Supplement to: "Parameters on the boundary in predictive regression"*

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Abstract

This document contains some supplemental material for Cavaliere, Georgiev and Zanelli (2024), CGZ hereafter. In particular, (i) we consider generalizations of some of the results in CGZ to the near-I(1) and to the stationary cases; (ii) we report additional Monte Carlo simulations.

S.1 Alternative data generating processes

The asymptotic theory in the paper is presented under the assumption that $x_{n,t}$ is a unit-root non-stationary process. Here we show that the choice of a bootstrap parameter space is fundamental for bootstrap validity also under alternative stochastic specifications for $x_{n,t}$, e.g., a near-unit root and a stationary specification. More importantly, a common definition of the bootstrap parameter space could be appropriate for all the considered specifications of $x_{n,t}$. Still, the functional forms of the limit distributions are not identical across the specifications of $x_{n,t}$ and, in the stationary case, we perform OLS estimation under the additional constraint $\hat{\delta} = 0$ in (3.2). The implications for bootstrap inference are discussed below.

S.1.1 Near-unit root regressor

Consider a modification of Assumption 1 where in part (c) the limit process becomes

$$(X, Z)' = \left(\int^{\cdot} e^{c(s-\cdot)} dW(s), Z \right)', \quad c > 0,$$

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for a Brownian motion $(W,Z)' \sim BM(0,\Omega)$. Thus, X is an Ornstein-Uhlenbeck process originating from a near-UR posited predicting variable $x_{n,t}$. The asymptotic distribution of $\hat{\theta}$ has a more complex structure than in the unit root case. Now $n^{1/2}(\hat{\theta}-\theta_0) \xrightarrow{w} M^{-1/2}\xi + v_c$ with $v_c := (0, c\omega_{xz}\omega_{xx}^{-1})'$ if $\theta_0 \in \operatorname{int}\Theta$. On the other hand,

$$n^{1/2}(\hat{\theta}-\theta_0) \xrightarrow{w} \arg\min_{\lambda\in\Lambda} ||\lambda - M^{-1/2}\xi - v_c||_M, \quad \Lambda := \{\lambda \in \mathbb{R}^2 : \dot{g}'\lambda \geq 0\} \tag{S.1}$$

if $g(\theta_0) = 0$. The limiting shift by v_c is due to the fact that $n^{1/2}\Delta x_{n,t}$ in the near-unit root case is not a sufficiently good proxy for the innovations driving $x_{n,t}$. Eqs. (3.5)–(3.6) for the standard bootstrap hold in the near-unit root case if X in the definition of M is understood as an Ornstein-Uhlenbeck process; therefore, $\theta_0 \in \partial\Theta$ induces the same kind of limiting bootstrap randomness as in the exact unit-root case. Additionally, the bootstrap limit distribution does not replicate the shift in the limit distribution of $n^{1/2}(\hat{\theta}-\theta_0)$ induced by the vector v_c, as a consequence of the conditional independence of the bootstrap innovations and the regressor $x_{n,t-1}$. This fact is not related to the position of θ_0 relative to Θ and requires separate treatment. Consider now the bootstrap estimator of Corollary 4.1 with the choice $g^* = g - |g|^{1+\kappa}$ for $\kappa > 0$. In the case where $x_{n,t}$ is near-unit root non-stationary, instead of (4.3) it holds that

$$(n^{1/2}(\hat{\theta}-\theta_0), (n^{1/2}(\hat{\theta}^* - \hat{\theta})|D_n)) \xrightarrow{w}_w \left(M^{-1/2}\xi + v_c, (M^{-1/2}\xi|M)\right)$$

if $g(\theta_0) > 0$, and

$$(n^{1/2}(\hat{\theta}-\theta_0), (n^{1/2}(\hat{\theta}^* - \hat{\theta})|D_n)) \xrightarrow{w}_w \Bigg(\arg\min_{\lambda\in\Lambda} ||\lambda - M^{-1/2}\xi - v_c||_M, \left(\arg\min_{\lambda\in\Lambda} ||\lambda - M^{-1/2}\xi||_M \Big| M\right)\Bigg)$$

if $g(\theta_0) = 0$, where X in the definition of M should again be read as an Ornstein-Uhlenbeck process. This means that g^* still does the job it is designed for (remove the random shift from the half-plane in the limiting bootstrap distribution). Nevertheless, bootstrap invalidity due to the limiting shift by v_c, not related to the position of θ_0 in Θ, remains to be tackled.

