_{t=1}^T z_t^2(\hat{\rho}_T)\\ &= T^{-1}\sum_{t=1}^T \{z_t(\rho_T) - (\hat{\rho}_T - \rho_T)y_{t-1}\}^2\\ &= T^{-1}\sum_{t=1}^T z_t^2(\rho_T) - 2T(\hat{\rho}_T-\rho_T)T^{-2}\sum_{t=1}^T y_{t-1}z_t(\rho_T) + T^2(\hat{\rho}_T-\rho_T)^2T^{-3}\sum_{t=1}^T y_{t-1}^2,\end{aligned}$$

for which we have $T^{-1}\sum_{t=1}^T z_t^2(\rho_T) = \hat{\sigma}^2_{\varepsilon,T}(\rho_T)$, $T^2(\hat{\rho}_T-\rho_T)^2T^{-3}\sum_{t=1}^T y_{t-1}^2 = O_p(T^{-1})$ by Lemma C.1 and the CMT, and

$$\begin{aligned}T^{-2}\sum_{t=1}^T y_{t-1}z_t(\rho_T) &= T^{-2}\sum_{t=1}^T y_{t-1}(cT^{-3/4}y_{t-1}v_t + \varepsilon_t)\\ &= cT^{-5/4}\int_0^1 Y_T^2(r)dW_{T,v}(r) + T^{-1}\sigma_\varepsilon\int_0^1 Y_T(r)dW_{\varepsilon,T}(r) = O_p(T^{-1}).\end{aligned}$$

Therefore

$$\hat{\sigma}^2_{\varepsilon,T}(\hat{\rho}_T) = \hat{\sigma}^2_{\varepsilon,T}(\rho_T) + o_p(1) \xrightarrow{p} \sigma^2_\varepsilon,$$

given Lemma B.4(a).

To prove part (b), write $\hat{\sigma}^2_{\eta,T}(\hat{\rho}_T)$ as

$$\hat{\sigma}^2_{\eta,T}(\hat{\rho}_T) = T^{-1}\sum_{t=1}^T\{z_t^2(\hat{\rho}_T) - \hat{\sigma}^2_{\varepsilon,T}(\hat{\rho}_T)\}^2 = T^{-1}\sum_{t=1}^T z_t^4(\hat{\rho}_T) - \hat{\sigma}^4_{\varepsilon,T}(\hat{\rho}_T). \tag{C.1}$$

The first term is

$$\begin{aligned}&T^{-1}\sum_{t=1}^T z_t^4(\hat{\rho}_T)\\ &= T^{-1}\sum_{t=1}^T\{z_t(\rho_T) - (\hat{\rho}_T-\rho_T)y_{t-1}\}^4\\ &= T^{-1}\sum_{t=1}^T z_t^4(\rho_T) - 4T(\hat{\rho}_T-\rho_T)T^{-2}\sum_{t=1}^T z_t^3(\rho_T)y_{t-1} + 6T^2(\hat{\rho}_T-\rho_T)^2T^{-3}\sum_{t=1}^T z_t^2(\rho_T)y_{t-1}^2\\ &- 4T^3(\hat{\rho}_T-\rho_T)^3T^{-4}\sum_{t=1}^T z_t(\rho_T)y_{t-1}^3 + T^4(\hat{\rho}_T-\rho_T)^4T^{-5}\sum_{t=1}^T y_{t-1}^4.\end{aligned}$$

Straightforward calculations reveal $T^{-2}\sum_{t=1}^{T} z_t^3(\rho_T)y_{t-1} = O_p(T^{-1/2})$, $T^{-3}\sum_{t=1}^{T} z_t^2(\rho_T)y_{t-1}^2 = O_p(T^{-1})$, $T^{-4}\sum_{t=1}^{T} z_t(\rho_T)y_{t-1}^3 = O_p(T^{-2})$, and $T^{-5}\sum_{t=1}^{T} y_{t-1}^4 = O_p(T^{-2})$, which gives

$$T^{-1}\sum_{t=1}^{T} z_t^4(\hat\rho_T) = T^{-1}\sum_{t=1}^{T} z_t^4(\rho_T) + O_p(T^{-1/2}).$$

Substituting this and $\hat\sigma_{\varepsilon,T}^2(\hat\rho_T) = \hat\sigma_{\varepsilon,T}^2(\rho_T) + o_p(1)$ into (C.1), we arrive at

$$\begin{aligned}\hat\sigma_{\eta,T}^2(\hat\rho_T) &= T^{-1}\sum_{t=1}^{T} z_t^4(\rho_T) - \hat\sigma_{\varepsilon,T}^4(\rho_T) + o_p(1)\\ &= \hat\sigma_{\eta,T}^2(\rho_T) + o_p(1) \xrightarrow{p} \sigma_\eta^2,\end{aligned}$$

by Lemma B.4(b).

