# Supplement to "Theory of Low Frequency Contamination from Nonstationarity and Misspecification: Consequences for HAR Inference"

Alessandro Casini*
University of Rome Tor Vergata

Taosong Deng†
Hunan University

Pierre Perron‡
Boston University

14th September 2024

**Abstract**

This supplemental material is for online publication only. Section S.A introduces the notion of long memory segmented locally stationary processes and presents the theoretical results referenced in Section 3. Section S.B contains the proofs of the results in the paper and Section S.C contains additional figures.

*Corresponding author at: Dep. of Economics and Finance, University of Rome Tor Vergata, Via Columbia 2, Rome, 00133, IT. Email: alessandro.casini@uniroma2.it.

†College of Finance and Statistics, Hunan University, 109 Shijiachong Road, Yuelu District, Changsha, Hunan 41006, China. Email: tsdeng@hnu.edu.cn.

‡Dep. of Economics, Boston University, 270 Bay State Road, Boston, MA 02215, US. Email: perron@bu.edu.

# S.A Results on Low Frequency Bias for the Sample Autocovariance and the Periodogram

In Section S.A.1 we define the long memory SLS processes. In Section S.A.2 and S.A.3 we present results on the low frequency bias for the sample autocovariance and the periodogram, respectively.

## S.A.1 Long Memory Segmented Locally Stationary Processes

Define the backward difference operator $\Delta V_t = \Delta^1 V_t = V_t - V_{t-1}$ and $\Delta^l V_t$ recursively. Long memory features can be expressed as a "pole" in the spectral density at frequency zero. That is, for a stationary process, long memory implies that $f(\omega) \sim \omega^{-2\vartheta}$ as $\omega \to 0$ where $\vartheta \in (0, 1/2)$ is the long memory parameter. In what follows, $l$ is some non-negative integer.

**Definition S.1.** A sequence of stochastic processes $\{V_{t,T}\}$ is called long memory segmented locally stationary with $m_0 + 1$ regimes, transfer function $A^0$ and trend $\mu.$ if there exists a representation

$$\Delta^l V_t = \mu_j(t/T) + \int_{-\pi}^{\pi} \exp(i\omega t) A^0_{j,t,T}(\omega)\, d\xi(\omega), \qquad \left(t = T^0_{j-1} + 1, \ldots, T^0_j\right), \tag{S.1}$$

for $j = 1, \ldots, m_0 + 1$, where by convention $T^0_0 = 0$ and $T^0_{m_0+1} = T$, (i) and (iii) of Definition 1 hold, and (ii) of Definition 1 is replaced by

(ii) There exist two constants $L_2 > 0$ and $D < 1/2$ (which depend on $j$) and a piecewise continuous function $A : [0, 1] \times \mathbb{R} \to \mathbb{C}$ such that, for each $j = 1, \ldots, m_0 + 1$, there exists a $2\pi$-periodic function $A_j : (\lambda^0_{j-1}, \lambda^0_j] \times \mathbb{R} \to \mathbb{C}$ with $A_j(u, -\omega) = \overline{A_j(u, \omega)}$,

$$A(u, \omega) = A_j(u, \omega) \text{ for } \lambda^0_{j-1} < u \le \lambda^0_j, \tag{S.2}$$

$$\sup_{1 \le j \le m_0+1} \sup_{T^0_{j-1} < t \le T^0_j,\, \omega} \left| A^0_{j,t,T}(\omega) - A_j(t/T, \omega) \right| \le L_2 T^{-1} |\omega|^{-D}, \tag{S.3}$$

and

$$\sup_{0 \le v \le u \le 1,\, u \ne \lambda^0_j\, (j=1,\ldots,m_0+1,),\, \omega} |A(u, \omega) - A(v, \omega)| \le L_2 |u - v| |\omega|^{-D}. \tag{S.4}$$

The spectral density of $\{V_{t,T}\}$ is given by $f_j(u, \omega) = |1 - \exp(-i\omega)|^{-2l} |A_j(u, \omega)|^{-2}$ for $j = 1, \ldots, m_0 + 1$. We say that the process $\{V_{t,T}\}$ has local memory parameter $\vartheta(u) \in (-\infty, l + 1/2)$ at time $u \in [0, 1]$ if it satisfies (S.1)-(S.4), and its generalized spectral density $f_j(u, \omega)$ $(j = 1, \ldots, m_0 + 1)$ satisfies the following condition,

$$f_j(u, \omega) = \left|1 - e^{-i\omega}\right|^{-2\vartheta_j(u)} f^*_j(u, \omega), \tag{S.5}$$

with $f^*_j(u, \omega) > 0$ and

$$\left| f^*_j(u, \omega) - f^*_j(u, 0) \right| \le L_4 f^*_j(u, \omega) |\omega|^{\nu}, \qquad \omega \in [-\pi, \pi], \tag{S.6}$$

where $L_4 > 0$ and $\nu \in (0, 2]$.

Definition S.1 extends Definition 1 and Assumption 1 by requiring the bound on the smoothness of $A(\cdot, \omega)$ to depend also on $|\omega|^{-D}$ thereby allowing a singularity at $\omega = 0$. Casini (2023) showed that $f_j(u, \omega) = |A_j(u, \omega)|^2$ for $j = 1, \ldots, m_0+1$. Using similar arguments, we obtain the form $f_j(u, \omega)$ given in (S.5). See Roueff and von Sachs (2011) for a definition of long memory local stationarity. Definition S.1 extends their definition to allow for $m_0$ discontinuities. We have assumed that breaks in the long memory parameter occur at the same locations as the breaks in the spectrum. This can be relaxed but would provide no added value in this paper.

**Example S.1.** A time-varying AR fractionally integrated moving average $(p, \vartheta, q)$ process with $m_0$ structural breaks satisfies Definition S.1 with $\vartheta_j : [0, 1] \to (-\infty, l + 1/2)$, $\sigma_j : [0, 1] \to \mathbb{R}_+$, $\phi_j = [\phi_1, \ldots, \phi_p]' : [0, 1] \to \mathbb{R}^q$ and $\theta_j = [\theta_1, \ldots, \theta_q]' : [0, 1] \to \mathbb{R}^p$ are left-Lipschitz functions for each $j = 1, \ldots, m_0 + 1$ such that $1 - \sum_{k=1}^{p} \phi_{j,k}(u) z^k$ does not vanish for all $u \in [0, 1]$ and $z \in \mathbb{C}$ such that $|z| \leq 1$. Using the latter condition, the local transfer function $A_j(u; \cdot)$ defines for each $j$ a causal autoregressive fractionally integrated moving average (ARFIMA$(p, \vartheta(u) - l, q)$ process whose spectral density satisfies the conditions (S.5) and (S.6) with $\nu = 2$. Using Lemma 3 in Roueff and von Sachs (2011), condition (S.4) holds with $D > \sup_{1 \leq j \leq m_0+1} \sup_{\lambda_{j-1}^0 < u \leq \lambda_j^0, \omega} \vartheta_j(u) - l$.

Definition S.1 implies that $\rho_V(u, k) \triangleq \mathrm{Corr}(V_{\lfloor Tu \rfloor}, V_{\lfloor Tu \rfloor + k}) \sim C k^{2\vartheta_j(u)-1}$ for $\lambda_{j-1}^0 < u < \lambda_j^0$ and large $k$ where $C > 0$. This means that the rescaled time-$u$ autocorrelation function (ACF$(u)$) has a power law decay which implies $\sum_{k=-\infty}^{\infty} |\rho_V(u, k)| = \infty$ if $\vartheta_j(u) \in (0, 1/2)$.

## S.A.2 The Sample Autocovariance Under Nonstationarity

We now establish some asymptotic properties of the sample autocovariance under nonstationarity. We consider the case $k \geq 0$ only; the case $k < 0$ is similar.

**Theorem S.1.** *Assume that $\{V_{t,T}\}$ satisfies Definition 1. Under Assumptions 1-2,*

$$\widehat{\Gamma}(k) \geq \int_0^1 c(u, k)\, du + d^* + o_{\mathrm{a.s.}}(1), \tag{S.7}$$

*where $d^* = 2^{-1} \sum_{j_1 \neq j_2} r_{j_1} r_{j_2} (\overline{\mu}_{j_2} - \overline{\mu}_{j_1})^2$. Further, as $k \to \infty$, $\widehat{\Gamma}(k) \geq d^*$ $\mathbb{P}$-a.s. If in addition it holds that $\mu_j(t/T) = \mu_j$ for $j = 1, \ldots, m_0 + 1$, then*

$$\widehat{\Gamma}(k) = \int_0^1 c(u, k)\, du + d^*_{\mathrm{Sta}} + o_{\mathrm{a.s.}}(1),$$

*where $d^*_{\mathrm{Sta}} = 2^{-1} \sum_{j_1 \neq j_2} r_{j_1} r_{j_2} (\mu_{j_2} - \mu_{j_1})^2$ and, as $k \to \infty$, $\widehat{\Gamma}(k) = d^*_{\mathrm{Sta}} + o_{\mathrm{a.s.}}(1)$.*

## S.A.3 The Periodogram Under Nonstationarity

Classical LRV estimators are weighted averages of periodogram ordinates around the zero frequency. Thus, it is useful to study the behavior of the periodogram as the frequency $\omega$ approaches zero. We now establish some properties of the asymptotic bias of the periodogram under nonstationarity. We consider the Fourier frequencies $\omega_l = 2\pi l/T \in (-\pi, \pi)$ for an integer $l \neq 0$ (mod $T$) and exclude $\omega_l = 0$ for mathematical convenience.

**Assumption S.1.** *(i) For each $j = 1, \ldots, m_0 + 1$ there exists a $B_j \in \mathbb{R}$ such that*

$$\left| \sum_{j=1}^{m_0+1} \sum_{t=\lfloor T\lambda_{j-1}^0 \rfloor + 1}^{\lfloor T\lambda_j^0 \rfloor} \mu_j (t/T) \exp(-i\omega_l t) \right|^2 \geq \left| \sum_{j=1}^{m_0+1} B_j \sum_{t=\lfloor T\lambda_{j-1}^0 \rfloor + 1}^{\lfloor T\lambda_j^0 \rfloor} \exp(-i\omega_l t) \right|^2, \qquad \omega_l \in (-\pi, \pi),$$

*where $B_{j_1} \neq B_{j_2}$ for $j_1 \neq j_2$; (ii) $|\Gamma(u, k)| = C_{u,k} k^{-m}$ for all $u \in [0, 1]$ and all $k \geq C_3 T^{\kappa}$ for some $C_3 < \infty$ , $C_{u,k} < \infty$ (which depends on $u$ and $k$), $0 < \kappa < 1/2$, and $m > 2$.*

Part (i) is easily satisfied (e.g., the special case with $\mu_j (t/T) = \mu_j$). Part (ii) is satisfied if $\{V_t\}$ is strong mixing with mixing parameters of size $-2\nu/(\nu - 1/2)$ for some $\nu > 1$ such that $\sup_{t\geq 1} \mathbb{E} |V_t|^{4\nu} < \infty$. This is less stringent than the size condition $-3\nu/(\nu - 1)$ for some $\nu > 1$ sufficient for Assumption 2-(i).

**Theorem S.2.** *Assume that $\{V_{t,T}\}$ satisfies Definition 1. Under Assumptions 1-2 and S.1,*

$$\mathbb{E}(I_T(\omega_l)) = 2\pi \int_0^1 f(u, \omega_l)\, du + \frac{1}{T\omega_l^2} \left| \left[ B_1 - B_{m_0+1} - \sum_{j=1}^{m_0} (B_j - B_{j+1}) \exp\left(-2\pi i l \lambda_l^0\right) \right] \right|^2 + o(1). \tag{S.8}$$

*Under Assumptions 1-2 and S.1-(ii), if $\mu_j (t/T) = \mu_j$ for each $j = 1, \ldots, m_0 + 1$, then*

$$\mathbb{E}(I_T(\omega_l)) = 2\pi \int_0^1 f(u, \omega_l)\, du + \frac{1}{T\omega_l^2} \left| \left[ \mu_j - \mu_{m_0+1} - \sum_{j=1}^{m_0} (\mu_j - \mu_{j+1}) \exp\left(-2\pi i l \lambda_j^0\right) \right] \right|^2 + o(1).$$

*In either case, if $T\omega_l^2 \to 0$ as $T \to \infty$ then $\mathbb{E}(I_T(\omega_l)) \to \infty$ for many values in $\{\omega_l\}$ as $\omega_l \to 0$.*

The theorem suggests that for small frequencies $\omega_l$ close to 0, the periodogram attains very large values. This follows because the first term of (S.8) is bounded for all $\omega_j$. Since $B_1, \ldots, B_{m_0+1}$ are fixed, the order of the second term of (S.8) is $O((T\omega_j^2)^{-1})$. Note that as $\omega_l \to 0$ there are some values $l$ for which the corresponding term involving $|\cdot|^2$ on the right-hand side of (S.8) is equal to zero. In such cases, $\mathbb{E}(I_T(\omega_l)) \geq 2\pi \int_0^1 f(u, \omega_l)\, du > 0$. For other values of $\{l\}$ as $\omega_l \to 0$, the second term of (S.8) diverges to infinity. Thus, considering the behavior of $\{\mathbb{E}(I_T(\omega_l))\}$ as $\omega_l \to 0$, it generally takes unbounded values except for some $\omega_l$ for which $\mathbb{E}(I_T(\omega_l))$ is bounded below by $2\pi \int_0^1 f(u, \omega_l)\, du > 0$. A SLS process with long memory has an unbounded local spectral density $f(u, \omega)$ as $\omega \to 0$ for some $u \in [0, 1]$. Since $f(\cdot, \cdot)$ cannot be negative, it follows that $\int_0^1 f(u, \omega)\, du$ is also unbounded as $\omega \to 0$. Theorem S.2 suggests that nonstationarity consisting of time-varying first moment results in a periodogram sharing features of a long memory series.

# S.B Mathematical Appendix

## S.B.1 Proofs of the Results in Sections 3 and S.A

### S.B.1.1 Proof of Theorem S.1

Let $\overline{V}_j = (Tr_j)^{-1} \sum_{t=\lfloor T\lambda_{j-1}^0 \rfloor+1}^{\lfloor T\lambda_j^0 \rfloor} V_t$, $\mu_{2,j}(u) = \mathbb{E}(V_{\lfloor Tu \rfloor})^2$ for $T_{j-1}^0 \leq Tu \leq T_j^0$ and $\overline{\mu}_{2,j} = r_j^{-1} \int_{\lambda_{j-1}^0}^{\lambda_j^0} \mu_{2,j}(u)\, du$. By Assumption 1-2-(i), the latter implying ergodicity, it follows for fixed $k \geq 0$ that

$$
\begin{aligned}
\widehat{\Gamma}(k) &= \sum_{j=1}^{m_0+1} r_j \frac{1}{Tr_j} \sum_{t=\lfloor T\lambda_{j-1}^0 \rfloor+1+k}^{\lfloor T\lambda_j^0 \rfloor} V_t V_{t-k} - \left( \sum_{j=1}^{m_0+1} r_j \frac{1}{Tr_j} \sum_{t=\lfloor T\lambda_{j-1}^0 \rfloor+1}^{\lfloor T\lambda_j^0 \rfloor} V_t \right)^2 \\
&= \sum_{j=1}^{m_0+1} \int_{\lambda_{j-1}^0}^{\lambda_j^0} c(u,\, k)\, du + \sum_{j=1}^{m_0+1} r_j \frac{1}{Tr_j} \sum_{t=\lfloor T\lambda_{j-1}^0 \rfloor+1+k}^{\lfloor T\lambda_j^0 \rfloor} \mathbb{E}(V_t)\, \mathbb{E}(V_{t-k}) \\
&\quad - \left( \sum_{j=1}^{m_0+1} r_j \frac{1}{Tr_j} \sum_{t=\lfloor T\lambda_{j-1}^0 \rfloor+1}^{\lfloor T\lambda_j^0 \rfloor} V_t \right)^2 + O\left(T^{-1}\right) + o_{\text{a.s.}}(1) \\
&= \int_0^1 c(u,\, k)\, du + \sum_{j=1}^{m_0+1} r_j \frac{1}{Tr_j} \sum_{t=\lfloor T\lambda_{j-1}^0 \rfloor+1+k}^{\lfloor T\lambda_j^0 \rfloor} \mathbb{E}(V_t)\, \mathbb{E}(V_{t-k}) \\
&\quad - \left( \sum_{j=1}^{m_0+1} r_j \overline{V}_j \right)^2 + O\left(T^{-1}\right) + o_{\text{a.s.}}(1) \\
&= \int_0^1 c(u,\, k)\, du + \sum_{j=1}^{m_0+1} r_j \frac{1}{Tr_j} \sum_{t=\lfloor T\lambda_{j-1}^0 \rfloor+1+k}^{\lfloor T\lambda_j^0 \rfloor} \mu^2(t/T) - \left( \sum_{j=1}^{m_0+1} r_j \overline{V}_j \right)^2 + O\left(T^{-1}\right) + o_{\text{a.s.}}(1),
\end{aligned}
$$

where we have used $\mathbb{E}(V_{t-k}) - \mathbb{E}(V_t) = O(k/T)$ by local stationarity in the third equality. Note that by ergodicity and an approximation to Riemann sums, we have

$$
\begin{aligned}
\sum_{j=1}^{m_0+1} r_j \overline{V}_j - \sum_{j=1}^{m_0+1} r_j \overline{\mu}_j &= \sum_{j=1}^{m_0+1} r_j \overline{V}_j - \sum_{j=1}^{m_0+1} r_j \mathbb{E}\left(\overline{V}_j\right) + \sum_{j=1}^{m_0+1} r_j \mathbb{E}\left(\overline{V}_j\right) - \sum_{j=1}^{m_0+1} r_j \overline{\mu}_j \\
&= o_{\text{a.s.}}(1) + O\left(T^{-1}\right). 
\end{aligned} \tag{S.9}
$$

Basic manipulations show that

$$
\begin{aligned}
&\sum_{j_2 \neq j_1} r_{j_1} r_{j_2} \left(\overline{\mu}_{j_2} - \overline{\mu}_{j_1}\right)^2 \\
&\qquad = \sum_{j_2 \neq j_1} r_{j_1} r_{j_2} \left(\overline{\mu}_{j_2}^2 + \overline{\mu}_{j_1}^2 - 2\overline{\mu}_{j_2} \overline{\mu}_{j_1}\right)
\end{aligned}
$$

$$= \sum_{1\leq j_2\leq m_0+1} r_{j_2}\overline{\mu}_{j_2}^2 \left(1-r_{j_2}\right) + \sum_{1\leq j_1\leq m_0+1} r_{j_1}\overline{\mu}_{j_1}^2 \left(1-r_{j_1}\right) - 2\sum_{j_1\neq j_2} r_{j_1} r_{j_2} \overline{\mu}_{j_2}\overline{\mu}_{j_1}$$

$$= 2\sum_{1\leq j\leq m_0+1} r_j\overline{\mu}_j^2 - 2\sum_{1\leq j\leq m_0+1} r_j^2\overline{\mu}_j^2 - 2\sum_{j_1\neq j_2} r_{j_1} r_{j_2}\overline{\mu}_{j_2}\overline{\mu}_{j_1}. \tag{S.10}$$

Note that

$$\left(Tr_j - k\right) \sum_{t=\lfloor T\lambda_{j-1}^0 \rfloor+1+k}^{\lfloor T\lambda_j^0 \rfloor} \mu^2\left(t/T\right) \geq \left( \sum_{t=\lfloor T\lambda_{j-1}^0 \rfloor+1+k}^{\lfloor T\lambda_j^0 \rfloor} \mu\left(t/T\right) \right)^2. \tag{S.11}$$

Thus,

$$\sum_{j=1}^{m_0+1} r_j \frac{1}{Tr_j} \sum_{t=\lfloor T\lambda_{j-1}^0 \rfloor+1+k}^{\lfloor T\lambda_j^0 \rfloor} \mu^2\left(t/T\right) = \sum_{j=1}^{m_0+1} r_j \frac{1}{Tr_j\left(Tr_j-k\right)} \left(Tr_j - k\right) \sum_{t=\lfloor T\lambda_{j-1}^0 \rfloor+1+k}^{\lfloor T\lambda_j^0 \rfloor} \mu^2\left(t/T\right)$$

$$\geq \sum_{j=1}^{m_0+1} r_j \frac{1}{Tr_j\left(Tr_j-k\right)} \left( \sum_{t=\lfloor T\lambda_{j-1}^0 \rfloor+1+k}^{\lfloor T\lambda_j^0 \rfloor} \mu\left(t/T\right) \right)^2$$

$$= \sum_{1\leq j\leq m_0+1} r_j\overline{\mu}_j^2 + o\left(1\right). \tag{S.12}$$

Using (S.9)-(S.12) we have,

$$\widehat{\Gamma}\left(k\right) = \int_0^1 c\left(u,\, k\right) du + \sum_{j=1}^{m_0+1} r_j \frac{1}{Tr_j} \sum_{t=\lfloor T\lambda_{j-1}^0 \rfloor+1+k}^{\lfloor T\lambda_j^0 \rfloor} \mu^2\left(t/T\right) - \left( \sum_{j=1}^{m_0+1} r_j\overline{V}_j \right)^2 + o_{\text{a.s.}}\left(1\right)$$

$$\geq \int_0^1 c\left(u,\, k\right) du + \sum_{j=1}^{m_0+1} r_j\overline{\mu}_{2,j} - \left( \sum_{j=1}^{m_0+1} r_j\overline{V}_j \right)^2 + O\left(T^{-1}\right) + o_{\text{a.s.}}\left(1\right)$$

$$= \int_0^1 c\left(u,\, k\right) du + 2^{-1}\sum_{j_1\neq j_2} r_{j_1} r_{j_2} \left(\overline{\mu}_{j_2} - \overline{\mu}_{j_1}\right)^2 + O\left(T^{-1}\right) + o_{\text{a.s.}}\left(1\right). \tag{S.13}$$

The claim that $\widehat{\Gamma}\left(k\right) \geq d$ $\mathbb{P}$-a.s. as $k\to\infty$ follows from Assumption 2-(i) since this implies that $c\left(u,\, k\right)\to 0$ as $k\to\infty$ and from the fact that the second term on the right-hand side of (S.13) does not depend on $k$. If in addition it holds that $\mu_j\left(t/T\right) = \mu_j$ for $j = 1, \ldots, m_0+1$, then (S.11) holds with equality and the result follows as a special case of (S.13). □

### S.B.1.2 Proof of Theorem S.2

**Lemma S.1.** *Assume that $\{V_{t,T}\}$ satisfies Definition 1. Under Assumptions 1-2 and S.1-(ii),*

$$\sum_{j_1 \neq j_2} \frac{1}{T} \sum_{t=\lfloor T\lambda^0_{j_1-1} \rfloor+1}^{\lfloor T\lambda^0_{j_1} \rfloor} \sum_{s=\lfloor T\lambda^0_{j_2-1} \rfloor+1}^{\lfloor T\lambda^0_{j_2} \rfloor} \mathbb{E}\left(\left(V_t-\mu\left(t/T\right)\right)\left(V_s-\mu\left(s/T\right)\right)\exp\left(-i\omega_l\left(t-s\right)\right)=o\left(1\right).$$