S.1.2 Stationary regressor

If $x_{n,t} = x_t$ is stationary, then the inclusion of $\Delta x_{n,t} = \Delta x_t$ among the regressors of (3.2) will, in general, compromise the consistency of $\hat{\theta}$ for the true value θ_0 in the predictive regression (3.1). Assume, however, that $n^{-1}\sum_{t=1}^n \tilde{x}_t\tilde{x}_t' \xrightarrow{p} M$ for $\tilde{x}_t := (1, x_t)'$ and a non-random positive definite matrix M, and that the unconstrained OLS estimator of θ from the predictive regression (3.1) is consistent at the $n^{-1/2}$ rate and has asymptotic $N(0, \omega_{zz}M^{-1})$ distribution. Then, the constrained OLS estimator $\hat{\theta}$ of (3.1) subject to $g(\hat{\theta}) \geq 0$ (equivalently, the constrained OLS estimator of (3.2) subject to $g(\hat{\theta}) \geq 0$, $\hat{\delta} = 0$) satisfies $n^{1/2}(\hat{\theta}-\theta_0) \xrightarrow{w} \ell_{st}(\theta_0) = \tilde{\ell}_{st} := M^{-1/2}\zeta$ with $\zeta \sim N(0, \omega_{zz}I_2)$ in the case where $\theta_0 \in \operatorname{int}\Theta$, and

$$n^{1/2}(\hat{\theta}-\theta_0) \xrightarrow{w} \ell_{st}(\theta_0) = \ell_{st} := \arg\min_{\lambda\in\Lambda} ||\lambda - M^{-1/2}\zeta||_M, \quad \Lambda := \{\lambda \in \mathbb{R}^2 : \dot{g}'\lambda \geq 0\}$$

in the case where $g(\theta_0) = 0$. In the stationary case with a non-random limiting M, the limiting behavior of the standard bootstrap is entirely analogous to the introductory location model example, as the possibility that $\theta_0 \in \partial\Theta$ is the only source of bootstrap randomness in the limit. For $\hat{\theta}$ defined in the previous paragraph, it holds that $n^{1/2}(\hat{\theta}^* - \hat{\theta}) \xrightarrow{w^*}_p M^{-1/2}\zeta^*$ with $\zeta^* \sim N(0, \omega_{zz} I_2)$ in the case where $\theta_0 \in \operatorname{int}\Theta$, such that the limit bootstrap distribution is non-random in this case, and

$$n^{1/2}(\hat{\theta}^* - \hat{\theta}) \xrightarrow{w^*}_w \left(\arg\min_{\lambda \in \Lambda_\ell^*} ||\lambda - M^{-1/2}\zeta^*||_M\right)\Big|\ell, \quad \Lambda_\ell^* := \{\lambda \in \mathbb{R}^2 : \dot{g}'\lambda \geq -\dot{g}'\ell\},$$

with $\zeta^*|\ell \sim N(0, \omega_{zz} I_2)$ in the case where $g(\theta_0) = 0$. We conclude that the same discrepancy between Λ and Λ_ℓ^* emerges in the case $g(\theta_0) = 0$ irrespective of the stochastic properties of the regressor. Consider now the bootstrap estimator of Corollary 4.1 with the choice $g^* = g - |g|^{1+\kappa}$ for $\kappa > 0$. For a stationary $x_{n,t}$ and a non-random M, the original and the bootstrap estimators satisfy

$$(n^{1/2}(\hat{\theta} - \theta_0), (n^{1/2}(\hat{\theta}^* - \hat{\theta})|D_n)) \xrightarrow{w}_p (\ell_{st}(\theta_0), \ell_{st}(\theta_0))$$

and bootstrap validity is restored as in Corollary 4.1, in particular because the random shift from the half-plane in the limiting bootstrap distribution is again removed.