To prove part (c), it suffices to show $T^{-1}\sum_{t=1}^{T} z_t(\hat\rho_T)\{z_t^2(\hat\rho_T) - \hat\sigma_{\varepsilon,T}^2(\hat\rho_T)\} \xrightarrow{p} \mathbb{E}[\varepsilon_t^3]$, given that $\hat\sigma_{\varepsilon,T}^2(\hat\rho_T) \xrightarrow{p} \sigma_\varepsilon^2$ and $\hat\sigma_{\eta,T}^2(\hat\rho_T) \xrightarrow{p} \sigma_\eta^2$. Now, we have

$$\begin{aligned}&T^{-1}\sum_{t=1}^{T} z_t(\hat\rho_T)\{z_t^2(\hat\rho_T) - \hat\sigma_{\varepsilon,T}^2(\hat\rho_T)\}\\ &= T^{-1}\sum_{t=1}^{T}\{z_t(\rho_T) - (\hat\rho_T-\rho_T)y_{t-1}\}\{z_t^2(\rho_T) - 2(\hat\rho_T-\rho_T)y_{t-1}z_t(\rho_T) + (\hat\rho_T-\rho_T)^2y_{t-1}^2 - \hat\sigma_{\varepsilon,T}^2(\hat\rho_T)\}\\ &= T^{-1}\sum_{t=1}^{T} z_t(\rho_T)\{z_t^2(\rho_T) - \hat\sigma_{\varepsilon,T}^2(\rho_T)\} + o_p(1) \xrightarrow{p} \mathbb{E}[\varepsilon_t^3],\end{aligned}$$

in view of the last line of the proof of Lemma B.4(c). □

Appendix D: Proofs of Results in Section 4

Lemma D.1. *Consider model* (2)*. Under Assumption 3 and under* $\mathrm{H}_0 : \omega_T = 0$*, the following results hold:*

(a) $y_{\lfloor Tr\rfloor}/\sqrt{T} \Rightarrow \sigma_{\varepsilon,lr}J_a(r)$*, where* J_a *satisfies* $dJ_a(r) = aJ_a(r)dr + dW_{\varepsilon,lr}(r)$*.*

(b)

$$T^{-3/2}\sum_{t=1}^{T} x_{2,t-1}z_t^2(\rho_T) \Rightarrow \sigma_{\eta,lr}\sigma_{\varepsilon,lr}^2\int_0^1 \tilde J_{a,2}(r)dW_{\eta,lr}(r) + 2\sigma_{\varepsilon,lr}\Lambda_{\varepsilon\eta}\int_0^1 J_a(r)dr,$$

(c)

$$T^{-1}\sum_{t=1}^{T} x_{1,t-1} z_t^2(\rho_T) \Rightarrow \sigma_{\eta,lr}\sigma_{\varepsilon,lr}\int_0^1 \tilde{J}_{a,1}(r)dW_{\eta,lr}(r) + \Lambda_{\varepsilon\eta},$$

(d) $\hat{\sigma}^2_{\tilde{\xi}^*}(\rho_T) \xrightarrow{p} \sigma^2_\eta$.

Proof. (a) Since $\mathrm{H}_0 : \omega_T = 0$ holds, $y_t = \rho_T y_{t-1} + \varepsilon_t$, and thus part (a) immediately follows by a standard argument under Assumption 3 (e.g., Phillips, 1987b).

(b) Noting that $z_t^2(\rho_T) = (y_t - \rho_T y_{t-1})^2 = \sigma_\varepsilon^2 + \eta_t$, we have

$$\begin{aligned} T^{-3/2}\sum_{t=1}^{T} x_{2,t-1} z_t^2(\rho_T) &= \sigma^2_{\varepsilon,lr}\sigma_{\eta,lr} T^{-3/2}\sum_{t=1}^{T}\Big(\frac{y_{t-1}^2}{\sigma^2_{\varepsilon,lr}} - T^{-1}\sum_{s=1}^{T}\frac{y_{s-1}^2}{\sigma^2_{\varepsilon,lr}}\Big)\frac{\eta_t}{\sigma_{\eta,lr}} \\ &= \sigma^2_{\varepsilon,lr}\sigma_{\eta,lr}\left\{\sum_{t=1}^{T}\Big(\frac{y_{t-1}}{\sigma_{\varepsilon,lr}\sqrt{T}}\Big)^2\frac{\eta_t}{\sigma_{\eta,lr}\sqrt{T}} - \sum_{s=1}^{T}\Big(\frac{y_{s-1}}{\sigma_{\varepsilon,lr}\sqrt{T}}\Big)^2\frac{1}{T}\times\sum_{t=1}^{T}\frac{\eta_t}{\sigma_{\eta,lr}\sqrt{T}}\right\}. \end{aligned}$$

The second term in the parentheses weakly converges to $\int_0^1 J_a^2(r)drW_{\eta,lr}(1)$ by part (a), the FCLT and the CMT. For the first term, we can apply Theorem 3.1 of Liang et al. (2016) to obtain

$$\sum_{t=1}^{T}\Big(\frac{y_{t-1}}{\sigma_{\varepsilon,lr}\sqrt{T}}\Big)^2\frac{\eta_t}{\sigma_{\eta,lr}\sqrt{T}} \Rightarrow \int_0^1 J_a^2(r)dW_{\eta,lr}(r) + 2\frac{\Lambda_{\varepsilon\eta}}{\sigma_{\varepsilon,lr}\sigma_{\eta,lr}}\int_0^1 J_a(r)dr.$$

Combining the above results yields

$$T^{-3/2}\sum_{t=1}^{T} x_{2,t-1} z_t^2(\rho_T) \Rightarrow \sigma^2_{\varepsilon,lr}\sigma_{\eta,lr}\int_0^1 \tilde{J}_{a,2}(r)dW_{\eta,lr}(r) + 2\sigma_{\varepsilon,lr}\Lambda_{\varepsilon\eta}\int_0^1 J_a(r)dr.$$

(c) Similarly to part (b), we have

$$T^{-1}\sum_{t=1}^{T} x_{1,t-1} z_t^2(\rho_T) = \sigma_{\varepsilon,lr}\sigma_{\eta,lr}\left\{\sum_{t=1}^{T}\frac{y_{t-1}}{\sigma_{\varepsilon,lr}\sqrt{T}}\frac{\eta_t}{\sigma_{\eta,lr}\sqrt{T}} - \sum_{s=1}^{T}\frac{y_{s-1}}{\sigma_{\varepsilon,lr}\sqrt{T}}\frac{1}{T}\times\sum_{t=1}^{T}\frac{\eta_t}{\sigma_{\eta,lr}\sqrt{T}}\right\}.$$

Applying part (a), Theorem 3.1 of Liang et al. (2016) and the CMT completes the proof.

