Online Supplement to the Paper: A Nonparametric Test for Instantaneous Causality with Time-varying Variances

Jilin Wu*, Ruike Wu†, and Zhijie Xiao‡

*Department of Finance, School of Economics, Gregory and Paula Chow Institute for studies in Economics, and Wang Yanan Institute for Studies in Economics (WISE), Xiamen University**
Department of Finance, School of Economics, Xiamen University†
Department of Economics, Boston College‡

This Appendix consists of four parts. Appendix A offers detailed proofs for Theorems 1-4 in the main paper; Appendix B presents Brown's central limit theorem (CLT) and Yoshihara's lemma, which are essential prerequisites for Appendix A; Appendix C reports the simulation results for the cases where the true lag length in the VAR model is assumed to be known; Appendix D reports additional simulation results for the time-varying coefficient VAR model in Section 6 of the main paper.

Appendix A: Proofs of the main results

Lemma A1. Under both $\mathbb{H}_0$ and $\mathbb{H}_A$, we have $Th^{1/2}\tilde{\lambda}_T = \frac{1}{Th^{1/2}}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}m_t'm_s + o_p(1)$, where $m_t = vec(u_{1t}u_{2t}')$.

Proof of Lemma A1. We rewrite $\hat{m}_t = m_t + (\hat{m}_t - m_t)$, then $Th^{1/2}\tilde{\lambda}_T$ has the following decomposition

$$\begin{aligned} Th^{1/2}\tilde{\lambda}_T &= \frac{1}{Th^{1/2}}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}m_t'm_s + \frac{1}{Th^{1/2}}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}(\hat{m}_t - m_t)'(\hat{m}_s - m_s) \\ &+ \frac{2}{Th^{1/2}}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}(\hat{m}_s - m_s)' m_t \\ &= U_{1T} + U_{2T} + 2U_{3T}. \end{aligned} \tag{A1}$$

Denote $M_t = X_{t-1}' \otimes I_d$ and divide $M_t = (M_{1t}', M_{2t}')'$, where M_{it} is a $d_i \times d^2 p$ matrix. Because $\hat{u}_t = Y_t - (X_{t-1}' \otimes I_d)\hat{\mathbf{\Pi}}$, then

$$\hat{u}_{it} = u_{it} - M_{it}(\hat{\mathbf{\Pi}} - \mathbf{\Pi}), i = 1, 2.$$

As a result, $\hat{m}_t - m_t$ can be rewritten as

$$\begin{aligned} \hat{m}_t - m_t &= -vec\left(u_{1t}(\hat{\mathbf{\Pi}}-\mathbf{\Pi})'M_{2t}'\right) - vec\left(M_{1t}(\hat{\mathbf{\Pi}}-\mathbf{\Pi})u_{2t}'\right) \\ &+ vec\left(M_{1t}(\hat{\mathbf{\Pi}}-\mathbf{\Pi})(\hat{\mathbf{\Pi}}-\mathbf{\Pi})'M_{2t}'\right) \\ &= -(M_{2t}\otimes u_{1t})(\hat{\mathbf{\Pi}}-\mathbf{\Pi}) - (u_{2t}\otimes M_{1t})(\hat{\mathbf{\Pi}}-\mathbf{\Pi}) \\ &+ (M_{2t}\otimes M_{1t})\, vec((\hat{\mathbf{\Pi}}-\mathbf{\Pi})(\hat{\mathbf{\Pi}}-\mathbf{\Pi})'). \end{aligned} \tag{A2}$$

Based on (A2), U_{2T} has the following decomposition

$$\begin{aligned} U_{2T} &= (\hat{\mathbf{\Pi}}-\mathbf{\Pi})'(U_{21T} + U_{22T} + U_{24T} + U_{25T})(\hat{\mathbf{\Pi}}-\mathbf{\Pi}) \\ &- (\hat{\mathbf{\Pi}}-\mathbf{\Pi})'(U_{23T} + U_{26T}) vec((\hat{\mathbf{\Pi}}-\mathbf{\Pi})(\hat{\mathbf{\Pi}}-\mathbf{\Pi})') \\ &- \left[vec((\hat{\mathbf{\Pi}}-\mathbf{\Pi})(\hat{\mathbf{\Pi}}-\mathbf{\Pi})')\right]'(U_{27T} + U_{28T})(\hat{\mathbf{\Pi}}-\mathbf{\Pi}) \\ &+ \left[vec((\hat{\mathbf{\Pi}}-\mathbf{\Pi})(\hat{\mathbf{\Pi}}-\mathbf{\Pi})')\right]' U_{29T} vec((\hat{\mathbf{\Pi}}-\mathbf{\Pi})(\hat{\mathbf{\Pi}}-\mathbf{\Pi})'), \end{aligned} \tag{A3}$$

where

$$U_{21T} = \frac{1}{Th^{1/2}} \sum_{t=p+1}^{T} \sum_{s\neq t}^{T} k_{s,t} \left(M_{2t}' M_{2s}\right) \otimes \left(u_{1t}' u_{1s}\right);$$

$$U_{22T} = \frac{1}{Th^{1/2}} \sum_{t=p+1}^{T} \sum_{s\neq t}^{T} k_{s,t} \left(M_{2t}' u_{2s}\right) \otimes \left(u_{1t}' M_{1s}\right);$$

$$U_{23T} = \frac{1}{Th^{1/2}} \sum_{t=p+1}^{T} \sum_{s\neq t}^{T} k_{s,t} \left(M_{2t}' M_{2s}\right) \otimes \left(u_{1t}' M_{1s}\right);$$

$$U_{24T} = \frac{1}{Th^{1/2}} \sum_{t=p+1}^{T} \sum_{s\neq t}^{T} k_{s,t} \left(u_{2t}' M_{2s}\right) \otimes \left(M_{1t}' u_{1s}\right);$$

$$U_{25T} = \frac{1}{Th^{1/2}} \sum_{t=p+1}^{T} \sum_{s\neq t}^{T} k_{s,t} \left(u_{2t}' u_{2s}\right) \otimes \left(M_{1t}' M_{1s}\right);$$

$$U_{26T} = \frac{1}{Th^{1/2}} \sum_{t=p+1}^{T} \sum_{s\neq t}^{T} k_{s,t} \left(u_{2t}' M_{2s}\right) \otimes \left(M_{1t}' M_{1s}\right);$$

$$U_{27T} = \frac{1}{Th^{1/2}} \sum_{t=p+1}^{T} \sum_{s\neq t}^{T} k_{s,t} \left(M_{2t}' M_{2s}\right) \otimes \left(M_{1t}' u_{1s}\right);$$

$$U_{28T} = \frac{1}{Th^{1/2}} \sum_{t=p+1}^{T} \sum_{s\neq t}^{T} k_{s,t} \left(M_{2t}' u_{2s}\right) \otimes \left(M_{1t}' M_{1s}\right);$$

$$U_{29T} = \frac{1}{Th^{1/2}} \sum_{t=p+1}^{T} \sum_{s\neq t}^{T} k_{s,t} \left(M_{2t}' M_{2s}\right) \otimes \left(M_{1t}' M_{1s}\right). \tag{A4}$$

For U_{21T},

$$\begin{aligned} E\left\|U_{21T}\right\| &\leq \frac{1}{Th^{1/2}} \sum_{t=p+1}^{T} \sum_{s\neq t}^{T} k_{s,t} E\left\|\left(M_{2t}' M_{2s}\right) \otimes \left(u_{1t}' u_{1s}\right)\right\| \\ &\leq \frac{1}{Th^{1/2}} \sum_{t=p+1}^{T} \sum_{s\neq t}^{T} k_{s,t} E\left\|\left(X_{t-1} X_{s-1}' \otimes I_d\right) \otimes \left(u_{1t}' u_{1s}\right)\right\| \\ &\leq \frac{d^{1/2}}{Th^{1/2}} \sum_{t=p+1}^{T} \sum_{s\neq t}^{T} k_{s,t} \left(E\left\|X_{t-1}\right\|^4 E\left\|X_{s-1}\right\|^4 E\left\|u_{1t}\right\|^4 E\left\|u_{1s}\right\|^4\right)^{1/4} \\ &\leq \frac{C}{Th^{1/2}} \sum_{t=p+1}^{T} \sum_{s\neq t}^{T} k_{s,t} = O\left(Th^{1/2}\right), \end{aligned} \tag{A5}$$

where the second inequality holds because M_{2t} is the subpart of M_t, the third inequality is obtained by using Cauchy-Schwarz inequality, and the last inequality holds because $\sup_t E\left\|X_{t-1}\right\|^4 < C$ and $\sup_t E\left\|u_{1t}\right\|^4 < C$, which are implied by $\sup_t E\left\|u_{it}\right\|^8 < C, i = 1, 2$, in Assumption 2(iii), and $\frac{1}{Th}\sum_{s\neq t}^{T} k_{s,t} = O(1)$. Hence $U_{21T} = O_p\left(Th^{1/2}\right)$.

In the similar ways, we can show $U_{22T} = U_{23T} = \ldots = U_{29T} = O_p\left(Th^{1/2}\right)$. In addition, $\sqrt{T}\left(\hat{\mathbf{\Pi}} - \mathbf{\Pi}\right) = O_p(1)$ under Assumptions 1-2, see Proposition 3.1 of Patilea and Rassi (2012). As a result, $U_{2T} = O_p\left(h^{1/2}\right)$.

Based on (A2), U_{3T} has the following decomposition

$$U_{3T} = -(\hat{\mathbf{\Pi}} - \mathbf{\Pi})'(U_{31T} + U_{32T}) + \left[vec((\hat{\mathbf{\Pi}} - \mathbf{\Pi})(\hat{\mathbf{\Pi}} - \mathbf{\Pi})')\right]' U_{33T}, \tag{A6}$$

where

$$U_{31T}=\frac{1}{Th^{1/2}}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T}k_{s,t}\left(M_{2s}'\otimes u_{1s}'\right)vec\left(u_{1t}u_{2t}'\right);$$

$$U_{32T}=\frac{1}{Th^{1/2}}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T}k_{s,t}\left(u_{2s}'\otimes M_{1s}'\right)vec\left(u_{1t}u_{2t}'\right);$$

$$U_{33T}=\frac{1}{Th^{1/2}}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T}k_{s,t}\left(M_{2s}'\otimes M_{1s}'\right)vec\left(u_{1t}u_{2t}'\right). \tag{A7}$$