*Proof.* Let $\overline{r}_{j_1,j_2}=\max\left\{r_{j_1},\, r_{j_2}\right\}$ and $\underline{r}_{j_1,j_2}=\min\left\{r_{j_1},\, r_{j_2}\right\}$. We consider the case of adjacent regimes (i.e., $j_2=j_1+1$) which also provides an upper bound for non-adjacent regimes due to the short memory property. For any $k=s-t=1,\ldots,\, \lfloor T\underline{r}_{j_1,j_2}\rfloor$ there are $k$ pairs in the above sum. The double sum above (over $t$ and $s$) can be split into

$$\begin{aligned} &T^{-1}\sum_{k=1}^{\lfloor CT^{\kappa}\rfloor}\left|\Gamma_{\{1:\lfloor CT^{\kappa}\rfloor\}}\left(\cdot,\, k\right)\right|+T^{-1}\sum_{k=\lfloor CT^{\kappa}\rfloor+1}^{\lfloor hT\rfloor}\left|\Gamma_{\{\lfloor CT^{\kappa}\rfloor+1:\lfloor hT\rfloor\}}\left(\cdot,\, k\right)\right| \\ &\quad+T^{-1}\sum_{k=\lfloor hT\rfloor+1}^{\lfloor T\underline{r}_{j_1,j_2}\rfloor-1}\left|\Gamma_{\left\{\lfloor hT\rfloor+1:\lfloor T\underline{r}_{j_1,j_2}\rfloor-1\right\}}\left(\cdot,\, k\right)\right|+T^{-1}\sum_{k=\lfloor T\underline{r}_{j_1,j_2}\rfloor}^{\lfloor T\overline{r}_{j_1,j_2}\rfloor}\left|\Gamma_{\left\{\underline{r}_{j_1,j_2}:\underline{r}_{j_1,j_2}\right\}}\left(\cdot,\, k\right)\right| \end{aligned} \tag{S.14}$$

where $C>0$, $0<h<1$ with $\lfloor hT\rfloor<\lfloor T\underline{r}_{j_1,j_2}\rfloor-1$, and $\Gamma_S\left(\cdot,\, k\right)$ is the sum of the autocovariances at lag $k$ computed at the time points corresponding to $k\in S$. Note that the term $\left|\exp\left(-i\omega_l\left(\pm k\right)\right)\right|$ can be bounded by some constant. The sums run over only $k>0$ because by symmetry $\Gamma_u\left(k\right)=\Gamma_{u-k/T}\left(-k\right)$. Consider the first sum in (S.14). This is of order $O\left(T^{-1}T^{2\kappa}\right)$ which goes to zero given $\kappa<1/2$. The second sum is also negligible using the following arguments. By Assumption S.1-(ii), $\left|\Gamma\left(u,\, k\right)\right|=C_{u,k}k^{-m}$ with $m>2$ and choosing $C$ large enough yields that the second sum of (S.14) converges to zero. In the third sum, the number of summands grows at rate $O\left(T\right)$ and for each lag $k$ there are $O\left(T\right)$ autocovariances. However, by Assumption S.1-(ii) each autocovariance is $O\left(T^{-m}\right)$. Thus, the bound is $O\left(T^{-1}T^{2-m}\right)$ which goes to zero as $T\rightarrow\infty$. The difference between the arguments used for the third sum and fourth sums is that now we do not have $O\left(T\right)$ autocovariances for each lag $k$. Thus, the bound for the fourth sum cannot be greater than the bound for the third sum. Thus, the fourth sum also converges to zero. □

*Proof of Theorem S.2.* We have,

$$\begin{aligned} I_T\left(\omega_l\right) &= \left|\frac{1}{\sqrt{T}}\sum_{j=1}^{m_0+1}\sum_{t=\lfloor T\lambda^0_{j-1}\rfloor+1}^{\lfloor T\lambda^0_j\rfloor}\exp\left(-i\omega_l t\right)V_t\right|^2 \\ &= \left|\frac{1}{\sqrt{T}}\sum_{j=1}^{m_0+1}\sum_{t=\lfloor T\lambda^0_{j-1}\rfloor+1}^{\lfloor T\lambda^0_j\rfloor}\left(X_t-\mu\left(t/T\right)\right)\exp\left(-i\omega_l t\right)+\frac{1}{\sqrt{T}}\sum_{j=1}^{m_0+1}\sum_{t=\lfloor T\lambda^0_{j-1}\rfloor+1}^{\lfloor T\lambda^0_j\rfloor}\mu\left(t/T\right)\exp\left(-i\omega_l t\right)\right|^2. \end{aligned}$$

From Assumption S.1,

$$\begin{aligned}
&\left|\sum_{j=1}^{m_0+1}\sum_{t=\lfloor T\lambda_{j-1}^0\rfloor+1}^{\lfloor T\lambda_j^0\rfloor}\mu\left(t/T\right)\exp\left(-i\omega_l t\right)\right|^2\\
&\qquad\geq\left|\sum_{j=1}^{m_0+1}B_j\sum_{t=\lfloor T\lambda_{j-1}^0\rfloor+1}^{\lfloor T\lambda_j^0\rfloor}\exp\left(-i\omega_l t\right)\right|^2\\
&\qquad=\left|\sum_{j=1}^{m_0+1}B_j\exp\left(-i\omega_l\left(\left\lfloor T\lambda_{j-1}^0\right\rfloor+1\right)\right)\sum_{t=0}^{\lfloor T\lambda_j^0\rfloor-\lfloor T\lambda_{j-1}^0\rfloor-1}\exp\left(-i\omega_l t\right)\right|^2\\
&\qquad=\left|\frac{\exp\left(-i\omega_l\right)}{1-\exp\left(-i\omega_l\right)}\sum_{j=1}^{m_0+1}B_j\exp\left(-i\omega_l\left(\left\lfloor T\lambda_{j-1}^0\right\rfloor\right)\right)\left(1-\exp\left(-i\omega_l\left(\left\lfloor T\lambda_j^0\right\rfloor-\left\lfloor T\lambda_{j-1}^0\right\rfloor\right)\right)\right)\right|^2\\
&\qquad=\left|\frac{\exp\left(-i\omega_l\right)}{1-\exp\left(-i\omega_l\right)}\sum_{j=1}^{m_0+1}B_j\left(\exp\left(-i\omega_l\left(\left\lfloor T\lambda_{j-1}^0\right\rfloor\right)\right)-\exp\left(-i\omega_l\left\lfloor T\lambda_j^0\right\rfloor\right)\right)\right|^2,
\end{aligned}$$

using the formula for the first $n$-th terms of a geometric series $\sum_{k=0}^{n-1}ar^k=a\sum_{k=0}^{n-1}r^k=a\left(1-r^n\right)/\left(1-r\right).$ Then, using summation by parts,

$$\begin{aligned}
&\frac{\exp\left(-i\omega_j\right)}{1-\exp\left(-i\omega_j\right)}\sum_{j=1}^{m_0+1}B_j\left(\exp\left(-i\omega_l\left(\left\lfloor T\lambda_{j-1}^0\right\rfloor\right)\right)-\exp\left(-i\omega_l\left\lfloor T\lambda_j^0\right\rfloor\right)\right)\\
&\qquad\qquad=\frac{\exp\left(-i\omega_j\right)}{1-\exp\left(-i\omega_j\right)}\left[B_1-B_{m_0+1}-\sum_{j=1}^{m_0}\left(B_j-B_{j+1}\right)\exp\left(-i\omega_l\left\lfloor T\lambda_j^0\right\rfloor\right)\right].
\end{aligned}$$

By Lemma S.1, it is sufficient to consider the cross-products within each regime $j$,

$$\begin{aligned}
\mathbb{E}\left(I_T\left(\omega_l\right)\right)\geq&\sum_{j=1}^{m_0+1}r_j\frac{1}{Tr_j}\mathbb{E}\sum_{t=\lfloor T\lambda_{j-1}^0\rfloor+1}^{\lfloor T\lambda_j^0\rfloor}\sum_{s=\lfloor T\lambda_{j-1}^0\rfloor+1}^{\lfloor T\lambda_j^0\rfloor}\left(V_t-\mu\left(t/T\right)\right)\left(V_s-\mu\left(s/T\right)\right)\exp\left(-i\omega_l\left(t-s\right)\right)\\
&+\sum\sum_{j_1\neq j_2}\frac{1}{T}\mathbb{E}\sum_{t=\lfloor T\lambda_{j_1-1}^0\rfloor+1}^{\lfloor T\lambda_{j_1}^0\rfloor}\sum_{s=\lfloor T\lambda_{j_2-1}^0\rfloor+1}^{\lfloor T\lambda_{j_2}^0\rfloor}\left(V_t-\mu\left(t/T\right)\right)\left(V_s-\mu\left(s/T\right)\right)\exp\left(-i\omega_l\left(t-s\right)\right)\\
&+\left|\frac{1}{\sqrt{T}}\frac{\exp\left(-i\omega_l\right)}{1-\exp\left(-i\omega_l\right)}\sum_{j=1}^{m_0+1}B_j\left(\exp\left(-i\omega_l\left(\left\lfloor T\lambda_{j-1}^0\right\rfloor\right)\right)-\exp\left(-i\omega_l\left\lfloor T\lambda_j^0\right\rfloor\right)\right)\right|^2+o\left(1\right)\\
=&\sum_{j=1}^{m_0+1}\left(\mathbb{E}\frac{1}{T}\sum_{t=\lfloor T\lambda_{j-1}^0\rfloor+1}^{\lfloor T\lambda_j^0\rfloor}\left(V_t-\mu\left(t/T\right)\right)^2+\frac{2}{Tr_j}\sum_{k=1}^{\lfloor Tr_j\rfloor-1}\sum_{t=\lfloor T\lambda_{j-1}^0\rfloor+k+1}^{\lfloor T\lambda_j^0\rfloor}\Gamma_{t/T}\left(k\right)\exp\left(-i\omega_l k\right)\right)
\end{aligned}$$

$$+\left|\frac{1}{\sqrt{T}}\frac{\exp\left(-i\omega_l\right)}{1-\exp\left(-i\omega_l\right)}\sum_{j=1}^{m_0+1}B_j\left(\exp\left(-i\omega_l\left(\left\lfloor T\lambda_{j-1}^0\right\rfloor\right)\right)-\exp\left(-i\omega_l\left\lfloor T\lambda_j^0\right\rfloor\right)\right)\right|^2+o\left(1\right).$$

Next, using the definition of $f\left(u,\,\omega_l\right)$, $e^{-2i\omega_l}=1$ by Euler's formula and letting $\omega_l\rightarrow 0$ we have,

$$\begin{aligned}\mathbb{E}\left(I_T\left(\omega_l\right)\right)\geq&\sum_{j=1}^{m_0+1}\left(\int_{\lambda_{j-1}^0}^{\lambda_j^0}c\left(u,\,0\right)du+2\sum_{k=1}^{\infty}\int_{\lambda_{j-1}^0}^{\lambda_j^0}c\left(u,\,k\right)\exp\left(-i\omega_l k\right)du\right)\\&+\frac{1}{T}\frac{1}{\left|1-\exp\left(-i\omega_l\right)\right|^2}\left|\left[B_1-B_{m_0+1}-\left(1+o\left(1\right)\right)\sum_{j=1}^{m_0}\left(B_j-B_{j+1}\right)\exp\left(-2\pi i l\lambda_j^0\right)\right]\right|^2+o\left(1\right)\\=&\,2\pi\sum_{j=1}^{m_0+1}\int_{\lambda_{j-1}^0}^{\lambda_j^0}f\left(u,\,\omega_l\right)du\\&+\frac{1}{T}\frac{1}{\left|1-\exp\left(-i\omega_l\right)\right|^2}\left|\left[B_1-B_{m_0+1}-\left(1+o\left(1\right)\right)\sum_{j=1}^{m_0}\left(B_j-B_{j+1}\right)\exp\left(-2\pi i l\lambda_j^0\right)\right]\right|^2+o\left(1\right)\\=&\,2\pi\int_0^1f\left(u,\,\omega_l\right)du+\frac{1}{T\omega_l^2}\left|\left[B_1-B_{m_0+1}-\sum_{j=1}^{m_0}\left(B_j-B_{j+1}\right)\exp\left(-2\pi i l\lambda_j^0\right)\right]\right|^2+o\left(1\right).\end{aligned}\tag{S.15}$$

By Assumption 1-(ii), the first term of (S.15) is bounded for all frequencies $\omega_j$. Since $B_1,\ldots,\,B_{m_0+1}$ are fixed, if $T\omega_l^2\rightarrow 0$ then the order of the second term of (S.15) is $O((T\omega_l^2)^{-1})$. Note that as $\omega_l\rightarrow 0$ there are some values of $l$ for which the corresponding term involving $|\cdot|^2$ on the right-hand side of (S.15) is equal to zero [see the argument in Mikosch and Stărica (2004)]. In such a case, $\mathbb{E}\left(I_T\left(\omega_l\right)\right)\geq 2\pi\int_0^1f\left(u,\,\omega_l\right)du>0$. For the other values of $\{l\}$ as $\omega_l\rightarrow 0$, the second term of (S.15) diverges to infinity. The outcome is that there are frequencies close to $\omega_l=0$ for which $\mathbb{E}\left(I_T\left(\omega_l\right)\right)\rightarrow\infty$. □

### S.B.1.3 Proof of Theorem 1

We consider the case $k\geq 0$. The case $k<0$ follows similarly. Consider any $u\in(0,\,1)$ such that $T_j^0\notin\mathbf{S}\left(u,\,k,\,n_{2,T}\right)$ for all $j=1,\ldots,\,m_0$. Theorem S.B.3 in Casini (2023) showed that

$$\mathbb{E}\left[\widehat{c}_T\left(u,\,k\right)\right]=c\left(u_0,\,k\right)+\frac{1}{2}\left(n_{2,T}/T\right)^2\left[\frac{\partial^2}{\partial^2u}c\left(u,\,k\right)\right]+o\left(\left(n_{2,T}/T\right)^2\right)+O\left(1/n_{2,T}\right).\tag{S.16}$$

Since $n_{2,T}\rightarrow\infty$ and $n_{2,T}/T\rightarrow 0$, $\mathbb{E}\left[\widehat{c}_T\left(u,\,k\right)\right]=c\left(u_0,\,k\right)+o\left(1\right)$. The same aforementioned theorem shows that $n_{2,T}\mathrm{Var}\left[\widehat{c}_T\left(u,\,k\right)\right]=O_{\mathbb{P}}\left(1\right)$. This combined with (S.16) yields part (i) of the theorem.

Next, we consider case (ii-a) with $n_{j,L}\left(u,\,k,\,n_{2,T}\right)/n_{2,T}\rightarrow\gamma\in(0,\,1)$. We have,

$$\widehat{c}_T\left(u,\,k\right)=n_{2,T}^{-1}\sum_{s=0}^{n_{2,T}}V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1}V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1-k}-\left(n_{2,T}^{-1}\sum_{s=0}^{n_{2,T}}V_{\lfloor Tu\rfloor-n_{2,T}/2+s+1}\right)^2$$

$$
\begin{aligned}
&= n_{2,T}^{-1} \sum_{s=0}^{T_j^0-(\lfloor Tu\rfloor+k/2-n_{2,T}/2+1)} V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1} V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1-k} \\
&\quad + n_{2,T}^{-1} \sum_{s=T_j^0-(\lfloor Tu\rfloor+k/2-n_{2,T}/2)}^{n_{2,T}} V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1} V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1-k} \\
&\quad - \Bigg( n_{2,T}^{-1} \sum_{s=0}^{T_j^0-(\lfloor Tu\rfloor+k/2-n_{2,T}/2+1)} V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1} \\
&\quad + n_{2,T}^{-1} \sum_{s=T_j^0-(\lfloor Tu\rfloor+k/2-n_{2,T}/2)}^{n_{2,T}} V_{\lfloor Tu\rfloor-n_{2,T}/2+s+1} \Bigg)^2 \\
&= n_{2,T}^{-1} \sum_{s=0}^{T_j^0-(\lfloor Tu\rfloor+k/2-n_{2,T}/2+1)} \Big( V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1} V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1-k} \\
&\quad - \mathbb{E}\left(V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1}\right) \mathbb{E}\left(V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1-k}\right)\Big) \\
&\quad + n_{2,T}^{-1} \sum_{s=T_j^0-(\lfloor Tu\rfloor+k/2-n_{2,T}/2)}^{n_{2,T}} \Big( V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1} V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1-k} \\
&\quad - \mathbb{E}\left(V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1}\right) \mathbb{E}\left(V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1-k}\right)\Big) \\
&\quad + n_{2,T}^{-1} \sum_{s=0}^{T_j^0-(\lfloor Tu\rfloor+k/2-n_{2,T}/2+1)} \mathbb{E}\left(V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1}\right) \mathbb{E}\left(V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1-k}\right) \\
&\quad + n_{2,T}^{-1} \sum_{s=T_j^0-(\lfloor Tu\rfloor+k/2-n_{2,T}/2)}^{n_{2,T}} \mathbb{E}\left(V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1}\right) \mathbb{E}\left(V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1-k}\right) \\
&\quad - \Bigg( n_{2,T}^{-1} \sum_{s=0}^{T_j^0-(\lfloor Tu\rfloor+k/2-n_{2,T}/2+1)} V_{\lfloor Tu\rfloor-n_{2,T}/2+s+1} \\
&\quad + n_{2,T}^{-1} \sum_{s=T_j^0-(\lfloor Tu\rfloor+k/2-n_{2,T}/2)}^{n_{2,T}} V_{\lfloor Tu\rfloor-n_{2,T}/2+s+1} \Bigg)^2 + o_{\mathbb{P}}(1) \qquad \text{(S.17)} \\
&\geq \gamma c\left(\lambda_j^0, k\right) + (1-\gamma)\, c(u, k) + \gamma \mu_j \left(\lambda_j^0\right)^2 + (1-\gamma)\, \mu_{j+1}(u)^2 \\
&\quad - \left(\gamma \mu_j \left(\lambda_j^0\right) + (1-\gamma)\, \mu_{j+1}(u)\right)^2 + o_{\mathbb{P}}(1) \\
&= \gamma c\left(\lambda_j^0, k\right) + (1-\gamma)\, c(u, k) + \gamma(1-\gamma)\left(\mu_j\left(\lambda_j^0\right) - \mu_{j+1}(u)\right)^2 + o_{\mathbb{P}}(1). \qquad \text{(S.18)}
\end{aligned}
$$

Consider the case (ii-b) with $n_{j,L}(u, k, n_{2,T})/n_{2,T} \to 0$. The other sub-case follows by symmetry. Eq. (S.17) continues to hold. The first term, third term and the first summation of the last term on the right-hand side of (S.17) are negligible. Thus, using ergodicity, implied by Assumptions 2-2-(i),

$$\begin{aligned}\widehat{c}_T(u,k) &= c(u,k) + n_{2,T}^{-1}\sum_{s=T_j^0-(\lfloor Tu\rfloor+k/2-n_{2,T}/2)}^{n_{2,T}} \mathbb{E}\left(V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1}\right)\mathbb{E}\left(V_{\lfloor Tu\rfloor+k/2-n_{2,T}/2+s+1}\right)\\ &\quad -\mu_{j+1}(u)^2+o_{\mathbb{P}}(1)\\ &= c(u,k)+\mu_{j+1}(u)^2-\mu_{j+1}(u)^2+o_{\mathbb{P}}(1) = c(u,k)+o_{\mathbb{P}}(1),\end{aligned}$$

where we have used the smoothness of $\mathbb{E}(V_t)$ implied by local stationarity. The second claim of the lemma follows from Assumption 2-(i) since this implies that $\sup_{u\in[0,1]} c(u,k)\to 0$ as $k\to\infty$ and the fact that the third term on the right-hand side of (S.18) does not depend on $k$. Thus, $\widehat{\Gamma}_{\mathrm{DK}}(k)\geq d_T^*+o_{\mathbb{P}}(1)$ where $d_T^*=(n_{2,T}/T)\,\gamma(1-\gamma)\left(\mu_j\left(\lambda_j^0\right)-\mu_{j+1}(u)\right)^2>0$ and $d_T^*\to 0$ since $n_{2,T}/T\to 0$. The factor $n_{2,T}/T$ in $d_T^*$ follows because the neighborhood $(\lambda_j^0-n_{2,T}/T,\,\lambda_j^0+n_{2,T}/T)$ includes $O(n_{2,T}/n_T)$ blocks which are then averaged out. □

### S.B.1.4 Proof of Theorem 2

Consider first any $u\in(0,1)$ such that $T_j^0\notin \mathbf{S}(u,0,n_T)$ for all $j=1,\ldots,m_0$. Theorem 3.3 in Casini and Perron (2024) shows that

$$\begin{aligned}\mathbb{E}\left(I_{\mathrm{L},T}(u,\omega_l)\right) &= \left|\frac{1}{\sqrt{n_T}}\sum_{s=0}^{n_T-1}V_{\lfloor Tu\rfloor-n_T/2+s+1,T}\exp(-i\omega_l s)\right|^2\\ &= f(u,\omega_l)+\frac{1}{6}\left(\frac{n_T}{T}\right)^2\frac{\partial^2}{\partial u^2}f(u,\omega_l)+o\left(\left(\frac{n_T}{T}\right)^2\right)+O\left(\frac{\log(n_T)}{n_T}\right).\end{aligned}\tag{S.19}$$

By Assumption 1 the absolute value of the first term on the right-hand side is bounded for all frequencies $\omega_l$. By Assumption 3-(iii) $\left|(\partial^2/\partial u^2)f(u,\omega_l)\right|$ is bounded and, since $n_T/T\to 0$, the second term converges to zero. Similarly, the third and fourth terms are negligible. Thus, $\mathbb{E}\left(I_{\mathrm{L},T}(u,\omega_l)\right)$ is bounded below by $f(u,\omega_l)>0$ as $\omega_l\to 0$ which establishes part (i). Now we consider part (ii). We begin with case (a). We only focus on the sub-case $n_{j,L}(u,0,n_T)/n_T\to\gamma$ with $\gamma\in(0,1)$. We have

$$\begin{aligned}I_{\mathrm{L},T}(\omega_l) &=\\ &\left|\frac{1}{\sqrt{n_T}}\left(\sum_{s=0}^{T_j^0-(\lfloor Tu\rfloor-n_T/2+1)}V_{\lfloor Tu\rfloor-n_T/2+s+1,T}\exp(-i\omega_l s)+\sum_{s=T_j^0-(\lfloor Tu\rfloor-n_T/2)}^{n_T-1}V_{\lfloor Tu\rfloor-n_T/2+s+1,T}\exp(-i\omega_l s)\right|^2\right.\\ &=\frac{1}{n_T}\left|\sum_{s=0}^{T_j^0-(\lfloor Tu\rfloor-n_T/2+1)}\left(V_{\lfloor Tu\rfloor-n_T/2+s+1,T}-\mu\left(\left(\lfloor Tu\rfloor-n_T/2+s+1\right)/T\right)\right)\exp(-i\omega_l s)\right.\\ &\quad+\sum_{s=T_j^0-(\lfloor Tu\rfloor-n_T/2)}^{n_T-1}\left(V_{\lfloor Tu\rfloor-n_T/2+s+1,T}-\mu\left(\left(\lfloor Tu\rfloor-n_T/2+s+1\right)/T\right)\right)\exp(-i\omega_l s)\\ &\quad\left.+\sum_{s=0}^{n_T-1}\mu\left(\left(\lfloor Tu\rfloor-n_T/2+s+1\right)/T\right)\exp(-i\omega_l s)\right|^2.\end{aligned}\tag{S.20}$$

Using Assumption 3, we have

$$\left|\sum_{s=0}^{n_T-1}\mu\left(\left(\lfloor Tu\rfloor-n_T/2+s+1\right)/T\right)\exp\left(-i\omega_l s\right)\right|^2\geq$$
$$\left|B_j\sum_{s=0}^{T_j^0-(\lfloor Tu\rfloor-n_T/2+1)}\exp\left(-i\omega_l s\right)+B_{j+1}\sum_{s=T_j^0-(\lfloor Tu\rfloor-n_T/2)}^{n_T-1}\exp\left(-i\omega_l s\right)\right|^2. \tag{S.21}$$

Note that

$$\begin{aligned}&B_j\sum_{s=0}^{T_j^0-(\lfloor Tu\rfloor-n_T/2+1)}\exp\left(-i\omega_l s\right)+B_{j+1}\sum_{s=T_j^0-(\lfloor Tu\rfloor-n_T/2)}^{n_T-1}\exp\left(-i\omega_l s\right)\\&\quad=B_j\sum_{s=0}^{T_j^0-(\lfloor Tu\rfloor-n_T/2+1)}\exp\left(-i\omega_l s\right)\\&\quad\quad+B_{j+1}\exp\left(-i\omega_l\left(T_j^0-\left(\lfloor Tu\rfloor-n_T/2\right)\right)\right)\sum_{s=0}^{n_T-1-\left(T_j^0-(\lfloor Tu\rfloor-n_T/2)\right)}\exp\left(-i\omega_l s\right).\end{aligned} \tag{S.22}$$

Focusing on the second term on the right-hand side above,

$$\begin{aligned}&n_T^{-1}\left|B_{j+1}\sum_{s=T_j^0-(\lfloor Tu\rfloor-n_T/2)}^{n_T-1}\exp\left(-i\omega_l s\right)\right|^2\\&=n_T^{-1}\left|B_{j+1}\exp\left(-i\omega_l\left(T_j^0-\left(\lfloor Tu\rfloor-n_T/2\right)\right)\right)\sum_{s=0}^{n_T-1-\left(T_j^0-(\lfloor Tu\rfloor-n_T/2)\right)}\exp\left(-i\omega_l s\right)\right|^2\\&=n_T^{-1}\left|B_{j+1}\exp\left(-i\omega_l\left(T_j^0-\left(\lfloor Tu\rfloor-n_T/2\right)\right)\right)\frac{1-\exp\left(-i\omega_l\left(n_T-\left(T_j^0-\left(\lfloor Tu\rfloor-n_T/2\right)\right)\right)\right)}{1-\exp\left(-i\omega_l\right)}\right|^2\\&=n_T^{-1}\left|B_{j+1}\frac{\exp\left(-i\omega_l\left(T_j^0-\left(\lfloor Tu\rfloor-n_T/2\right)\right)\right)-\exp\left(-i\omega_l n_T\right)}{1-\exp\left(-i\omega_l\right)}\right|^2.\end{aligned} \tag{S.23}$$

We show that the above equation diverges to infinity as $\omega_l\to 0$ with $n_T\omega_l^2\to 0$. If $n_T\omega_l\to a\in(0,\infty)$ then $\mathrm{Re}\left(\exp\left(-i\omega_l n_T\right)\right)\neq 1$ and the order is determined by the denominator. As in the proof of Theorem S.2, $|1-\exp(-i\omega_l)|^2=\omega_l^2$. Since $n_T\omega_l^2\to 0$, the right-hand side above diverges. If $n_T\omega_l\to 0$, we apply L'Hôpital's rule to obtain

$$n_T^{-1}\left|B_{j+1}\frac{-i\left(T_j^0-\left(\lfloor Tu\rfloor-n_T/2\right)\right)+in_T}{i}\right|^2$$

$$
\begin{aligned}
&= n_T^{-1} B_{j+1}^2 \left( - \left( T_j^0 - \left( \lfloor Tu \rfloor - n_T/2 \right) \right)^2 + n_T^2 - \left( T_j^0 - \left( \lfloor Tu \rfloor - n_T/2 \right) \right) n_T \right) \\
&= O\left( n_T^2 / n_T \right) = O\left( n_T \right),
\end{aligned}
$$

which shows that the right-hand side of (S.23) diverges. A similar argument can be applied to the first term on the right-hand side of (S.22) and to the product of the latter term and the complex conjugate of the second term on the right-hand side of (S.22).