(d) As in part (c) of Lemma B.5, we have

$$\begin{aligned} \hat{\sigma}^2_{\tilde{\xi}^*}(\rho_T) = {} & T^{-1}\sum_{t=1}^{T}(\tilde{\xi}_t^*)^2 - T^{-1}\begin{pmatrix}\sum_{t=1}^T x_{1,t-1}\xi_t^* & \sum_{t=1}^T x_{2,t-1}\xi_t^*\end{pmatrix} \\ & \times\begin{pmatrix}\sum_{t=1}^T (x_{1,t-1})^2 & \sum_{t=1}^T x_{1,t-1}x_{2,t-1} \\ \sum_{t=1}^T x_{2,t-1}x_{1,t-1} & \sum_{t=1}^T (x_{2,t-1})^2\end{pmatrix}^{-1}\begin{pmatrix}\sum_{t=1}^T x_{1,t-1}\xi_t^* \\ \sum_{t=1}^T x_{2,t-1}\xi_t^*\end{pmatrix}. \end{aligned}$$

The first term becomes

$$T^{-1}\sum_{t=1}^{T}(\widetilde{\xi}_t^*)^2 = T^{-1}\sum_{t=1}^{T}(\varepsilon_t^2-\sigma_\varepsilon^2)^2 - \left[T^{-1}\sum_{t=1}^{T}(\varepsilon_t^2-\sigma_\varepsilon^2)\right]^2$$
$$\xrightarrow{p} \sigma_\eta^2. \quad \text{(the short-run variance of } \eta_t\text{)}$$

The second term satisfies

$$\begin{aligned}
&T^{-1}\begin{pmatrix}\sum_{t=1}^T x_{1,t-1}\xi_t^* & \sum_{t=1}^T x_{2,t-1}\xi_t^*\end{pmatrix}\begin{pmatrix}\sum_{t=1}^T (x_{1,t-1})^2 & \sum_{t=1}^T x_{1,t-1}x_{2,t-1}\\ \sum_{t=1}^T x_{2,t-1}x_{1,t-1} & \sum_{t=1}^T (x_{2,t-1})^2\end{pmatrix}^{-1}\begin{pmatrix}\sum_{t=1}^T x_{1,t-1}\xi_t^*\\ \sum_{t=1}^T x_{2,t-1}\xi_t^*\end{pmatrix}\\
&= T^{-1}\begin{pmatrix}T^{-1}\sum_{t=1}^T x_{1,t-1}(\varepsilon_t^2-\sigma_\varepsilon^2) & T^{-3/2}\sum_{t=1}^T x_{2,t-1}(\varepsilon_t^2-\sigma_\varepsilon^2)\end{pmatrix}\\
&\times\begin{pmatrix}T^{-2}\sum_{t=1}^T (x_{1,t-1})^2 & T^{-5/2}\sum_{t=1}^T x_{1,t-1}x_{2,t-1}\\ T^{-5/2}\sum_{t=1}^T x_{2,t-1}x_{1,t-1} & T^{-3}\sum_{t=1}^T (x_{2,t-1})^2\end{pmatrix}^{-1}\begin{pmatrix}T^{-1}\sum_{t=1}^T x_{1,t-1}(\varepsilon_t^2-\sigma_\varepsilon^2)\\ T^{-3/2}\sum_{t=1}^T x_{2,t-1}(\varepsilon_t^2-\sigma_\varepsilon^2)\end{pmatrix}\\
&= O_p(T^{-1}),
\end{aligned}$$

by Theorem 3.1 of Liang et al. (2016). Hence, we obtain $\hat{\sigma}^2_{\widetilde{\xi}^*}(\rho_T) \xrightarrow{p} \sigma_\eta^2$. □

Proof of (16)**.** As in the proof of Theorem 2, we have

$$\begin{aligned}
W_T(\rho_T) = \hat{\sigma}^{-2}_{\widetilde{\xi}^*}&\begin{pmatrix}T^{-1}\sum_{t=1}^T x_{1,t-1}z_t^2(\rho_T)\\ T^{-3/2}\sum_{t=1}^T x_{2,t-1}z_t^2(\rho_T)\end{pmatrix}'\begin{pmatrix}T^{-2}\sum_{t=1}^T (x_{1,t-1})^2 & T^{-5/2}\sum_{t=1}^T x_{1,t-1}x_{2,t-1}\\ T^{-5/2}\sum_{t=1}^T x_{2,t-1}x_{1,t-1} & T^{-3}\sum_{t=1}^T (x_{2,t-1})^2\end{pmatrix}^{-1}\\
&\times\begin{pmatrix}T^{-1}\sum_{t=1}^T x_{1,t-1}z_t^2(\rho_T)\\ T^{-3/2}\sum_{t=1}^T x_{2,t-1}z_t^2(\rho_T)\end{pmatrix}.
\end{aligned}$$

Given the above equation, (16) is an immediate consequence of Lemma D.1 and the CMT. □