For U_{33T}, its proof is the same as that of U_{21T}, so $U_{33T}=O_p\left(Th^{1/2}\right)$. The proofs of U_{31T} and U_{32T} are similar, so we only prove U_{31T} here. For U_{31T}, let $Z_d=[\mathbf{0}_{d_2\times d_1},I_{d_2}]$, if $s<t$, we have

$$\begin{aligned}
E\left\|U_{31T}\right\|^2 &= E\left\|\frac{1}{Th^{1/2}}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T}k_{s,t}\left[\left((X_{s-1}\otimes I_d)Z_d'\right)\otimes u_{1s}'\right]vec\left(u_{1t}u_{2t}'\right)\right\|^2\\
&\leq \frac{1}{T^2h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T}\sum_{s'\neq t}^{T}k_{s,t}k_{s',t}tr\left\{E\left[\left[\left(X_{s'-1}'X_{s-1}\otimes I_d\right)\otimes\left(u_{1s'}u_{1s}'\right)\right]vec\left(u_{1t}u_{2t}'\right)\left[vec\left(u_{1t}u_{2t}'\right)\right]'\right]\right\}\\
&+\frac{2}{T^2h}\sum_{t=p+2}^{T}\sum_{t'\neq t}^{T}\sum_{s<t,s<t'}^{T}k_{s,t}k_{s,t'}tr\left\{\left[E\left(X_{s-1}'X_{s-1}\otimes I_d\right)\otimes E\left(u_{1s}u_{1s}'\right)\right]E\left[vec\left(u_{1t}u_{2t}'\right)\right]E\left[vec\left(u_{1t'}u_{2t'}'\right)\right]'\right\}\\
&\leq \frac{d^{1/2}}{T^2h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T}\sum_{s'\neq t}^{T}k_{s,t}k_{s',t}E\left(\left\|X_{s'-1}X_{s-1}\right\|\left\|u_{1s'}u_{1s}'\right\|\left\|vec\left(u_{1t}u_{2t}'\right)\right\|^2\right)\\
&+\frac{2d^{1/2}}{T^2h}\sum_{t=p+2}^{T}\sum_{t'\neq t}^{T}\sum_{s<t,s<t'}^{T}k_{s,t}k_{s,t'}E\left\|X_{s'-1}X_{s-1}\right\|E\left\|u_{1s}u_{1s}'\right\|E\left\|vec\left(u_{1t}u_{2t}'\right)\right\|E\left\|vec\left(u_{1t'}u_{2t'}'\right)\right\|\\
&\leq \frac{C}{T^2h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T}\sum_{s'\neq t}^{T}k_{s,t}k_{s',t}+\frac{C}{T^2h}\sum_{t=p+2}^{T}\sum_{t'\neq t}^{T}\sum_{s<t,s<t'}^{T}k_{s,t}k_{s,t'}\\
&= O\left(Th\right),
\end{aligned}\tag{A8}$$

where the last inequality is obtained by using Cauchy-Schwarz inequality as well as $\sup_t E\left\|X_t\right\|^8<C$, $\sup_t E\left\|u_{it}\right\|^4<C$ and $\sup_t E\left\|u_{it}\right\|^2<C$, $i=1,2$, which are implied by $\sup_t E\left\|u_{it}\right\|^8<C, i=1,2$, in Assumption 2(iii). The last line holds because $\frac{1}{T^2h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T}\sum_{s'\neq t}^{T}k_{s,t}k_{s',t}=O\left(Th\right)$ and $\frac{1}{T^2h}\sum_{t=p+2}^{T}\sum_{t'\neq t}^{T}\sum_{s<t,s<t'}^{T}k_{s,t}k_{s,t'}=O\left(Th\right)$. On the other hand, if $s>t$, by using the similar arguments to those of showing the case of $s<t$ in (A8), we also obtain $E\left\|U_{31T}\right\|^2=O\left(Th\right)$. The two cases of $s<t$ and $s>t$ mean that $U_{31T}=O_p\left(T^{1/2}h^{1/2}\right)$. Similarly, we can show $U_{32T}=O_p\left(T^{1/2}h^{1/2}\right)$.

The orders of magnitude of U_{31T},U_{32T} and U_{33T} as well as $\sqrt{T}\left(\hat{\mathbf{\Pi}}-\mathbf{\Pi}\right)=O_p\left(1\right)$ give $U_{3T}=O_p\left(h^{1/2}\right)$. Combining the results of U_{2T} and U_{3T}, we obtain Lemma A1. □

Lemma A2. Under $\mathbb{H}_0$, we have

$$R_T=\frac{1}{Th^{1/2}}\sum_{t=p+2}^{T}\sum_{s=p+1}^{t-1}k_{s,t}m_t'm_s\xrightarrow{d}\mathbb{N}\left(0,\varphi^2\right),$$

where $\varphi^2=\left(vec\left(\Upsilon\right)\right)'\left(\int_0^1\left(\Omega\left(r\right)\otimes\Omega\left(r\right)\right)dr\right)vec\left(\Upsilon\right)\int_0^1k^2(u)du$.

Proof of Lemma A2. We denote $W_t=\frac{1}{Th^{1/2}}\sum_{s=p+1}^{t-1}k_{s,t}m_t'm_s$, and let $s_T^2=E(R_T^2)$. Since W_t is an m.d.s. with respect to $\mathcal{F}_{t-1}$ under the null, we apply Brown's (1971) martingale central limit theorem to prove the asymptotic normality, which states that $s_T^{-1}R_T\xrightarrow{d}\mathbb{N}\left(0,1\right)$ if

$$(i)\;\; s_T^{-2}\sum_{t=p+2}^{T}E\left(W_t^2\middle|\mathcal{F}_{t-1}\right)\xrightarrow{p}1;(ii)\;\; s_T^{-2}\sum_{t=p+2}^{T}E\left[W_t^2I\left(\left|W_t\right|>\eta s_T\right)\right]\to 0,\;\forall\,\eta>0. \tag{A9}$$

After some calculation, it is not hard to find that proving the two conditions in (A9) is equivalent to checking whether

$$(i)\ \ s_T^{-2}\sum_{t=p+2}^{T}W_t^2-1\xrightarrow{p}0 \text{ and } (ii)\ \ \sum_{t=p+2}^{T}E(W_t^4)\to 0 \tag{A10}$$

hold or not[1].

In order to prove (i) in (A10), it suffices to show $E\left(\sum_{t=p+2}^{T}W_t^2-s_T^2\right)^2=E\left(\sum_{t=p+2}^{T}W_t^2\right)^2-s_T^4=o(1)$. First it is easy to have

$$s_T^2=\frac{1}{T^2h}\sum_{t=p+2}^{T}\sum_{s=p+1}^{t-1}k_{s,t}^2E\left(m_t'm_s\right)^2+\frac{2}{T^2h}\sum_{t=p+3}^{T}\sum_{s'=p+2}^{t-1}\sum_{s=p+1}^{s'-1}k_{s,t}k_{s',t}E\left(m_t'm_sm_t'm_{s'}\right)=S_1+2S_2. \tag{A11}$$

For S_2, we consider the following two cases: (a) $s'-s\geq t-s'$, and (b) $t-s'\geq s'-s$. For (a), using the inequality of Yoshihara (1976), we have

$$\begin{aligned}
|S_2| &\leq \frac{1}{T^2h}\sum_{s=p+1}^{T-2}\sum_{s'=s+1}^{T-1}\sum_{t=s'+1}^{T}k_{s,t}k_{s',t}\left|E\left(m_t'm_sm_t'm_{s'}\right)\right| \\
&\leq \frac{k^2(0)}{T^2h}\sum_{s=p+1}^{T-2}\sum_{s'=s+1}^{T-1}\sum_{t=s'+1}^{T}M_1^{1/(1+\delta)}\beta\left(s'-s\right)^{\delta/(1+\delta)} \\
&\leq \frac{k^2(0)M_1^{1/(1+\delta)}}{T^2h}\sum_{s=p+1}^{T-2}\sum_{s'=s+1}^{T-1}\left(s'-s\right)\beta\left(s'-s\right)^{\delta/(1+\delta)} \\
&= O\left(\frac{1}{Th}\right),
\end{aligned} \tag{A12}$$

where we have used $M_1=\max\left\{\sup_{t,s,s'}E\left|m_t'm_sm_t'm_{s'}\right|^{1+\delta},\sup_{t,s'}E\left\|m_t'm_{s'}m_t'\right\|^{1+\delta}\sup_sE\left\|m_s\right\|^{1+\delta}\right\}<C$ (implied by $\sup_{t,s,t',s'}E\left|u_{it}'u_{is}u_{it'}'u_{is'}\right|^{4(1+\delta)}<C$, $i=1,2$, in Assumption 2(iii)), and $\sum_{s'=s+1}^{T-1}\left(s'-s\right)\beta\left(s'-s\right)^{\delta/(1+\delta)}<\infty$ by Assumption 2(ii). The case (b) is similar, so $S_2=O(\frac{1}{Th})$.

For S_1, we rewrite it as

$$S_1=\frac{1}{T^2h}\sum_{t=p+2}^{T}\sum_{s=p+1}^{t-1}k_{s,t}^2tr\left\{E\left(m_tm_t'\right)E\left(m_sm_s'\right)\right\}+\frac{1}{T^2h}\sum_{t=p+2}^{T}\sum_{s=p+1}^{t-1}k_{s,t}^2tr\left\{cov\left(m_tm_t',m_sm_s'\right)\right\}. \tag{A13}$$

The second term on the R.H.S in (A13) is negligible by using the mixing condition, whose proof is similar to that of S_2 in (A12). For the first term in (A13) , we have $E\left(m_tm_t'\right)=\left(G_{2t}\otimes G_{1t}\right)\Upsilon\left(G_{2t}'\otimes G_{1t}'\right)$ since $u_{it}=G_{it}\varepsilon_t$, which leads to

$$tr\left[E\left(m_tm_t'\right)E\left(m_sm_s'\right)\right]=\left(vec\left(\Upsilon\right)\right)'\left(\left(G_{2t}'G_{2s}\otimes G_{1t}'G_{1s}\right)\otimes\left(G_{2t}'G_{2s}\otimes G_{1t}'G_{1s}\right)\right)vec\left(\Upsilon\right).$$

[1] (i)

$$\begin{aligned}
E\left(\sum_{t=p+2}^{T}E\left(W_t^2\middle|\mathcal{F}_{t-1}\right)-s_T^2\right)^2 &= E\left[\left(\sum_{t=p+2}^{T}W_t^2-s_T^2\right)-\sum_{t=p+2}^{T}\left(W_t^2-E\left(W_t^2\middle|\mathcal{F}_{t-1}\right)\right)\right]^2 \\
&\leq 2E\left(\sum_{t=p+2}^{T}W_t^2-s_T^2\right)^2+2E\left(\sum_{t=p+2}^{T}\left(W_t^2-E\left(W_t^2\middle|\mathcal{F}_{t-1}\right)\right)\right)^2 \\
&\leq 2E\left(\sum_{t=p+2}^{T}W_t^2-s_T^2\right)^2+2\sum_{t=p+2}^{T}E\left(W_t^2-E\left(W_t^2\middle|\mathcal{F}_{t-1}\right)\right)^2 \\
&\leq 2E(\sum_{t=p+2}^{T}W_t^2-s_T^2)^2+2\sum_{t=p+2}^{T}E\left(W_t^4\right),
\end{aligned}$$

where from the first line to the second line we use $(a_1-\alpha_2)^2\leq 2(\alpha_1^2+\alpha_2^2)$, and from the second line to the third line we use the fact that $W_t^2-E\left(W_t^2\middle|\mathcal{F}_{t-1}\right)$ is an m.d.s.

(ii) By using $s_T^{-2}\sum_{t=p+2}^{T}E\left[W_t^2I\left(|W_t|>\eta s_T\right)\right]<\eta^{-2}s_T^{-4}\sum_{t=p+2}^{T}E\left[W_t^4I\left(|W_t|>\eta s_T\right)\right]\leq\eta^{-2}s_T^{-4}\sum_{t=p+2}^{T}E\left(W_t^4\right),\forall\eta>0$.

As a result we obtain

$$s_T^2 \to (vec(\Upsilon))' \left(\int_0^1 (\Omega(r) \otimes \Omega(r))\, dr \right) vec(\Upsilon) \int_0^1 k^2(u)du, \tag{A14}$$

where $\Omega(r) = G_2'(r) G_2(r) \otimes G_1'(r) G_1(r)$.