S.1.3 Concluding remarks

An inferential framework that would be asymptotically valid in the unit root, near-unit root, and stationary cases, allowing the researcher to remain agnostic to the stochastic properties of the regressor, could be based on two main ingredients. First, the definition of the bootstrap parameter space in a way such that it approximates sufficiently well the geometry of the original parameter space; e.g., by setting $g^* = g - |g|^{1+\kappa}$ in the definition of Θ^* for some $\kappa > 0$, see above. This definition is independent of the stochastic properties of the regressor. Second, the use of an estimator (different from our choice of OLS) that gives rise to limit distributions that (a) in the near-unit root case depend on c only through the process X (and thus, the matrix M), but are free from shifts in the direction of v_c, and (b) allow for a common treatment of the contemporaneous correlation between the innovations of the predictive regression and the shocks driving $x_{n,t}$ (vs. the inclusion or omission of $\Delta x_{n.t}$ in the estimated eq. (3.2)). We conjecture that constrained versions of both the IVX (extended instrumental variables) estimator and the associated bootstrap schemes as discussed in Demetrescu et al. (2023) would give rise to asymptotically valid bootstrap inference. A detailed discussion is beyond the scope of this appendix due to our focus on issues attributable to the boundary of the parameter space.

S.2 Additional Monte Carlo simulations

In this section, we present additional numerical results in support of the theoretical arguments provided in CGZ. In particular, Tables S.1 and S.2 refer to the same testing procedure considered in Tables 1 and 2 in CGZ, respectively, but focus on the case

$g^* = g_2^* := g - n^{-\kappa}|g|$. Furthermore, in Tables S.3 and S.4 we present the simulated ERPs of bootstrap tests under local alternatives such that $\theta_0 \in \text{int}(\Theta)$, using $g^* = g_1^*$ and $g^* = g_2^*$, respectively.

References

Cavaliere, G., I. Georgiev and E. Zanelli (2024): Parameter on the boundary in predictive regression, *Econometric Theory*, forthcoming.

Demetrescu, M., I. Georgiev, A.M.R. Taylor and P.M.M. Rodrigues (2023): Extensions to IVX methods of inference for return predictability, *Journal of Econometrics* 237 (Issue 2, Part C).

TABLE S1: *Empirical rejection probabilities (ERPs) of bootstrap tests under the null.*

Nominal level: 0.05

		$\theta_0 = (0,0)'$					$\theta_0 = (-0.75, 0.75)'$					$\theta_0 = (-1.50, 1.50)'$				
		b_1	b_2				b_1	b_2				b_1	b_2			
			κ					κ					κ			
dist.	n		0.05	0.10	0.20	0.40		0.05	0.10	0.20	0.40		0.05	0.10	0.20	0.40
ξ_1	100	4.2	4.9	5.3	5.5	5.6	6.9	7.0	7.3	8.3	9.6	6.3	6.4	6.6	7.3	9.6
	400	3.9	4.8	5.1	5.3	5.3	5.5	5.7	6.0	7.1	9.2	5.3	5.3	5.3	5.7	8.6
	800	3.7	4.7	5.0	5.2	5.2	5.2	5.3	5.6	6.7	9.4	5.2	5.2	5.2	5.3	8.4
ξ_2	100	4.2	4.9	5.3	5.6	5.7	7.1	7.3	7.5	8.4	9.9	6.2	6.4	6.6	7.2	9.5
	400	3.8	4.6	5.0	5.1	5.2	5.7	6.0	6.3	7.3	9.4	5.3	5.3	5.3	5.7	8.7
	800	3.6	4.5	4.8	4.9	4.9	5.1	5.2	5.5	6.7	9.3	5.1	5.1	5.1	5.3	8.6
ξ_3	100	4.3	4.9	5.3	5.6	5.7	7.1	7.2	7.4	8.5	9.9	6.4	6.5	6.7	7.4	9.8
	400	3.7	4.6	4.9	5.1	5.1	5.5	5.8	6.1	7.2	9.3	5.2	5.2	5.2	5.6	8.6
	800	3.7	4.6	5.0	5.1	5.2	5.1	5.2	5.4	6.5	9.1	5.1	5.1	5.1	5.3	8.4