Proof of Theorem 4. First note that under the null of $\omega_T = 0$,

$$\begin{aligned}
&\hat{\theta}_T(\rho_T) - (X'X)^{-1}U_T(\rho_T)\\
&= (X'X)^{-1}\left\{\begin{pmatrix}\sum_{t=1}^T x_{1,t-1}\eta_t\\ \sum_{t=1}^T x_{2,t-1}\eta_t\end{pmatrix} - \begin{pmatrix}T\hat{\Lambda}_{\varepsilon\eta}(\rho_T)\\ 2\hat{\Lambda}_{\varepsilon\eta}(\rho_T)\sum_{t=1}^T y_{t-1}\end{pmatrix} - \begin{pmatrix}\hat{\psi}_{lr,T}(\rho_T)\hat{\sigma}_{\tilde{\xi}^*,lr}(\rho_T)\hat{\sigma}^{-1}_{\varepsilon,lr}(\rho_T)\sum_{t=1}^T x_{1,t-1}\varepsilon_t\\ \hat{\psi}_{lr,T}(\rho_T)\hat{\sigma}_{\tilde{\xi}^*,lr}(\rho_T)\hat{\sigma}^{-1}_{\varepsilon,lr}(\rho_T)\sum_{t=1}^T x_{2,t-1}\varepsilon_t\end{pmatrix}\right\}\\
&=: (X'X)^{-1}F_T.
\end{aligned}$$

Therefore, using Lemma D.1, $W_T^{lr}(\rho_T)$ can be written as

$$
\begin{aligned}
W_T^{lr}(\rho_T) &= \frac{F_T'(X'X)^{-1}F_T}{\hat{\sigma}^2_{\tilde{\xi}^*,lr}(\rho_T)(1-\hat{\psi}^2_{lr,T})} \\
&= \{\hat{\sigma}^2_{\tilde{\xi}^*,lr}(\rho_T)(1-\hat{\psi}^2_{lr,T})\}^{-1}\left\{\begin{pmatrix} T^{-1}\sum_{t=1}^T x_{1,t-1}\eta_t \\ T^{-3/2}\sum_{t=1}^T x_{2,t-1}\eta_t \end{pmatrix} - \begin{pmatrix} \hat{\Lambda}_{\varepsilon\eta}(\rho_T) \\ 2\hat{\Lambda}_{\varepsilon\eta}(\rho_T)T^{-3/2}\sum_{t=1}^T y_{t-1} \end{pmatrix}\right. \\
&\qquad \left. - \begin{pmatrix} \hat{\psi}_{lr,T}(\rho_T)\hat{\sigma}_{\tilde{\xi}^*,lr}(\rho_T)\hat{\sigma}^{-1}_{\varepsilon,lr}(\rho_T)T^{-1}\sum_{t=1}^T x_{1,t-1}\varepsilon_t \\ \hat{\psi}_{lr,T}(\rho_T)\hat{\sigma}_{\tilde{\xi}^*,lr}(\rho_T)\hat{\sigma}^{-1}_{\varepsilon,lr}(\rho_T)T^{-3/2}\sum_{t=1}^T x_{2,t-1}\varepsilon_t \end{pmatrix}\right\}' \\
&\quad \times \begin{pmatrix} T^{-2}\sum_{t=1}^T x_{1,t-1}^2 & T^{-5/2}\sum_{t=1}^T x_{1,t-1}x_{2,t-1} \\ T^{-5/2}\sum_{t=1}^T x_{2,t-1}x_{1,t-1} & T^{-3}\sum_{t=1}^T x_{2,t-1}^2 \end{pmatrix}^{-1} \\
&\quad \times \left\{\begin{pmatrix} T^{-1}\sum_{t=1}^T x_{1,t-1}\eta_t \\ T^{-3/2}\sum_{t=1}^T x_{2,t-1}\eta_t \end{pmatrix} - \begin{pmatrix} \hat{\Lambda}_{\varepsilon\eta}(\rho_T) \\ 2\hat{\Lambda}_{\varepsilon\eta}(\rho_T)T^{-3/2}\sum_{t=1}^T y_{t-1} \end{pmatrix}\right. \\
&\qquad \left. - \begin{pmatrix} \hat{\psi}_{lr,T}(\rho_T)\hat{\sigma}_{\tilde{\xi}^*,lr}(\rho_T)\hat{\sigma}^{-1}_{\varepsilon,lr}(\rho_T)T^{-1}\sum_{t=1}^T x_{1,t-1}\varepsilon_t \\ \hat{\psi}_{lr,T}(\rho_T)\hat{\sigma}_{\tilde{\xi}^*,lr}(\rho_T)\hat{\sigma}^{-1}_{\varepsilon,lr}(\rho_T)T^{-3/2}\sum_{t=1}^T x_{2,t-1}\varepsilon_t \end{pmatrix}\right\} \\
&\Rightarrow \{\sigma^2_{\eta,lr}(1-\psi^2_{lr})\}^{-1}\left\{\begin{pmatrix} \sigma_{\eta,lr}\sigma_{\varepsilon,lr}\int_0^1 \tilde{J}_{a,1}(r)dW_{\eta,lr}(r) \\ \sigma_{\eta,lr}\sigma^2_{\varepsilon,lr}\int_0^1 \tilde{J}_{a,2}(r)dW_{\eta,lr}(r) \end{pmatrix} - \begin{pmatrix} \psi_{lr}\sigma_{\eta,lr}\sigma_{\varepsilon,lr}\int_0^1 \tilde{J}_{a,1}(r)dW_{\varepsilon,lr} \\ \psi_{lr}\sigma_{\eta,lr}\sigma^2_{\varepsilon,lr}\int_0^1 \tilde{J}_{a,2}(r)dW_{\varepsilon,lr} \end{pmatrix}\right\}' \\
&\quad \times \begin{pmatrix} \sigma^2_{\varepsilon,lr}\int_0^1 \tilde{J}^2_{a,1}(r)dr & \sigma^3_{\varepsilon,lr}\int_0^1 \tilde{J}_{a,1}(r)\tilde{J}_{a,2}(r)dr \\ \sigma^3_{\varepsilon,lr}\int_0^1 \tilde{J}_{a,2}(r)\tilde{J}_{a,1}(r)dr & \sigma^4_{\varepsilon,lr}\int_0^1 \tilde{J}^2_{a,2}(r)dr \end{pmatrix}^{-1} \\
&\quad \times \left\{\begin{pmatrix} \sigma_{\eta,lr}\sigma_{\varepsilon,lr}\int_0^1 \tilde{J}_{a,1}(r)dW_{\eta,lr}(r) \\ \sigma_{\eta,lr}\sigma^2_{\varepsilon,lr}\int_0^1 \tilde{J}_{a,2}(r)dW_{\eta,lr}(r) \end{pmatrix} - \begin{pmatrix} \psi_{lr}\sigma_{\eta,lr}\sigma_{\varepsilon,lr}\int_0^1 \tilde{J}_{a,1}(r)dW_{\varepsilon,lr} \\ \psi_{lr}\sigma_{\eta,lr}\sigma^2_{\varepsilon,lr}\int_0^1 \tilde{J}_{a,2}(r)dW_{\varepsilon,lr} \end{pmatrix}\right\} \\
&\overset{d}{=} \begin{pmatrix} \int_0^1 \tilde{J}_{a,1}(r)dW_1(r) \\ \int_0^1 \tilde{J}_{a,2}(r)dW_1(r) \end{pmatrix}' \begin{pmatrix} \int_0^1 \left(\tilde{J}_{a,1}\right)^2(r)dr & \int_0^1 \tilde{J}_{a,1}(r)\tilde{J}_{a,2}(r)dr \\ \int_0^1 \tilde{J}_{a,2}(r)\tilde{J}_{a,1}(r)dr & \int_0^1 \left(\tilde{J}_{a,2}\right)^2(r)dr \end{pmatrix}^{-1} \begin{pmatrix} \int_0^1 \tilde{J}_{a,1}(r)dW_1(r) \\ \int_0^1 \tilde{J}_{a,2}(r)dW_1(r) \end{pmatrix},
\end{aligned}
$$