From (A11) and (A12), we know $s_T^2 = \frac{1}{T^2 h} \sum_{t=p+2}^{T} \sum_{s=p+1}^{t-1} k_{s,t}^2 E(m_t' m_s)^2 + o(1)$, then

$$\begin{aligned}
E\left(\sum_{t=p+2}^{T} W_t^2\right)^2 - s_T^4 \approx{}& \frac{1}{T^4 h^2} \sum_{t,t'=p+2}^{T} \sum_{s=p+1}^{t-1} \sum_{r=p+1}^{t'-1} k_{s,t}^2 k_{r,t'}^2 \left[E\left(m_t' m_s m_{t'}' m_r\right)^2 - E\left(m_t' m_s\right)^2 E\left(m_{t'}' m_r\right)^2 \right] \\
&+ \frac{2}{T^4 h^2} \sum_{t,t'=p+2}^{T} \sum_{s=p+1}^{t-1} \sum_{s' \neq s}^{t-1} k_{s,t} k_{s',t} k_{s,t'} k_{s',t'} E(m_t' m_s m_t' m_{s'} m_{t'}' m_s m_{t'}' m_{s'}) \\
&+ \frac{2}{T^4 h^2} \sum_{t,t'=p+2}^{T} \sum^{t-1} \sum^{t'-1} \sum_{s \neq r \neq r'}^{t'-1} k_{s,t}^2 k_{r,t'} k_{r',t'} E(\left(m_t' m_s\right)^2 m_{t'}' m_r m_{t'}' m_{r'}) \\
&+ \frac{4}{T^4 h^2} \sum_{t,t'=p+2}^{T} \sum_{s=1}^{t-1} \sum_{s \neq r}^{t'-1} k_{s,t}^2 k_{r,t'} k_{s,t'} E(\left(m_t' m_s\right)^2 m_{t'}' m_r m_{t'}' m_s) \\
&+ \frac{4}{T^4 h^2} \sum_{t,t'=p+2}^{T} \sum^{t-1} \sum^{t-1} \sum_{s \neq s' \neq r'}^{t'-1} k_{s,t} k_{s',t} k_{s,t'} k_{r',t'} E(m_t' m_s m_t' m_{s'} m_{t'}' m_s m_{t'}' m_{r'}) \\
&+ \frac{1}{T^4 h^2} \sum_{t,t'=p+2}^{T} \sum^{t-1} \sum^{t-1} \sum^{t'-1} \sum_{s \neq s' \neq r \neq r'}^{t'-1} k_{s,t} k_{s',t} k_{r,t'} k_{r',t'} E(m_t' m_s m_t' m_{s'} m_{t'}' m_r m_{t'}' m_{r'}) \\
\approx{}& D_1 + 2D_2 + 2D_3 + 4D_4 + 4D_5 + D_6.
\end{aligned} \tag{A15}$$

We first consider D_1 and rewrite it as

$$\begin{aligned}
D_1 ={}& \frac{1}{T^4 h^2} \sum_{t=p+2}^{T} \sum_{s=p+1}^{t-1} \sum_{r=p+1}^{t-1} k_{s,t}^2 k_{r,t}^2 \left[E\left(m_t' m_s m_t' m_r\right)^2 - E\left(m_t' m_s\right)^2 E\left(m_t' m_r\right)^2 \right] \\
&+ \frac{2}{T^4 h^2} \sum_{t=p+2}^{T} \sum_{t'<t}^{T} \sum_{s=p+1}^{t-1} \sum_{r=p+1}^{t'-1} k_{s,t}^2 k_{r,t'}^2 \left[E\left(m_t' m_s m_{t'}' m_r\right)^2 - E\left(m_t' m_s\right)^2 E\left(m_{t'}' m_r\right)^2 \right] \\
={}& D_{11} + 2D_{12}.
\end{aligned} \tag{A16}$$

For D_{12}, suppose that $r < t' < s < t$. By using the inequality of Yoshihara (1976) we have

$$\begin{aligned}
|D_{12}| \leq{}& \frac{1}{T^4 h^2} \sum_{t=s+1}^{T} \sum_{s=t'+1}^{T-1} \sum_{t'=r+1}^{T-2} \sum_{r=p+1}^{T-3} k_{s,t}^2 k_{r,t'}^2 \left| E\left(m_t' m_s m_{t'}' m_r\right)^2 - E\left(m_t' m_s\right)^2 E\left(m_{t'}' m_r\right)^2 \right| \\
\leq{}& \frac{k^2(0)}{T^4 h^2} \sum_{t=s+1}^{T} k_{s,t}^2 \sum_{s=t'+1}^{T-1} \sum_{t'=r+1}^{T-2} \sum_{r=p+1}^{T-3} M_2^{1/(1+\delta)} \beta\left(s - t'\right)^{\delta/(1+\delta)} \\
\leq{}& \frac{C}{Th} \left(\frac{1}{Th} \sum_{t=s+1}^{T} k_{s,t}^2 \right) \left(\sum_{s=t'+1}^{T-1} \beta\left(s - t'\right)^{\delta/(1+\delta)} \right) \\
={}& O\left(\frac{1}{Th}\right),
\end{aligned} \tag{A17}$$

where $M_2 = \max\left\{ \sup_{t,s,t',r} E\left|m_t' m_s m_{t'}' m_r\right|^{2(1+\delta)}, \sup_{t,s} E\left|m_t' m_s\right|^{2(1+\delta)} \sup_{t',r} E\left|m_{t'}' m_r\right|^{2(1+\delta)} \right\} < C$ (implied by $\sup_{t,s,t',s'} E\left|u_{it}' u_{is} u_{it'}' u_{is'}\right|^{4(1+\delta)} < C$, $i = 1, 2$, in Assumption 2(iii)), $\frac{1}{Th} \sum_{t=s+1}^{T} k_{s,t}^2 = O(1)$, and $\sum_{s=t'+1}^{T-1} \beta\left(s - t'\right)^{\delta/(1+\delta)} < C$ by Assumption 2(ii). The cases for $r < s < t' < t$ and $s < r < t' < t$ are similar. For D_{11}, by using the similar arguments to those of showing $|D_{12}| = O\left(\frac{1}{Th}\right)$, we can obtain $|D_{11}| = O\left(\frac{1}{T^2 h}\right)$. Thus, $D_1 = O\left(\frac{1}{Th}\right)$.

The proofs of D_2, D_3, D_4, D_5 and D_6 are similar, here we only show that $D_6 = o(1)$. Now rewrite

$$\begin{aligned}
D_6 &= \frac{1}{T^4h^2} \sum_{t=p+2}^{T} \sum^{t-1} \sum^{t-1} \sum^{t-1} \sum_{s\neq s'\neq r\neq r'}^{t-1} k_{s,t}k_{s',t}k_{r,t}k_{r',t} E(m_t' m_s m_t' m_{s'} m_t' m_r m_t' m_{r'}) \\
&+ \frac{2}{T^4h^2} \sum_{t=p+2}^{T} \sum_{t'<t}^{T} \sum^{t-1} \sum^{t-1} \sum^{t'-1} \sum_{s\neq s'\neq r\neq r'}^{t'-1} k_{s,t}k_{s',t}k_{r,t'}k_{r',t'} E(m_t' m_s m_t' m_{s'} m_{t'}' m_r m_{t'}' m_{r'}) \\
&= D_{61} + 2D_{62}. \qquad \text{(A18)}
\end{aligned}$$

Without loss of generality, suppose that $s < s'$ and $r < r'$ so that $s' < s < t$ and $r' < r < t'$. Denote $r' < s' < r < s < t' < t$ as Case 1 and consider the following subcases:

Case 1(a): $s' - r' \geq \max\{r - s', s - r, t' - s, t - t'\}$;
Case 1(b): $r - s' \geq \max\{s' - r', s - r, t' - s, t - t'\}$;
Case 1(c): $s - r \geq \max\{s' - r', r - s', t' - s, t - t'\}$;
Case 1(d): $t' - s \geq \max\{s' - r', r - s', s - r, t - t'\}$;
Case 1(e): $t - t' \geq \max\{s' - r', r - s', s - r, t' - s\}$.

For Case 1(a),

$$\begin{aligned}
|D_{62}| &\leq \frac{1}{T^4h^2} \sum_{r'=p+1}^{T-5} \sum_{s'=r'+1}^{T-4} \sum_{r=s'+1}^{T-3} \sum_{s=r+1}^{T-2} \sum_{t'=s+1}^{T-1} \sum_{t=t'+1}^{T} k_{s,t}k_{s',t}k_{r,t'}k_{r',t'} \left|E(m_t' m_s m_t' m_{s'} m_{t'}' m_r m_{t'}' m_{r'})\right| \\
&\leq \frac{k^4(0)}{T^4h^2} \sum_{r'=p+1}^{T-5} \sum_{s'=r'+1}^{T-4} \sum_{r=s'+1}^{T-3} \sum_{s=r+1}^{T-2} \sum_{t'=s+1}^{T-1} \sum_{t=t'+1}^{T} M_3^{1/(1+\delta)} \beta\left(s' - r'\right)^{\delta/(1+\delta)} \\
&\leq \frac{C}{T^2h^2} \sum_{r'=p+1}^{T-5} \sum_{s'=r'+1}^{T-4} \left(s' - r'\right)^2 \beta\left(s' - r'\right)^{\delta/(1+\delta)} \\
&= O\left(\frac{1}{Th^2}\right), \qquad \text{(A19)}
\end{aligned}$$

where $\sum_{s'=r'+1}^{T-4} \left(s' - r'\right)^2 \beta\left(s' - r'\right)^{\delta/(1+\delta)} < C$ by Assumption 2(ii), and

$$\begin{aligned}
M_3 = \max \Bigg\{ &\sup_{r,r',s,s',t,t'} E\left|m_t' m_s m_t' m_{s'} m_{t'}' m_r m_{t'}' m_{r'}\right|^{1+\delta}, \\
&\sup_{r,s,s',t,t'} E\left\|m_t' m_s m_t' m_{s'} m_{t'}' m_r m_{t'}'\right\|^{1+\delta} \sup_{r'} E\left\|m_{r'}\right\|^{1+\delta} \Bigg\} < C,
\end{aligned}$$

which is implied by $\sup_{t,s,t',s'} E\left|u_{it}' u_{is} u_{it'}' u_{is'}\right|^{4(1+\delta)} < C,\ i = 1, 2,$ in Assumption 2(iii).

The proofs of Cases 1(b)-(d) are similar to that of Case 1(a), and $|D_{62}|$ is still $O\left(\frac{1}{Th^2}\right)$. For Case 1(e),

$$\begin{aligned}
\left|E(m_t' m_s m_t' m_{s'} m_{t'}' m_r m_{t'}' m_{r'})\right| &\leq \left\|cov\left(m_t m_t', m_s m_{r'}' m_{t'} m_r' m_{t'} m_{s'}'\right)\right\| \\
&+ E\left\|m_t\right\|^2 \left\|E\left(m_s m_{r'}' m_{t'} m_r' m_{t'} m_{s'}'\right)\right\|.
\end{aligned}$$

The proof of $|D_{62}|$ that involves $\|cov\left(m_t m_t', m_s m_{r'}' m_{t'} m_r' m_{t'} m_{s'}'\right)\|$ is still similar to that of Case 1(a), but for the proof of $|D_{62}|$ that involves $\|E\left(m_s m_{r'}' m_{t'} m_r' m_{t'} m_{s'}'\right)\|$, we consider the second-order statistic of $\{s' - r', r - s', s - r, t' - s\}$ and repeat the arguments for Cases 1(a)-(d). Finally, we obtain $|D_{62}| = O\left(\frac{1}{Th^2}\right)$ in Case 1(e). Thus, $D_{62} = o(1)$. The proof of D_{61} is similar to that of D_{62}, and then $D_{61} = o(1)$.

By using the similar arguments to those of showing $D_6 = o(1)$, we can show that D_2, D_3, D_4 and D_5 are negligible. Combining these results, we have $E\left(\sum_{t=p+2}^{T} W_t^2 - s_T^2\right)^2 = o(1)$, which means that $s_T^{-2} \sum_{t=p+2}^{T} W_t^2 \xrightarrow{p} 1$.