Nominal level: 0.10

		$\theta_0 = (0,0)'$					$\theta_0 = (-0.75, 0.75)'$					$\theta_0 = (-1.50, 1.50)'$				
		b_1	b_2				b_1	b_2				b_1	b_2			
			κ					κ					κ			
dist.	n		0.05	0.10	0.20	0.40		0.05	0.10	0.20	0.40		0.05	0.10	0.20	0.40
ξ_1	100	8.0	9.1	9.9	10.5	10.7	13.0	13.3	13.7	15.4	18.6	11.5	11.7	12.0	12.9	17.2
	400	7.7	9.2	9.9	10.3	10.5	10.4	10.6	11.1	12.9	17.6	10.3	10.3	10.3	10.7	15.9
	800	7.4	9.0	9.7	10.0	10.1	10.4	10.4	10.7	12.2	18.1	10.1	10.1	10.1	10.2	15.5
ξ_2	100	8.1	9.2	9.9	10.5	10.7	13.2	13.5	13.9	15.6	18.7	11.3	11.5	11.8	12.7	16.9
	400	7.5	9.0	9.7	10.2	10.3	10.7	11.0	11.4	13.2	18.0	10.2	10.3	10.3	10.7	15.9
	800	7.2	8.9	9.5	9.9	10.0	10.2	10.3	10.5	12.0	17.7	10.3	10.3	10.3	10.4	15.7
ξ_3	100	8.3	9.4	10.2	10.8	11.0	13.3	13.7	14.1	15.8	19.0	11.7	11.9	12.2	13.2	17.5
	400	7.6	9.1	9.8	10.2	10.3	10.4	10.6	11.1	13.1	17.7	10.2	10.2	10.2	10.6	15.9
	800	7.4	9.0	9.6	10.0	10.1	10.1	10.1	10.4	11.9	17.6	10.0	10.0	10.0	10.1	15.5

Note: bootstrap tests are based on a standard fixed-regressor wild bootstrap (b_1) and on the proposed corrected wild bootstrap method (b_2) of Section 4, using $g^* = g - n^{-\kappa}|g|$. ERPs are estimated using 50,000 Monte Carlo replications and 999 bootstrap repetitions. The column "dist." shows the distributions of ε_t: $\xi_1 \sim iidN(0,1)$, $\xi_2 \sim ARCH(1)$ and $\xi_3 = \sqrt{0.5}v_t + \sqrt{0.5}\varepsilon_{x,t}$, where $v_t \sim iidN(0,1)$ and $\varepsilon_{x,t}$ is the error term of the predictive variable $x_{n,t}$.

Table S2: *Empirical rejection probabilities (ERPs) of bootstrap tests under local alternatives.*

Nominal level: 0.05

		$a_0 = (-3,0)'$					$a_0 = (3,0)'$					$a_0 = (5,0)'$				
		b_1	b_2				b_1	b_2				b_1	b_2			
			κ					κ					κ			
dist.	n		0.05	0.10	0.20	0.40		0.05	0.10	0.20	0.40		0.05	0.10	0.20	0.40
ξ_1	100	21.0	21.1	21.3	21.5	21.5	40.6	40.8	40.9	41.0	41.0	68.0	68.0	68.0	68.0	68.0
	400	18.9	19.1	19.3	19.5	19.5	38.5	38.7	38.8	38.8	38.8	64.9	64.9	64.9	64.9	64.9
	800	18.6	18.8	19.0	19.1	19.1	37.6	37.8	37.9	37.9	37.9	64.0	64.0	64.0	64.0	64.0
ξ_2	100	21.7	21.9	22.0	22.2	22.3	41.9	42.1	42.2	42.2	42.3	68.5	68.5	68.5	68.5	68.5
	400	19.2	19.4	19.6	19.7	19.8	38.3	38.6	38.7	38.7	38.7	64.7	64.8	64.8	64.8	64.8
	800	18.6	18.8	19.0	19.1	19.1	37.8	38.0	38.1	38.1	38.1	64.2	64.2	64.2	64.2	64.2
ξ_3	100	20.6	20.7	20.9	21.2	21.3	40.8	41.0	41.1	41.1	41.1	67.3	67.3	67.3	67.3	67.3
	400	19.0	19.1	19.3	19.4	19.4	38.1	38.3	38.4	38.5	38.5	65.0	65.0	65.0	65.0	65.0
	800	18.3	18.5	18.7	18.8	18.9	37.7	38.0	38.0	38.1	38.1	63.5	63.5	63.5	63.5	63.5