It remains to consider case (b) and the sub-case $n_{j,L}\left(u,\, 0,\, n_T\right)/n_T \to 0$. The other sub-case follows by symmetry. We have (S.20) and (S.21). Note that,

$$
\begin{aligned}
&\left| \frac{1}{\sqrt{n_T}} B_{j+1} \sum_{s=T_j^0-(\lfloor Tu \rfloor - n_T/2)}^{n_T-1} \exp\left(-i\omega_l s\right) \right|^2 \\
&= \left| \frac{1}{\sqrt{n_T}} B_{j+1} \sum_{s=0}^{n_T-1} \exp\left(-i\omega_l s\right) - \frac{1}{\sqrt{n_T}} B_{j+1} \sum_{s=0}^{T_j^0-(\lfloor Tu \rfloor - n_T/2)-1} \exp\left(-i\omega_l s\right) \right|^2 \\
&= \left| - \frac{1}{\sqrt{n_T}} B_{j+1} \sum_{s=0}^{T_j^0-(\lfloor Tu \rfloor - n_T/2)-1} \exp\left(-i\omega_l s\right) \right|^2 \to 0.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\mathbb{E}\left(I_{\mathrm{LT}}\left(\omega_l\right)\right) = \frac{1}{n_T} \Bigg| &\left( \sum_{s=0}^{T_j^0-(\lfloor Tu \rfloor - n_T/2+1)} \left( V_{\lfloor Tu \rfloor - n_T/2+s+1,T} - \mu\left(\left(\lfloor Tu \rfloor - n_T/2 + s + 1\right)/T\right) \right) \exp\left(-i\omega_l s\right) \right) \\
&+ \sum_{s=T_j^0-(\lfloor Tu \rfloor - n_T/2)}^{n_T-1} \left( V_{\lfloor Tu \rfloor - n_T/2+s+1,T} - \mu\left(\left(\lfloor Tu \rfloor - n_T/2 + s + 1\right)/T\right) \right) \exp\left(-i\omega_l s\right) \Bigg|^2 + o\left(1\right).
\end{aligned}
$$

Note that the first sum above involves at most $C < \infty$ summands. So the first term is negligible. The expectation of the product of the first term and the conjugate of the second term is negligible by using arguments similar to the proof in Lemma S.1 with $n_T$ in place of $T$. Thus, the limit of $\mathbb{E}\left(I_T\left(\omega_l\right)\right)$ is equal to the right-hand side of (S.19) plus additional $o\left(1\right)$ terms. □

## S.B.2 Proofs of the Results in Section 4

We first introduce the multiple Fejér kernel as in Velasco and Robinson (2001),

$$
\Psi_T^{(n)}\left(x_1, \ldots, x_n\right) = \frac{1}{\left(2\pi\right)^{n-1} T} \sum_{t_1 \cdots t_n = 1}^{T} \exp\left\{ i \sum_{j=1}^{n} t_j x_j \right\},
$$

with $x_n = -\sum_{j=1}^{n-1} x_j$. Velasco and Robinson (2001) discussed the following properties. $\Psi_T^{(n)}(x_1, \ldots, x_n)$ is integrable in $\Pi^{n-1}$ and integrates to one for all $T$. For $\delta > 0$ and $T \geq 1$, we have

$$\int_{\mathbf{D}^c} \left| \Psi_T^{(n)}(x_1, \ldots, x_n) \right| dx_1 \ldots dx_{n-1} = O\left( \frac{\log^{n-1} T}{T \sin \delta/2} \right), \tag{S.24}$$

where $\mathbf{D}^c$ is the complement in $\Pi^{n-1}$ of the set $\mathbf{D} = \{x \in \Pi^{n-1} : |x_j| \leq \delta,\, j = 1, \ldots, n-1\}$. For $j = 1, \ldots, n-1$,

$$\int_\Pi \cdots \int_\Pi |x_j| |\Psi_T^{(n)}(x_1, \ldots, x_n)| dx_1 \cdots dx_n = O\left( T^{-1} \log^{n-1} T \right). \tag{S.25}$$

Recall that the Dirichlet kernel is defined as $D_T(x) = \sum_{t=1}^T \exp(itx)$. It satisfies the following two relations,

$$|D_T(x)| \leq \min\left\{T,\, 2|x|^{-1}\right\}; \qquad \int_\Pi |D_T(x)| dx = O(\log T). \tag{S.26}$$

Eq. (S.24)-(S.25) follow from

$$\left| \Psi_T^{(n)}(x_1, \ldots, x_n) \right| \leq \frac{1}{(2\pi)^{n-1} T} |D_T(x_1)| |D_T(x_2)| \cdots |D_T(x_n)| dx_1 \cdots dx_n. \tag{S.27}$$

### S.B.2.1 Preliminary Lemmas

**Lemma S.2.** *(Bhattacharya and Rao, 1975, pp. 97-98, 113). Let $\mathbb{Q}_1$ and $\mathbb{Q}_2$ be probability measures on $\mathbb{R}^2$ and $\mathscr{B}^2$ the class of all Borel subsets of $\mathbb{R}^2$. Let $\phi$ be a positive number. Then there exists a kernel probability measure $\mathbb{G}_\phi$ such that*

$$\sup_{\mathbf{B} \in \mathscr{B}^2} |\mathbb{Q}_1(\mathbf{B}) - \mathbb{Q}_2(\mathbf{B})| \leq \frac{2}{3} \|(\mathbb{Q}_1 - \mathbb{Q}_2) \bullet \mathbb{G}_\phi\| + \frac{4}{3} \sup_{\mathbf{B} \in \mathscr{B}^2} \mathbb{Q}_2\left( (\partial \mathbf{B})^{2\phi} \right),$$

*where $\mathbb{G}_\phi$ satisfies*

$$\mathbb{G}_\phi(\mathbf{B}(0, r)^c) = O\left( \left( \frac{\phi}{r} \right)^3 \right), \tag{S.28}$$

*and its Fourier transform $\widehat{\mathbb{G}}_\phi$ satisfies*

$$\widehat{\mathbb{G}}_\phi(\mathbf{t}) = 0 \qquad \text{for} \quad \|\mathbf{t}\| \geq 8 \times 2^{4/3}/\pi^{1/3} \phi. \tag{S.29}$$

*Here $(\partial \mathbf{B})^{2\phi}$ is a neighborhood of radius $2\phi$ of the boundary of $\mathbf{B}$, $\|\cdot\|$ is the variation norm, and $\bullet$ means convolution.*

**Lemma S.3.** *Let Assumptions 4, 6-7 hold. For $s \geq 2$ with $\epsilon_T(2s) \to 0$, we have*

$$\operatorname{Tr}\left( (\Sigma_V W_{b_1})^s \right) = T (2\pi)^{2s-1} \sum_{j=0}^{d_f} L_j(s)\, b_{1,T}^{1+j-s} + O\left( T b_{1,T}^{1-s} \epsilon_T(2s) \right),$$

*where* $\epsilon_T(2s) = (Tb_{1,T})^{-1} \log^{2s-1} T,\ L_j(s) = (1/j)!\mu_j(K^s)(d^j/d\omega^j)(f(u,0)\,du)^s$ *with* $|L_j(s)| < \infty$ *and* $L_j(s)$ *differs from zero only for* $j$ *even* $(j = 0, \ldots, d_f)$.

*Proof of Lemma S.3.* Let $r_{2s+1} = r_1$ and note that

$$\begin{aligned}
&\operatorname{Tr}((\Sigma_V W_{b_1})^s) \\
&\quad = \sum_{1 \le r_1, \ldots, r_{2s} \le T} \prod_{j=1}^{s} \mathbb{E}\left(V_{r_{2j-1}} V_{r_{2j}}\right) w(b_{1,T}(r_{2j} - r_{2j+1})) \\
&\quad = \sum_{1 \le r_1, \ldots, r_{2s} \le T} \prod_{j=1}^{s} \int_{\Pi} f(r_{2j-1}/T, \omega_{2j-1}) e^{i(r_{2j-1} - r_{2j})\omega_{2j-1}} \int_{\Pi} \widetilde{K}_{b_1}(\omega_{2j}) e^{i(r_{2j} - r_{2j+1})\omega_{2j}} d\omega \\
&\quad = \sum_{k_2, k_4, \ldots, k_{2s} = -T+1}^{T-1} \sum_{r_1 = |k_2|+1}^{T} \sum_{r_3 = |k_4|+1}^{T} \cdots \sum_{r_{2s-1} = |k_{2s}|+1}^{T} \prod_{j=1}^{s} \int_{\Pi} f(r_{2j-1}/T, \omega_{2j-1}) e^{ik_{2j}(\omega_{2j-1} - \omega_{2j})} \\
&\quad \times \int_{\Pi} \widetilde{K}_{b_1}(\omega_{2j}) e^{i((-k_{2j} - k_{2j+2})\omega_{2j})} d\omega \\
&\quad = \sum_{k_2, k_4, \ldots, k_{2s} = -T+1}^{T-1} \prod_{j=1}^{s} (T - |k_{2j}|) \int_{\Pi} \int_0^1 f(u_{2j-1}, \omega_{2j-1}) e^{ik_{2j}(\omega_{2j-1} - \omega_{2j})} \\
&\quad \times \int_{\Pi} \widetilde{K}_{b_1}(\omega_{2j}) e^{i((-k_{2j} - k_{2j+2})\omega_{2j})} du d\omega + O\left(T^{-1}\right) \\
&\quad = \sum_{1 \le r_1, \ldots, r_{2s} \le T} \prod_{j=1}^{s} (T - |k_{2j}|) \int_{\Pi} \int_0^1 f(u_{2j-1}, \omega_{2j-1}) \int_{\Pi} \widetilde{K}_{b_1}(\omega_{2j}) \exp\left\{ i \sum_{j=1}^{2s} \omega_j (r_j - r_{j+1}) \right\} du d\omega + O\left(T^{-1}\right) \\
&\quad = T(2\pi)^{2s-1} \int_{\Pi^{2s}} H_{b_1}(\omega, \mu) \widetilde{K}_{b_1}(\omega) \Psi_T^{(2s)}(\mu) d\omega d\mu + O\left(T^{-1}\right),
\end{aligned} \tag{S.30}$$

where $\Psi_T^{(2s)}(\mu) = \Psi_T^{(2s)}(\mu_1, \ldots, \mu_{2s})$,

$$\begin{aligned}
H_{b_1}(\omega, \mu) = & \int_0^1 \cdots \int_0^1 f(u_1, \omega - \mu_2 - \ldots - \mu_{2s}) \widetilde{K}_{b_1}(\omega - \mu_3 - \ldots - \mu_{2s}) \\
& \times f(u_3, \omega - \ldots - \mu_{2s}) \widetilde{K}_{b_1}(\omega - \mu_4 - \ldots - \mu_{2s}) \ldots f(u_{2s-1}, \omega - \mu_{2s}) du,
\end{aligned}$$

$d\mu = d\mu_2, \ldots, d\mu_{2s}$, $d\omega = d\omega_1, \ldots, \omega_{2s}$, $du = du_1, du_3, \ldots, du_{2s-1}$, and we have made the change in variables

$$\begin{cases} \mu_1 = \omega_1 - \omega_2 \\ \mu_2 = \omega_2 - \omega_1 \\ \cdots \\ \mu_{2s} = \omega_{2s} - \omega_{2s-1} \end{cases} \qquad \begin{cases} \omega_{2s-1} = \omega - \mu_{2s} \\ \omega_{2s-2} = \omega - \mu_{2s} - \mu_{2s-1} \\ \cdots \\ \omega_1 = \omega - \mu_{2s} - \ldots - \mu_s = \omega - \mu_1 \end{cases}$$

with $\sum_{j=1}^{2s} \mu_j = 0$, setting $\omega = \omega_{2s}$, and expressing all the $\omega_j$ in terms of $\omega$ and $\mu_j$, $j = 2, \ldots, 2s$.

Let

$$B = \left| \operatorname{Tr}((\Sigma_V W_{b_1})^s) - T(2\pi)^{2s-1} \int_{\Pi} \left( \int_0^1 f(u, \omega) du \right)^s \widetilde{K}_{b_1}^{s-1}(\omega) d\omega \right|.$$

Using (S.30) we have

$$B \leq T\left(2\pi\right)^{2s-1}\int_{\Pi^{2s}}\left|H_{b_1}\left(\omega,\,\mu\right)-\left(\int_0^1 f\left(u,\,\omega\right)du\right)^s\widetilde{K}_{b_1}^{s-1}\left(\omega\right)\right|\left|\widetilde{K}_{b_1}\left(\omega\right)\Psi_T^{(2s)}\left(\mu\right)\right|d\omega d\mu+O\left(T^{-1}\right). \tag{S.31}$$

We split the integral in (S.31) into two sets, for small and for large $\mu_j$. Define the set $\mathbf{M}=\{\mu\in\Pi^{2s-1}:\sup_j|\mu_j|\leq b_{1,T}/\left(2s\right)\}$. Since $K\left(\omega\right)$ takes small values for $|\omega|>\pi b_{1,T}$, for all $u$ all functions $f\left(u,\,\omega\right)$ are boundedly differentiable in $\omega$ in the set $\mathbf{M}$. We use the following inequality,

$$|A_1\cdots A_r-B_1\cdots B_r|\leq\sum_{q=0}^{r-1}|B_1\cdots B_q|\,|B_{q+1}-A_{q+1}|\,|A_{q+2}\cdots A_r|\,, \tag{S.32}$$

and $\sup_\omega|\widetilde{K}_{b_1}\left(\omega\right)|=O(b_{1,T}^{-1})$ to bound the integral in (S.31) over $\mathbf{M}$ by

$$O\left(Tb_{1,T}^{-s+1}\right)\sum_{q=0}^{s-1}\int_\Pi\int_{\mathbf{M}}\int_0^1|f\left(u_{2q+1},\,\omega-\mu_{2+2q}-\ldots-\mu_{2s}\right)-f\left(u_{2q+1},\,\omega\right)|\left|\widetilde{K}_{b_1}\left(\omega\right)\Psi_T^{(2s)}\left(\mu\right)\right|du_{2q+1}d\mu d\omega \tag{S.33}$$

$$+O\left(Tb_{1,T}^{-s+1}\right)\sum_{q=0}^{s-2}\int_\Pi\int_{\mathbf{M}}\left|\widetilde{K}_{b_1}\left(\omega-\mu_{3+2q}-\ldots-\mu_{2s}\right)-\widetilde{K}_{b_1}\left(\omega\right)\right|\left|\Psi_T^{(2s)}\left(\mu\right)\right|d\mu d\omega. \tag{S.34}$$

We apply the mean value theorem in (S.33) to yield,

$$\begin{aligned}&O\left(Tb_{1,T}^{1-s}\right)\int_\Pi\left|\widetilde{K}_{b_1}\left(\omega\right)\right|d\omega\sum_{q=0}^{2s}\int_{\mathbf{M}}|\mu_q||\Psi_T^{(2s)}\left(\mu\right)|d\mu\\&\qquad\leq O\left(Tb_{1,T}^{1-s}\right)\int_\Pi\left|\widetilde{K}_{b_1}\left(\omega\right)\right|d\omega\sum_{q=0}^{2s}\int_{\Pi^{2s-1}}|\mu_q||\Psi_T^{(2s)}\left(\mu\right)|d\mu\\&\qquad=O\left(b_{1,T}^{1-s}\log^{2s-1}T\right),\end{aligned}$$

where the equality follows from (S.25). Using the Lipschitz property of $K$ (cf. Assumption 7), the expression in (S.34) is of order $O(b_{1,T}^{-s}\log^{2s-1}T)$.

Let $\mathbf{M}^c$ denote the complement of $\mathbf{M}$ in $\Pi^{2s-1}$. We now study the contribution to $B$ corresponding to the set $\mathbf{M}^c$. This is bounded by

$$T\left(2\pi\right)^{2s-1}\int_\Pi\int_{\mathbf{M}^c}\left|H_{b_1}\left(\omega,\,\mu\right)\widetilde{K}_{b_1}\left(\omega\right)\right|\left|\Psi_T^{(2s)}\left(\mu\right)\right|d\omega d\mu \tag{S.35}$$

$$+\,T\left(2\pi\right)^{2s-1}\int_\Pi\left|\left(\int_0^1 f\left(u,\,\omega\right)du\right)^s\widetilde{K}_{b_1}^s\left(\omega\right)\right|d\omega\int_{\mathbf{M}^c}\left|\Psi_T^{(2s)}\left(\mu\right)\right|d\mu. \tag{S.36}$$

The expression in (S.36) is $O(b_{1,T}^{-s}\log^{2s-1}T)$ using (S.24) and

$$\int_{\Pi}\left|\left(\int_0^1 f(u,\omega)du\right)^s \widetilde{K}_{b_1}^s(\omega)\right|d\omega = O\left(b_{1,T}^{-s}\right).$$

Applying (S.27) the expression in (S.35) is bounded by

$$\int_{\mathbf{M}'}\prod_{j=1}^{s}\int_0^1\left|f\left(u_{2j-1},\omega_{2j-1}\right)\widetilde{K}_{b_1}\left(\omega_{2j}\right)D_T\left(\omega_{2j}-\omega_{2j-1}\right)D_T\left(\omega_{2j+1}-\omega_{2j}\right)\right|du_{2j-1}d\omega_{2j}d\omega_{2j-1}, \tag{S.37}$$

where $\mathbf{M}' = \{|\omega_2-\omega_1|>\nu_T\}\cup\{|\omega_3-\omega_2|>\nu_T\}\cup\ldots\cup\{|\omega_{2s}-\omega_{2s-1}|>\nu_T\}$ with $\nu_T = b_{1,T}/(2s)$ and $2s+1$ is to be interpreted as 1. Note that the integral in (S.37) differs from zero only if $|\omega_2|, |\omega_4|, \ldots, |\omega_{2s}| \leq b_{1,T}\pi$. Without loss of generality, we consider only the case where just one of the events in $\mathbf{M}'$ is satisfied, $|\omega_{2j}-\omega_{2j-1}|>\nu_T$, say, the other cases can be handled similarly.

From (S.26) it follows that $|D_T(\omega_{2j}-\omega_{2j-1})| = O(b_{1,T}^{-1})$ since $|\omega_{2j}-\omega_{2j-1}|>\nu_T = b_{1,T}/(2s)$, and $\int_{\Pi}|D_T(\omega_{2j}-\omega_{2j-1})\widetilde{K}_{b_1}(\omega_{2j})|d\omega_{2j} = O(b_{1,T}^{-1}\log T)$. For $\epsilon>0$, consider the following decomposition

$$\begin{aligned}&\int_{\Pi}\int_0^1\left|f\left(u_{2j-1},\omega_{2j-1}\right)D_T\left(\omega_{2j-1}-\omega_{2j-2}\right)\right|du_{2j-1}d\omega_{2j-1} \\ &\quad=\int_{|\omega_{2j-1}|\leq\epsilon}\int_0^1\left|f\left(u_{2j-1},\omega_{2j-1}\right)D_T\left(\omega_{2j-1}-\omega_{2j-2}\right)\right|du_{2j-1}d\omega_{2j-1} \\ &\qquad+\int_{|\omega_{2j-1}|>\epsilon}\int_0^1\left|f\left(u_{2j-1},\omega_{2j-1}\right)D_T\left(\omega_{2j-1}-\omega_{2j-2}\right)\right|du_{2j-1}d\omega_{2j-1}.\end{aligned} \tag{S.38}$$

By Assumption 4 $f(u_{2j-1},\omega_{2j-1})$ is bounded if $|\omega_{2j-1}|\leq\epsilon$. Then, the integral over $|\omega_{2j-1}|\leq\epsilon$ above is of order $O(\log T)$. On the other hand, if $|\omega_{2j-1}|>\epsilon$ (and recall that $|\omega_{2j-1}|\leq b_{1,T}\pi$), we yield as $T\to\infty$ $|\omega_{2j-1}-\omega_{2j-2}|>\epsilon/2$, say. Then, $|D_T(\omega_{2j-1}-\omega_{2j-2})| = O(1)$ by (S.26) and the second summand of (S.38) is finite in view of the integrability of $f(u,\omega)$ by Assumption 5. It follows that (S.38) is $O(\log T)$. There are other $s-1$ integrals of this type that can be handled in the same way. The remaining integral is of the form

$$\int_{\Pi}\int_{\Pi}\int_0^1\left|\widetilde{K}_{b_1}\left(\omega_{2s}\right)f\left(u_{2s-1},\omega_1\right)D_T\left(\omega_1-\omega_{2s}\right)\right|du_{2s-1}d\omega_1 d\omega_{2s} = O\left(\log T\right),$$

where $\omega_1 = \omega_{2s+1}$ and we have used the same argument as in (S.38) to show that the integral in $\omega_1$ is $O(\log T)$ for all $\omega_{2s}$ and that $\int_{\Pi}|\widetilde{K}_{b_1}(\omega_{2s})|d\omega_{2s} = O(1)$. Thus, (S.37) is $O(b_{1,T}^{-s}\log^{2s-1}T)$ and $B = O(b_{1,T}^{1-s}\log^{2s-1}T + b_{1,T}^{-s}\log^{2s-1}T + T^{-1}) = O(Tb_{1,T}^{1-s}\epsilon_T(2s))$.

Define $R_{b_1}(s) = \sum_{j=0}^{d_f}L_j(s)\,b_{1,T}^{1+j-s}$. Using the Lipschitz property of $f^{(d_f)}(u,\omega)$ for all $u$,

$$\begin{aligned}&\left|\int_{\Pi}\widetilde{K}_{b_1}^s(\omega)\left(\int_0^1 f(u,\omega)\,du\right)^s d\omega - R_{b_1}(s)\right| \\ &\quad\leq\int_{\Pi}\left|\widetilde{K}_{b_1}(\omega)\right|^{s-1}\left|\left(\int_0^1 f(u,\omega)\,du\right)^s - \sum_{j=0}^{d_f}\frac{1}{j!}\left(\frac{d}{d\omega}\right)^j\left(\int_0^1 f(u,0)\,du\right)^s\omega^j\right|\left|\widetilde{K}_{b_1}(\omega)\right|d\omega\end{aligned}$$

$$= O\left(\sup_{\omega\in\Pi}\left|\widetilde{K}_{b_1}(\omega)\right|^{s-1}\left|\int_{\Pi}|\omega|^{d_f+\varrho}\right|\left|\widetilde{K}_{b_1}(\omega)\right|d\omega\right) = O\left(b_{1,T}^{d_f+\varrho-s+1}\right),$$

where we have used $\sup_{\omega\in\Pi}|\widetilde{K}_{b_1}(\omega)| = O(b_{1,T}^{-1})$.