Now we turn to the proof of (ii) in (A10), and rewrite

$$\sum_{t=p+2}^{T} E(W_t^4) = \frac{1}{T^4h^2}\sum_{t=p+2}^{T}\sum_{s=p+1}^{t-1}\sum_{s'=p+1}^{t-1}\sum_{r=p+1}^{t-1}\sum_{r'=p+1}^{t-1} k_{s,t}k_{s',t}k_{r,t}k_{r',t}E(m_t'm_s m_t'm_{s'}m_t'm_r m_t'm_{r'}), \tag{A20}$$

whose proof is also similar to that of $E\left(\sum_{t=p+2}^{T} W_t^2\right)^2 - s_T^4 = o(1)$ in (A15), and we can obtain $\sum_{t=p+2}^{T} E(W_t^4) = o(1)$. Thus, $R_T \xrightarrow{d} \mathbb{N}\left(0, \varphi^2\right)$ holds. This completes the proof of Lemma A2.

□

Lemma A3. Under both $\mathbb{H}_0$ and $\mathbb{H}_A$, we have $\hat{\sigma}_T^2 = \frac{2}{T^2h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2\left(m_t'm_s\right)^2 + o_p(1)$, and $\hat{\sigma}_T^2 \xrightarrow{p} \sigma^2$.

Proof of Lemma A3. Rewriting $\hat{m}_t = m_t + (\hat{m}_t - m_t)$, we have the following decomposition

$$\begin{aligned}
\hat{\sigma}_T^2 =& \frac{2}{T^2h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2\left(m_t'm_s\right)^2 + \frac{4}{T^2h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2\left[m_t'\left(\hat{m}_s - m_s\right)\right]^2 \\
&+ \frac{8}{T^2h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2 m_s'm_t m_t'\left(\hat{m}_s - m_s\right) + \frac{2}{T^2h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2\left[\left(\hat{m}_t - m_t\right)'\left(\hat{m}_s - m_s\right)\right]^2 \\
&+ \frac{4}{T^2h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2 m_s'm_t\left(\hat{m}_t - m_t\right)'\left(\hat{m}_s - m_s\right) + \frac{4}{T^2h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2 m_t'\left(\hat{m}_s - m_s\right)\left(\hat{m}_t - m_t\right)'m_s \\
&+ \frac{8}{T^2h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2 m_t'\left(\hat{m}_s - m_s\right)\left(\hat{m}_t - m_t\right)'\left(\hat{m}_s - m_s\right) \\
=& B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7.
\end{aligned} \tag{A21}$$

We need to show that $B_2 = \cdots = B_7 = o_p(1)$ and $\hat{\sigma}_T^2 = B_1 + o_p(1)$. The proofs of $B_2, \cdots$, and B_7 are similar, we only prove B_2 here. Using (A2) and the inequality $\left(\sum_{i=1}^{d} a_i\right)^2 \leq d\sum_{i=1}^{d} a_i^2$, we have

$$\begin{aligned}
\left[m_t'\left(\hat{m}_s - m_s\right)\right]^2 \leq& 3\left[\left\|\left(u_{2t}'M_{2s}\right)\otimes\left(u_{1t}'u_{1s}\right)\right\|^2 + \left\|\left(u_{2t}'u_{2s}\right)\otimes\left(u_{1t}'M_{1s}\right)\right\|^2\right]\left\|\hat{\mathbf{\Pi}}-\mathbf{\Pi}\right\|^2 \\
&+ 3\left\|\left(u_{2t}'M_{2s}\right)\otimes\left(u_{1t}'M_{1s}\right)\right\|^2\left\|\hat{\mathbf{\Pi}}-\mathbf{\Pi}\right\|^4.
\end{aligned}$$

Then

$$B_2 \leq \left\|\hat{\mathbf{\Pi}}-\mathbf{\Pi}\right\|^2\left(B_{21} + B_{22}\right) + \left\|\hat{\mathbf{\Pi}}-\mathbf{\Pi}\right\|^4 B_{23},$$

where

$$B_{21} = \frac{12}{T^2h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2\left\|\left(u_{2t}'M_{2s}\right)\otimes\left(u_{1t}'u_{1s}\right)\right\|^2;$$

$$B_{22} = \frac{12}{T^2h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2\left\|\left(u_{2t}'u_{2s}\right)\otimes\left(u_{1t}'M_{1s}\right)\right\|^2;$$

$$B_{23} = \frac{12}{T^2h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2\left\|\left(u_{2t}'M_{2s}\right)\otimes\left(u_{1t}'M_{1s}\right)\right\|^2.$$

We can show that B_{21}, B_{22} and B_{23} are all $O_p(1)$, whose proofs are similar to that of proving U_{21T} in (A5). Thus, we have $B_2 = O_p(\frac{1}{T})$ since $\sqrt{T}\left(\hat{\mathbf{\Pi}}-\mathbf{\Pi}\right) = O_p(1)$ under Assumptions 1-2. By taking similar arguments to those of showing that B_2 is negligible, we also have $B_3 = \cdots = B_7 = o_p(1)$. Hence $\hat{\sigma}_T^2 = B_1 + o_p(1)$.

Because we have shown that

$$s_T^2 = \frac{1}{T^2h}\sum_{t=p+2}^{T}\sum_{s=p+1}^{t-1} k_{s,t}^2 E\left(m_t'm_s\right)^2 + o(1) \to \left(vec\left(\Upsilon\right)\right)'\left(\int_0^1\left(\Omega\left(r\right)\otimes\Omega\left(r\right)\right)dr\right)vec\left(\Upsilon\right)\int_0^1 k^2(u)du,$$

in the proof of $s_T^{-2}\sum_{t=p+2}^{T} W_t^2 - 1 \xrightarrow{p} 0$ in (A10), then

$$E\left(B_1\right) \to 2\left(vec\left(\Upsilon\right)\right)'\left(\int_0^1 \left(\Omega\left(r\right)\otimes\Omega\left(r\right)\right)dr\right)vec\left(\Upsilon\right)\int_{-1}^{1} k^2(u)du.$$

Based on the mean squared error of B_1, we can show $E\left[B_1 - E\left(B_1\right)\right]^2 = o(1)$, whose proof is similar to that of $E\left(\sum_{t=p+2}^{T} W_t^2 - s_T^2\right)^2 = o(1)$ in Lemma A2, so $\hat{\sigma}_T^2 \xrightarrow{p} \sigma^2$. This completes the proof of Lemma A3. □

Lemma A4. Under $\mathbb{H}_A$, we have $R_T = \frac{1}{Th^{1/2}}\sum_{t=p+2}^{T}\sum_{s=p+1}^{t-1} k_{s,t}m_t'm_s = O_p(Th^{1/2})$.

Proof of Lemma A4. Define $z_t = m_t - E\left(m_t\right)$, then

$$\begin{aligned}
R_T &= \frac{1}{Th^{1/2}}\sum_{t=p+2}^{T}\sum_{s=p+1}^{t-1} k_{s,t}z_t'z_s + \frac{2}{Th^{1/2}}\sum_{t=p+2}^{T}\sum_{s=p+1}^{t-1} k_{s,t}z_t'E\left(m_s\right) \\
&+ \frac{1}{Th^{1/2}}\sum_{t=p+2}^{T}\sum_{s=p+1}^{t-1} k_{s,t}E\left(m_t\right)'E\left(m_s\right) \\
&= R_{1T} + 2R_{2T} + R_{3T}.
\end{aligned} \tag{A22}$$

For R_{3T},

$$\begin{aligned}
R_{3T} &= \frac{1}{Th^{1/2}}\sum_{s=p+1}^{T-1}\sum_{t=s+1}^{T} k_{s,t}\left(vec\left(\Sigma_t^{12}\right)\right)'vec\left(\Sigma_s^{12}\right) \\
&= Th^{1/2}\left[\frac{1}{Th}\sum_{i=1}^{T} k\left(\frac{i}{Th}\right)\right]\frac{1}{T}\sum_{s=p+1}^{T-1}\left(vec\left(\Sigma^{12}\left(\frac{s}{T}\right)\right)\right)'vec\left(\Sigma^{12}(\frac{s}{T})\right) + o\left(1\right) \\
&\to \frac{1}{2}Th^{1/2}tr\left(\int_0^1 \Sigma^{12}\left(r\right)\Sigma^{12}\left(r\right)'dr\right),
\end{aligned} \tag{A23}$$

since $E\left(m_t\right) = vec\left(\Sigma_t^{12}\right)$, $\frac{1}{Th}\sum_{i=1}^{T} k\left(\frac{i}{Th}\right) \to \frac{1}{2}$, and $\frac{1}{T}\sum_{s=p+1}^{T-1}\left(vec\left(\Sigma^{12}\left(\frac{s}{T}\right)\right)\right)'vec\left(\Sigma^{12}(\frac{s}{T})\right) \to tr\left(\int_0^1 \Sigma^{12}\left(r\right)\Sigma^{12}\left(r\right)'dr\right)$.

Note that $z_t = vec\left[G_{1t}\left(\varepsilon_t\varepsilon_t' - I_d\right)G_{2t}'\right]$ is an m.d.s. with respect to $\mathcal{F}_{t-1}$, then

$$\begin{aligned}
\left\|R_{2T}\right\|^2 &= \frac{1}{T^2h}\sum_{t=p+2}^{T}\sum_{s=p+1}^{t-1}\sum_{s'=p+1}^{t-1} k_{s,t}k_{s',t}\left(vec\left(\Sigma_s^{12}\right)\right)'E\left(z_tz_t'\right)vec\left(\Sigma_{s'}^{12}\right) \\
&\le \frac{1}{T^2h}\sum_{t=p+2}^{T}\sum_{s=p+1}^{t-1}\sum_{s'=p+1}^{t-1} k_{s,t}k_{s',t}\left(\sup_s\left\|vec\left(\Sigma_s^{12}\right)\right\|\right)^2\sup_t E\left\|z_t\right\|^2 \\
&\le \frac{C}{T^2h}\sum_{t=p+2}^{T}\sum_{s=p+1}^{t-1} k_{s,t}^2 + \frac{C}{T^2h}\sum_{t=p+2}^{T}\sum_{s=p+1}^{t-1}\sum_{s'\neq s}^{t-1} k_{s,t}k_{s',t} \\
&= O\left(Th\right),
\end{aligned} \tag{A24}$$

where $\sup_s\left\|vec\left(\Sigma_s^{12}\right)\right\| < C$, $\sup_t E\left\|z_t\right\|^2 < C$, $\frac{1}{T^2h}\sum_{t=p+2}^{T}\sum_{s=p+1}^{t-1} k_{s,t}^2 = O\left(1\right)$ and $\frac{1}{T^3h^2}\sum_{t=p+2}^{T}\sum_{s=p+1}^{t-1}\sum_{s'\neq s}^{t-1} k_{s,t}$ $k_{s',t} = O\left(1\right)$. Hence $R_{2T} = O_p\left(T^{1/2}h^{1/2}\right)$.

Lastly, because $z_t = vec\left[G_{1t}\left(\varepsilon_t\varepsilon_t' - I_d\right)G_{2t}'\right]$ is an m.d.s. with respect to $\mathcal{F}_{t-1}$, we can show that R_{1T} has an asymptotically normal distribution by using Brown's (1971) martingale CLT, whose proof is similar to that of Lemma A2. Thus, $R_{1T} = O_p(1)$. Combining the results of R_{1T}, R_{2T} and R_{3T}, we then have $R_T = O_p(Th^{1/2})$. This completes the proof of Lemma A4. □

Lemma A5. Under $\mathbb{H}_{LA} : \Sigma^{12}\left(r\right) = T^{-1/2}h^{-1/4}\pi(r)$, we have

$$R_T = \frac{1}{Th^{1/2}} \sum_{t=p+2}^{T} \sum_{s=p+1}^{t-1} k_{s,t} m_t' m_s \xrightarrow{d} \mathbb{N}\left(\frac{1}{2} tr\left(\int_0^1 \pi(r)\pi(r)' dr\right), \varphi^2\right), \tag{A25}$$

where $\varphi^2 = (vec(\Upsilon))' \left(\int_0^1 (\Omega(r) \otimes \Omega(r)) dr\right) vec(\Upsilon) \int_0^1 k^2(u) du$.