Nominal level: 0.10

		$a_0 = (-3,0)'$					$a_0 = (3,0)'$					$a_0 = (5,0)'$				
		b_1	b_2				b_1	b_2				b_1	b_2			
			κ					κ					κ			
dist.	n		0.05	0.10	0.20	0.40		0.05	0.10	0.20	0.40		0.05	0.10	0.20	0.40
ξ_1	100	29.6	29.8	30.1	30.5	30.7	54.7	55.0	55.1	55.2	55.2	81.7	81.7	81.7	81.8	81.8
	400	27.0	27.3	27.8	28.1	28.2	52.2	52.5	52.6	52.7	52.7	79.6	79.6	79.6	79.6	79.6
	800	26.4	26.8	27.2	27.5	27.6	51.7	52.1	52.1	52.2	52.2	78.7	78.7	78.7	78.7	78.7
ξ_2	100	30.2	30.4	30.7	31.2	31.4	55.7	55.9	55.9	56.0	56.1	82.0	82.0	82.0	82.0	82.0
	400	27.1	27.4	27.9	28.2	28.3	51.8	52.0	52.1	52.2	52.2	79.3	79.3	79.3	79.3	79.3
	800	26.6	26.9	27.4	27.7	27.7	51.5	51.8	51.9	51.9	51.9	78.6	78.6	78.6	78.6	78.6
ξ_3	100	29.1	29.3	29.6	30.1	30.3	54.2	54.4	54.5	54.6	54.6	80.9	80.9	80.9	80.9	80.9
	400	26.7	27.0	27.4	27.8	27.8	51.7	52.0	52.1	52.2	52.2	79.4	79.4	79.4	79.4	79.4
	800	26.2	26.5	27.0	27.3	27.3	51.3	51.6	51.7	51.7	51.8	78.5	78.5	78.5	78.5	78.5

Note: bootstrap tests are based on a standard fixed-regressor wild bootstrap (b_1) and on the proposed corrected wild bootstrap method (b_2) of Section 4, using $g^* = g - n^{-\kappa}|g|$. ERPs are estimated using 50,000 Monte Carlo replications and 999 bootstrap repetitions. The column "dist." shows the distributions of ε_t: $\xi_1 \sim iidN(0,1)$, $\xi_2 \sim ARCH(1)$ and $\xi_3 = \sqrt{0.5}v_t + \sqrt{0.5}\varepsilon_{x,t}$, where $v_t \sim iidN(0,1)$ and $\varepsilon_{x,t}$ is the error term of the predictive variable $x_{n,t}$.

Table S3: *Empirical rejection probabilities (ERPs) of bootstrap tests under local alternatives.*