Note that $L_j(s)$ differs from zero for $j$ even because $L_j(s)$ depends on $\mu_j(K^s)$. □

**Lemma S.4.** *Let Assumptions 4 and 6-7 hold. For $s \geq 1$ with $\epsilon_T(2s+2) \to 0$, we have*

$$\mathbf{1}'(\Sigma_V W_{b_1})^s \Sigma_V \mathbf{1} = T(2\pi)^{2s+1}\left(\int_0^1 f(u,0)\,du\right)^{s+1}\left(\widetilde{K}_{b_1}(0)\right)^s + O\left(b_{1,T}^{-1-s}\log^{2s+1}T + T^{-1}\right).$$

*Proof of Lemma S.4.* We first write $\mathbf{1}'(\Sigma_V W_{b_1})^s\Sigma_V\mathbf{1}$ using an argument similar to the one used to derive (S.30), the only difference being that we also have the summation over two additional indexes. We write

$$\begin{aligned}
&\sum_{0\leq r_1,\ldots,r_{2s+2}\leq T}\mathbb{E}\left(V_{r_{2s+1}}V_{r_{2s+2}}\right)\Pi_{j=1}^s\left\{\mathbb{E}\left(V_{r_{2j-1}}V_{r_{2j}}\right)w\left(b_{1,T}\left(r_{2j}-r_{2j+1}\right)\right)\right\}\\
&\quad= \sum_r\int_{\Pi} f\left(r_{2s+1}/T,\,\omega_{2s+1}\right)e^{i(r_{2s+1}-r_{2s+2})\omega_{2s+1}}\Pi_{j=1}^s\\
&\qquad\times\left\{f\left(r_{2j-1}/T,\,\omega_{2j-1}\right)e^{i(r_{2j-1}-r_{2j})\omega_{2j-1}}\int_{\Pi}\widetilde{K}_{b_1}(\lambda_{2j})\,e^{i(r_{2j}-r_{2j+1})\lambda_{2j}}\right\}d\lambda d\omega\\
&\quad= T(2\pi)^{2s+1}\int_{\Pi^{2s+1}}S_{b_1}(\mu)\,\Psi_T^{(2s+2)}(\mu)\,d\mu + O\left(T^{-1}\right),
\end{aligned}\tag{S.39}$$

using a change of variable, where $\Psi_T^{(2s+2)}(\mu) = \Psi_T^{(2s+2)}(\mu_1,\ldots,\mu_{2s+1},-\sum_{j=1}^{2s+1}\mu_j)$,

$$S_{b_1}(\mu) = \int_0^1\cdots\int_0^1 f(u_1,\mu_1)\,\widetilde{K}_{b_1}(\mu_1+\mu_2)\ldots\widetilde{K}_{b_1}(\mu_1+\ldots+\mu_{2s})\,f(u_{2s+1},\mu_1+\ldots+\mu_{2s+1})\,du,$$

and $d\mu = d\mu_1\ldots d\mu_{2s+1}$, $du = du_1\ldots du_{2s+1}$ and $d\omega = d\omega_1\ldots d\omega_{2s+1}$. Proceeding as in the proof of Lemma S.3, we divide the range of integration in (S.39), $\Pi^{2s+1}$, into two sets, $\mathbf{M}$ and its complement $\mathbf{M}^c$, where $\mathbf{M} = \{|\mu_j| \leq \pi b_{1,T}/(2s+2),\ j=1,\ldots,2s+1\}$. We have

$$\begin{aligned}
&\left|\int_{\mathbf{M}}S_{b_1}(\mu)\,\Psi_T^{(2s+2)}(\mu)\,d\mu - \int_{\mathbf{M}}\left(\int_0^1 f(u,0)\,du\right)^{s+1}\widetilde{K}_{b_1}^s(0)\,\Psi_T^{(2s+2)}(\mu)\,d\mu\right|\\
&\quad= O\left(b_{1,T}^{-s-1}\right)\int_{\Pi^{2s+1}}\sum_{j=2}^{2s}|\mu_j|\left|\Psi_T^{(2s+2)}(\mu)\right|d\mu\\
&\quad= O\left(b_{1,T}^{-s-1}T^{-1}\log^{2s+1}T\right),
\end{aligned}\tag{S.40}$$

using (S.25), (S.32), Assumptions 4 and 7. On the other hand, the contribution from $\mathbf{M}^c$ is less than or equal to

$$\int_{\mathbf{M}^c}|S_{b_1}(\mu)|\left|\Psi_T^{(2s+2)}(\mu)\right|d\mu + O\left(b_{1,T}^{-s-1}T^{-1}\log^{2s+1}T\right),\tag{S.41}$$

where we have used (S.24). Using the same argument used for (S.37), the integral in (S.41) is less than or equal to

$$\frac{1}{T(2\pi)^{2s+1}}\int_{\mathbf{M}'}\prod_{j=1}^{s}\int_{0}^{1}\int_{0}^{1}\left[f\left(u_{2j-1},\,\omega_{2j-1}\right)\widetilde{K}_{b_1}\left(\omega_{2j}\right)D_T\left(\omega_{2j}-\omega_{2j-1}\right)\right. \\ \left.\times D_T\left(\omega_{2j+1}-\omega_{2j}\right)f\left(u_{2s+1},\,\omega_{2s+1}\right)D_T\left(\omega_1\right)D_T\left(-\omega_{2s-1}\right)\right]dud\omega, \tag{S.42}$$

where

$$\mathbf{M}'=\left\{|\omega_1|>\pi b_{1,T}/\left(2s+2\right)\right\}\cup\left\{|\omega_2-\omega_1|>\pi b_{1,T}/\left(2s+2\right)\right\}\cup\ldots\cup\left\{|\omega_{2s-1}-\omega_{2s}|>\pi b_{1,T}/\left(2s+2\right)\right\},$$

and (S.42) is nonzero only if $|\omega_2|,\,|\omega_4|,\ldots,\,|\omega_{2s}|\leq\pi b_{1,T}$.

If $|\omega_{j+1}-\omega_j|>\pi b_{1,T}/\left(2s+2\right)$ for at least one index $j\in\{1,\ldots,\,2s\}$ we can obtain a bound of order $(T^{-1}b_{1,T}^{-s-1}\log^{2s+1}T)$ for (S.42) as in Lemma S.3. The same bound is obtained for the case $|\omega_1|>\pi b_{1,T}/\left(2s+2\right)$ with a similar argument. Combining these results with (S.39)-(S.41) concludes the proof. □

**Lemma S.5.** *Let Assumptions 4, 6-7 and 11-12 hold. For $s\geq 2$ with $\epsilon_{Tb_{2,T}}\left(2s\right)\to 0$, we have*

$$\mathrm{Tr}\left(\left(\Sigma_{\widetilde{V}}W_{b_1}\right)^s\right)=Tb_{2,T}\left(2\pi\right)^{2s-1}\left(\sum_{j=0}^{d_f}L_j\left(s\right)b_{1,T}^{1+j-s}+b_{2,T}^2\sum_{j=0}^{d_f}\left(\left(L_{2,j}\left(s\right)+L_{3,j}\left(s\right)\right)b_{1,T}^{1+j-s}\right)\right) \\ +O\left(Tb_{2,T}b_{1,T}^{1-s}\epsilon_{Tb_{2,T}}\left(2s\right)+b_{1,T}^{-s}\frac{\log^{2s}\left(Tb_{2,T}\right)}{Tb_{2,T}}\right),$$

*where $\epsilon_{Tb_{2,T}}\left(2s\right)=\left(Tb_{2,T}\right)^{-1}\log^{2s-1}\left(Tb_{2,T}\right)$, $L_j\left(s\right)=\left(1/j\right)!\mu_j(K^s)\int_0^1K_2^s\left(x\right)dx\left(d^j/d\omega^j\right)\left(\int_0^1f\left(u,\,0\right)du\right)^s$ with $|L_j\left(s\right)|<\infty$, $L_j\left(s\right)$ differs from zero only for $j$ even, $L_{2,j}\left(s\right)$ depends on $\frac{\partial^2}{\partial u^2}\int_{\widetilde{\mathbf{C}}}f\left(u,\,\omega\right)du$, $K_2$, $\widetilde{K}_{b_1}$ and $s$ with $|L_{2,j}\left(s\right)|<\infty$, and $L_{3,j}\left(s\right)$ depends on $\Delta_f\left(\cdot\right)$, $\widetilde{K}_{b_1}$ and $s$ with $|L_{3,j}\left(s\right)|<\infty$.*

*Proof of Lemma S.5.* Let $r_{2s+1}=r_1$ and note that

$$\begin{aligned}\mathrm{Tr}\left(\left(\Sigma_{\widetilde{V}}W_{b_1}\right)^s\right)&=\int_0^1\cdots\int_0^1\sum_{1\leq r_1,\ldots,r_{2s}\leq T}\prod_{j=1}^{s}\mathbb{E}\left(\widetilde{V}_{r_{2j-1}}\left(u_j\right)\widetilde{V}_{r_{2j}}\left(u_j\right)\right)w\left(b_{1,T}\left(r_{2j}-r_{2j+1}\right)\right)du\\&=\int_0^1\cdots\int_0^1\sum_{1\leq r_1,\ldots,r_{2s}\leq T}\prod_{j=1}^{s}K_2\left(\frac{\left(Tu_j-\left(r_{2j-1}-\left(r_{2j}-r_{2j-1}\right)/2\right)\right)/T}{b_{2,T}}\right)\\&\quad\times\int_\Pi f\left(r_{2j-1}/T,\,\omega\right)e^{i\left(r_{2j-1}-r_{2j}\right)\omega_{2j-1}}d\omega\int_\Pi\widetilde{K}_{b_1}\left(\omega_{2j}\right)e^{i\left(r_{2j}-r_{2j+1}\right)\omega_{2j}}d\omega du\\&=\sum_{k_2,\,k_4,\ldots,\,k_{2s}=-\lfloor Tb_{2,T}\rfloor+1}^{\lfloor Tb_{2,T}\rfloor-1}\int_0^1\cdots\int_0^1\int_{\Pi^2}\prod_{j=1}^{s}\left(Tb_{2,T}-|k_{2j}|\right)f\left(u_{2j-1},\,\omega_{2j-1}\right)e^{i\left(\omega_{2j-1}-\omega_{2j}\right)k_{2j}}\\&\quad\times\widetilde{K}_{b_1}\left(\omega_{2j}\right)e^{i\left(-k_{2j}-k_{2j+2}\right)\omega_{2j}}d\omega du+O\left(b_{2,T}^2\right)+O\left(\frac{\log\left(Tb_{2,T}\right)}{Tb_{2,T}}\right)\\&=Tb_{2,T}\left(2\pi\right)^{2s-1}\int_{\Pi^{2s}}\left(H_{b_1}\left(\omega,\,\mu\right)\int_0^1K_2^s\left(x\right)dx+H_{2,b_1}\left(\omega,\,\mu\right)+H_{3,b_1}\left(\omega,\,\mu\right)\right)\end{aligned}\tag{S.43}$$

$$\times \widetilde{K}_{b_1}(\omega)\,\Psi^{(2s)}_{Tb_{2,T}}(\mu)\,d\omega d\mu + O\left(b_{2,T}^2 b_{1,T}^{-s}\log^{2s-1}(Tb_{2,T})\right) + O\left(b_{1,T}^{-s}\frac{\log^{2s}(Tb_{2,T})}{Tb_{2,T}}\right),$$

where $H_{b_1}(\omega,\,\mu)$, $d\omega$ and $d\mu$ are defined as in (S.30), $\Psi^{(2s)}_{Tb_{2,T}}(\mu)=\Psi^{(2s)}_{Tb_{2,T}}(\mu_1,\ldots,\,\mu_{2s})$,

$$\begin{aligned}H_{2,b_1}(\omega,\,\mu) = b_{2,T}^2&\left(\int_0^1 x^2 K_2(x)\,dx\right)\left(\int_0^1 K_2^{s-1}(x)\,dx\right)\\ &\times\sum_{j\in\mathbf{J}}\frac{\partial^2}{\partial u_j^2}\int_{\widetilde{\mathbf{C}}}\cdots\int_{\widetilde{\mathbf{C}}} f(u_1,\,\omega-\mu_2-\ldots-\mu_{2s})\,\widetilde{K}_{b_1}(\omega-\mu_3-\ldots-\mu_{2s})\\ &\times f(u_3,\,\omega-\ldots-\mu_{2s})\,\widetilde{K}_{b_1}(\omega-\mu_4-\ldots-\mu_{2s})\ldots f(u_{2s-1},\,\omega-\mu_{2s})\,du_1\cdots du_{2s-1},\end{aligned}$$

with $\mathbf{J}=\{1,\,3,\ldots,\,2s-1\}$, and $H_{3,b_1}(\omega,\,\mu)$ depends on the discontinuity points, i.e.,

$$\begin{aligned}H_{3,b_1}(\omega,\,\mu) = b_{2,T}^2&\left(\int_0^1 K_2^{s-1}(x)\,dx\right)\left(\mathbf{1}\left\{u_1=\lambda_j^0,\,j=1,\ldots,\,m_0\right\}\Delta_{f,j}(\omega-\mu_2-\ldots-\mu_{2s})\right)\\ &\times\widetilde{K}_{b_1}(\omega-\mu_3-\ldots-\mu_{2s})\,f(u_3,\,\omega-\ldots-\mu_{2s})\,\widetilde{K}_{b_1}(\omega-\mu_4-\ldots-\mu_{2s})\ldots f(u_{2s-1},\,\omega-\mu_{2s})\\ &\vdots\\ &+b_{2,T}^2\left(\int_0^1 K_2^{s-1}(x)\,dx\right) f(u_1,\,\omega-\mu_2-\ldots-\mu_{2s})\,\widetilde{K}_{b_1}(\omega-\mu_3-\ldots-\mu_{2s})\\ &\times f(u_3,\,\omega-\ldots-\mu_{2s})\,\widetilde{K}_{b_1}(\omega-\mu_4-\ldots-\mu_{2s})\ldots\\ &\times\mathbf{1}\left\{u_{2s-1}=\lambda_j^0,\,j=1,\ldots,\,m_0\right\}\Delta_{f,j}(\omega-\mu_{2s}),\end{aligned}$$

with

$$\Delta_{f,j}(\omega)=\int_0^1\left(\frac{\partial}{\partial u_-}f\left(\lambda_j^0,\,\omega\right)\int_0^{1-s} xK_2(x)\,dx+\frac{\partial}{\partial u_+}f\left(\lambda_j^0,\,\omega\right)\int_{1-s}^1 xK_2(x)\,dx\right)ds. \tag{S.44}$$

Let

$$B=\left|Tb_{2,T}(2\pi)^{2s-1}\int_0^1 K_2^s(x)\,dx\int_{\Pi^{2s}}\left(H_{b_1}(\omega,\,\mu)\,\widetilde{K}_{b_1}(\omega)\,\Psi^{(2s)}_{Tb_{2,T}}(\mu)-\left(\int_0^1 f(u,\,\omega)\,du\right)^s\widetilde{K}^s_{b_1}(\omega)\right)d\omega d\mu\right|.$$

Using (S.43) we have

$$B\leq Tb_{2,T}(2\pi)^{2s-1}\int_0^1 K_2^s(x)\,dx\int_{\Pi^{2s}}\left|H_{b_1}(\omega,\,\mu)-\left(\int_0^1 f(u,\,\omega)\,du\right)^s\widetilde{K}^{s-1}_{b_1}(\omega)\right|\left|\widetilde{K}_{b_1}(\omega)\,\Psi^{(2s)}_{Tb_{2,T}}(\mu)\right|d\omega d\mu. \tag{S.45}$$

We split the integral in (S.45) into two sets, for small and for large $\mu_j$. Define the set $\mathbf{M}=\{\mu\in\Pi^{2s-1}:$

$\sup_j |\mu_j| \leq b_{1,T}/(2s)\}$. Proceeding as in (S.33)-(S.34), we have

$$O\left(Tb_{2,T}b_{1,T}^{-s+1}\right)\sum_{q=0}^{s-1}\int_{\Pi}\int_{\mathbf{M}}\int_{0}^{1}\left|f\left(u,\,\omega-\mu_{2+2q}-\ldots-\mu_{2s}\right)-f\left(u,\,\omega\right)\right|\left|\widetilde{K}_{b_1}\left(\omega\right)\Psi_{Tb_{2,T}}^{(2s)}\left(\mu\right)\right|dud\omega d\mu \tag{S.46}$$

$$+O\left(Tb_{2,T}b_{1,T}^{-s+1}\right)\sum_{q=0}^{s-2}\int_{\Pi}\int_{\mathbf{M}}\left|\widetilde{K}_{b_1}\left(\omega-\mu_{2+2q}-\ldots-\mu_{2s}\right)-\widetilde{K}_{b_1}\left(\omega\right)\right|\left|\Psi_{Tb_{2,T}}^{(2s)}\left(\mu\right)\right|d\omega d\mu. \tag{S.47}$$

We apply the mean value theorem in (S.46) and use (S.25) to yield,

$$\begin{aligned}
&O\left(Tb_{2,T}b_{1,T}^{-s+1}\right)\int_{\Pi}\left|\widetilde{K}_{b_1}\left(\omega\right)\right|d\omega\sum_{q=0}^{2s}\int_{\mathbf{M}}|\mu_q|\left|\Psi_{Tb_{2,T}}^{(2s)}\left(\mu\right)\right|d\mu\\
&\qquad\leq O\left(Tb_{2,T}b_{1,T}^{-s+1}\right)\int_{\Pi}\left|\widetilde{K}_{b_1}\left(\omega\right)\right|d\omega\sum_{q=0}^{2s}\int_{\Pi^{2s-1}}|\mu_q|\left|\Psi_{Tb_{2,T}}^{(2s)}\left(\mu\right)\right|d\mu\\
&\qquad=O\left(b_{1,T}^{-s+1}\log^{2s-1}\left(Tb_{2,T}\right)\right).
\end{aligned}$$

On the other hand, using the Lipschitz property of $K$ (cf. Assumption 7), the expression in (S.47) is of order $O(b_{1,T}^{-s}\log^{2s-1}(Tb_{2,T}))$.

Let $\mathbf{M}^c$ denote the complement of $\mathbf{M}$ in $\Pi^{2s-1}$. The contribution to $B$ corresponding to the set $\mathbf{M}^c$ is bounded by

$$Tb_{2,T}\left(2\pi\right)^{2s-1}\int_{\Pi}\int_{\mathbf{M}^c}\left|H_{b_1}\left(\omega,\,\mu\right)\widetilde{K}_{b_1}\left(\omega\right)\right|\left|\Psi_{Tb_{2,T}}^{(2s)}\left(\mu\right)\right|d\omega d\mu \tag{S.48}$$

$$+\,Tb_{2,T}\left(2\pi\right)^{2s-1}\int_{\Pi}\left|\left(\int_{0}^{1}f\left(u,\,\omega\right)du\right)^{s}\widetilde{K}_{b_1}^{s}\left(\omega\right)\right|d\omega\int_{\mathbf{M}^c}\left|\Psi_{Tb_{2,T}}^{(2s)}\left(\mu\right)\right|d\mu. \tag{S.49}$$

The expression in (S.49) is $O(b_{1,T}^{-s}\log^{2s-1}(Tb_{2,T}))$ using (S.24) and

$$\int_{\Pi}\left|\left(\int_{0}^{1}f\left(u,\,\omega\right)\right)^{s}\widetilde{K}_{b_1}^{s}\left(\omega\right)\right|d\omega=O\left(b_{1,T}^{-s}\right).$$

The expression in (S.48) is bounded by

$$\int_{\mathbf{M}'}\prod_{j=1}^{s}\int_{0}^{1}\left|f\left(u_{2j-1},\,\omega_{2j-1}\right)\widetilde{K}_{b_1}\left(\omega_{2j}\right)D_{Tb_{2,T}}\left(\omega_{2j}-\omega_{2j-1}\right)D_{Tb_{2,T}}\left(\omega_{2j+1}-\omega_{2j}\right)\right|du_{2j-1}d\omega_{2j}d\omega_{2j-1}, \tag{S.50}$$

where $\mathbf{M}'$ is defined after (S.37).

From (S.26) it follows that $|D_{Tb_{2,T}}\left(\omega_{2j}-\omega_{2j-1}\right)|=O(b_{1,T}^{-1})$ since $|\omega_{2j}-\omega_{2j-1}|>\nu_T=b_{1,T}/(2s)$, and $\int_{\Pi}|D_{Tb_{2,T}}\left(\omega_{2j}-\omega_{2j+1}\right)\widetilde{K}_{b_1}\left(\omega_{2j}\right)|d\omega_{2j}=O(b_{1,T}^{-1}\log(Tb_{2,T}))$. For $\epsilon>0$, consider the following decomposition

$$\begin{aligned}&\int_{\Pi}\int_0^1\left|f\left(u_{2j-1},\,\omega_{2j-1}\right)D_{Tb_{2,T}}\left(\omega_{2j-1}-\omega_{2j-2}\right)\right|du_{2j-1}d\omega_{2j-1}\\&\quad=\int_{|\omega_{2j-1}|\leq\epsilon}\int_0^1\left|f\left(u_{2j-1},\,\omega_{2j-1}\right)D_{Tb_{2,T}}\left(\omega_{2j-1}-\omega_{2j-2}\right)\right|du_{2j-1}d\omega_{2j-1}\\&\qquad+\int_{|\omega_{2j-1}|>\epsilon}\int_0^1\left|f\left(u_{2j-1},\,\omega_{2j-1}\right)D_{Tb_{2,T}}\left(\omega_{2j-1}-\omega_{2j-2}\right)\right|du_{2j-1}d\omega_{2j-1}.\end{aligned}\tag{S.51}$$

By Assumption 4 $f(u_{2j-1},\,\omega_{2j-1})$ is bounded if $|\omega_{2j-1}|\leq\epsilon$. Then the integral over $|\omega_{2j-1}|\leq\epsilon$ above is of order $O(\log(Tb_{2,T}))$. On the other hand, if $|\omega_{2j-1}|>\epsilon$ we have $|D_{Tb_{2,T}}(\omega_{2j-1}-\omega_{2j-2})|=O\left(1\right)$ by (S.26) and the second summand of (S.51) is finite in view of the integrability of $f\left(u,\,\omega\right)$ by Assumption 5. It follows that (S.51) is $O(\log(Tb_{2,T}))$. There are other $s-1$ integrals of this type that can be handled in the same way. The remaining integral is of the form

$$\int_{\Pi}\int_{\Pi}\int_0^1\left|\widetilde{K}_{b_1}\left(\omega_{2s}\right)f\left(u_{2s-1},\,\omega_1\right)D_{Tb_{2,T}}\left(\omega_1-\omega_{2s}\right)\right|du_{2s-1}d\omega_1d\omega_{2s}=O\left(\log\left(Tb_{2,T}\right)\right),$$

where $\omega_1=\omega_{2s+1}$ and we have used the same argument as in (S.51) to show that the integral in $\omega_1$ is $O(\log(Tb_{2,T}))$ for all $\omega_{2s}$ and that $\int_{\Pi}|\widetilde{K}_{b_1}\left(\omega_{2s}\right)|d\omega_{2s}=O\left(1\right)$. Thus, (S.50) is $O(b_{1,T}^{-s}\log^{2s-1}Tb_{2,T})$ and $B=O(b_{1,T}^{1-s}\log^{2s-1}(Tb_{2,T})+b_{1,T}^{-s}\log^{2s-1}(Tb_{2,T}))=O(Tb_{2,T}b_{1,T}^{1-s}\epsilon_{Tb_{2,T}}\left(2s\right))$.

Next, let

$$B_2=Tb_{2,T}\left(2\pi\right)^{2s-1}\int_{\Pi^{2s}}\left|H_{2,b_1}\left(\omega,\,\mu\right)-b_{2,T}^2\Lambda_2\left(f'',\,\widetilde{\mathbf{C}},\,s\right)\widetilde{K}_{b_1}^{s-1}\left(\omega\right)\right|\left|\widetilde{K}_{b_1}\left(\omega\right)\Psi_{Tb_{2,T}}^{(2s)}\left(\mu\right)\right|d\omega d\mu,$$

where $\Lambda_2(f'',\,\widetilde{\mathbf{C}},\,s)$ depends on $f\left(u,\,\omega\right)$, the second partial derivative of $f\left(u,\,\omega\right)$ in $u$ at the continuity points in $\widetilde{\mathbf{C}}$ and $s$. By Assumption 12, for $j\in\mathbf{J}$ and $u_j\in\widetilde{\mathbf{C}}$ $(\partial^2/\partial u_j^2)f\left(u_j,\,\omega_j\right)$ has similar smoothness properties in $\omega_j$ to those of $f\left(u_j,\,\omega_j\right)$. Thus, the proof used above to bound $B$ can be repeated which then results in $B_2=O(Tb_{2,T}^3b_{1,T}^{1-s}\epsilon_{Tb_{2,T}}\left(2s\right))$.