Proof of Lemma A5. The proof here is the same as that of Lemma A4 only by letting $\Sigma^{12}(r) = T^{-1/2}h^{-1/4}\pi(r)$, and R_T still can be decomposed as the summation of R_{1T}, R_{2T} and R_{3T}. It is clear that $R_{3T} \to \frac{1}{2} tr\left(\int_0^1 \pi(r)\pi(r)' dr\right)$ by (A23), and $R_{2T} = O_p\left(h^{1/4}\right)$ by (A24). Specifically, we can show $R_{1T} \xrightarrow{d} \mathbb{N}(0, \varphi^2)$, whose proof is similar to that of Lemma A2. Combining the results of R_{1T}, R_{2T} and R_{3T}, we then have $R_T \xrightarrow{d} \mathbb{N}\left(\frac{1}{2} tr\left(\int_0^1 \pi(r)\pi(r)' dr\right), \varphi^2\right)$. This completes the proof of Lemma A5. □

The proofs of the following Lemmas A6-A8 are similar to those of Lemmas A1-A3. Let $\xrightarrow{p^*}$ and $\xrightarrow{d^*}$ represent the convergence in probability and in distribution respectively under the bootstrap law. Let $o_{p^*}(1)$ and $O_{p^*}(1)$ be the corresponding versions of $o_p(1)$ and $O_p(1)$ in the bootstrap probability space. Denote by $E^*(\cdot)$ the conditional expectation given the original data $\{Y_t\}_{t=1}^T$. In addition, we let $Th^{1/2}\tilde{\lambda}_T^* = \frac{1}{Th^{1/2}} \sum_{t=p+1}^T \sum_{s\neq t} k_{s,t} \hat{m}_t^{*\prime} \hat{m}_s^*$ and $\hat{\sigma}_T^{*2} = \frac{2}{T^2 h} \sum_{t=p+1}^T \sum_{s\neq t}^T k_{s,t}^2 \left(\hat{m}_t^{*\prime} \hat{m}_s^*\right)^2$, where $\hat{m}_t^* = \xi_t \hat{m}_t$. □

Lemma A6. Under both $\mathbb{H}_0$ and $\mathbb{H}_A$, we have $Th^{1/2}\tilde{\lambda}_T^* = \frac{1}{Th^{1/2}} \sum_{t=p+1}^T \sum_{s\neq t} k_{s,t} m_t^{*\prime} m_s^* + o_{p^*}(1)$, where $m_t^* = \xi_t m_t$.

Proof of Lemma A6. Similar to (A1), $Th^{1/2}\tilde{\lambda}_T^*$ has the following decomposition

$$\begin{aligned} Th^{1/2}\tilde{\lambda}_T^* &= \frac{1}{Th^{1/2}} \sum_{t=p+1}^{T} \sum_{s\neq t}^{T} k_{s,t} m_t^{*\prime} m_s^* + \frac{1}{Th^{1/2}} \sum_{t=p+1}^{T} \sum_{s\neq t}^{T} k_{s,t} \left(\hat{m}_t^* - m_t^*\right)' \left(\hat{m}_s^* - m_s^*\right) \\ &+ \frac{2}{Th^{1/2}} \sum_{t=p+1}^{T} \sum_{s\neq t}^{T} k_{s,t} \left(\hat{m}_s^* - m_s^*\right)' m_t^* \\ &= U_{1T}^* + U_{2T}^* + 2U_{3T}^*. \end{aligned} \tag{A26}$$

Because $\{\xi_t\}_{t=p+1}^T$ are generated by an i.i.d. standard normal distribution, and are also independent of $\{Y_t\}_{t=1}^T$, then $E(\xi_t) = 0$ and $E\left(\xi_t^2\right) = 1$. By taking the similar arguments to those of proving $U_{2T} = U_{3T} = O_p\left(h^{1/2}\right)$ in (A3)-(A8), we can show $U_{2T}^* = U_{3T}^* = O_{p^*}\left(h^{1/2}\right)$. Hence, $Th^{1/2}\tilde{\lambda}_T^* = \frac{1}{Th^{1/2}} \sum_{t=p+1}^T \sum_{s\neq t} k_{s,t} m_t^{*\prime} m_s^* + o_{p^*}(1)$. This completes the proof of Lemma A6. □

Lemma A7. Under both $\mathbb{H}_0$ and $\mathbb{H}_A$, we have

$$R_T^* = \frac{1}{Th^{1/2}} \sum_{t=p+2}^{T} \sum_{s=p+1}^{t-1} k_{s,t} m_t^{*\prime} m_s^* \xrightarrow{d^*} \mathbb{N}\left(0, \varphi^2\right),$$

where $\varphi^2 = (vec(\Upsilon))' \left(\int_0^1 (\Omega(r) \otimes \Omega(r)) dr\right) vec(\Upsilon) \int_0^1 k^2(u) du$.

Proof of Lemma A7. Denote $W_t^* = \frac{1}{Th^{1/2}} \sum_{s=p+1}^{t-1} k_{s,t} m_t^{*\prime} m_s^*$ and let $s_T^{*2} = E^*(R_T^{*2})$. It is obvious that W_t^* is an m.d.s. under both the null and the alternatives. Similarly, we still apply Brown's (1971) martingale CLT to show $s_T^{*-1} R_T^* \xrightarrow{d^*} N(0, 1)$. Similar to the proof of (A10) in Lemma A2, it is enough to prove the following two conditions:

$$(i) \quad (s_T^*)^{-2} \sum_{t=p+2}^{T} W_t^{*2} - 1 \xrightarrow{p^*} 0; \quad (ii) \quad \sum_{t=p+2}^{T} E^*(W_t^{*4}) \xrightarrow{p} 0. \tag{A27}$$

First, it is obvious that

$$s_T^{*2} = \frac{1}{T^2 h} \sum_{t=p+2}^{T} \sum_{s=p+1}^{t-1} k_{s,t}^2 E^* \left(m_t^{*\prime} m_s^*\right)^2 = \frac{1}{T^2 h} \sum_{t=p+2}^{T} \sum_{s=p+1}^{t-1} k_{s,t}^2 \left(m_t' m_s\right)^2,$$

which converges to

$$\left(vec\left(\Upsilon\right)\right)'\left(\int_0^1\left(\Omega\left(r\right)\otimes\Omega\left(r\right)\right)dr\right)vec\left(\Upsilon\right)\int_0^1 k^2(u)du$$

by using the result of Lemma A3.

In order to prove (i) in (A27), it suffices to show $E^*\left(\sum_{t=p+2}^T W_t^{*2}-s_T^{*2}\right)^2=E^*\left(\sum_{t=p+2}^T W_t^{*2}\right)^2-s_T^{*4}=o_p(1)$. Because $s_T^{*2}=\frac{1}{T^2h}\sum_{t=p+2}^T\sum_{s=p+1}^{t-1}k_{s,t}^2E^*\left(m_t^{*\prime}m_s^*\right)^2$, then

$$\begin{aligned}
E^*\left(\sum_{t=p+2}^T W_t^{*2}\right)^2-s_T^{*4}&=\frac{1}{T^4h^2}\sum_{t,t'=p+2}^T\sum_{s=p+1}^{t-1}\sum_{r=p+1}^{t'-1}k_{s,t}^2k_{r,t'}^2\left[E^*\left(m_t^{*\prime}m_s^*m_{t'}^{*\prime}m_r^*\right)^2-E^*\left(m_t^{*\prime}m_s^*\right)^2E^*\left(m_{t'}^{*\prime}m_r^*\right)^2\right]\\
&+\frac{2}{T^4h^2}\sum_{t,t'=p+2}^T\sum_{s=p+1}^{t-1}\sum_{s'\neq s}^{t-1}k_{s,t}k_{s',t}k_{s,t'}k_{s',t'}E^*\left(m_t^{*\prime}m_s^*m_t^{*\prime}m_{s'}^*m_{t'}^{*\prime}m_s^*m_{t'}^{*\prime}m_{s'}^*\right)\\
&=D_1^*+2D_2^*. \qquad\qquad \text{(A28)}
\end{aligned}$$

For D_1^*,

$$\begin{aligned}
D_1^*&=\frac{8}{T^4h^2}\sum_{t=p+2}^T\sum_{s=p+1}^{t-1}k_{s,t}^4\left(m_t'm_s\right)^4+\frac{2}{T^4h^2}\sum_{t=p+2}^T\sum_{s=p+1}^{t-1}\sum_{r\neq s}^{t-1}k_{s,t}^2k_{r,t}^2\left(m_t'm_sm_t'm_r\right)^2\\
&+\frac{2}{T^4h^2}\sum_{t=p+2}^T\sum_{t'\neq t}^T\sum_{s=p+1}^{t-1}k_{s,t}^2k_{s,t'}^2\left(m_t'm_sm_{t'}'m_s\right)^2\\
&=O_p\left(\frac{1}{T}\right), \qquad\qquad \text{(A29)}
\end{aligned}$$

since $\sup_{t,s}E\left|m_t'm_s\right|^4<C$, $\sup_{t,s,r}E\left|m_t'm_sm_t'm_r\right|^2<C$ and $\sup_{t,s,t'}E\left|m_t'm_sm_{t'}'m_s\right|^2<C$ (which are implied by $\sup_{t,s,t',s'}E\left|u_{it}'u_{is}u_{it'}'u_{is'}\right|^{4(1+\delta)}<C$, $i=1,2$, in Assumption 2(iii)), as well as

$$\frac{1}{T^4h^2}\sum_{t=p+2}^T\sum_{s=p+1}^{t-1}k_{s,t}^4=O\left(\frac{1}{T^2h}\right),\quad \frac{1}{T^4h^2}\sum_{t=p+2}^T\sum_{s=p+1}^{t-1}\sum_{r\neq s}^{t-1}k_{s,t}^2k_{r,t}^2=O\left(\frac{1}{T}\right),\text{ and }\frac{1}{T^4h^2}\sum_{t=p+2}^T\sum_{t'\neq t}^T\sum_{s=p+1}^{t-1}k_{s,t}^2k_{s,t'}^2=O\left(\frac{1}{T}\right).$$

For D_2^*,

$$\begin{aligned}
D_2^*&=\frac{3}{T^4h^2}\sum_{t=p+2}^T\sum_{s=p+1}^{t-1}\sum_{s'\neq s}^{t-1}k_{s,t}^2k_{s',t}^2\left(m_t'm_sm_t'm_{s'}\right)^2\\
&+\frac{1}{T^4h^2}\sum_{t=p+2}^T\sum_{t'\neq t}^T\sum_{s=1}^{t-1}\sum_{s'\neq s}^{t-1}k_{s,t}k_{s',t}k_{s,t'}k_{s',t'}\left(m_t'm_sm_t'm_{s'}m_{t'}'m_sm_{t'}'m_{s'}\right)\\
&=D_{21}^*+D_{22}^*. \qquad\qquad \text{(A30)}
\end{aligned}$$

Similarly, we can show $D_{21}^*=O_p\left(\frac{1}{T}\right)$, whose proof is the same as that of the second term in the first line of (A29), and $D_{22}^*=O_p\left(h^2\right)$, which holds because $\sup_{t,s,t',s'}E\left|m_t'm_sm_t'm_{s'}m_{t'}'m_sm_{t'}'m_{s'}\right|<C$ (implied by $\sup_{t,s,t',s'}E\left|u_{it}'u_{is}u_{it'}'u_{is'}\right|^{4(1+\delta)}<C$, $i=1,2$, in Assumption 2(iii)), and $\frac{1}{T^4h^2}\sum_{t=p+2}^T\sum_{t'\neq t}^T\sum_{s=1}^{t-1}\sum_{s'\neq s}^{t-1}k_{s,t}k_{s',t}k_{s,t'}$ $k_{s',t'}=O(h^2)$. Thus, $D_2^*=O_p\left(h^2\right)$. Combining the results of D_1^* and D_2^* , we have $E^*\left(\sum_{t=p+2}^T W_t^{*2}-s_T^{*2}\right)^2=o_p(1)$, which means that $\left(s_T^*\right)^{-2}\sum_{t=p+2}^T W_t^{*2}-1\xrightarrow{p^*}0$.