where the last equality follows from the fact that $W_{\eta,lr} \overset{d}{=} \psi_{lr}W_{\varepsilon,lr} + \sqrt{1-\psi^2_{lr}}W_1$. The weak limit is the chi-square distribution with 2 degrees of freedom since W_1 and $W_{\varepsilon,lr}$ are independent. □

Appendix E: Additional Simulation Results

E.1 Additional results for Section 2.5 (asymptotic power)

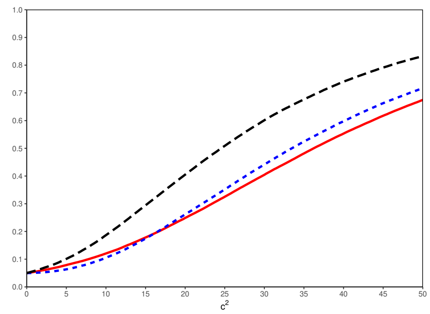
In this section, we give additional simulation results pertaining to Section 2.5, where the asymptotic power functions are computed for infeasible versions of the LN, augmented t, and augmented Wald tests. The simulation results for $a = -5, -10$ are shown in Figures E.1 and E.2, respectively. The general pattern of the power properties is the same as when $a = 0$: The LN test performs best for small q, and the Wald test performs best for moderate to large q. However, the powers of all the tests we consider get lower as a deviates from 0 (as long as $a < 0$). A similar tendency of the power properties of tests for $\mathrm{H}_0 : \omega_T^2 = 0$ has been observed through simulation by earlier work such as Nagakura (2009) and Horváth and Trapani (2019). Nonetheless, even when $a < 0$, the power function of the Wald test is increasing in q while the power function of the LN test is decreasing in q.

E.2 Additional results for Section 5.1 (finite-sample size)

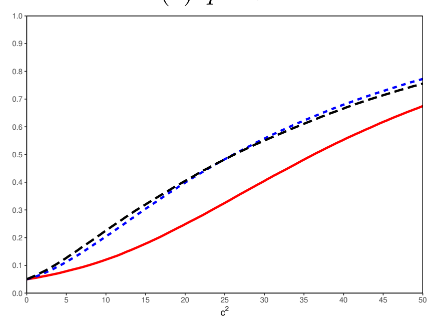
This section gives the finite-sample size of the Bonferroni-Wald test modified to allow for serial correlation and conditional heteroskedasticity in ε_t. We conduct the Bonferroni-Wald test as explained in Algorithm 3 in Section 4.2 (without refinement). y_t is generated by $y_t = \rho y_{t-1} + \varepsilon_t$ with $\rho \in [0.7, 1.01]$. Two DGPs are considered for ε_t: (i) ε_t is a GARCH(1,1) process of the form $\varepsilon_t = \sigma_t u_t$, $\sigma_t^2 = 0.1 + 0.85\sigma_{t-1}^2 + 0.05\varepsilon_{t-1}^2$, where $u_t \sim$ i.i.d. $N(0,1)$, and (ii) $\varepsilon_t = 0.4\varepsilon_{t-1} + u_t$ is an AR(1) process.

To proceed with the modified Bonferroni-Wald test, we need to specify lag length p to construct the confidence interval for ρ_T. For the first DGP, we set $p = 1$ since ε_t is serially uncorrelated. This means that the confidence interval for γ_T, $\mathrm{CI}_\gamma(\alpha_1)$, can be directly used as the confidence interval for ρ_T (Step 2 is skipped). For the second DGP, we use $p = 4$, 6, and 8 for $T = 200$, 500, and 1000, respectively. We set $\alpha_1 = \alpha_2 = 0.05$, resulting in the eventual significance level of $\alpha = \alpha_1 + \alpha_2 = 0.1$. We show the results in Figures E.3 and E.4.