Let

$$\begin{aligned}B_3=Tb_{2,T}\left(2\pi\right)^{2s-1}&\int_{\Pi^{2s}}\left|H_{3,b_1}\left(\omega,\,\mu\right)-b_{2,T}^2\Lambda_3\left(f',\,\left\{\lambda_j^0,\,j=1,\ldots,\,m_0\right\},\,s\right)\widetilde{K}_{b_1}^{s-1}\left(\omega\right)\right|\\&\times\left|\widetilde{K}_{b_1}\left(\omega\right)\Psi_{Tb_{2,T}}^{(2s)}\left(\mu\right)\right|d\omega d\mu,\end{aligned}$$

where $\Lambda_3(f',\,\{\lambda_j^0,\,j=1,\ldots,\,m_0\},\,s)$ depends on $f\left(u,\,\omega\right),\Delta_f\left(\cdot\right)$ and $s$. By Assumption 12, $\left(\partial/\partial u_{-}\right)f\left(u,\,\omega\right)$ and $\left(\partial/\partial u_{+}\right)f\left(u,\,\omega\right)$ for $u$ a discontinuity point have similar smoothness properties in $\omega$ to those of $f\left(u,\,\omega\right)$. Thus, the proof used above to bound $B$ can be repeated which then results in $B_3=O(Tb_{2,T}^3b_{1,T}^{1-s}\epsilon_{Tb_{2,T}}\left(2s\right))$.

The rest of the proof follows from the same arguments used in the last part of the proof of Lemma S.3. □

**Lemma S.6.** *Let Assumptions 4, 6-7 and 11-12 hold. For $s \geq 1$ with $\epsilon_T (2s+2) \to 0$, we have*

$$\mathbf{1}' \left(\Sigma_{\widetilde{V}} W_{b_1}\right)^s \Sigma_{\widetilde{V}} \mathbf{1} = T b_{2,T} (2\pi)^{2s+1} \left( \left( \left( \int_0^1 f(u, 0)\, du \right)^{s+1} \int_0^1 K_2^{s+1}(x)\, dx + b_{2,T}^2 \left( \widetilde{\Lambda}_2 \left(f'', \widetilde{\mathbf{C}}, s\right) + \widetilde{\Lambda}_3 \left(f', \left\{\lambda_j^0, j = 1, \ldots, m_0\right\}, s\right)\right)\right) \left(\widetilde{K}_{b_1}(0)\right)^s + O\left( b_{1,T}^{1-s} \log^{2s+1}(T b_{2,T}) + b_{1,T}^{-s} \frac{\log^{2s+1}(T b_{2,T})}{T b_{2,T}} \right)\right),$$

*where $\Lambda_2(f'', \widetilde{\mathbf{C}}, s)$ depends on $f(u, \omega)$, the second partial derivative of $f(u, \omega)$ in $u$ at the continuity points in $\widetilde{\mathbf{C}}$ and $s$, and $\widetilde{\Lambda}_3(f', \{\lambda_j^0, j = 1, \ldots, m_0\}, s)$ depends on $f(u, \omega)$, $\Delta_f(\cdot)$ and $s$.*

*Proof of Lemma S.6.* We first write $\mathbf{1}'(\Sigma_{\widetilde{V}} W_{b_1})^s \Sigma_{\widetilde{V}} \mathbf{1}$ using an argument similar to the one used to derive (S.39),

$$\begin{aligned}
&\int_0^1 \sum_{1 \leq r_1, \ldots, r_{2s+2} \leq T} \mathbb{E}\left(\widetilde{V}_{r_{2s+1}}(u_{s+1}) \widetilde{V}_{r_{2s+2}}(u_{s+1})\right) \int_0^1 \cdots \int_0^1 \Pi_{j=1}^s \\
&\quad \times \left\{ \mathbb{E}\left(\widetilde{V}_{r_{2j-1}}(u_j) \widetilde{V}_{r_{2j}}(u_j)\right) w\left(b_{1,T}(r_{2j} - r_{2j+1})\right)\right\} du \\
&= T b_{2,T} \sum_{k_{2s+2} = -\lfloor T b_{2,T} \rfloor + 1}^{\lfloor T b_{2,T} \rfloor - 1} \int_0^1 \int_\Pi f(u_{s+1}/T, \omega_{2s+1}) e^{-i k_{2s+2} \omega_{2s+1}} \Pi_{j=1}^s \int_0^1 \cdots \int_0^1 \\
&\quad \times \left\{ f(u_{2j-1}/T, \omega_{2j-1}) \sum_{k_2, k_4, \ldots, k_{2s} = -\lfloor T b_{2,T} \rfloor + 1}^{\lfloor T b_{2,T} \rfloor - 1} \frac{T b_{2,T} - |k_{2j}|}{T b_{2,T}} \int_\Pi \widetilde{K}_{b_1}(\omega_{2j}) e^{i(k_{2j} + k_{2j+1})\omega_{2j}} \right\} d\omega du \\
&= T b_{2,T} (2\pi)^{2s+1} \int_{\Pi^{2s+1}} \left( S_{b_1}(\mu) \int_0^1 K_2^{s+1}(x)\, dx + S_{2,b_1}(\mu) + S_{3,b_1}(\mu) \right) \Psi_{T b_{2,T}}^{(2s+2)}(\mu)\, d\mu \qquad \text{(S.52)} \\
&\quad + O\left(b_{2,T}^2 b_{1,T}^{-s} \log^{2s-1}(T b_{2,T})\right) + O\left(b_{1,T}^{-s} \frac{\log^{2s}(T b_{2,T})}{T b_{2,T}}\right),
\end{aligned}$$

where $\Psi_{T b_{2,T}}^{(2s+2)}(\mu)$, $S_{b_1}(\mu)$ and $d\mu = d\mu_1 \ldots d\mu_{2s+1}$ are defined as in (S.39),

$$\begin{aligned}
S_{2,b_1}(\mu) = b_{2,T}^2 &\left( \int_0^1 x^2 K_2(x)\, dx \right) \int_0^1 K_2^s(x)\, dx \sum_{j \in \mathbf{J}} \frac{\partial^2}{\partial u_j^2} \int_{\widetilde{\mathbf{C}}} \cdots \int_{\widetilde{\mathbf{C}}} f(u_1, \mu_1) \widetilde{K}_{b_1}(\mu_1 + \mu_2) \ldots \\
&\times \widetilde{K}_{b_1}(\mu_1 + \ldots + \mu_{2s}) f(u_{2s+1}, \mu_1 + \ldots + \mu_{2s+1})\, du,
\end{aligned}$$

with $\mathbf{J} = \{1, 3, \ldots, 2s+1\}$ and $S_{3,b_1}(\omega, \mu)$ depends on the discontinuity points, i.e.,

$$\begin{aligned}
S_{3,b_1}(\mu) = b_{2,T}^2 \int_0^1 K_2^s(x)\, dx &\left( \mathbf{1}\left\{u_1 = \lambda_j^0, j = 1, \ldots, m_0\right\} \Delta_{f,j}(\mu_1)\right) \widetilde{K}_{b_1}(\mu_1 + \mu_2) \\
&\ldots \widetilde{K}_{b_1}(\mu_1 + \ldots + \mu_{2s}) f(u_{2s-1}, \mu_1 + \ldots + \mu_{2s+1})
\end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&+ b_{2,T}^2 \int_0^1 K_2^s(x)\, dx f(u_1,\, \omega - \mu_2 - \ldots - \mu_{2s}) \widetilde{K}_{b_1}(\omega - \mu_3 - \ldots - \mu_{2s}) \\
&\times \widetilde{K}_{b_1}(\mu_1 + \ldots + \mu_{2s}) \mathbf{1}\left\{u_{2s-1} = \lambda_j^0,\, j = 1, \ldots, m_0\right\} \Delta_{f,j}(\mu_1 + \ldots + \mu_{2s+1}),
\end{aligned}
$$

with $\Delta_{f,j}(\omega)$ defined in (S.44). Proceeding as in the proof of Lemma S.4, we divide the range of integration of the integral involving $S_{b_1}(\mu)$ in (S.52), $\Pi^{2s+1}$, into two sets, $\mathbf{M}$ and its complement $\mathbf{M}^c$, where $\mathbf{M} = \{|\mu_j| \leq \pi b_{1,T}/(2s+2),\, j = 1, \ldots, 2s+1\}$. We have

$$
\begin{aligned}
&\left| \int_{\mathbf{M}} S_{b_1}(\mu) \Psi_{Tb_{2,T}}^{(2s+2)}(\mu)\, d\mu - \int_{\mathbf{M}} \left( \int_0^1 f(u,\, 0)\, du \right)^{s+1} \widetilde{K}_{b_1}^s(0) \Psi_{Tb_{2,T}}^{(2s+2)}(\mu)\, d\mu \right| \\
&= O\left(b_{1,T}^{-s-1}\right) \int_{\Pi^{2s+1}} \sum_{j=2}^{2s} |\mu_j| \left|\Psi_{Tb_{2,T}}^{(2s+2)}(\mu)\right| d\mu \\
&= O\left(b_{1,T}^{-s-1} (Tb_{2,T})^{-1} \log^{2s+1}(Tb_{2,T})\right), \qquad \text{(S.53)}
\end{aligned}
$$

using (S.25), (S.32), Assumptions 4 and 7. On the other hand, the contribution from $\mathbf{M}^c$ is less than or equal to

$$
Tb_{2,T}(2\pi)^{2s+1} \int_{\mathbf{M}^c} |S_{b_1}(\mu)| \left|\Psi_{Tb_{2,T}}^{(2s+2)}(\mu)\right| d\mu + O\left(b_{1,T}^{-s} \log^{2s+1}(Tb_{2,T})\right), \qquad \text{(S.54)}
$$

where we have used (S.24). Using the same argument used for (S.50), the expression in (S.54) is less than or equal to

$$
\begin{aligned}
&\int_{\mathbf{M}'} \prod_{j=1}^{s} \int_0^1 \int_0^1 \Big| f(u_{2j-1},\, \lambda_{2j-1}) \widetilde{K}_{b_1}(\lambda_{2j}) D_{Tb_{2,T}}(\lambda_{2j} - \lambda_{2j-1}) \qquad \text{(S.55)} \\
&\times D_{Tb_{2,T}}(\lambda_{2j+1} - \lambda_{2j}) f(u_{2s+1},\, \lambda_{2s+1}) D_{Tb_{2,T}}(\lambda_1) D_{Tb_{2,T}}(-\lambda_{2s-1}) \Big| du_{2s+1} du_{2j-1} d\lambda,
\end{aligned}
$$

where $\mathbf{M}' = \{|\lambda_1| > \pi b_{1,T}/(2s+2)\} \cup \{|\lambda_2 - \lambda_1| > \pi b_{1,T}/(2s+2)\} \cup \ldots \cup \{|\lambda_{2s-1} - \lambda_{2s}| > \pi b_{1,T}/(2s+2)\}$ and (S.55) is nonzero only if $|\lambda_2|, |\lambda_4|, \ldots, |\lambda_{2s}| \leq \pi b_{1,T}$.

If $|\lambda_{j+1} - \lambda_j| > \pi b_{1,T}/(2s+2)$ for at least one index $j \in \{1, \ldots, 2s\}$ we can obtain a bound of order $((Tb_{2,T})^{-1} b_{1,T}^{-s-1} \log^{2s+1}(Tb_{2,T}))$ for (S.55) as in Lemma S.5.

Next, we have

$$
\begin{aligned}
&Tb_{2,T}(2\pi)^{2s+1} \Big| \int_{\Pi^{2s}} (S_{b_2}(\mu) + S_{b_3}(\mu)) \Psi_{Tb_{2,T}}^{(2s+2)}(\mu)\, d\mu \qquad \text{(S.56)} \\
&- b_{2,T}^2 \int_{\Pi^{2s}} \left( \widetilde{\Lambda}_2\left(f'',\, \widetilde{\mathbf{C}},\, s\right) + \widetilde{\Lambda}_3\left(f',\, \left\{\lambda_j^0,\, j = 1, \ldots, m\right\},\, s\right) \right) \widetilde{K}_{b_1}^s(0) \Psi_{Tb_{2,T}}^{(2s+2)}(\mu)\, d\mu \Big|.
\end{aligned}
$$

By Assumption 12, $(\partial^2/\partial u^2) f(u,\omega)$ for $u \in \widetilde{\mathbf{C}}$, $(\partial/\partial u_-) f(u,\omega)$ and $(\partial/\partial u_+) f(u,\omega)$ for $u$ a discontinuity point have similar smoothness properties in $\omega$ to those of $f(u,\omega)$. Thus, the proof used above to bound (S.53) can be repeated which then results in (S.56) being $O(b_{2,T}^2 b_{1,T}^{-s-1} \log^{2s+1}(Tb_{2,T})$. $\square$

**Lemma S.7.** *Let Assumptions 4, 5 ($p > 1$), 6-7 and 10 ($0 < q < 1$) hold. Then, $||\Sigma_V W_{b_1}|| \leq C_1 \nu_{2,T}$ where $C_1$ depends on $f(\cdot, \cdot)$ and $K$, $0 < C_1 < \infty$ and $\nu_{2,T} = \max\{b_{1,T}^{-1} \log^2 T,\, T^{(2-p)/2p} b_{1,T}^{-1/2} \log^2 T)\} \to \infty$.*

*Proof of Lemma S.7.* We have

$$\begin{aligned}
\|\Sigma_V W_{b_1}\| &= \sup_{\|x\|=1} \left| \sum_{j,h=1}^{T} x_j x_h \sum_{t=1}^{T} \sum_{s=1}^{T} \int_{\Pi^2} f(t/T,\, \lambda) \widetilde{K}_{b_1}(\omega) e^{it\lambda} e^{-is\omega} e^{i(h\omega - j\lambda)} d\lambda d\omega \right| + O\left(T^{-1}\right) \\
&= \sup_{\|x\|=1} \left| \sum_{t=1}^{T} f(t/T,\, \lambda) e^{it\lambda} \sum_{j,h} x_j x_h \int_{\Pi^2} \widetilde{K}_{b_1}(\omega) D_T(-\omega) e^{i(h\omega - j\lambda)} d\lambda d\omega \right| + O\left(T^{-1}\right) \\
&\leq \sup_{\|x\|=1} \left| \int_{\omega \leq \epsilon} \int_{\lambda} \sum_{t=1}^{T} f(t/T,\, \lambda) e^{it\lambda} D_T(-\omega) \sum_{j,h} x_j x_h \widetilde{K}_{b_1}(\omega) e^{i(h\omega - j\lambda)} d\lambda d\omega \right| \\
&\quad + \sup_{\|x\|=1} \left| \int_{\omega > \epsilon} \int_{\lambda} \sum_{t=1}^{T} f(t/T,\, \lambda) e^{it\lambda} D_T(-\omega) \sum_{j,h} x_j x_h \widetilde{K}_{b_1}(\omega) e^{i(h\omega - j\lambda)} d\lambda d\omega \right| + O\left(T^{-1}\right) \\
&\triangleq A_1 + o(1) + O\left(T^{-1}\right).
\end{aligned} \tag{S.57}$$

Let $L_{2,T} : \mathbb{R} \to \mathbb{R}$ be the periodic extension with period $2\pi$ of

$$L_{2,T}(\omega) = \begin{cases} T, & |\omega| \leq 1/T, \\ 1/|\omega|, & 1/T \leq |\omega| \leq |\pi|. \end{cases}$$

Lemma S.A.1-2 in Casini and Perron (2024) showed that

$$\left| \sum_{t=1}^{T} f(t/T,\, \lambda) e^{-it\lambda} \right| \leq L_{2,T}(\lambda), \tag{S.58}$$

and $\int_{\Pi} L_{2,T}(\lambda) d\lambda \leq C_L \log T$ for $T > 1$ and $C_L > 0$ being a constant independent of $T$. Let $X_T(\omega) = \sum_{j=1}^{T} x_j e^{ij\omega}$. Then, the contribution to $A_1$ from $|\lambda| \leq \epsilon$ is bounded by

$$\begin{aligned}
&\sup_{\|x\|=1} \int_{\omega \leq \epsilon} \int_{\lambda} \left| \sum_{t=1}^{T} f(t/T,\, \lambda) e^{it\lambda} \right| |D_T(-\omega)| \, |X_T(\omega)| \, |X_T(\lambda)| \left| \widetilde{K}_{b_1}(\omega) \right| d\lambda d\omega \\
&\quad \leq \sup_{\|x\|=1} b_{1,T}^{-1} \sup_{\omega \in \Pi} |K(\omega)| \int_{\Pi} L_{2,T}(\lambda) \left( \int_{\Pi} |D_T(-\omega)| \, |X_T(\omega)| \, |X_T(\lambda)| \right) d\lambda d\omega \\
&\quad \leq \sup_{\|x\|=1} b_{1,T}^{-1} \sup_{\omega \in \Pi} |K(\omega)| \left( \int_{\Pi} L_{2,T}(\lambda)^2 d\lambda \right)^{1/2} \left( \int_{\Pi} |X_T(\lambda)|^2 d\lambda \right)^{1/2} \\
&\quad\quad \times \left( \int_{\Pi} |D_T(-\omega)|^2 d\omega \right)^{1/2} \left( \int_{\Pi} |X_T(\omega)|^2 d\omega \right)^{1/2} \\
&\quad \leq 2\pi C_2 b_{1,T}^{-1} \sup_{\omega \in \Pi} |K(\omega)| \log^2 T,
\end{aligned} \tag{S.59}$$

where $0 < C_2 < \infty$ and we have used $\sup_{\omega\in\Pi} |K(\omega)| = O(b_{1,T}^{-1})$, $(\int_\omega |X_T(\omega)|^2 d\omega) = 2\pi$ and (S.58). For $|\lambda| > \epsilon$ the contribution to $A_1$ is bounded by

$$\begin{aligned}
&\sup_{\|x\|=1} \int_{\omega\leq\epsilon} \sum_{t=1}^{T} \left(\int_\Pi (f(t/T,\lambda))^p d\lambda\right)^{1/p} \left(\int_\Pi |e^{it\lambda} X_T(\lambda)|^{\frac{p}{p-1}} d\lambda\right)^{(p-1)/p} \left|D_T(-\omega) X_T(\omega) \widetilde{K}_{b_1}(\omega)\right| d\omega d\omega \\
&\leq C_2 \sup_{\|x\|=1} \sum_{t=1}^{T} \left(\int_\Pi |e^{it\lambda} X_T(\lambda)|^{\frac{p}{p-1}} d\lambda\right)^{(p-1)/p} \int_{\omega\leq\epsilon} \left|D_T(-\omega) X_T(\omega) \widetilde{K}_{b_1}(\omega)\right| d\omega \\
&\leq C_2 \sup_{\|x\|=1} \sum_{t=1}^{T} \left(\int_\Pi |e^{it\lambda}|^{\frac{p}{p-1}} d\lambda\right)^{(p-1)/p} \int_{\omega\leq\epsilon} \left(\int_\Pi |X_T(\lambda)|^{\frac{p}{p-1}} d\lambda\right)^{(p-1)/p} \\
&\quad \times \left(\int_\Pi |D_T(-\omega)| d\omega\right) \left(\int_\Pi |X_T(\omega)|^2 d\omega\right)^{1/2} \left(\int_\Pi \left|\widetilde{K}_{b_1}(\omega)\right|^2 d\omega\right)^{1/2} \\
&\leq \sqrt{2\pi} C_2 \left(\sup_\omega |K(\omega)|\right)^{1/2} \|K\|_1 (2\pi)^{(p-1)/p} T^{\frac{2-p}{2p}} b_{1,T}^{-1} \log^2 T,
\end{aligned} \tag{S.60}$$

where $0 < C_2 < \infty$ and we have used $\sup_{x,\lambda} |X_T(\lambda)| \leq \sqrt{T}$ and

$$\begin{aligned}
\left(\int_\Pi |X_T(\lambda)|^{\frac{p}{p-1}} d\lambda\right)^{(p-1)/p} &= \left(\int_\Pi |X_T(\lambda)|^{2+\frac{2-p}{p-1}} d\lambda\right)^{(p-1)/p} \\
&= \left(\int_\Pi |X_T(\lambda)|^2 |X_T(\lambda)|^{\frac{2-p}{p-1}} d\lambda\right)^{(p-1)/p} \\
&\leq \left(\int_\Pi |X_T(\lambda)|^2 T^{\frac{1}{2}\left(\frac{2-p}{p-1}\right)} d\lambda\right)^{(p-1)/p} \\
&\leq (2\pi)^{(p-1)/p} T^{\frac{2-p}{2p}}.
\end{aligned}$$

From (S.59)-(S.60) we have $A_1 \leq C_1 \nu_{2,T}$ for some $C_1$ such that $0 < C_1 < \infty$. □

**Lemma S.8.** *Let Assumptions 4, 5 (for some $p > 1$), 6, 7 and $b_{1,T} + T^{-1} b_{1,T}^{-1} \log^3 T \to 0$ hold. Then, there exists $c_2 > 0$ such for $\|\mathbf{t}\| > c_1 m_T$ with $c_1 > 0$ we have $|\psi(\mathbf{t})| \leq \exp\{-c_2 m_T^2\}$, where $m_T = \min\{(Tb_{1,T})^{-1/2} \log T, T^{(p-1)/p}\} \to \infty$.*

*Proof of Lemma S.8.* The proof is similar to the proof of Lemma 15 in Velasco and Robinson (2001) with the difference that reference to Lemma 16 there is changed to reference to Lemma S.7. □

**Lemma S.9.** *Let Assumptions 4, 5 ($p > 1$), 6-7, 10 ($0 < q < 1$) and 11-12 hold. Then, $||\Sigma_{\widetilde{V}} W_{b_1}|| \leq C_1 \nu_{2,T}$ where $C_1$ depends on $f(u,\omega)$ and $K$, $0 < C_1 < \infty$ and $\nu_{2,T} = \max\{b_{1,T}^{-1} \log(Tb_{2T}), (Tb_{2,T})^{(2-p)/2p} b_{1,T}^{-1/2})\} \to \infty$.*

*Proof of Lemma S.9.* The proof is similar to the proof of Lemma S.7. □

**Lemma S.10.** *Let Assumptions 4, 5 ($p > 1$), 6-7, 11-12 and $b_{1,T} + (Tb_{1,T} b_{2,T})^{-1} \log^3 T \to 0$ hold. Then, there exists a $c_4 > 0$ such for $\|\mathbf{t}\| > c_3 m_{2,T}$ with $c_3 > 0$ we have $|\psi(t_1, t_2)| \leq \exp(-c_4 m_{2,T}^2)$, where $m_{2,T} = \min\{(Tb_{2,T} b_{1,T})^{1/2} / \log(Tb_{2,T}), (Tb_{2,T})^{(p-1)/p}\} \to \infty$.*

*Proof of Lemma S.10.* Following Bentkus and Rudzkis (1982) and Velasco and Robinson (2001) we first study the characteristic function of $\widehat{J}_{\mathrm{DK},T}$. Define $\tau(t_2) = \mathbb{E}(\exp(it_2 v_2)) = \tau'(t_2)\exp(-it_2 \Upsilon_{2,T})$, where

$$\tau'(t_2) = \left| I - \frac{2it_2}{\sqrt{Tb_{2,T}/b_{1,T}}\mathsf{V}_{2,T}J_T}\Sigma_{\widetilde{V}}W_{b_1}\right|^{-1/2} = \prod_{j=1}^{T}\left(1 - 2it_2\frac{\widetilde{\lambda}_j}{\sqrt{Tb_{2,T}/b_{1,T}}\mathsf{V}_{2,T}J_T}\right)^{-1/2},$$

and $\widetilde{\lambda}_j$ are the eigenvalues of $\Sigma_{\widetilde{V}}W_{b_1}$. Note that

$$1 = \mathrm{Var}(v_2) = \frac{b_{1,T}}{Tb_{2,T}}\frac{1}{\mathsf{V}_{2,T}^2 J_T^2}2\mathrm{Tr}\left[(\Sigma_{\widetilde{V}}W_{b_1})^2\right] = \frac{b_{1,T}}{Tb_{2,T}}\frac{2}{\mathsf{V}_{2,T}^2 J_T^2}\sum_{j=1}^{T}\widetilde{\lambda}_j^2,$$

where we have used the normality of $\{V_t\}$ and the relationship between the trace and the eigenvalues. Rearranging yields $\sum_{j=1}^{T}\widetilde{\lambda}_j^2 = 2^{-1}b_{1,T}^{-1}Tb_{2,T}\mathsf{V}_{2,T}^2 J_T^2 = O(b_{1,T}^{-1}Tb_{2,T})$. Further, we have $\max_j |\widetilde{\lambda}_j| = \sup_{\|x\|=1}|\Sigma_{\widetilde{V}}W_{b_1}x,\, x| = ||\Sigma_{\widetilde{V}}W_{b_1}||$. We can apply Lemma S.9 to yield

$$\max_j\left|\widetilde{\lambda}_j\right| \leq C_1\nu_{2,T}, \qquad \nu_{2,T} = \max\left\{b_{1,T}^{-1}\log(Tb_{2T}),\, (Tb_{2,T})^{(2-p)/2p}b_{1,T}^{-1/2}\right\} \to \infty,$$

where $C_1 > 0$ is such that $C_1 < \infty$. Let $g_j = \widetilde{\lambda}_j(C_1\nu_{2,T})^{-1}$ and note that for $T$ large enough we have $|g_j| \leq 1$. Using $\sum_{j=1}^{T} g_j^2 = (2C_1^2\nu_{2,T}^2)\mathsf{V}_{2,T}^2 J_T^2 b_{1,T}^{-1}Tb_{2,T}$ we yield

$$\begin{aligned}
|\tau(t_2)| &\leq \prod_{j=1}^{T}\left(1 + 4t^2\frac{C_1^2\nu_{2,T}^2}{b_{1,T}^{-1}Tb_{2,T}\mathsf{V}_{2,T}^2 J_T^2}\right)^{-(1/4)g_j^2} \\
&= \left(1 + t_2^2\frac{\nu_{2,T}^2}{b_{1,T}^{-1}Tb_{2,T}}\frac{4C_1^2}{\mathsf{V}_{2,T}^2 J_T^2}\right)^{-(1/8)C_1^{-2}\mathsf{V}_{2,T}^2 J_T^2 b_{1,T}^{-1}Tb_{2,T}\nu_{2,T}^{-2}} \\
&= \left(1 + t_2^2\frac{\nu_{2,T}^2}{b_{1,T}^{-1}Tb_{2,T}}\left[C_2 + O\left(b_{1,T}^2 + \epsilon_{Tb_{2,T}}(2)\right)\right]\right)^{-(1/2)\left(C_2^{-1}+O\left(b_{1,T}^2+\epsilon_{Tb_{2,T}}(2)\right)\right)Tb_{2,T}b_{1,T}^{-1}\nu_{2,T}^{-2}},
\end{aligned}$$

where $C_2 = C_1^2/(\pi^3 4(\int_0^1 f(u,0)\,du)^2\|K\|_2^2\|K_2\|_2^2)$ and we have applied $(1+at) \geq (1+t)^a$ which is valid for $t \geq 0$ and $0 \leq a \leq 1$. Thus, for all $\eta > 0$, we have

$$|\tau(t_2)| \leq \left(1 + \eta_1^2\right)^{-\eta_2\left(Tb_{2,T}b_{1,T}^{-1}\nu_{2,T}^{-2}\right)}, \tag{S.61}$$

for $|t_2| > \eta\sqrt{Tb_{2,T}b_{1,T}^{-1}}\nu_{2,T}^{-1}$ and for $\eta_1 > 0$ and $\eta_2 > 0$ depending on $\eta$.