Now we turn to prove (ii) in (A27), and rewrite

$$\sum_{t=p+2}^T E^*(W_t^{*4})=\frac{1}{T^4h^2}\sum_{t=p+2}^T\sum_{s=p+1}^{t-1}\sum_{s'=p+1}^{t-1}\sum_{r=p+1}^{t-1}\sum_{r'=p+1}^{t-1}k_{s,t}k_{s',t}k_{r,t}k_{r',t}E^*\left(m_t^{*\prime}m_s^*m_t^{*\prime}m_{s'}^*m_t^{*\prime}m_r^*m_t^{*\prime}m_{r'}^*\right), \qquad \text{(A31)}$$

whose proof is also similar to that of $E^*\left(\sum_{t=p+2}^T W_t^{*2}-s_T^{*2}\right)^2=o_p(1)$ in (A28). We can obtain $\sum_{t=p+2}^T E^*(W_t^{*4})=o_p\left(1\right)$. As a result, $R_T^*\xrightarrow{d^*}\mathbb{N}\left(0,\varphi^2\right)$. This completes the proof of Lemma A7. □

Lemma A8. Under both $\mathbb{H}_0$ and $\mathbb{H}_A$,we have $\hat{\sigma}_T^{*2} = \frac{2}{T^2 h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2 \left(m_t^{*\prime} m_s^*\right)^2 + o_{p^*}(1)$, and $\hat{\sigma}_T^{*2} \xrightarrow{p^*} \sigma^2$.

Proof of Lemma A8. Similar to the decomposition of $\hat{\sigma}_T^2$ in (A21), we have

$$\begin{aligned}
\hat{\sigma}_T^{*2} &= \frac{2}{T^2 h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2 \left(m_t^{*\prime} m_s^*\right)^2 + \frac{4}{T^2 h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2 \left[m_t^{*\prime}\left(\hat{m}_s^* - m_s^*\right)\right]^2 \\
&+ \frac{8}{T^2 h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2 \left[m_s^{*\prime} m_t^* m_t^{*\prime}\left(\hat{m}_s^* - m_s^*\right)\right] + \frac{2}{T^2 h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2 \left[\left(\hat{m}_t^* - m_t^*\right)'\left(\hat{m}_s^* - m_s^*\right)\right]^2 \\
&+ \frac{4}{T^2 h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2 \left[m_s^{*\prime} m_t^*\left(\hat{m}_t^* - m_t^*\right)'\left(\hat{m}_s^* - m_s^*\right)\right] + \frac{4}{T^2 h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2 \left[m_t^{*\prime}\left(\hat{m}_s^* - m_s^*\right)\left(\hat{m}_t^* - m_t^*\right)' m_s^*\right] \\
&+ \frac{8}{T^2 h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2 \left[m_t^{*\prime}\left(\hat{m}_s^* - m_s^*\right)\left(\hat{m}_t^* - m_t^*\right)'\left(\hat{m}_s^* - m_s^*\right)\right] \\
&= B_1^* + B_2^* + B_3^* + B_4^* + B_5^* + B_6^* + B_7^*. \qquad \text{(A32)}
\end{aligned}$$

We need to show that $B_2^* = \cdots = B_7^* = o_{p^*}(1)$ and $\hat{\sigma}_T^{*2} = B_1^* + o_{p^*}(1)$. The proofs of $B_2^*, \cdots$, and B_7^* are similar to those of $B_2, \cdots$, and B_7 in (A21) since $E(\xi_t^2\xi_s^2) = 1$ for $t \neq s$, and $E(\xi_t^4) < \infty, t = p+1, \ldots, T$. As a result, we obtain $B_2^* = \cdots = B_7^* = o_{p^*}(1)$, and have

$$\hat{\sigma}_T^{*2} = \frac{2}{T^2 h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2 \left(m_t^{*\prime} m_s^*\right)^2 + o_{p^*}(1).$$

Furthermore, note that $E^*(B_1^*) = B_1 = \frac{2}{T^2 h}\sum_{t=p+1}^{T}\sum_{s\neq t}^{T} k_{s,t}^2 \left(m_t' m_s\right)^2 \xrightarrow{p} \sigma^2$, which has been proved in Lemma A3. Based on the mean squared error of B_1^*, we can show $E^*\left[B_1^* - E^*\left(B_1^*\right)\right]^2 = o_p(1)$, whose proof is similar to that of $E^*\left(\sum_{t=p+2}^{T} W_t^{*2} - s_T^{*2}\right)^2 = o_p(1)$ in Lemma A7, so $\hat{\sigma}_T^{*2} \xrightarrow{p^*} \sigma^2$. This completes the proof of Lemma A8. □

Proof of Theorem 1. Under $\mathbb{H}_0$, by using Lemmas A1-A3, we immediately have Theorem 1. □

Proof of Theorem 2. Under $\mathbb{H}_A$, by using Lemmas A1,A3 and A4, we immediately have Theorem 2. □

Proof of Theorem 3. Under $\mathbb{H}_{LA}$, by using Lemmas A1,A3 and A5, we immediately have Theorem 3. □

Proof of Theorem 4. Under $\mathbb{H}_0$ and $\mathbb{H}_A$, by using Lemmas A6-A8, we immediately have Theorem 4. □

Appendix B: Brown's CLT and Yoshihara's Lemma

In the following, we introduce Brown's (1971) martingale central limit theorem and a useful inequality of mixing processes given by Yoshihara (1976).

Brown's Martingale Central Limit Theorem (1971) Let $\{S_t, \mathcal{F}_t, t = 1, 2, \cdots, T\}$ be a martingale on the probability space $\{\Omega, \mathcal{F}, P\}$, with $S_0 = 0$, and $X_t = S_t - S_{t-1}, t = 1, 2, \cdots, T$. $\mathcal{F}_0$ need not be the trivial σ-field $\{\phi, \Omega\}$. Let $\phi_t(u) = E(e^{iuX_t}|\mathcal{F}_{t-1}) = E_{t-1}(e^{iuX_t})$, where E_{t-1} denotes $E(\cdot|\mathcal{F}_{t-1})$, and let $\sigma_t^2 = E_{t-1}(X_t^2)$, $V_T^2 = \sum_{t=1}^{T}\sigma_t^2$, $s_T^2 = EV_T^2 = ES_T^2$, $f_T(u) = \Pi_{t=1}^{T}\phi_t(u/S_T)$ and $b_T = s_T^{-2} max_{t\leq T}\sigma_t^2$, for $t = 1, 2, \cdots, T$. For martingales satisfy

$$V_T^2 s_T^{-2} \xrightarrow{p} 1, as\ T \to \infty,$$

and Lindeberg condition

$$s_T^{-2}\sum_{t=1}^{T} E\left[X_t^2 I(|X_t| \geq \varepsilon s_T)\right] \to 0,$$

for all $\varepsilon > 0$, then we have $f_T(u) \xrightarrow{p} e^{-\frac{1}{2}u^2}$ as $T \to \infty$, for all u and $b_T \xrightarrow{p} 0$ as $T \to \infty$, and for all x,

$$lim_{T\to\infty} P(S_T/s_T \leq x) = \Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy.$$

Yoshihara's Lemma (1976) Let $x_{t_1}, x_{t_2}, \ldots, x_{t_k}$ (with $t_1 < t_2 < \cdots < t_k$) be absolutely regular random vectors with mixing coefficients β. Let $h\left(x_{t_1}, x_{t_2}, \ldots, x_{t_k}\right)$ be a Borel measurable function, and let there be a $\delta > 0$ such that $P = \max\left\{P_1, P_2\right\} < \infty$,where

$$P_1 = \sup_{t_1, t_2, \ldots, t_k} \int \left|h\left(x_{t_1}, x_{t_2}, \ldots, x_{t_k}\right)\right|^{1+\delta} dF\left(x_{t_1}, x_{t_2}, \ldots, x_{t_k}\right),$$

$$P_2 = \sup_{t_1, t_2, \ldots, t_k} \int \left|h\left(x_{t_1}, x_{t_2}, \ldots, x_{t_k}\right)\right|^{1+\delta} dF\left(x_{t_1}, \ldots, x_{t_j}\right) dF\left(x_{t_{j+1}}, \ldots, x_{t_k}\right).$$

Then for all $\tau = t_{j+1} - t_j$, we have

$$\begin{aligned} & \left| \int h\left(x_{t_1}, x_{t_2}, \ldots, x_{t_k}\right) dF\left(x_{t_1}, x_{t_2}, \ldots, x_{t_k}\right) \right. \\ & \left. - \int h\left(x_{t_1}, x_{t_2}, \ldots, x_{t_k}\right) dF\left(x_{t_1}, \ldots, x_{t_j}\right) dF\left(x_{t_{j+1}}, \ldots, x_{t_k}\right) \right| \\ & \leq 4 P^{1/(1+\delta)} \beta\left(\tau\right)^{\delta/(1+\delta)}. \end{aligned}$$

References

[1] Brown, B. M., 1971. Martingale central limit theorems. Annals of Mathematical Statistics 42, 59–66.

[2] Yoshihara, K., 1976. Limiting behavior of U-statistics for stationary, absolutely regular processes. Zeitschrift fur Wahrscheinlichkeitstheorie und verwandte Gebiete 35, 237–252.

Appendix C: The simulation results for the VAR model with known lag length

In this part, we provide some additional simulation results for the cases where the true lag length $p = 2$ is assumed to be known. The entire Monte Carlo experiment designs are the same as those in Section 4 of the main paper. Tables 1-3 and Tables 4-6 display the rejection frequencies under the null and under the alternatives respectively when the true lag order $p = 2$ is known. By comparison with the testing results of p chosen by $\tilde{Q}_m^{OLS}$ in the main paper, we find that the choice of lag length by the corrected portmanteau test has negligible impacts on the testing size and power performance of the proposed test in $DGPS$.1-3 and $DGPP$.1-3, which is even more pronounced as we gradually increase the sample size T.

	J_{T1}^B	J_{T2}^B	J_{T3}^B	$J_{T,cv}^B$	S_b	S_w	S_{st}
			1% asymptotic nominal level				
$T = 100$	0.015	0.012	0.012	0.012	0.005	0.005	0.010
$T = 200$	0.007	0.008	0.010	0.009	0.007	0.008	0.007
$T = 500$	0.011	0.007	0.007	0.007	0.011	0.010	0.006
$T = 800$	0.008	0.008	0.007	0.008	0.006	0.005	0.003
			5% asymptotic nominal level				
$T = 100$	0.052	0.046	0.047	0.044	0.037	0.046	0.048
$T = 200$	0.056	0.052	0.042	0.049	0.033	0.035	0.035
$T = 500$	0.044	0.046	0.047	0.044	0.045	0.045	0.043
$T = 800$	0.057	0.052	0.048	0.046	0.040	0.046	0.039
			10% asymptotic nominal level				
$T = 100$	0.095	0.089	0.086	0.092	0.097	0.096	0.096
$T = 200$	0.091	0.087	0.089	0.085	0.078	0.082	0.073
$T = 500$	0.100	0.087	0.084	0.090	0.092	0.098	0.097
$T = 800$	0.097	0.100	0.097	0.102	0.088	0.095	0.089

Table C1: Empirical sizes of the tests under $DGPS$.1 with the lag length $p = 2$.