Nominal level: 0.05

		$a_0 = (-3,1)'$					$a_0 = (2,2)'$					$a_0 = (3,4)'$				
		b_1	b_2				b_1	b_2				b_1	b_2			
			κ					κ					κ			
dist.	n		0.25	0.50	1.0	2.0		0.25	0.50	1.0	2.0		0.25	0.50	1.0	2.0
ξ_1	100	12.8	12.9	13.0	13.2	13.4	48.4	49.6	50.1	50.3	50.4	73.0	73.9	74.4	74.7	74.7
	400	11.4	11.6	11.9	12.2	12.3	45.4	47.2	47.5	47.6	47.6	70.0	71.6	72.0	72.0	72.0
	800	10.9	11.2	11.6	11.7	11.8	44.8	46.9	47.1	47.1	47.2	69.3	71.1	71.4	71.4	71.4
ξ_2	100	13.1	13.2	13.3	13.5	13.6	49.6	50.8	51.3	51.6	51.6	73.2	74.1	74.7	75.0	75.0
	400	11.4	11.6	11.8	12.1	12.2	46.1	48.0	48.3	48.3	48.3	70.2	71.8	72.2	72.3	72.3
	800	11.0	11.3	11.7	11.9	11.9	45.2	47.2	47.4	47.4	47.4	69.6	71.5	71.7	71.7	71.7
ξ_3	100	12.3	12.4	12.5	12.7	12.9	48.1	49.3	49.9	50.1	50.1	72.4	73.2	73.8	74.1	74.1
	400	11.4	11.6	11.9	12.2	12.3	46.0	47.8	48.2	48.2	48.3	69.9	71.5	72.0	72.0	72.0
	800	11.1	11.4	11.8	12.0	12.1	45.0	46.9	47.1	47.1	47.1	69.4	71.3	71.6	71.6	71.6

Nominal level: 0.10

		$a_0 = (-3,1)'$					$a_0 = (2,2)'$					$a_0 = (3,4)'$				
		b_1	b_2				b_1	b_2				b_1	b_2			
			κ					κ					κ			
dist.	n		0.25	0.50	1.0	2.0		0.25	0.50	1.0	2.0		0.25	0.50	1.0	2.0
ξ_1	100	21.2	21.5	21.6	22.0	22.4	58.8	60.4	61.1	61.5	61.5	80.7	81.6	82.2	82.5	82.5
	400	19.2	19.6	20.2	21.0	21.2	56.0	58.2	58.6	58.7	58.7	78.2	79.9	80.3	80.4	80.4
	800	18.3	18.9	19.7	20.2	20.2	55.8	58.1	58.5	58.5	58.5	77.8	79.8	80.1	80.1	80.2
ξ_2	100	21.8	22.0	22.1	22.5	23.0	59.6	61.1	61.8	62.1	62.2	81.0	81.9	82.5	82.9	82.9
	400	19.1	19.5	20.1	20.7	21.0	56.8	59.0	59.5	59.6	59.6	78.6	80.4	80.8	80.8	80.9
	800	18.9	19.5	20.2	20.7	20.8	56.0	58.4	58.7	58.8	58.8	78.0	79.9	80.2	80.3	80.3
ξ_3	100	20.6	20.8	20.9	21.3	21.8	58.5	60.1	60.8	61.1	61.2	80.2	81.2	81.7	82.0	82.1
	400	19.1	19.5	20.1	20.8	21.0	56.6	58.7	59.2	59.3	59.3	78.3	80.1	80.5	80.6	80.6
	800	18.7	19.2	20.0	20.5	20.6	55.7	58.2	58.5	58.6	58.6	77.8	79.5	79.9	79.9	79.9

Note: bootstrap tests are based on a standard fixed-regressor wild bootstrap (b_1) and on the proposed corrected wild bootstrap method (b_2) of Section 4, using $g^* = g - |g|^{1+\kappa}$. ERPs are estimated using 50,000 Monte Carlo replications and 999 bootstrap repetitions. The column "dist." shows the distributions of ε_t: $\xi_1 \sim iidN(0,1)$, $\xi_2 \sim ARCH(1)$ and $\xi_3 = \sqrt{0.5}v_t + \sqrt{0.5}\varepsilon_{x,t}$, where $v_t \sim iidN(0,1)$ and $\varepsilon_{x,t}$ is the error term of the predictive variable $x_{n,t}$.