Next, we consider the joint characteristic function $\psi_T(t_1, t_2)$. Its modulus is equal to

$$|\psi_T(t_1, t_2)| = |\tau(t_2)|\exp\left(-\frac{1}{2}t_1^2\xi_{2,T}'\mathscr{R}\left(I - 2it_2\Sigma_{\widetilde{V}}Q_{2,T}\right)^{-1}\Sigma_{\widetilde{V}}\xi_{2,T}\right), \tag{S.62}$$

where $\mathscr{R}(A)$ stands for the real part of $A$. From Anderson (1958, p. 161) $\mathscr{R}(\Sigma_{\widetilde{V}}^{-1} - 2it_2Q_{2,T})^{-1} = \mathscr{R}(I - 2it_2Q_{2,T})^{-1}\Sigma_{\widetilde{V}}$ is positive definite since $t_2Q_{2,T}$ is real. Then $\xi_{2,T}'\mathscr{R}(I - 2it_2\Sigma_{\widetilde{V}}Q_{2,T})^{-1}\Sigma_{\widetilde{V}}\xi_{2,T} > 0$ for all $t_2 \in \mathbb{R}$. Thus, $|t_2| \leq d\sqrt{Tb_{2,T}b_{1,T}^{-1}}/\nu_{2,T}$ for all $d > 0$ and $\xi_{2,T}'\mathscr{R}(I - 2it_2\Sigma_{\widetilde{V}}Q_{2,T})^{-1}\Sigma_{\widetilde{V}}\xi_{2,T} > \epsilon$ for some $\epsilon > 0$ depending on $d$ because $||\Sigma_{\widetilde{V}}Q_{2,T}|| = O(Tb_{2,T}b_{1,T}^{-1})^{-1/2}||\Sigma_{\widetilde{V}}W_{b_1}|| = (O(Tb_{2,T}b_{1,T}^{-1})^{-1/2}\nu_{2,T})$, and $||\xi_{2,T}|| = (\sqrt{Tb_{2,T}J_T})^{-1}\sqrt{1^2 + 1^2 + \ldots + 1^2} = 1/\sqrt{b_{2,T}J_T}$, with $J_T \to 2\pi \int_0^1 f(u, 0)\, du$, $0 < f(u, 0) < \infty$ for all $u$ by Assumption 4. Then, for $|t_1|\sqrt{2} > d_1\sqrt{Tb_{2,T}b_{1,T}^{-1}}/\nu_{2,T}$ and $|t_2|\sqrt{2} \leq d_1\sqrt{Tb_{2,T}b_{1,T}^{-1}}/\nu_{2,T}$ and some $\epsilon_1 > 0$ depending on $d_1$,

$$\exp\left(-\frac{1}{2}t_1^2\xi_{2,T}'\mathscr{R}\left(I - 2it_2\Sigma_{\widetilde{V}}Q_{2,T}\right)^{-1}\Sigma_{\widetilde{V}}\xi_{2,T}\right) \leq \exp\left(-\frac{1}{2}t_1^2\epsilon_1\right) \leq \exp\left(-\frac{1}{4}d_1^2\epsilon_1\frac{Tb_{2,T}b_{1,T}^{-1}}{\nu_{2,T}^2}\right). \tag{S.63}$$

From (S.61)-(S.63), there exists a $d_2 > 0$ such that $|\psi_T(\mathbf{t})| \leq \exp(-d_2(Tb_{2,T}b_{1,T}^{-1}/\nu_{2,T}^2))$ for $\{\mathbf{t} : ||\mathbf{t}|| > d_1\sqrt{Tb_{2,T}b_{1,T}^{-1}}/\nu_{2,T}\} \subset \mathbf{B}_1 \cup \mathbf{B}_2$ where $\mathbf{B}_1 = \{\mathbf{t} \in \mathbb{R}^2 : |t_2| > (d_1/\sqrt{2})\sqrt{Tb_{2,T}b_{1,T}^{-1}}/\nu_{2,T}\}$ and $\mathbf{B}_2 = \{\mathbf{t} \in \mathbb{R}^2 : |t_2| \leq (d_1/\sqrt{2})\sqrt{Tb_{2,T}b_{1,T}^{-1}}/\nu_{2,T}$ and $|t_1| > (d_1/\sqrt{2})\sqrt{Tb_{2,T}b_{1,T}^{-1}}/\nu_{2,T}\}$, and the lemma follows because $Tb_{2,T}b_{1,T}^{-1}/\nu_{2,T}^2 = m_{2,T}^2 \to \infty$. □

### S.B.2.2 Additional Lemmas Used for the Proofs of Theorems 3-4

We first present a result about the limit of $J_T$ and a result about the bias of $\widehat{J}_{\text{HAC},T}$.

**Lemma S.11.** *Let Assumption 4 with $d_f = 1$ and $\varrho = 0$ hold. Then, $J_T - 2\pi\int_0^1 f(u, 0)\, du = O\left(T^{-1}\log T\right)$. If in addition Assumption 2-(i) holds, then the order is $O(T^{-1})$.*

**Lemma S.12.** *Let Assumptions 4, 6, 8, and 9 hold. Then,*

$$\mathbb{E}\left(\widehat{J}_{\text{HAC},T}\right) - 2\pi\int_0^1 f(u, 0)\, du - 2\pi\frac{\int_0^1 f^{(d_f)}(u, 0)\, du}{d_f!}\mu_{d_f}(K)\, b_{1,T}^{d_f} = O\left(T^{-1}\log T + b_{1,T}^{d_f+\varrho}\right).$$

We now study the cumulants of the normalized spectral estimate $h_2$.

**Lemma S.13.** *Let Assumptions 4, 6-7 hold. For $s > 2$ with $\epsilon_T(s) = b_{1,T}^{d_f+\varrho} + T^{-1}b_{1,T}\log^{2s-1}T \to 0$, we have*

$$\overline{\kappa}_T(0, s) \triangleq \kappa_T(0, s)\left(\frac{T}{b_{1,T}}\right)^{(s-2)/2} = \sum_{j=0}^{d_f}\Xi_j(0, s)\, b_{1,T}^j + O\left(\epsilon_T(s)\right),$$

*where $\Xi_j(0, s)$ is bounded and depends on $K$ and $f^{(j)}(u, 0)$ $(j = 0, \ldots, d_f)$.*

A few examples of $\Xi_j(0, s)$ are $\Xi_0(0, s) = (4\pi)^{(s-2)/2}(s-1)!\int_\Pi K^s(\omega)\, d\omega\, \|K\|_2^{-s}$ and $\Xi_1(2, s) = 0$. If $(\partial/\partial\omega)(\int_0^1 f(u, \omega)\, du)|_{\omega=0} = 0$ then $\Xi_j(0, s) = 0$ for $j \geq 1$. In order to develop an Edgeworth expansion to approximate the distribution of $\mathbf{h}$, we need to study the cross-cumulants of $\mathbf{h}$.

**Lemma S.14.** *Let Assumptions 4 and 6-7 hold. For $s > 0$ with $\epsilon_T(s+2) \to 0$, we have*

$$\overline{\kappa}_T(2, s) \triangleq \kappa_T(2, s)\left(Tb_{1,T}\right)^{s/2} = \sum_{j=0}^{d_f}\Xi_j(2, s)b_{1,T}^j + O\left(\epsilon_T(s+2)\right),$$

*where* $\Xi_j(2,\, s)$ *is bounded and depends on* $K$ *and* $f^{(j)}(u,\, 0)$ $(j=0,\ldots,\, d_f)$.

For example, we have $\Xi_0(2,\, s) = (4\pi)^{s/2}\, s! K^s(0) \|K\|_2^{-s}$ and $\Xi_1(2,\, s)=0$. Using Lemmas S.13-S.14 we can substitute out $\mathsf{B}_T$ and $\mathsf{V}_T$ in $Z_T$ and, by only focusing on the leading terms, we define the following linear stochastic approximation,

$$\widetilde{Z}_T \triangleq h_1 \left(1 - 2^{-1}\overline{c}_1 b_{1,T}^{d_f} - 2^{-1}\sqrt{4\pi}\, \|K_2\|\, h_2 \left(Tb_{1,T}\right)^{-1/2}\right).$$

**Lemma S.15.** *Let Assumptions 4, 5* ($p>1$), *6-8 and 10* ($q=1/(1+2d_f)$) *hold. Then,* $Z_T$ *has the same Edgeworth expansion as* $\widetilde{Z}_T$ *uniformly for convex Borel sets up to order* $O((Tb_{1,T})^{-1/2})$.

Note that the condition $q=1/(1+2d_f)$ is sufficient for the consistency of $\widehat{J}_{\mathrm{HAC},T}$. Indeed, for $d_f=2$ it implies that $b_{1,T}=T^{-1/5}$ which coincides with the MSE-optimal bandwidth choice for the quadratic spectral kernel [cf. Andrews (1991)].[1]

### S.B.2.3 Proof of Lemma S.11

Note that $J_T=\sum_{k=-T+1}^{T-1}\Gamma_T(k)$ where $\Gamma_T(k)=T^{-1}\sum_{t=|k|+1}^{T}\mathbb{E}(V_tV_{t-|k|})$. We have

$$\begin{aligned} J_T &= \sum_{k=-T+1}^{T-1} \frac{1}{T} \sum_{t=|k|+1}^{T} \int_{\Pi} f(t/T,\, \omega)\, e^{ik\omega} d\omega \\ &= \sum_{k=-T+1}^{T-1} \frac{T-|k|}{T} \int_{|k|/T}^{1} \int_{\Pi} f(u,\, \omega)\, e^{ik\omega} d\omega du + O\left(T^{-1}\right) \\ &= 2\pi \int_0^1 \int_{\Pi} f(u,\, \omega)\, \Psi_T^{(2)}(\omega)\, d\omega du + O\left(T^{-1}\right). \end{aligned}$$

Since $\int_{\Pi}\Psi_T^{(2)}(\omega)\, d\omega=1$, we can apply the mean value theorem for $f(u,\,\omega)$ in a small interval $[-\epsilon,\,\epsilon]$, $\epsilon>0$, for some $|\eta|\leq 1$ depending on $\omega$,

$$\begin{aligned} \left| J_T - 2\pi \int_0^1 f(u,\, 0)\, du \right| &\leq 2\pi \left( \int_{|\omega|\leq\epsilon} + \int_{|\omega|>\epsilon} \right) \int_0^1 \int_{\Pi} |f(u,\, \omega) - f(u,\, 0)| \left|\Psi_T^{(2)}(\omega)\right| d\omega du + O\left(T^{-1}\right) \\ &= O\left( \int_{|\omega|\leq\epsilon} \int_0^1 |\omega|\, |f^{(1)}(u,\, \omega\eta)|\, \left|\Psi_T^{(2)}(\omega)\right| du d\omega \right. \\ &\quad \left. + \left( \int_0^1 (||f(u,\, \omega)||_1 + f(u,\, 0))\, du \right) T^{-1} \right) + O\left(T^{-1}\right) \\ &= O\left(T^{-1}\log T\right) + O\left(T^{-1}\right), \end{aligned}$$

where we have used Assumption 4,

$$\left|\Psi_T^{(2)}(\omega)\right| \leq \frac{1}{2\pi T} |D_T(\omega)|\, |D_T(-\omega)| \leq \frac{1}{\pi T} \left|\omega^{-2}\right|,$$

[1] Note that the MSE bounds under nonstationarity in Section 8 in Andrews (1991), which are used to determine the optimal bandwidth, are not correctly stated [cf. Casini (2022)].

from (S.26)-(S.27) and $|\Psi_T^{(2)}(\omega)| \leq O\left((T)^{-1}\right)$ if $|\omega| > \epsilon$.

For the second result in the lemma, note that

$$J_T = \sum_{k=-T+1}^{T-1} T^{-1} \sum_{t=|k|+1}^{T} \mathbb{E}\left(V_t V_{t-|k|}\right) = -\sum_{k=-T+1}^{T-1} T^{-1} \sum_{t=1}^{|k|} \mathbb{E}\left(V_t V_{t-|k|}\right) + \sum_{k=-T+1}^{T-1} T^{-1} \sum_{t=1}^{T} \mathbb{E}\left(V_t V_{t-|k|}\right).$$

Then,

$$\begin{aligned}\left|J_T - 2\pi \int_0^1 f(u, 0)\, du\right| &\leq \left|\sum_{k=-T+1}^{T-1} T^{-1} \sum_{t=1}^{T} \mathbb{E}\left(V_t V_{t-|k|}\right) - 2\pi \int_0^1 f(u, 0)\, du\right| + \left|\sum_{k=-T+1}^{T-1} T^{-1} \sum_{t=1}^{k} \mathbb{E}\left(V_t V_{t-|k|}\right)\right|, \\ &= O\left(T^{-1}\right),\end{aligned}$$

using Assumption 2-(i). □

### S.B.2.4 Proof of Lemma S.12

We can write $\widehat{J}_{\mathrm{HAC},T} = 2\pi \int_\Pi \widetilde{K}_{b_1}(\omega)\, I_T(\omega)\, d\omega$. Note that

$$\mathbb{E}\left(I_T(\omega)\right) = \int_0^1 \int_\Pi f(u, \lambda)\, \Psi_T^{(2)}(\omega - \lambda)\, d\lambda du + O\left(T^{-1}\right).$$

Thus, we obtain

$$\mathbb{E}\left(\widehat{J}_{\mathrm{HAC},T}\right) = 2\pi \int_\Pi \widetilde{K}_{b_1}(\omega) \int_0^1 \int_\Pi f(u, \alpha + \omega)\, \Psi_T^{(2)}(\alpha)\, d\alpha du d\omega + O\left(T^{-1}\right).$$

Then, using $\int_\Pi \Psi_T^{(2)}(\omega)\, d\omega = 1$ and $\int_\Pi \widetilde{K}_{b_1}(\omega)\, d\omega = 1$ we have

$$\begin{aligned}&\mathbb{E}\left(\widehat{J}_{\mathrm{HAC},T}\right) - 2\pi \int_0^1 f(u, 0)\, du - 2\pi b_{1,T}^{d_f} \mu_{d_f}(K) \int_0^1 \frac{f^{(d_f)}(u, 0)}{d_f!} du \\ &\quad = 2\pi \int_\Pi \widetilde{K}_{b_1}(\omega) \int_0^1 \int_\Pi \Psi_T^{(2)}(\alpha)\left(f(u, \omega + \alpha) - f(u, \omega)\right) d\alpha du d\omega \\ &\qquad + \int_\Pi \widetilde{K}_{b_1}(\omega) \int_0^1 \left[f(u, \omega) - f(u, 0) - b_{1,T}^{d_f} \mu_{d_f}(K) \frac{f^{(d_f)}(u, 0)}{d_f!}\right] du d\omega + O\left(T^{-1}\right) \\ &\quad \triangleq A_1 + A_2 + O\left(T^{-1}\right).\end{aligned}$$

For $\epsilon > 0$, we introduce the sets $\mathbf{A} = \{|\alpha|, |\omega| \leq \epsilon/2\}$ and its complement $\mathbf{A}^c$, both defined in $\Pi^2$. Let $A_{11}$ and $A_{12}$ be the contributions to $A_1$ corresponding to $\mathbf{A}$ and $\mathbf{A}^c$, respectively. Then, applying the mean value theorem we have

$$\begin{aligned}|A_{11}| &= 2\pi \int_{|\omega| \leq \epsilon/2} \left|\widetilde{K}_{b_1}(\omega)\, d\omega\right| d\omega \int_{|\alpha| \leq \epsilon/2} \left|\Psi_T^{(2)}(\alpha)\right| |\alpha|\, d\alpha \int_0^1 \sup_{|\omega| \leq \epsilon} \left|f^{(1)}(u, \omega)\right| du \\ &= O\left(T^{-1} \log T\right),\end{aligned}$$

where we have used (S.26)-(S.27) and Assumption 4. Let $\mathbf{B}_1 = \{|\alpha| > \epsilon/2\}$ and $\mathbf{B}_2 = \{|\omega| > \epsilon/2,\, |\alpha| \leq \epsilon/2\}$ and note that $\mathbf{A}^c \subset \{\mathbf{B}_1 \cup \mathbf{B}_2\}$. The contribution to $A_{12}$ from $\mathbf{B}_1$ is

$$\begin{aligned}
&\left|\int_{|\alpha|>\epsilon/2} \Psi_T^{(2)}(\alpha) \int_{\Pi} \widetilde{K}_{b_1}(\omega) \int_0^1 \left(f(u,\, \omega+\alpha) - f(u,\, \omega)\right) du d\omega d\alpha\right| \\
&\quad = O\left(T^{-1} \int_{\Pi^2} \int_0^1 \left|\widetilde{K}_{b_1}(\omega) \left(f(u,\, \omega+\alpha) - f(u,\, \omega)\right)\right| du d\omega d\alpha\right) \\
&\quad = O\left(T^{-1} \left(1 + \int_{|\omega|\leq\epsilon} \int_0^1 \left|\widetilde{K}_{b_1}(\omega) f(u,\, \omega)\right| du d\omega\right)\right) \\
&\quad = O\left(T^{-1} \int_{\Pi} \left|\widetilde{K}_{b_1}(\omega)\right| d\omega\right), \qquad \text{(S.64)}
\end{aligned}$$

using (S.26)-(S.27) and Assumption 4. Since $\widetilde{K}_{b_1}(\omega)$ is of reduced magnitude for $\omega > \epsilon/2$, the contribution to $A_{12}$ from $\mathbf{B}_2$ is, for large $T$,

$$\left|\int_{|\omega|>\epsilon/2} \int_{|\alpha|\leq\epsilon/2} \widetilde{K}_{b_1}(\omega) \Psi_T^{(2)}(\alpha) \int_0^1 \left(f(u,\, \omega+\alpha) - f(u,\, \omega)\right) du d\alpha d\omega\right| = 0, \qquad \text{(S.65)}$$

This implies that $A_{12} = O\left(T^{-1}\right)$.