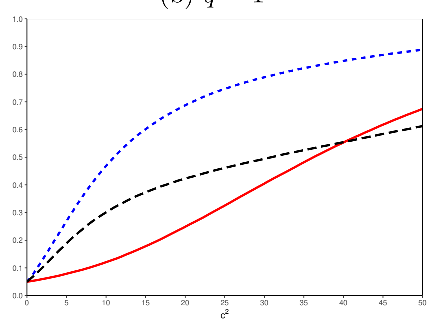
For the first DGP in which ε_t is a GARCH process (Figure E.3), the type 1 error is around or below the 10% nominal level over $\rho \in [0.7, 1.01]$ for $T = 200, 500$. However, it increases as ρ deviates from 1 if $\rho \leq 0.9$ and $T = 1000$, exceeding the nominal level eventually. This is due to the identification problem emerging as y_t takes on stationarity ($a = T(\rho - 1) \to -\infty$); see Remark 2. Therefore, the Bonferroni test will be valid only in a close neighborhood of



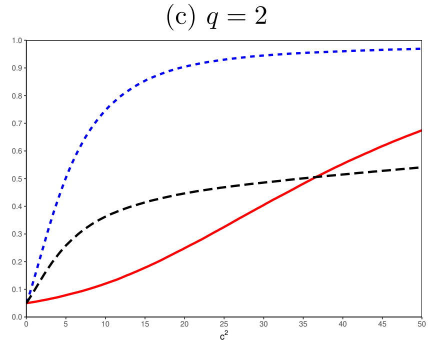
(a) $q = 0$



(b) $q = 1$

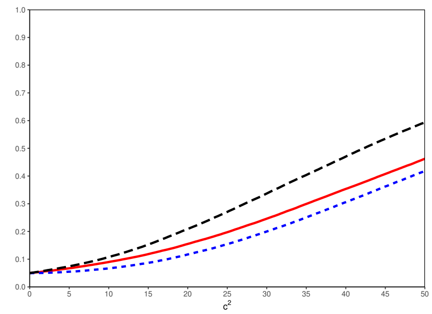


(c) $q = 2$

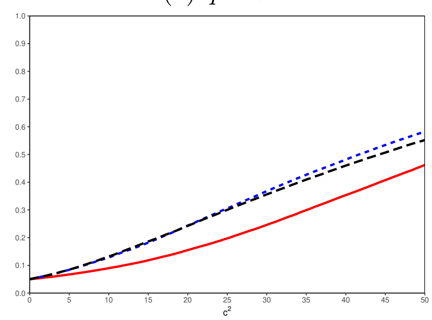


(d) $q = 3$

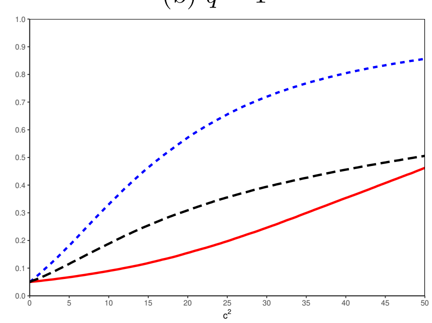
Figure E.1: Asymptotic power for $a = -5$



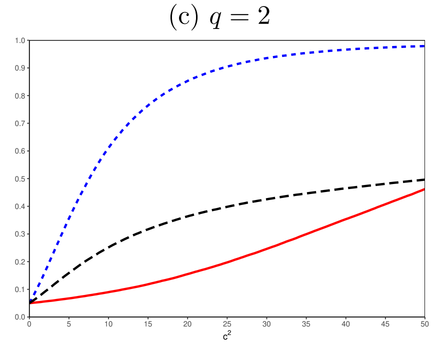
(a) $q = 0$



(b) $q = 1$



(c) $q = 2$



(d) $q = 3$

Figure E.2: Asymptotic power for $a = -10$

⋯⋯: Augmented Wald, ——: Augmented t, - - -: LN

$\rho = 1$ when the disturbance is conditionally heteroskedastic. For the case of ε_t being an AR(1) process (Figure E.4), the type 1 error is stable between 2-5% and does not increase as $a = T(\rho_T - 1) \to -\infty$.

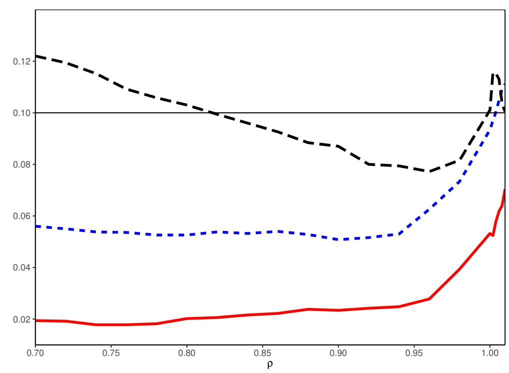


Figure E.3: Finite-sample rejection rates of the Bonferroni-Wald test under the null, (i)ε_t GARCH

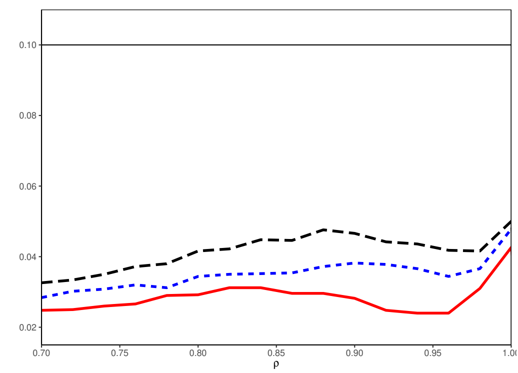


Figure E.4: Finite-sample rejection rates of the Bonferroni-Wald test under the null, (ii)ε_t AR