	J_{T1}^B	J_{T2}^B	J_{T3}^B	$J_{T,cv}^B$	S_b	S_w	S_{st}
			1% asymptotic nominal level				
$T = 100$	0.010	0.009	0.011	0.010	0.005	0.006	0.007
$T = 200$	0.016	0.013	0.012	0.011	0.007	0.008	0.009
$T = 500$	0.015	0.016	0.012	0.015	0.011	0.006	0.008
$T = 800$	0.011	0.008	0.008	0.008	0.005	0.005	0.004
			5% asymptotic nominal level				
$T = 100$	0.051	0.047	0.050	0.049	0.042	0.047	0.052
$T = 200$	0.039	0.037	0.035	0.038	0.035	0.038	0.031
$T = 500$	0.049	0.047	0.044	0.048	0.045	0.052	0.048
$T = 800$	0.051	0.044	0.046	0.045	0.040	0.044	0.034
			10% asymptotic nominal level				
$T = 100$	0.089	0.090	0.087	0.089	0.094	0.097	0.093
$T = 200$	0.078	0.074	0.077	0.078	0.084	0.084	0.076
$T = 500$	0.089	0.096	0.103	0.092	0.101	0.113	0.108
$T = 800$	0.091	0.094	0.100	0.100	0.098	0.101	0.090

Table C2: Empirical sizes of the tests under $DGPS$.2 with the lag length $p = 2$.

	J_{T1}^B	J_{T2}^B	J_{T3}^B	$J_{T,cv}^B$	S_b	S_w	S_{st}
			1% asymptotic nominal level				
$T = 100$	0.011	0.010	0.008	0.011	0.005	0.007	0.009
$T = 200$	0.015	0.012	0.011	0.011	0.008	0.007	0.010
$T = 500$	0.013	0.013	0.015	0.016	0.012	0.009	0.011
$T = 800$	0.008	0.006	0.008	0.008	0.006	0.007	0.007
			5% asymptotic nominal level				
$T = 100$	0.053	0.050	0.044	0.048	0.048	0.049	0.055
$T = 200$	0.045	0.044	0.042	0.043	0.038	0.037	0.044
$T = 500$	0.039	0.043	0.039	0.039	0.046	0.052	0.053
$T = 800$	0.043	0.039	0.040	0.044	0.043	0.043	0.051
			10% asymptotic nominal level				
$T = 100$	0.086	0.081	0.079	0.082	0.094	0.097	0.102
$T = 200$	0.078	0.076	0.084	0.078	0.088	0.089	0.093
$T = 500$	0.088	0.090	0.098	0.093	0.101	0.106	0.116
$T = 800$	0.085	0.090	0.092	0.096	0.087	0.098	0.100

Table C3: Empirical sizes of the tests under $DGPS.3$ with the lag length $p = 2$.

	J_{T1}^B	J_{T2}^B	J_{T3}^B	$J_{T,cv}^B$	S_b	S_w	S_{st}
			1% asymptotic nominal level				
$T = 100$	0.319	0.344	0.316	0.314	0.009	0.012	0.011
$T = 200$	0.658	0.718	0.715	0.708	0.062	0.005	0.010
$T = 500$	1.000	1.000	1.000	1.000	0.857	0.007	0.028
$T = 800$	1.000	1.000	1.000	1.000	0.897	0.012	0.017
			5% asymptotic nominal level				
$T = 100$	0.547	0.589	0.580	0.575	0.109	0.042	0.060
$T = 200$	0.850	0.874	0.878	0.873	0.296	0.062	0.079
$T = 500$	1.000	1.000	1.000	1.000	0.992	0.046	0.091
$T = 800$	1.000	1.000	1.000	1.000	0.995	0.051	0.059
			10% asymptotic nominal level				
$T = 100$	0.675	0.707	0.711	0.693	0.235	0.091	0.117
$T = 200$	0.910	0.924	0.926	0.926	0.502	0.101	0.134
$T = 500$	1.000	1.000	1.000	1.000	1.000	0.095	0.132
$T = 800$	1.000	1.000	1.000	1.000	1.000	0.092	0.116

Table C4: Empirical power of the tests under $DGPP.1$ with the lag length $p = 2$.

	J^B_{T1}	J^B_{T2}	J^B_{T3}	$J^B_{T,cv}$	S_b	S_w	S_{st}
			1% asymptotic nominal level				
$T = 100$	0.158	0.189	0.182	0.181	0.114	0.183	0.174
$T = 200$	0.406	0.443	0.469	0.476	0.298	0.432	0.407
$T = 500$	0.899	0.925	0.941	0.933	0.765	0.840	0.853
$T = 800$	0.994	0.994	0.997	1.000	0.949	0.986	0.988
			5% asymptotic nominal level				
$T = 100$	0.363	0.396	0.399	0.399	0.302	0.385	0.369
$T = 200$	0.660	0.703	0.720	0.715	0.549	0.670	0.686
$T = 500$	0.975	0.982	0.987	0.986	0.914	0.933	0.941
$T = 800$	1.000	1.000	1.000	1.000	0.990	0.993	0.995
			10% asymptotic nominal level				
$T = 100$	0.483	0.517	0.529	0.522	0.417	0.495	0.509
$T = 200$	0.775	0.800	0.823	0.822	0.676	0.770	0.794
$T = 500$	0.992	0.995	0.994	0.993	0.950	0.968	0.971
$T = 800$	1.000	1.000	1.000	1.000	0.996	0.999	0.999

Table C5: Empirical power of the tests under $DGPP.2$ with the lag length $p = 2$.

	J^B_{T1}	J^B_{T2}	J^B_{T3}	$J^B_{T,cv}$	S_b	S_w	S_{st}
			1% asymptotic nominal level				
$T = 100$	0.123	0.118	0.099	0.095	0.015	0.009	0.018
$T = 200$	0.357	0.344	0.318	0.313	0.044	0.006	0.007
$T = 500$	0.935	0.935	0.923	0.929	0.298	0.013	0.018
$T = 800$	0.999	0.997	0.995	0.994	0.568	0.013	0.018
			5% asymptotic nominal level				
$T = 100$	0.310	0.313	0.289	0.272	0.098	0.051	0.058
$T = 200$	0.651	0.653	0.618	0.623	0.199	0.052	0.062
$T = 500$	0.988	0.987	0.987	0.982	0.649	0.051	0.063
$T = 800$	1.000	1.000	1.000	1.000	0.890	0.053	0.065
			10% asymptotic nominal level				
$T = 100$	0.448	0.431	0.408	0.402	0.211	0.102	0.115
$T = 200$	0.776	0.785	0.767	0.756	0.373	0.101	0.122
$T = 500$	0.995	0.997	0.997	0.996	0.820	0.089	0.115
$T = 800$	1.000	1.000	1.000	1.000	0.958	0.094	0.120

Table C6: Empirical power of the tests under $DGPP.3$ with the lag length $p = 2$.

Appendix D: The simulation results for Section 6 of the main paper

This part provides the simulation results for the time-varying coefficient VAR models introduced in Section 6 of the main paper. The whole Monte Carlo experiment designs for the covariance structure are the same as those in Section 4 of the main paper except that we employ a time-varying VAR(1) model in the conditional mean. Specifically, the time-varying VAR(1) model is specified as

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = A(t/T) \begin{pmatrix} Y_{1t-1} \\ Y_{2t-1} \end{pmatrix} + \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix},$$

in which two types of the coefficient $A(\cdot)$ are taken into account as follows:

$TVC.1$–Monotonic linear change:

$$A(r) = \begin{pmatrix} 0.04 & -0.6 \\ 0.39 & -0.16 \end{pmatrix} + \begin{pmatrix} 0.6 & -0.4 \\ -0.4 & 0.6 \end{pmatrix} r, r \in [0,1].$$

$TVC.2$–Non-monotonic smooth change:

$$A(r) = \begin{pmatrix} 0.64sin(2\pi r) & -sin(2\pi r) \\ -0.01cos(2\pi r) & 0.44cos(2\pi r) \end{pmatrix}, r \in [0,1].$$

It is easy to verify that the time-varying VAR(1) models generated by $TVC.1$ and $TVC.2$ satisfy the locally stationary conditions since the eigenvalues of the matrix $I_2 - A(r)z$ all lie outside the unit circle uniformly in $r \in [0,1]$. The innovation $u_t = (u_{1t}, u_{2t})'$ is normally distributed with the variance-covariance structure $\Sigma(\cdot)$ specified as in $DGPS.1$-3 and $DGPP.1$-3 of the main paper. We still employ the Epanechnikov kernel $k(u) = \frac{3}{4}\left(1-u^2\right) I\left(|u| \leq 1\right)$ in both the nonparametric estimation of $A(r)$ and the construction of the test statistic J_T. The testing procedure here involves the two bandwidths b and h, and their orders of magnitude, especially the relationship of the second bandwidth b with the first one h, depend on the sample size T, which requires theoretical justifications. Because this Monte Carlo simulation is just to show the validity of our proposed test in the context of time-varying coefficient models, for simplicity we let the bandwidth $b = 0.1$ for the local linear estimation of $A(r)$, and consider three fixed bandwidths $h = 0.1, 0.2, 0.3$ for J_T. The corresponding three bootstrapped test statistics are then denoted as $\check{J}_{T1}^B$, $\check{J}_{T2}^B$ and $\check{J}_{T3}^B$, respectively. We generate 1000 data sets of $\{Y_t\}_{t=1}^T$ for each $T = 200, 500$ and 800, and use $B = 299$ bootstrap iterations for each simulated data set. For comparison, the results for S_b, S_w and S_{st} are also tabulated. In addition, because the lag selection in the time-varying VAR model is much more complicated, we simply assume the true lag length $p = 1$ to be known.

The empirical size and power performance of all the tests is reported in Tables 1-12. Specifically, Tables 1-3 and Tables 6-9 are the empirical sizes for the $TVC.1$ and $TVC.2$ models, respectively. We find that the three bootstrapped tests $\check{J}_{T1}^B$, $\check{J}_{T2}^B$ and $\check{J}_{T3}^B$ exhibit reasonable sizes in all cases, and their estimated sizes are quite close to the nominal ones at the three significance levels. In contrast, the tests S_b, S_w and S_{st} are no longer reliable because they suffer serious size distortion under the null and the degree of distortion does not ameliorate as we increase the sample size T. Tables 4-6 and Tables 10-12 report the empirical power of all the tests at the 1%, 5% and 10% levels (in which the empirical critical values of S_w and S_{st} based on the asymptotic distributions are adjusted). The results show that our bootstrapped tests are more powerful against the given alternatives than the other three tests in all cases.

	$\check{J}_{T1}^B$	$\check{J}_{T2}^B$	$\check{J}_{T3}^B$	S_b	S_w	S_{st}
			1% asymptotic nominal level			
$T = 200$	0.016	0.017	0.018	0.034	0.042	0.043
$T = 500$	0.012	0.013	0.012	0.104	0.117	0.120
$T = 800$	0.009	0.010	0.011	0.164	0.209	0.204
			5% asymptotic nominal level			
$T = 200$	0.071	0.060	0.058	0.115	0.119	0.118
$T = 500$	0.054	0.053	0.057	0.268	0.280	0.282
$T = 800$	0.050	0.044	0.047	0.372	0.415	0.408
			10% asymptotic nominal level			
$T = 200$	0.118	0.110	0.108	0.193	0.211	0.203
$T = 500$	0.099	0.096	0.098	0.363	0.388	0.384
$T = 800$	0.105	0.103	0.097	0.487	0.527	0.521

Table D1: Empirical sizes of the tests under $TVC.1$-$DGPS.1$.