As for $A_2$ we apply a Taylor's expansion of $f(u,\, \omega)$ around $\omega = 0$ and we split the integral into two parts for $|\omega| \leq \epsilon$ and $|\omega| > \epsilon$, denoted as $A_{21}$ and $A_{22}$, respectively. We have for $|\eta| \leq 1$ depending on $\omega$,

$$\begin{aligned}
A_{21} &= \int_{|\omega|\leq\epsilon} \widetilde{K}_{b_1}(\omega) \int_0^1 \left(\sum_{j=1}^{d_f-1} f^{(j)}(u,\, 0) \frac{\omega^j}{j!} + f^{(d_f)}(u,\, \eta\omega) \frac{\omega^{d_f}}{d_f!} - \frac{f^{(d_f)}(u,\, 0)}{d_f!} \mu_{d_f}(K)\, b_{1,T}^{d_f}\right) du d\omega \\
&= \sum_{j=1}^{d_f-1} \int_{\Pi} \omega^j \widetilde{K}_{b_1}(\omega)\, d\omega \int_0^1 f^{(j)}(u,\, 0) \frac{1}{j!} du \\
&\quad + d_f^{-1} \int_{|\omega|\leq b_{1,T}\pi} \omega^{d_f} \widetilde{K}_{b_1}(\omega) \int_0^1 \left(f^{(d_f)}(u,\, \eta\omega) - f^{(d_f)}(u,\, 0)\right) du d\omega \\
&= O\left(\int_{|\omega|\leq b_{1,T}\pi} \left|\widetilde{K}_{b_1}(\omega)\right| |\omega|^{d_f+\varrho}\, d\omega\right) = O\left(b_{1,T}^{d_f+\varrho}\right),
\end{aligned}$$

where we have used Assumption 8 and the fact that as $b_{1,T} \to 0$ the integration is within $[-\epsilon,\, \epsilon]$ and that by Assumption 4 $f^{(d_f)}(u,\, 0)$ is Lipschitz continuous of order $\varrho$ for all $u \in [0,\, 1]$. We can use the same argument used for $A_{12}$ to show that $A_{22} = 0$. $\square$

### S.B.2.5 Proof of Lemma S.13

From the definition of $Q_T$, we have

$$\kappa_T(0,\, s) = 2^{s-1}(s-1)!\, (\mathsf{V}_T J_T)^{-s} \left(T/b_{1,T}\right)^{-s/2} \mathrm{Tr}((\Sigma_V W_{b_1})^s),$$

for $s>1$. By Lemma S.3,

$$\overline{\kappa}_T(0,\, s)=\kappa_T(0,\, s)\left(b_{1,T}T\right)^{(s-2)/2}=\frac{2^{s-1}\left(s-1\right)!\left(2\pi\right)^{2s-1}}{\left(\mathsf{V}_T J_T\right)^{s}}\left(\sum_{j=0}^{d_f}L_j\left(s\right)b_{1,T}^{j}+O\left(\epsilon_T\left(2s\right)\right)\right). \tag{S.66}$$

Using again Lemma S.3 with $s=2$ to evaluate $\mathsf{V}_T^2$ yields

$$\begin{aligned}\mathsf{V}_T^2\frac{J_T^2}{4\pi^2}&=\frac{1}{4\pi^2}Tb_{1,T}\mathrm{Var}\left(\widehat{J}_{\mathrm{HAC},T}\right)=\frac{1}{4\pi^2}Tb_{1,T}\mathrm{Var}\left(\mathbf{V}'\frac{W_{b_1}}{T}\mathbf{V}\right)\\&=\frac{2b_{1,T}}{4\pi^2T}\mathrm{Tr}\left(W_{b_1}^2\Sigma_V^2\right)=\frac{2b_{1,T}}{4\pi^2T}\left(T\left(2\pi\right)^3\sum_{j=0}^{d_f}L_j\left(2\right)b_{1,T}^{j-1}+Tb_{1,T}^{-1}\epsilon_T\left(2\right)\right)\\&=4\pi\sum_{j=0}^{d_f}L_j\left(2\right)b_{1,T}^{j}+\epsilon_T\left(2\right),\end{aligned}$$

where we have use the normality of $V_t$. Lemma S.3 implies that $0<L_0\left(2\right)<\infty$ and $L_j\left(2\right)$ are fixed constants independent of $T$. Then

$$\left(\mathsf{V}_T\frac{J_T}{2\pi}\right)^{-s}=\left(4\pi\right)^{-s/2}\sum_{j=0}^{d_f}H_j\left(s\right)b_{1,T}^{j}+O\left(\epsilon_T\left(s\right)\right), \tag{S.67}$$

where $H_0\left(s\right)=L_0\left(2\right)^{-s/2}$ and so on. Denoting $c\left(0,\, s\right)=\left(4\pi\right)^{(s-2)/2}\left(s-1\right)!$ and using (S.66)-(S.67) we yield the following expression for the cumulants, $\overline{\kappa}_T(0,\, s)=c\left(0,\, s\right)\sum_{j=0}^{d_f}P_j\left(s\right)b_{1,T}^{j}+O\left(\epsilon_T\left(s\right)\right)$, where $P_j\left(s\right)=\sum_{t=0}^{j}H_t\left(s\right)L_{j-t}\left(s\right)$ are constants not depending on $T$ with $P_1\left(s\right)=0$, $P_2\left(s\right)=H_0\left(s\right)L_2\left(s\right)+J_2\left(s\right)L_0\left(s\right)$, and so on. Setting $\Xi_j\left(0,\, s\right)=c\left(0,\, s\right)P_j\left(s\right)$ the lemma follows. □

### S.B.2.6 Proof of Lemma S.14

Note that for $s>0$ we have

$$\kappa_T(2,\, s)=2^s s!\xi_T'\left(\Sigma_V Q_T\right)^s\Sigma_V\xi_T=2^s s!\frac{1}{TJ_T}\frac{b_{1,T}^{s/2}}{T^{s/2}\mathsf{V}_T^s J_T^s}\mathbf{1}'\left(W_{b_1}\Sigma_V\right)^s\Sigma_V\mathbf{1}.$$

From Lemma S.4,

$$\begin{aligned}\overline{\kappa}_T(2,\, s)&=\left(Tb_{1,T}\right)^{s/2}2^s s!\frac{1}{TJ_T}\frac{b_{1,T}^{s/2}}{T^{s/2}\mathsf{V}_T^s J_T^s}\mathbf{1}'\left(W_{b_1}\Sigma_V\right)^s\Sigma_V\mathbf{1}\\&=\left(Tb_{1,T}\right)^{s/2}2^s s!\frac{1}{TJ_T}\frac{b_{1,T}^{s/2}}{T^{s/2}\mathsf{V}_T^s J_T^s}\left(T\left(2\pi\right)^{2s+1}\left(\int_0^1 f\left(u,\,0\right)du\right)^{s+1}\left(\widetilde{K}_{b_1}\left(0\right)\right)^s\right.\\&\quad\left.+\,O\left(b_{1,T}^{-1-s}\log^{2s+1}T\right)\right)\\&=\left(\frac{2\pi}{J_T\mathsf{V}_T}\right)^s\frac{2\pi\int_0^1 f\left(u,\,0\right)du}{J_T}\left(4\pi\right)^s s!\left(\int_0^1 f\left(u,\,0\right)du\right)^s K\left(0\right)^s+O\left(\epsilon_T\left(s+2\right)\right),\end{aligned}$$

where we have used the fact that $\widetilde{K}_{b_1}(0) = b_{1,T}^{-1} K(0)$. Using Lemma S.11 and eq. (S.67), we yield

$$\begin{aligned}\overline{\kappa}_T(2,\, s) &= \left(\frac{2\pi}{J_T \mathsf{V}_T}\right)^s \left(1 + O\left(T^{-1}\log T\right)\right)(4\pi)^s\, s! \left(\int_0^1 f(u,\,0)\,du\right)^s K(0)^s + O\left(\epsilon_T(s+2)\right)\\ &= (4\pi)^{-s/2}(4\pi)^s\, s! \left(\int_0^1 f(u,\,0)\,du\right)^s K(0)^s \sum_{j=0}^{d_f} H_j(s)\, b_{1,T}^j + O\left(\epsilon_T(s+2)\right),\end{aligned}$$

where the $H_s(j)$ are as in the proof of Lemma S.13. The lemma follows by setting $\Xi_j(2,\, s) = (4\pi)^{-s/2}(4\pi)^s\, s!$ $(\int_0^1 f(u,\,0)\,du)^s K(0)^s\, H_j(s)$. $\square$

### S.B.2.7 Proof of Theorem 3

We first construct the approximation for $\psi_T(\mathbf{t})$. It follows from Velasco and Robinson (2001) and Taniguchi and Puri (1996) that only the cumulants $\kappa_T(0,\, s)$ and $\kappa_T(2,\, s)$ are nonzero, and that the cumulant generating function is given by

$$\log \psi_T(\mathbf{t}) = \frac{1}{2}\|i\mathbf{t}\|^2 + \sum_{s=3}^{\tau+1} \frac{(Tb_{1,T})^{(2-s)/2}}{s!} \sum_{|\mathbf{r}|=s} \frac{s!}{r_1! r_2!} \overline{\kappa}_T(r_1,\, r_2)(it_1)^{r_1}(it_2)^{r_2} + R_T(\tau), \tag{S.68}$$

where $\mathbf{r} = (r_1,\, r_2)'$ with $r_1 \in \{0,\, 2\}$ and $|\mathbf{r}| = r_1 + r_2$, and

$$\begin{aligned}R_T(\tau) &= (Tb_{1,T})^{-\tau/2}\left(R_{0,\tau+2}(it_2)^{\tau+2} + R_{2,\tau}(it_1)^2(it_2)^\tau\right), && \tau \text{ even},\\ R_T(\tau) &= (Tb_{1,T})^{-\tau/2}\frac{1}{(\tau+2)!}\left(\overline{\kappa}_T(0,\, \tau+2)(it_2)^{\tau+2} + \frac{(\tau+2)(\tau+1)}{2}\overline{\kappa}_T(2,\, \tau)(it_1)^2(it_2)^\tau\right)\\ &\quad + (Tb_{1,T})^{-\tau/2}\left(R_{0,\tau+3}(it_2)^{\tau+3} + R_{2,\tau+1}(it_1)^2(it_2)^{\tau+1}\right), && \tau \text{ odd},\end{aligned}$$

where the $R_{0,j}$ and $R_{2,j}$ are bounded. Using Lemmas S.13-S.14, we have

$$\begin{aligned}\log \psi_T(\mathbf{t}) &= \frac{1}{2}\|i\mathbf{t}\|^2 + \sum_{s=3}^{\tau+1}\frac{(Tb_{1,T})^{(2-s)/2}}{s!}\left(\overline{\kappa}_T(0,\, s)(it_2)^s + \frac{s(s-1)}{2}\overline{\kappa}_T(2,\, s-2)(it_1)^2(it_2)^{s-2}\right) + R_T(\tau)\\ &= \frac{1}{2}\|i\mathbf{t}\|^2 + \sum_{s=3}^{\tau+1}(Tb_{1,T})^{(2-s)/2}\left(B_T(s,\, \mathbf{t}) + \left\{(it_2)^s + (it_1)^2(it_2)^{s-2}\right\} O(\epsilon_T(s))\right) + R_T(\tau),\end{aligned}$$

where

$$B_T(s,\, \mathbf{t}) = \frac{1}{s!}\sum_{j=0}^{d_f} b_{1,T}^j \left\{\Xi_j(0,\, s)(it_2)^s + \frac{s(s-1)}{2}\Xi_j(2,\, s-2)(it_1)^2(it_2)^{s-2}\right\}.$$

The approximation of the characteristic function of $\mathbf{u}$ using its cumulant generating function is

$$\mathcal{A}_T(\mathbf{t},\,\tau) = \exp\left\{\frac{1}{2}\|i\mathbf{t}\|^2\right\}\left[1+\sum_{j=3}^{\tau+1}(Tb_{1,T})^{(2-j)/2}\sum_{\mathbf{r}}\prod_{n=3}^{\tau+1}[B_T(n,\,\mathbf{t})]^{r_n}\frac{1}{r_3!\cdots r_{\tau+1}!}\right],$$

where $\mathbf{r}=(r_3,\ldots,\,r_{\tau+1})'$, $r_n\in\{0,\,1,\ldots\}$, and the summation is over all $\mathbf{r}$ satisfying $\sum_{n=3}^{\tau+1}(n-2)\,r_n = j-2$. To obtain a second-order Edgeworth expansion we set $\tau=2$ and we include in $\mathcal{A}_T(\mathbf{t},\,2)$ terms up to order $(Tb_{1,T})^{-1/2}$,

$$\mathcal{A}_T(\mathbf{t},\,2) = \exp\left\{\frac{1}{2}\|i\mathbf{t}\|^2\right\}\left(1+\overline{B}_T(3,\,\mathbf{t})\,(Tb_{1,T})^{-1/2}\right), \tag{S.69}$$

where in $\overline{B}_T(3,\,\mathbf{t})$ includes only the leading term in $b_{1,T}^j$ $(j=0)$ in the expansion for the cumulant of order three. Note that the characteristic function of $\mathbb{Q}_T^{(2)}(\cdot)$ is $\mathcal{A}_T(\mathbf{t},\,2)$.

The rest of the proof consists of studying the distance between the true distribution and its Edgeworth approximation. Lemma S.16 studies the Edgeworth approximation for the characteristic function for $\|\mathbf{t}\|\le c_1\sqrt{Tb_{1,T}}$, whereas Lemma S.8 analyzes its tail behavior. The desired result follows from the same steps as in Theorem 1 of Velasco and Robinson (2001) which relies on Lemma S.2. □

**Lemma S.16.** *Let Assumptions 4, 6-7 and $b_{1,T}+(Tb_{1,T})^{-1}\log^5 T\to 0$ hold. There exists $\delta_1>0$ such that, for $\|\mathbf{t}\|\le\delta_1\sqrt{Tb_{1,T}}$ and a number $d_1>0$,*

$$|\psi_T(\mathbf{t})-A_T(\mathbf{t},\,2)|\le\exp\left\{-d_1\|\mathbf{t}\|^2\right\}\widetilde{F}(\|\mathbf{t}\|)\,O\left((Tb_{1,T})^{-1/2}\left(b_{1,T}^2+\epsilon_T(3)\right)+\frac{1}{Tb_{1,T}}\right),$$

*where $\widetilde{F}(\|\mathbf{t}\|)$ is a polynomial in $\mathbf{t}$ with bounded coefficients and $\mathcal{A}_T(\mathbf{t},\,2)$ is defined as in (S.69).*

*Proof of Lemma S.16.* It is similar to the proof of Lemma 14 in Velasco and Robinson (2001). □

### S.B.2.8 Proof of Lemma S.15

It is similar to the proof of Lemma 5 in Velasco and Robinson (2001). □

### S.B.2.9 Proof of Theorem 4

Consider the transformation $\mathbf{s}=(s_1,\,s_2)'=(\widetilde{Z}_T(h_1,\,h_2),\,h_2)'=\Delta_T(\mathbf{h})$ say, and its inverse $\mathbf{h}=\Delta_T^{-1}(\mathbf{s})=(h_1^{\dagger}(s_1,\,s_2),\,s_2)'$. Let $\mathbf{L}_T=\{\mathbf{h}:\,|h_i|<l_1T^{\gamma},\,0<\gamma<d_f/(3(1+2d_f)),\,i=1,\,2\}$, where $l_i$ are some fixed constants. Using $(1+x)^{-1}=1-x+x^2-x^3+\ldots$ for $|x|<1$, we have uniformly in the set $\mathbf{L}_T$,

$$h_1^{\dagger}(\mathbf{s}) = s_1\left[1+\frac{1}{2}\overline{c}_1 b_{1,T}^{d_f}+\frac{1}{2}\sqrt{4\pi}\,\|K_2\|\,s_2\,(Tb_{1,T})^{-1/2}\right]+o\left((Tb_{1,T})^{-1/2}\right).$$

We have $\mathbb{P}(Z_T\in\mathbf{C})=\mathbb{P}(\mathbf{h}\in\Delta_T^{-1}(\mathbf{C}\times\mathbb{R}))$ and from Theorem 3,

$$\sup_{\mathbf{C}}\left|\mathbb{P}\left(\mathbf{h}\in\Delta_T^{-1}(\mathbf{C}\times\mathbb{R})\right)-\mathbb{Q}_T^{(2)}\left(Z_T^{-1}(\mathbf{C}\times\mathbb{R})\right)\right| = o\left((Tb_{1,T})^{-1/2}\right)+\text{cost}\sup_{\mathbf{C}}\mathbb{Q}_T^{(2)}\left(\left(\partial\Delta_T^{-1}(\mathbf{C}\times\mathbb{R})\right)^{2\phi_T}\right),$$

where $\phi_T = (Tb_{1,T})^{-\varpi}$ with $1/2 < \varpi < 1$. The rest of the proof is similar to the proof of Theorem 2 in Velasco and Robinson (2001). □

### S.B.3 Additional Lemmas Used for the Proofs of Theorems 5-6

**Lemma S.17.** *Let Assumptions 4, 6, 8-9 and 11-12 hold. Then,*

$$\begin{aligned}\mathbb{E}\left(\widehat{J}^{*}_{\mathrm{DK},T}\right) &- 2\pi \int_0^1 f(u,\, 0)\, du - 2\pi \frac{\int_0^1 f^{(d_f)}(u,\, 0)\, du}{d_f!} \mu_{d_f}(K)\, b_{1,T}^{d_f} \\ &- \pi b_{2,T}^2 \int_0^1 x^2 K_2(x)\, dx \int_{\widetilde{\mathbf{C}}} \frac{\partial^2}{\partial u^2} f(u,\, 0)\, du - 2\pi b_{2,T}^2 \Delta_f(0) \\ &= O\left(b_{1,T}^{d_f+\varrho} + (Tb_{2,T})^{-1} \log(Tb_{2,T})\right) + o\left(b_{2,T}^2\right).\end{aligned}$$

The term $2\pi b_{2,T}^2 \Delta_f(0)$ in Lemma S.17 is the contribution to the bias due to the local time-smoothing in the neighborhoods involving a discontinuity point.

We now consider the cumulants of the normalized spectral estimate $v_2$.

**Lemma S.18.** *Let Assumptions 4, 6-7 and 11-12 hold. For $s > 2$ with $\epsilon_{Tb_{2,T}}(s) = b_{1,T}^{d_f+\varrho} + (Tb_{2,T}b_{1,T})^{-1}$ $\log^{2s-1}(Tb_{2,T}) \to 0$, we have*

$$\begin{aligned}\overline{\kappa}_{2,T}(0,\, s) &\triangleq \kappa_{2,T}(0,\, s)\, (Tb_{1,T}b_{2,T})^{(s-2)/2} \\ &= \sum_{j=0}^{d_f} \Xi_{2,j}(0,\, s)\, b_{1,T}^j + b_{2,T}^2 \sum_{j=0}^{d_f} \left(\widetilde{\Xi}_{2,j}(0,\, s) + \widetilde{\Xi}_{3,j}(0,\, s)\right) b_{1,T}^j + O\left(\epsilon_{Tb_{2,T}}(s)\right),\end{aligned}$$

*where $\Xi_{2,j}(0,\, s)$ is bounded and depends on $K$, $K_2$ and on $f^{(j)}(u,\, 0)$ $(j = 0, \ldots, d_f)$, $\widetilde{\Xi}_{2,j}(0,\, s)$ is bounded and depends on $K$, $K_2$, $f^{(j)}(u,\, 0)$ and $(\partial^2/\partial u^2)\, f(u,\, \omega)$ and $\widetilde{\Xi}_{3,j}(0,\, s)$ is bounded and depends on $K$, $K_2$, $f^{(j)}(u,\, 0)$ and $\Delta_f(\omega)$.*

We now consider the cross-cumulants of $\mathbf{v}$.

**Lemma S.19.** *Let Assumptions 4, 6-7 and 11-12 hold. For $s > 0$ with $\epsilon_{Tb_{2,T}}(s+2) \to 0$,*

$$\begin{aligned}\overline{\kappa}_{2,T}(2,\, s) \triangleq \kappa_{2,T}(2,\, s)\, (Tb_{2,T}b_{1,T})^{s/2} &= \sum_{j=0}^{d_f} \left(\Xi_{2,j}(2,\, s) + b_{2,T}^2 \left(\widetilde{\Xi}_{2,j}(2,\, s) + \widetilde{\Xi}_{3,j}(2,\, s)\right)\right) b_{1,T}^j \\ &+ O\left(\epsilon_{Tb_{2,T}}(s+2)\right),\end{aligned}$$

*where $\Xi_{2,j}(2,\, s)$ is bounded and depends on $K$, $K_2$ and $f^{(j)}(u,\, 0)$ $(j = 0, \ldots, d_f)$, $\widetilde{\Xi}_{2,j}(2,\, s)$ is bounded and depends on $K$, $K_2$, $f^{(j)}(u,\, 0)$ and $(\partial^2/\partial u^2)\, f(u,\, \omega)$, and $\widetilde{\Xi}_{3,j}(2,\, s)$ is bounded and depends on $K$, $K_2$, $f^{(j)}(u,\, 0)$ and $\Delta_f(\omega)$.*

### S.B.3.1 Proof of Lemma S.17

For $r \in \widetilde{\mathbf{C}}$, using a second-order Taylor's expansion as in the proof of Theorem 7.3 in Casini and Perron (2024), we yield

$$\begin{aligned}
\mathbb{E}\left(\widetilde{I}_T(r,\,\omega)\right) &= \mathbb{E}\left(\frac{1}{2\pi T b_{2,T}}\left|\sum_{t=1}^{T}\exp(-i\omega t)\,\widetilde{V}_t(r)\right|^2\right) \\
&= \frac{1}{2\pi}\frac{1}{Tb_{2,T}}\sum_{k=-\lfloor Tb_{2,T}\rfloor+1}^{\lfloor Tb_{2,T}\rfloor-1}\sum_{t=|k|+1}^{T}\int_{\Pi}K_2\left(\frac{(Tr-(t-k/2))/T}{b_{2,T}}\right)f((t+k/2)/T,\,\lambda)\,e^{ik(\omega-\lambda)}d\lambda \\
&\quad + O\left((Tb_{2,T})^{-1}\log(Tb_{2,T})\right) \\
&= \int_{\Pi} f(r,\,\lambda)\,\Psi^{(2)}_{Tb_{2,T}}(\omega-\lambda)\,d\lambda \\
&\quad + \frac{b_{2,T}^2}{2}\int_0^1 x^2K_2(x)\,dx\frac{\partial^2}{\partial u^2}f(u,\,\omega)\,|_{u=r} + o\left(b_{2,T}^2\right) + O\left((Tb_{2,T})^{-1}\log(Tb_{2,T})\right).
\end{aligned}$$

In a neighborhood of a break point $\lambda_j^0$, let $r=\lambda_j^0+sb_{2,T}$ for some $s\in(0,\,1)$. Then,

$$\begin{aligned}
\mathbb{E}\left(\widetilde{I}_T(r,\,\omega)\right) &= \int_{\Pi} f(r,\,\lambda)\,\Psi^{(2)}_{Tb_{2,T}}(\omega-\lambda)\,d\lambda \\
&\quad + b_{2,T}\left(\int_0^{1-s} xK_2(x)\,dx\frac{\partial}{\partial u_-}f\left(\lambda_j^0,\,\omega\right) + \int_{1-s}^{1} xK_2(x)\,dx\frac{\partial}{\partial u_+}f\left(\lambda_j^0,\,\omega\right)\right).
\end{aligned}$$

When integrating the last term above over $r$ we have

$$b_{2,T}^2\sum_{j=1}^{m_0}\int_0^1\left(\frac{\partial}{\partial u_-}f\left(\lambda_j^0,\,\omega\right)\int_0^{1-s}xK_2(x)\,dx + \frac{\partial}{\partial u_+}f\left(\lambda_j^0,\,\omega\right)\int_{1-s}^{1}xK_2(x)\,dx\right)ds.$$

Thus, we obtain

$$\begin{aligned}
\mathbb{E}\left(\widehat{J}^*_{\mathrm{DK},T}\right) &= 2\pi\int_{\Pi}\widetilde{K}_{b_1}(\omega)\int_0^1\int_{\Pi} f(u,\,\alpha+\omega)\,\Psi^{(2)}_T(\alpha)\,d\lambda du d\omega \\
&\quad + \pi b_{2,T}^2\int_0^1 x^2K_2(x)\,dx\int_{\Pi}\widetilde{K}_{b_1}(\omega)\int_{\widetilde{\mathbf{C}}}\frac{\partial^2}{\partial u^2}f(u,\,\omega)\,dud\omega \\
&\quad + 2\pi b_{2,T}^2\int_{\Pi}\widetilde{K}_{b_1}(\omega)\,\Delta_f(\omega)\,d\omega + o\left(b_{2,T}^2\right) + O\left((Tb_{2,T})^{-1}\log(Tb_{2,T})\right).
\end{aligned}$$

Then, using $\int_{\Pi}\Psi^{(2)}_T(\omega)\,d\omega=1$, $\int_{\Pi}\widetilde{K}_{b_1}(\omega)\,d\omega=1$, Assumption 12 and similar arguments as in the proof of Lemma S.12 applied to the terms involving $\frac{\partial^2}{\partial u^2}f(u,\,\omega)$ and $\Delta_f(\omega)$, we have

$$\mathbb{E}\left(\widehat{J}^*_{\mathrm{DK},T}\right) - 2\pi\int_0^1 f(u,\,0)\,du - 2\pi b_{1,T}^{d_f}\mu_{d_f}(K)\int_0^1\frac{f^{(d_f)}(u,\,0)}{d_f!}du$$

$$
\begin{aligned}
&\quad - \pi b_{2,T}^{2} \int_{0}^{1} x^{2} K_{2}(x)\, dx \int_{\widetilde{\mathbf{C}}} \frac{\partial^{2}}{\partial u^{2}} f(u,\, 0)\, du - 2\pi b_{2,T}^{2} \Delta_{f}(0) \\
&= 2\pi \int_{\Pi} \widetilde{K}_{b_1}(\omega) \int_{0}^{1} \int_{\Pi} \Psi_{T}^{(2)}(\alpha) \left(f(u,\, \omega + \alpha) - f(u,\, \omega)\right) d\alpha du d\omega \\
&\quad + 2\pi \int_{\Pi} \widetilde{K}_{b_1}(\omega) \int_{0}^{1} \left[ f(u,\, \omega) - f(u,\, 0) - b_{1,T}^{d_f} \mu_{d}(K) \frac{f^{(d_f)}(u,\, 0)}{d_f!} \right] du d\omega \\
&\quad + o\left(b_{2,T}^{2}\right) + O\left((Tb_{2,T})^{-1} \log (Tb_{2,T})\right) + o\left(b_{2,T}^{2} b_{1,T}^{q_2}\right) \\
&\triangleq A_1 + A_2 + o\left(b_{2,T}^{2}\right) + O\left((Tb_{2,T})^{-1} \log (Tb_{2,T})\right).
\end{aligned}
$$

To conclude the proof, note that by Lemma S.12 we have $|A_1| + |A_2| = O\left(T^{-1} \log T\right) + O(b_{1,T}^{d_f + \varrho})$. $\square$

### S.B.3.2 Proof of Lemma S.18

We have

$$
\kappa_{2,T}(0,\, s) = 2^{s-1}(s-1)!\, (\mathsf{V}_{2,T} J_T)^{-s} (Tb_{2,T}/b_{1,T})^{-s/2} \operatorname{Tr}((\Sigma_{\widetilde{V}} W_{b_1})^{s}),
$$

for $s > 1$. By Lemma S.5,

$$
\begin{aligned}
\overline{\kappa}_{2,T}(0,\, s) &= \kappa_{2,T}(0,\, s)\, (Tb_{1,T} b_{2,T})^{(s-2)/2} \qquad \text{(S.70)} \\
&= \frac{2^{s-1}(s-1)!\,(2\pi)^{2s-1}}{(\mathsf{V}_{2,T} J_T)^{s}} \left( \sum_{j=0}^{d_f} L_j(s)\, b_{1,T}^{j} + b_{2,T}^{2} \sum_{j=0}^{d_f} \left( (L_{2,j}(s) + L_{3,j}(s))\, b_{1,T}^{j} \right) + O\left(\epsilon_{Tb_{2,T}}(s)\right) \right).
\end{aligned}
$$