TABLE S4: *Empirical rejection probabilities (ERPs) of bootstrap tests under local alternatives.*

Nominal level: 0.05

		$a_0 = (-3,1)'$					$a_0 = (2,2)'$					$a_0 = (3,4)'$				
		b_1	b_2				b_1	b_2				b_1	b_2			
			κ					κ					κ			
dist.	n		0.05	0.10	0.20	0.40		0.05	0.10	0.20	0.40		0.05	0.10	0.20	0.40
ξ_1	100	12.8	13.0	13.2	13.6	13.7	48.4	49.6	50.1	50.4	50.4	73.0	74.0	74.5	74.7	74.7
	400	11.4	11.6	12.0	12.2	12.3	45.4	47.0	47.4	47.6	47.6	70.0	71.4	71.9	72.0	72.0
	800	10.9	11.1	11.5	11.7	11.8	44.8	46.5	47.0	47.1	47.2	69.3	70.8	71.3	71.4	71.4
ξ_2	100	13.1	13.3	13.5	13.9	14.0	49.6	50.8	51.3	51.6	51.6	73.2	74.2	74.7	75.0	75.0
	400	11.4	11.6	11.9	12.1	12.2	46.1	47.7	48.1	48.3	48.3	70.2	71.6	72.1	72.3	72.3
	800	11.0	11.3	11.7	11.8	11.9	45.2	46.9	47.3	47.4	47.4	69.6	71.2	71.6	71.7	71.7
ξ_3	100	12.3	12.4	12.8	13.2	13.3	48.1	49.3	49.9	50.1	50.2	72.4	73.4	73.9	74.1	74.2
	400	11.4	11.7	12.0	12.2	12.3	46.0	47.6	48.0	48.2	48.3	69.9	71.4	71.8	72.0	72.0
	800	11.1	11.4	11.8	12.0	12.1	45.0	46.5	47.0	47.1	47.2	69.4	71.0	71.5	71.6	71.6

Nominal level: 0.10

		$a_0 = (-3,1)'$					$a_0 = (2,2)'$					$a_0 = (3,4)'$				
		b_1	b_2				b_1	b_2				b_1	b_2			
			κ					κ					κ			
dist.	n		0.05	0.10	0.20	0.40		0.05	0.10	0.20	0.40		0.05	0.10	0.20	0.40
ξ_1	100	21.2	21.5	21.9	22.7	23.0	58.8	60.2	60.9	61.4	61.5	80.7	81.6	82.1	82.4	82.5
	400	19.2	19.6	20.3	21.0	21.2	56.0	57.7	58.3	58.6	58.7	78.2	79.6	80.1	80.4	80.4
	800	18.3	18.8	19.6	20.1	20.2	55.8	57.7	58.2	58.5	58.5	77.8	79.4	79.9	80.1	80.1
ξ_2	100	21.8	22.0	22.5	23.3	23.7	59.6	61.0	61.6	62.1	62.2	81.0	81.9	82.5	82.9	82.9
	400	19.1	19.5	20.1	20.8	21.0	56.8	58.5	59.2	59.5	59.6	78.6	80.0	80.6	80.8	80.8
	800	18.9	19.4	20.1	20.7	20.8	56.0	57.9	58.5	58.7	58.8	78.0	79.5	80.1	80.3	80.3
ξ_3	100	20.6	20.8	21.3	22.2	22.6	58.5	59.9	60.6	61.1	61.2	80.2	81.1	81.7	82.0	82.1
	400	19.1	19.5	20.2	20.8	21.0	56.6	58.3	58.9	59.2	59.3	78.3	79.7	80.3	80.5	80.6
	800	18.7	19.1	19.9	20.5	20.6	55.7	57.7	58.3	58.6	58.6	77.8	79.2	79.7	79.9	79.9

Note: bootstrap tests are based on a standard fixed-regressor wild bootstrap (b_1) and on the proposed corrected wild bootstrap method (b_2) of Section 4, using $g^* = g - n^{-\kappa}|g|$. ERPs are estimated using 50,000 Monte Carlo replications and 999 bootstrap repetitions. The column "dist." shows the distributions of ε_t: $\xi_1 \sim iidN(0,1)$, $\xi_2 \sim ARCH(1)$ and $\xi_3 = \sqrt{0.5}v_t + \sqrt{0.5}\varepsilon_{x,t}$, where $v_t \sim iidN(0,1)$ and $\varepsilon_{x,t}$ is the error term of the predictive variable $x_{n,t}$.