—: $T = 200$, ⋯⋯: $T = 500$, - - -: $T = 1000$

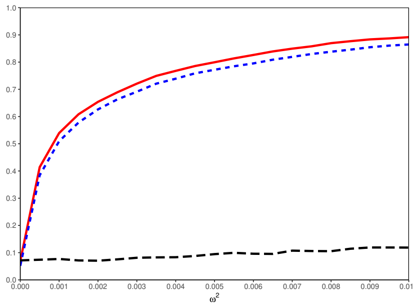
E.3 Additional results for Section 5.2 (finite-sample power)

Figures E.5-E.7 display the finite-sample power functions of the infeasible Wald, Bonferroni Wald (following Algorithm 2), LN, and HT tests for $\rho = 1.01, 0.98, 0.95$. For the case $\rho = 1.01$, where all but the LN test are performed, the infeasible and Bonferroni-Wald tests have good power, and their power functions are almost identical. In contrast, the power function of the HT test stays around the nominal level 0.05 over $\omega^2 \in (0, 0.01]$.

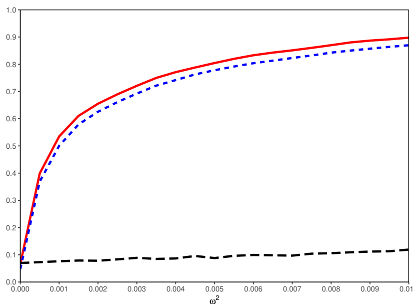
Turning to the cases $\rho < 1$, it is noticeable that for each ρ, the power functions are similar in shape to those for $\rho = 1$, but all tests lose their power as $1 - \rho$ gets larger. This finding is in line with the asymptotic analysis given in Section E.1.

E.3.1 Power for the case where v_t is serially correlated

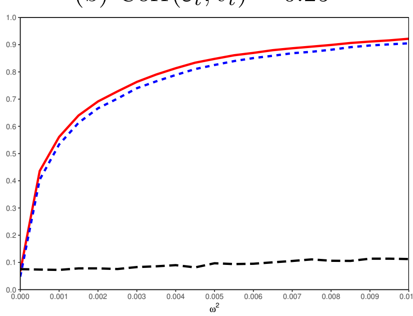
Throughout the article, we have assumed that the random part $\{v_t\}$ of the coefficient is an i.i.d. sequence. To investigate how tests behave without the i.i.d. assumption on v_t, we generate y_t based on the same DGP as that of Section 5.2 except that v_t is generated as $v_t = 0.4v_{t-1} + u_t$ with $u_t \sim$ i.i.d. $N(0, 1)$. In this setting, we calculate the power functions



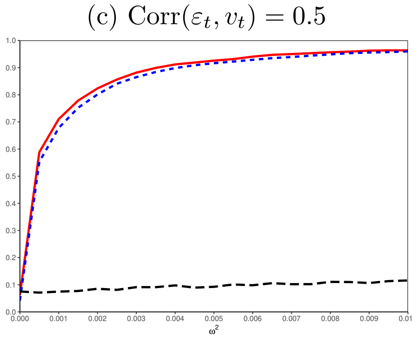
(a) $\text{Corr}(\varepsilon_t, v_t) = 0$



(b) $\text{Corr}(\varepsilon_t, v_t) = 0.25$

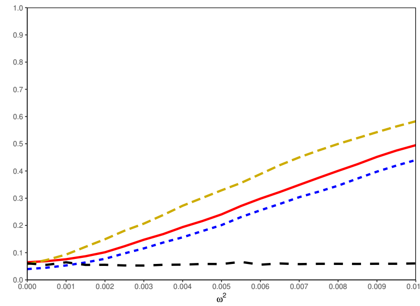


(c) $\text{Corr}(\varepsilon_t, v_t) = 0.5$

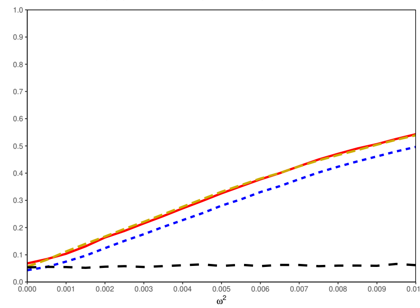


(d) $\text{Corr}(\varepsilon_t, v_t) = 0.75$

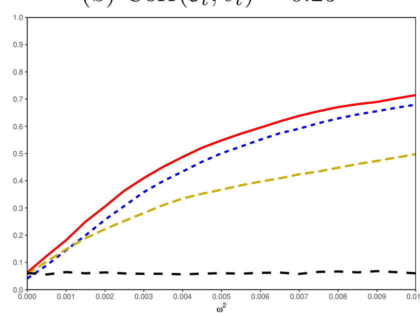
Figure E.5: Power for $T = 200$, $\rho = 1.01$, ε_t i.i.d



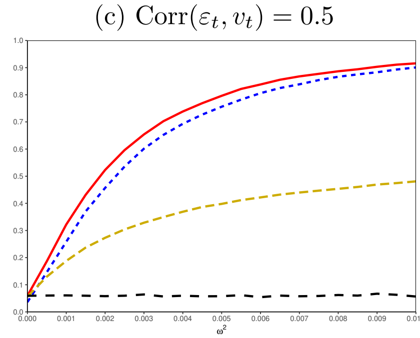
(a) $\text{Corr}(\varepsilon_t, v_t) = 0$