	$\check{J}_{T1}^B$	$\check{J}_{T2}^B$	$\check{J}_{T3}^B$	S_b	S_w	S_{st}
			1% asymptotic nominal level			
$T = 200$	0.018	0.018	0.018	0.023	0.024	0.020
$T = 500$	0.010	0.005	0.010	0.051	0.070	0.060
$T = 800$	0.011	0.011	0.009	0.074	0.108	0.097
			5% asymptotic nominal level			
$T = 200$	0.071	0.061	0.053	0.090	0.093	0.082
$T = 500$	0.043	0.052	0.057	0.162	0.200	0.180
$T = 800$	0.051	0.050	0.045	0.228	0.274	0.252
			10% asymptotic nominal level			
$T = 200$	0.126	0.118	0.112	0.151	0.169	0.153
$T = 500$	0.096	0.103	0.095	0.269	0.313	0.298
$T = 800$	0.109	0.097	0.092	0.339	0.387	0.372

Table D2: Empirical sizes of the tests under $TVC.1$-$DGPS.2$.

	$\check{J}_{T1}^B$	$\check{J}_{T2}^B$	$\check{J}_{T3}^B$	S_b	S_w	S_{st}
			1% asymptotic nominal level			
$T = 200$	0.019	0.016	0.015	0.030	0.029	0.033
$T = 500$	0.013	0.008	0.008	0.069	0.091	0.099
$T = 800$	0.013	0.008	0.011	0.112	0.150	0.159
			5% asymptotic nominal level			
$T = 200$	0.067	0.058	0.061	0.092	0.108	0.116
$T = 500$	0.047	0.046	0.048	0.196	0.239	0.239
$T = 800$	0.052	0.050	0.049	0.271	0.330	0.338
			10% asymptotic nominal level			
$T = 200$	0.115	0.104	0.108	0.162	0.191	0.189
$T = 500$	0.098	0.109	0.101	0.297	0.348	0.352
$T = 800$	0.103	0.104	0.104	0.385	0.439	0.450

Table D3: Empirical sizes of the tests under $TVC.1$-$DGPS.3$.

	$\check{J}^B_{T1}$	$\check{J}^B_{T2}$	$\check{J}^B_{T3}$	S_b	S_w	S_{st}
			1% asymptotic nominal level			
$T = 200$	0.579	0.685	0.719	0.037	0.006	0.008
$T = 500$	0.982	0.991	0.993	0.162	0.009	0.010
$T = 800$	1.000	1.000	1.000	0.373	0.005	0.010
			5% asymptotic nominal level			
$T = 200$	0.775	0.868	0.897	0.162	0.035	0.055
$T = 500$	0.996	0.999	0.999	0.494	0.028	0.041
$T = 800$	1.000	1.000	1.000	0.831	0.026	0.046
			10% asymptotic nominal level			
$T = 200$	0.867	0.919	0.934	0.304	0.089	0.123
$T = 500$	0.999	1.000	1.000	0.740	0.065	0.086
$T = 800$	1.000	1.000	1.000	0.972	0.059	0.093

Table D4: Empirical power of the tests under $TVC.1$-$DGPP.1$.

	$\check{J}^B_{T1}$	$\check{J}^B_{T2}$	$\check{J}^B_{T3}$	S_b	S_w	S_{st}
			1% asymptotic nominal level			
$T = 200$	0.354	0.439	0.461	0.147	0.086	0.143
$T = 500$	0.851	0.924	0.937	0.424	0.178	0.225
$T = 800$	0.983	0.994	0.994	0.676	0.313	0.341
			5% asymptotic nominal level			
$T = 200$	0.579	0.695	0.711	0.321	0.263	0.291
$T = 500$	0.964	0.988	0.990	0.689	0.438	0.457
$T = 800$	0.997	0.999	1.000	0.865	0.617	0.640
			10% asymptotic nominal level			
$T = 200$	0.715	0.801	0.811	0.441	0.379	0.408
$T = 500$	0.988	0.997	0.997	0.780	0.579	0.603
$T = 800$	0.999	1.000	1.000	0.931	0.731	0.747

Table D5: Empirical power of the tests under $TVC.1$-$DGPP.2$.

	$\check{J}^B_{T1}$	$\check{J}^B_{T2}$	$\check{J}^B_{T3}$	S_b	S_w	S_{st}
			1% asymptotic nominal level			
$T = 200$	0.365	0.417	0.384	0.112	0.009	0.010
$T = 500$	0.903	0.931	0.914	0.524	0.004	0.011
$T = 800$	0.997	0.998	0.998	0.834	0.003	0.006
			5% asymptotic nominal level			
$T = 200$	0.617	0.675	0.659	0.365	0.046	0.059
$T = 500$	0.984	0.989	0.989	0.850	0.036	0.047
$T = 800$	1.000	1.000	1.000	0.987	0.031	0.044
			10% asymptotic nominal level			
$T = 200$	0.736	0.802	0.795	0.545	0.081	0.108
$T = 500$	0.993	0.997	0.995	0.929	0.068	0.085
$T = 800$	1.000	1.000	1.000	0.997	0.063	0.083

Table D6: Empirical power of the tests under $TVC.1$-$DGPP.3$.

	$\check{J}^B_{T1}$	$\check{J}^B_{T2}$	$\check{J}^B_{T3}$	S_b	S_w	S_{st}
			1% asymptotic nominal level			
$T = 200$	0.014	0.013	0.018	0.038	0.027	0.041
$T = 500$	0.014	0.015	0.012	0.100	0.069	0.108
$T = 800$	0.010	0.011	0.011	0.164	0.104	0.151
			5% asymptotic nominal level			
$T = 200$	0.064	0.069	0.067	0.128	0.108	0.131
$T = 500$	0.057	0.052	0.057	0.247	0.181	0.226
$T = 800$	0.053	0.041	0.044	0.372	0.257	0.321
			10% asymptotic nominal level			
$T = 200$	0.123	0.107	0.106	0.219	0.183	0.225
$T = 500$	0.102	0.102	0.095	0.354	0.271	0.317
$T = 800$	0.107	0.098	0.093	0.518	0.392	0.445

Table D7: Empirical sizes of the tests under $TVC.2$-$DGPS.1$.

	$\check{J}^B_{T1}$	$\check{J}^B_{T2}$	$\check{J}^B_{T3}$	S_b	S_w	S_{st}
			1% asymptotic nominal level			
$T = 200$	0.015	0.013	0.015	0.026	0.020	0.024
$T = 500$	0.010	0.007	0.008	0.061	0.034	0.027
$T = 800$	0.013	0.018	0.013	0.071	0.046	0.048
			5% asymptotic nominal level			
$T = 200$	0.064	0.064	0.061	0.098	0.076	0.070
$T = 500$	0.049	0.048	0.045	0.159	0.106	0.108
$T = 800$	0.057	0.051	0.048	0.224	0.132	0.134
			10% asymptotic nominal level			
$T = 200$	0.113	0.111	0.110	0.162	0.138	0.128
$T = 500$	0.101	0.101	0.088	0.245	0.188	0.183
$T = 800$	0.097	0.101	0.090	0.320	0.221	0.215

Table D8: Empirical sizes of the tests under $TVC.2$-$DGPS.2$.

	$\check{J}^B_{T1}$	$\check{J}^B_{T2}$	$\check{J}^B_{T3}$	S_b	S_w	S_{st}
			1% asymptotic nominal level			
$T = 200$	0.015	0.017	0.013	0.026	0.019	0.019
$T = 500$	0.011	0.010	0.011	0.042	0.018	0.017
$T = 800$	0.012	0.012	0.011	0.058	0.027	0.025
			5% asymptotic nominal level			
$T = 200$	0.058	0.062	0.068	0.089	0.065	0.065
$T = 500$	0.052	0.042	0.051	0.139	0.067	0.067
$T = 800$	0.053	0.050	0.048	0.171	0.081	0.074
			10% asymptotic nominal level			
$T = 200$	0.111	0.106	0.103	0.163	0.126	0.115
$T = 500$	0.101	0.108	0.099	0.220	0.137	0.132
$T = 800$	0.101	0.099	0.100	0.280	0.134	0.130

Table D9: Empirical sizes of the tests under $TVC.2$-$DGPS.3$.

	$\check{J}^B_{T1}$	$\check{J}^B_{T2}$	$\check{J}^B_{T3}$	S_b	S_w	S_{st}
			1% asymptotic nominal level			
$T = 200$	0.569	0.682	0.718	0.025	0.012	0.015
$T = 500$	0.981	0.992	0.994	0.054	0.005	0.006
$T = 800$	1.000	1.000	1.000	0.114	0.006	0.008
			5% asymptotic nominal level			
$T = 200$	0.789	0.866	0.888	0.093	0.045	0.053
$T = 500$	0.997	1.000	1.000	0.217	0.046	0.053
$T = 800$	1.000	1.000	1.000	0.349	0.066	0.080
			10% asymptotic nominal level			
$T = 200$	0.863	0.934	0.940	0.173	0.080	0.088
$T = 500$	1.000	1.000	1.000	0.352	0.100	0.110
$T = 800$	1.000	1.000	1.000	0.544	0.138	0.158

Table D10: Empirical power of the tests under $TVC.2$-$DGPP.1$.

	$\check{J}^B_{T1}$	$\check{J}^B_{T2}$	$\check{J}^B_{T3}$	S_b	S_w	S_{st}
			1% asymptotic nominal level			
$T = 200$	0.344	0.435	0.448	0.074	0.066	0.062
$T = 500$	0.856	0.915	0.933	0.192	0.119	0.129
$T = 800$	0.982	0.996	0.996	0.339	0.154	0.200
			5% asymptotic nominal level			
$T = 200$	0.584	0.676	0.708	0.197	0.174	0.178
$T = 500$	0.963	0.987	0.989	0.418	0.323	0.334
$T = 800$	0.997	0.999	1.000	0.563	0.431	0.425
			10% asymptotic nominal level			
$T = 200$	0.700	0.787	0.816	0.312	0.283	0.305
$T = 500$	0.987	0.995	0.997	0.533	0.456	0.451
$T = 800$	0.999	1.000	1.000	0.688	0.585	0.588

Table D11: Empirical power of the tests under $TVC.2$-$DGPP.2$.

	$\check{J}^B_{T1}$	$\check{J}^B_{T2}$	$\check{J}^B_{T3}$	S_b	S_w	S_{st}
			1% asymptotic nominal level			
$T = 200$	0.351	0.403	0.365	0.098	0.007	0.011
$T = 500$	0.895	0.934	0.910	0.372	0.011	0.018
$T = 800$	0.998	0.999	0.998	0.670	0.012	0.021
			5% asymptotic nominal level			
$T = 200$	0.614	0.663	0.650	0.284	0.054	0.064
$T = 500$	0.982	0.991	0.987	0.684	0.078	0.097
$T = 800$	1.000	1.000	1.000	0.917	0.062	0.080
			10% asymptotic nominal level			
$T = 200$	0.719	0.801	0.798	0.455	0.110	0.133
$T = 500$	0.993	0.997	0.995	0.843	0.115	0.127
$T = 800$	1.000	1.000	1.000	0.973	0.148	0.161

Table D12: Empirical power of the tests under $TVC.2$-$DGPP.3$.