(b) $\text{Corr}(\varepsilon_t, v_t) = 0.25$



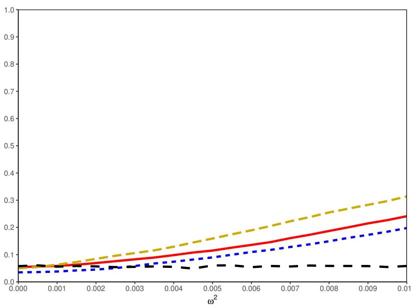
(c) $\text{Corr}(\varepsilon_t, v_t) = 0.5$



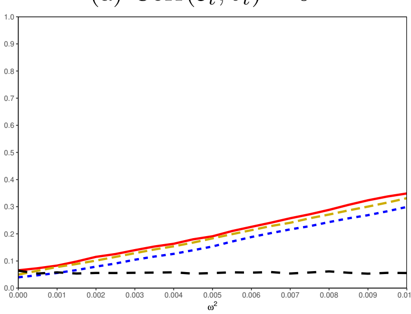
(d) $\text{Corr}(\varepsilon_t, v_t) = 0.75$

Figure E.6: Power for $T = 200$, $\rho = 0.98$, ε_t i.i.d

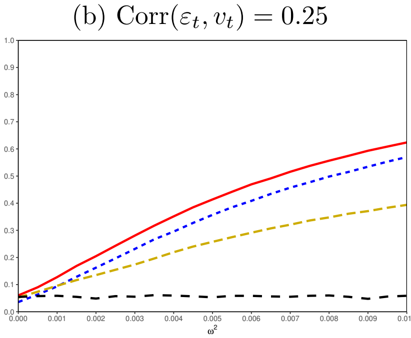
⋯⋯: Bonf. Wald, ——: Inf. Wald, - - -:LN , - - -: HT



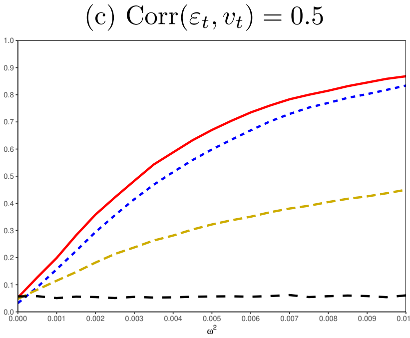
(a) $\mathrm{Corr}(\varepsilon_t, v_t) = 0$



(b) $\mathrm{Corr}(\varepsilon_t, v_t) = 0.25$

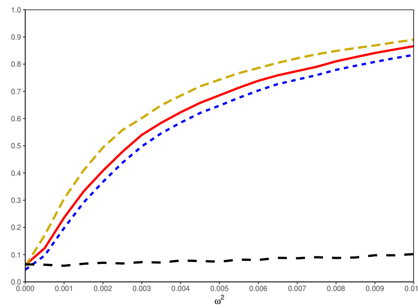


(c) $\mathrm{Corr}(\varepsilon_t, v_t) = 0.5$

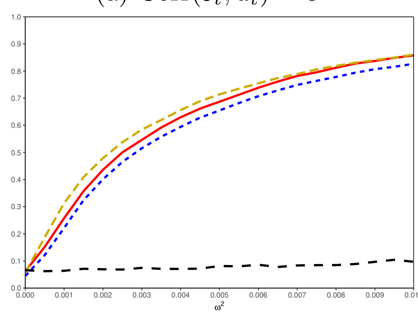


(d) $\mathrm{Corr}(\varepsilon_t, v_t) = 0.75$

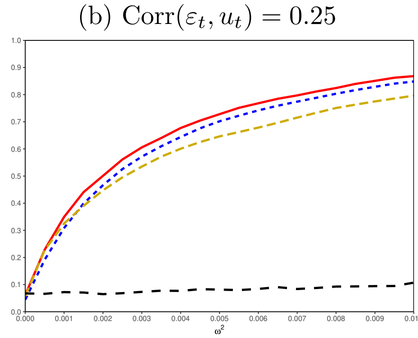
Figure E.7: Power for $T = 200$, $\rho = 0.95$, ε_t i.i.d



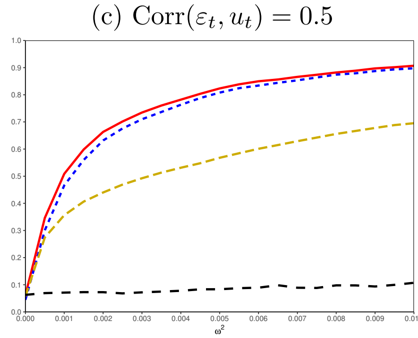
(a) $\mathrm{Corr}(\varepsilon_t, u_t) = 0$



(b) $\mathrm{Corr}(\varepsilon_t, u_t) = 0.25$



(c) $\mathrm{Corr}(\varepsilon_t, u_t) = 0.5$



(d) $\mathrm{Corr}(\varepsilon_t, u_t) = 0.75$

Figure E.8: Power for $T = 200$, $\rho = 1$, v_t AR(1)

⋯⋯: Bonf. Wald, ——: Inf. Wald, - - -:LN , - - -: HT of the infeasible Wald, Bonferroni-Wald, LN, and HT tests. We report the result for $\rho = 1$ only. The results for the case of $\rho = 1.01, 0.98, 0.95$ are available upon request. The result is given in Figure E.8 along with the correlation between ε_t and u_t (rather than v_t). The power functions and power rankings are quite similar to those obtained in the case of v_t being i.i.d. (see Figure 3). Hence, the power analysis given in Section 5.2 is robust to the property of v_t.

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