Using Lemma S.5 to evaluate $\mathsf{V}_{2,T}^{2}$ yields

$$
\begin{aligned}
\mathsf{V}_{2,T}^{2} \frac{J_T^{2}}{4\pi^{2}} &= \frac{1}{4\pi^{2}} Tb_{1,T} b_{2,T} \operatorname{Var}\left(\widehat{J}_{\mathrm{DK},T}^{*}\right) = Tb_{1,T} b_{2,T} \operatorname{Var}\left( \int_{0}^{1} \widetilde{\mathbf{V}}(r)' \frac{W_{b_1}}{Tb_{2,T}} \widetilde{\mathbf{V}}(r)\, dr \right) \\
&= \frac{2b_{1,T}}{4\pi^{2} Tb_{2,T}} \operatorname{Tr}\left( W_{b_1}^{2} \Sigma_{\widetilde{V}}^{2} \right) \\
&= \frac{2b_{1,T}}{4\pi^{2}} (2\pi)^{3} \left( \sum_{j=0}^{d_f} L_j(2)\, b_{1,T}^{j-1} + b_{2,T}^{2} \sum_{j=0}^{d_f} \left( (L_{2,j}(s) + L_{3,j}(s))\, b_{1,T}^{j-1} \right) \right) + Tb_{2,T} b_{1,T}^{-1} O\left(\epsilon_{Tb_{2,T}}(2)\right) \\
&= 4\pi \left( \sum_{j=0}^{d_f} L_j(2)\, b_{1,T}^{j} + b_{2,T}^{2} \sum_{j=0}^{d_f} \left( (L_{2,j}(s) + L_{3,j}(s))\, b_{1,T}^{j} \right) \right) + O\left(\epsilon_{Tb_{2,T}}(2)\right),
\end{aligned}
$$

where we have use the normality of $\{V_t\}$. Since Lemma S.5 implies that $0 < L_0(2) < \infty$ and $L_j(2)$ are fixed constants independent of $T$, we then have

$$
\left( \mathsf{V}_{2,T} \frac{J_T}{2\pi} \right)^{-s} = (4\pi)^{-s/2} \sum_{j=0}^{d_f} H_j(2)\, b_{1,T}^{j} + O\left(\epsilon_{Tb_{2,T}}(2)\right), \qquad \text{(S.71)}
$$

where $H_0(s) = L_0(2)^{-s/2}$ and so on. Using (S.70)-(S.71) we yield

$$\overline{\kappa}_{2,T}(0,\, s) = c(0,\, s)\left(\sum_{j=0}^{d_f} P_{2,j}(s)\, b_{1,T}^{j} + b_{2,T}^{2} \sum_{j=0}^{d_f}\left(\left(\widetilde{P}_{2,j}(s) + \widetilde{P}_{3,j}(s)\right) b_{1,T}^{j}\right)\right) + O\left(\epsilon_{Tb_{2,T}}(2)\right),$$

where $c(0,\, s) = (4\pi)^{(s-2)/2}(s-1)!$, $P_{2,j}(s) = \sum_{t=0}^{j} H_t(s)\, L_{j-t}(s)$ are constants not depending on $T$ with $P_{2,1}(s) = 0$, $P_{2,2}(s) = H_0(s)\, L_2(s) + H_2(s)\, L_0(s)$ and so on, and $\widetilde{P}_{2,j}(s) = \sum_{t=0}^{j} H_t(s)\, L_{2,j-t}(s)$ and $\widetilde{P}_{3,j}(s) = \sum_{t=0}^{j} H_t(s)\, L_{3,j-t}(s)$. The lemma follows from setting $\Xi_{2,j}(0,\, s) = c(0,\, s)\, P_{2,j}(s)$, $\widetilde{\Xi}_{2,j}(0,\, s) = c(0,\, s)\, \widetilde{P}_{2,j}(s)$ and $\widetilde{\widetilde{\Xi}}_{2,j}(0,\, s) = c(0,\, s)\, \widetilde{P}_{3,j}(s)$. □

### S.B.3.3 Proof of Lemma S.19

For $s > 0$ we have

$$\kappa_{2,T}(2,\, s) = 2^s s! \xi_T' \left(\Sigma_{\widetilde{V}} Q_{2,T}\right)^s \Sigma_{\widetilde{V}} \xi_T = 2^s s! \frac{1}{Tb_{2,T} J_T} \frac{b_{1,T}^{s/2}}{(Tb_{2,T})^{s/2} \mathsf{V}_{2,T}^{s} J_T^{s}} \mathbf{1}' \left(W_{b_1} \Sigma_{\widetilde{V}}\right)^s \Sigma_{\widetilde{V}} \mathbf{1}.$$

From Lemma S.6, we have

$$\begin{aligned}
\overline{\kappa}_{2,T}(2,\, s) &= (Tb_{1,T} b_{2,T})^{s/2}\, 2^s s! \frac{1}{Tb_{2,T} J_T} \frac{b_{1,T}^{s/2}}{(Tb_{2,T})^{s/2} \mathsf{V}_{2,T}^{s} J_T^{s}} \mathbf{1}' \left(W_{b_1} \Sigma_{\widetilde{V}}\right)^s \Sigma_{\widetilde{V}} \mathbf{1} \\
&= (Tb_{1,T} b_{2,T})^{s/2}\, 2^s s! \frac{1}{Tb_{2,T} J_T} \frac{b_{1,T}^{s/2}}{(Tb_{2,T})^{s/2} \mathsf{V}_{2,T}^{s} J_T^{s}} \\
&\quad \times \Bigg(Tb_{2,T}(2\pi)^{2s+1}\left(\left(\left(\int_0^1 f(u,\, 0)\, du\right)^{s+1} \int_0^1 K_2^{s+1}(x)\, dx + b_{2,T}^2 \widetilde{\Lambda}_2\left(f'',\, \widetilde{\mathbf{C}},\, s\right)\right.\right. \\
&\quad \left.\left. + b_{2,T}^2 \widetilde{\Lambda}_3\left(f',\, \left\{\lambda_j^0,\, j = 1, \ldots,\, m_0\right\},\, s\right)\right) \left(\widetilde{K}_{b_1}(0)\right)^s \\
&\quad + O\left(b_{1,T}^{1-s} \log^{2s+1}(Tb_{2,T}) + b_{1,T}^{-s} \frac{\log^{2s+1}(Tb_{2,T})}{Tb_{2,T}}\right)\Bigg) \\
&= \left(\frac{2\pi}{J_T \mathsf{V}_{2,T}}\right)^s \frac{2\pi \int_0^1 f(u,\, 0)\, du}{J_T} (4\pi)^s s! \left(\left(\int_0^1 f(u,\, 0)\, du\right)^s \int_0^1 K_2^{s+1}(x)\, dx + b_{2,T}^2 \left(\widetilde{\Lambda}_2^* + \widetilde{\Lambda}_3^*\right)\right) K(0)^s \\
&\quad + O\left(\epsilon_{Tb_{2,T}}(s+2)\right),
\end{aligned}$$

where $\widetilde{\Lambda}_2^*$ and $\widetilde{\Lambda}_3^*$ are equal to $\widetilde{\Lambda}_2$ and $\widetilde{\Lambda}_3$, respectively, without the factor $\int_0^1 f(u,\, 0)\, du$, and we have used $\widetilde{K}_{b_1}(0) = b_{1,T}^{-1} K(0)$. Using Lemma S.11 and (S.71), we yield

$$\begin{aligned}
\overline{\kappa}_{2,T}(2,\, s) &= \left(\frac{J_T \mathsf{V}_{2,T}}{2\pi}\right)^{-s} \left(1 + O\left((Tb_{2,T})^{-1} \log(Tb_{2,T})\right)\right) \\
&\quad \times (4\pi)^s s! \left(\left(\int_0^1 f(u,\, 0)\, du\right)^s \int_0^1 K_2^{s+1}(x)\, dx + b_{2,T}^2 \left(\widetilde{\Lambda}_2^* + \widetilde{\Lambda}_3^*\right)\right) K(0)^s + O\left(\epsilon_{Tb_{2,T}}(s+2)\right)
\end{aligned}$$

$$= (4\pi)^{-s/2} (4\pi)^s s! \left( \left( \int_0^1 f(u,0)\, du \right)^s \int_0^1 K_2^{s+1}(x)\, dx + b_{2,T}^2 \left( \widetilde{\Lambda}_2^* + \widetilde{\Lambda}_3^* \right) \right) K(0)^s \sum_{j=0}^{d_f} H_j(s)\, b_{1,T}^j$$
$$+ O\left( \epsilon_{Tb_{2,T}} (s+2) \right),$$

where the $H_j(s)$ are as in (S.71). Letting

$$\Xi_{2,j}(2,s) = (4\pi)^{-s/2} (4\pi)^s s! \left( \int_0^1 f(u,0)\, du \right)^s K(0)^s \int_0^1 K_2^{s+1}(x)\, dx H_j(s)$$
$$\widetilde{\Xi}_{2,j}(2,s) = (4\pi)^{-s/2} (4\pi)^s s! \widetilde{\Lambda}_2^* K(0)^s \int_0^1 K_2^s(x)\, dx H_j(s)$$
$$\widetilde{\Xi}_{3,j}(2,s) = (4\pi)^{-s/2} (4\pi)^s s! \widetilde{\Lambda}_3^* K(0)^s \int_0^1 K_2^s(x)\, dx H_j(s),$$

the lemma follows. □

### S.B.3.4 Proof of Theorem 5

It follows from Velasco and Robinson (2001) and Taniguchi (1987) that only the cumulants $\kappa_{2,T}(0,s)$ and $\kappa_{2,T}(2,s)$ are nonzero, and that the cumulant generating function is given by

$$\log \psi_T(\mathbf{t}) = \frac{1}{2} \|i\mathbf{t}\|^2 + \sum_{s=3}^{\tau+1} \frac{(Tb_{1,T}b_{2,T})^{(2-s)/2}}{s!} \sum_{|\mathbf{r}|=s} \frac{s!}{r_1! r_2!} \overline{\kappa}_{2,T}(r_1, r_2) (it_1)^{r_1} (it_2)^{r_2} + R_T^*(\tau), \tag{S.72}$$

where $\mathbf{r} = (r_1, r_2)'$, with $r_1 \in \{0, 2\}$ and $|\mathbf{r}| = r_1 + r_2$, and

$$R_T^*(\tau) = (Tb_{1,T}b_{2,T})^{-\tau/2} \left[ R'_{0,\tau+2} (it_2)^{\tau+2} + R'_{2,\tau} (it_1)^2 (it_2)^{\tau} \right], \qquad \tau \text{ even},$$
$$R_T^*(\tau) = (Tb_{1,T}b_{2,T})^{-\tau/2} \frac{1}{(\tau+2)!} \left[ \overline{\kappa}_{2,T}(0, \tau+2) (it_2)^{\tau+2} + \frac{(\tau+2)(\tau+1)}{2} \overline{\kappa}_{2,T}(2,\tau) (it_1)^2 (it_2)^{\tau} \right]$$
$$+ (Tb_{1,T}b_{2,T})^{-\tau/2} \left[ R'_{0,\tau+3} (it_2)^{\tau+3} + R'_{2,\tau+1} (it_1)^2 (it_2)^{\tau+1} \right], \qquad \tau \text{ odd},$$

where the $R'_{0,j}$ and $R_{2,j}$ are bounded. Using Lemmas S.18-S.19, we have

$$\log \psi_T(\mathbf{t}) = \frac{1}{2} \|i\mathbf{t}\|^2 + \sum_{s=3}^{\tau+1} \frac{(Tb_{1,T}b_{2,T})^{(2-s)/2}}{s!} \left( \overline{\kappa}_{2,T}(0,s) (it_2)^s + \frac{s(s-1)}{2} \overline{\kappa}_{2,T}(2, s-2) (it_1)^2 (it_2)^{s-2} \right)$$
$$+ R_T^*(\tau)$$
$$= \frac{1}{2} \|i\mathbf{t}\|^2 + \sum_{s=3}^{\tau+1} (Tb_{1,T}b_{2,T})^{(2-s)/2} \left[ B_{2,T}(s, \mathbf{t}) + \left\{ (it_2)^s + (it_1)^2 (it_2)^{s-2} \right\} O(\epsilon_T(s)) \right] + R_T^*(\tau),$$

where

$$B_{2,T}(s,\,\mathbf{t})=\frac{1}{s!}\sum_{j=0}^{d_f}b_{1,T}^{j}\Big\{\Big(\Xi_{2,j}(0,\,s)+b_{2,T}^{2}\left(\widetilde{\Xi}_{2,j}(0,\,s)+\widetilde{\Xi}_{3,j}(0,\,s)\right)\Big)(it_2)^{s}$$
$$+\frac{s\,(s-1)}{2}\Big(\Xi_{2,j}(2,\,s-2)+b_{2,T}^{2}\left(\widetilde{\Xi}_{2,j}(2,\,s-2)+\widetilde{\Xi}_{3,j}(2,\,s-2)\right)\Big)(it_1)^{2}(it_2)^{s-2}\Big\}.$$

The approximation of the characteristic function of $\mathbf{v}$ using its cumulant generating function is

$$\mathcal{A}_{2,T}(\mathbf{t},\,\tau)=\exp\left(\frac{1}{2}\|i\mathbf{t}\|^{2}\right)\left[1+\sum_{j=3}^{\tau+1}(Tb_{1,T}b_{2,T})^{(2-j)/2}\sum_{\mathbf{r}}\prod_{n=3}^{\tau+1}(B_{2,T}(n,\,\mathbf{t}))^{r_n}\frac{1}{r_3!\ldots r_{\tau+1}!}\right],$$

where $\mathbf{r}=(r_3,\ldots,r_{\tau+1})'$, $r_n\in\{0,1,\ldots\}$, and the summation is over all $\mathbf{r}$ satisfying $\sum_{n=3}^{\tau+1}(n-2)\,r_n=j-2$. To obtain a second-order Edgeworth expansion we set $\tau=2$ and we include in $\mathcal{A}_{2,T}(\mathbf{t},\,2)$ the terms up to order $(Tb_{1,T}b_{2,T})^{-1/2}$,

$$\mathcal{A}_{2,T}(\mathbf{t},\,2)=\exp\left(\frac{1}{2}\|i\mathbf{t}\|^{2}\right)\left[1+\overline{B}_{2,T}(3,\,\mathbf{t})\,(Tb_{1,T}b_{2,T})^{-1/2}\right],\tag{S.73}$$

where $\overline{B}_{2,T}(3,\,\mathbf{t})$ includes only the leading term in $b_{1,T}^{j}$ $(j=0)$ in the expansion for the cumulant of order three. Note that the characteristic function of $\mathbb{Q}_{2,T}^{(2)}(\cdot)$ is $\mathcal{A}_{2,T}(\mathbf{t},\,2)$. We use Lemma S.2 with kernel $\mathbb{G}$ to bound the distance between $\mathbb{P}_T$ and $\mathbb{Q}_{2,T}^{(2)}$. First,

$$\left\|\left(\mathbb{P}_T-\mathbb{Q}_{2,T}^{(2)}\right)\bullet\mathbb{G}_{\phi_T}\right\|_{\mathrm{TV}}\leq 2\sup_{\mathbf{B}\subset\mathbf{B}(0,\,r_T)}\left|\left(\mathbb{P}_T-\mathbb{Q}_{2,T}^{(2)}\right)\bullet\mathbb{G}_{\phi_T}\right|+2\sup_{\mathbf{B}\subset\mathbf{B}(0,\,r_T)^c}\left|\left(\mathbb{P}_T-\mathbb{Q}_{2,T}^{(2)}\right)\bullet\mathbb{G}_{\phi_T}\right|,$$

where $\mathbf{B}(0,\,r_T)$ is a neighborhood around 0 with radius $r_T$, $r_T=(Tb_{1,T}b_{2,T})^{a}$ with $a>0$, and $\|\cdot\|_{\mathrm{TV}}$ denotes the total variation norm. For $\mathbf{B}\subset\mathbf{B}(0,\,r_T)^c$ we have uniformly

$$\begin{aligned}\left|\left(\mathbb{P}_T-\mathbb{Q}_{2,T}^{(2)}\right)\bullet\mathbb{G}_{\phi_T}\right|&\leq\left|\mathbb{P}_T\bullet\mathbb{G}_{\phi_T}\right|+\left|\mathbb{Q}_{2,T}^{(2)}\bullet\mathbb{G}_{\phi_T}\right|\\&\leq\mathbb{P}\left(\|\mathbf{v}\|\geq r_T/2\right)+2\mathbb{G}_{\phi_T}\left(\mathbf{B}(0,\,r_T/2)^c\right)+2\mathbb{Q}_{2,T}^{(2)}\left(\mathbf{B}(0,\,r_T/2)^c\right).\end{aligned}$$

By definition of $q_{2,T}^{(2)}(\mathbf{v})$ it follows that $\mathbb{Q}_{2,T}^{(2)}(\mathbf{B}(0,\,r_T/2)^c)=o((Tb_{1,T}b_{2,T})^{-1/2})$. In view of the definition of $v_2$, we have $\mathbb{P}\{\|\mathbf{v}\|\geq r_T/2\}=o((Tb_{1,T}b_{2,T})^{-1/2})$. By Lemma S.2,

$$\mathbb{G}_{\phi_T}\left(\mathbf{B}(0,\,r_T/2)^c\right)=O\left((\phi_T/r_T)^{3}\right)=O\left((Tb_{1,T}b_{2,T})^{-3(\varpi+a)}\right)=o\left((Tb_{1,T}b_{2,T})^{-1/2}\right).$$

For $\mathbf{B}\subset\mathbf{B}(0,\,r_T)$ we have by Fourier inversion

$$\left|\left(\mathbb{P}_T-\mathbb{Q}_{2,T}^{(2)}\right)\bullet\mathbb{G}_{\phi_T}\right|\leq(2\pi)^{-1}\pi r_T^{2}\int\left|\left(\widehat{\mathbb{P}}_T-\widehat{\mathbb{Q}}_{2,T}^{(2)}\right)(\mathbf{t})\,\widehat{\mathbb{G}}_{\phi_T}(\mathbf{t})\right|d\mathbf{t},\tag{S.74}$$

where $\widehat{\mathbb{P}}_T$ denotes the characteristic function of $\mathbb{P}_T$ (i.e., $\widehat{\mathbb{P}}_T = \psi_T(\mathbf{t})$) and $\widehat{\mathbb{Q}}^{(2)}_{2,T} = \mathcal{A}_{2,T}(\mathbf{t}, 2)$. Let $a' = 8 \times 2^{4/3}\pi^{-1/3}$. Using Lemma S.20, a bound for (S.74) is given by

$$O\left((Tb_{1,T}b_{2,T})^{2a-1/2}\right)\left[b^2_{1,T} + \epsilon_{Tb_{2,T}}(3)\right]\int_{\|\mathbf{t}\|\leq c_2\sqrt{Tb_{1,T}b_{2,T}}}\left|e^{-d_2\|\mathbf{t}\|^2}F(\|\mathbf{t}\|)\right|\left|\widehat{\mathbb{G}}_{\phi_T}(\|\mathbf{t}\|)\right|d\mathbf{t} \tag{S.75}$$

$$+O(Tb_{1,T}b_{2,T})^{2a}\int_{c_2\sqrt{Tb_{1,T}b_{2,T}}<\|\mathbf{t}\|\leq a'(Tb_{1,T}b_{2,T})^{\varpi}}\int\left|\left(\widehat{\mathbb{P}}_T - \widehat{\mathbb{Q}}^{(2)}_{2,T}\right)(\mathbf{t})\,\widehat{\mathbb{G}}_{\phi_T}(\mathbf{t})\right|d\mathbf{t}. \tag{S.76}$$

The integral over $\|\mathbf{t}\| > a'(Tb_{1,T}b_{2,T})^{\varpi}$ is equal to zero from (S.29). Choosing $a \leq 1/4$ (S.75) is $o(((Tb_{1,T}b_{2,T}))^{-1/2})$.

By Lemma S.10, for $c_2 m_{2,T} < \|\mathbf{t}\|$ the expression in (S.76) is bounded by

$$O\left((Tb_{1,T}b_{2,T})^{2a}\right)\int_{c_2\sqrt{Tb_{1,T}b_{2,T}}<\|\mathbf{t}\|\leq a'(Tb_{1,T}b_{2,T})^{\varpi}} e^{-d_3 m^2_{2,T}}d\mathbf{t} + o\left((Tb_{1,T}b_{2,T})^{-1/2}\right),$$

for some $d_3 > 0$. This implies that (S.76) is bounded by $O(((Tb_{1,T}b_{2,T})^{2(\varpi+a)})e^{-d_3 m^2_{2,T}}) + o((Tb_{1,T}b_{2,T})^{-1/2})$ since by Assumptions 10-11 it holds $m_{2,T} \geq \epsilon(Tb_{2,T})^{\epsilon}$ for some $\epsilon > 0$ depending on $q$ and $p$. □

**Lemma S.20.** *Let Assumptions 4, 6-7, 11-12 and* $b_{1,T} + (Tb_{1,T}b_{2,T})^{-1}\log^5(Tb_{2,T}) \to 0$ *hold. Then there exists a* $c_2 > 0$ *such that, for* $\|\mathbf{t}\| \leq c_2\sqrt{Tb_{1,T}b_{2,T}}$ *and a* $d_2 > 0$,

$$|\psi_T(\mathbf{t}) - \mathcal{A}_{2,T}(\mathbf{t}, 2)| \leq \exp\left(-d_2\|\mathbf{t}\|^2\right)\widetilde{F}(\|\mathbf{t}\|)\,O\left((Tb_{1,T}b_{2,T})^{-1/2}\left(b^2_{1,T} + \epsilon_{Tb_{2,T}}(3)\right) + \frac{1}{Tb_{1,T}b_{2,T}}\right),$$

*where* $\widetilde{F}(\|\mathbf{t}\|)$ *is a polynomial in* $\mathbf{t}$ *with bounded coefficients and* $\mathcal{A}_{2,T}(\mathbf{t}, 2)$ *is defined in* (S.73).

*Proof of Lemma S.20.* From Feller (1971, p. 535) for complex $\alpha$ and $\beta$ it holds that $|e^a - 1 - b| \leq e^{\gamma}(|a-b| + |b|^2/2)$, where $\gamma = \max\{|a|, |b|\}$. We set

$$a = \log\psi(\mathbf{t}) - \frac{1}{2}\|i\mathbf{t}\|^2 = (Tb_{1,T}b_{1,T})^{-1/2}\sum_{|\mathbf{r}|=3}\frac{s!}{r_1!r_2!}\overline{\kappa}_{2,T}(r_1, r_2)(it_1)^{r_1}(it_2)^{r_2} + R^*_T(2),$$

where the right-hand side follows from (S.72). Let $b = (Tb_{1,T}b_{1,T})^{-1/2}\overline{B}_{2,T}(3, \mathbf{t})$ where $\overline{B}_{2,T}(3, \mathbf{t})$ is defined after (S.73). Using Lemmas S.18-S.19 for $s = 3$ we have

$$|a-b| \leq \Bigg|(Tb_{1,T}b_{1,T})^{-1/2}O\left(b^2_{1,T} + \epsilon_{Tb_{2,T}}(3)\right)\left((it_2)^3 + (it_1)^2(it_2)\right) + \frac{1}{Tb_{1,T}b_{2,T}}\left(R'_{0,4}(it_2)^4 + R'_{2,2}(it_1)^2(it_1)^2\right)\Bigg| \tag{S.77}$$

$$\leq P_1(\|\mathbf{t}\|)\,O\left((Tb_{1,T}b_{1,T})^{-1/2}\left(b^2_{1,T} + \epsilon_{Tb_{2,T}}(3)\right) + \frac{1}{Tb_{1,T}b_{2,T}}\right),$$

where $P_1$ is a polynomial of degree of 4. Note that $|b|^2/2 \leq P_2(\|\mathbf{t}\|)\,O(Tb_{1,T}b_{1,T})^{-1})$ where $P_2$ is a polynomial of degree 6. Then, for some polynomial $P$

$$|a-b|+\frac{|b|^2}{2}\leq P\left(\|\mathbf{t}\|\right)O\left(\left(Tb_{1,T}b_{1,T}\right)^{-1/2}\left(b_{1,T}^2+\epsilon_{Tb_{2,T}}\left(3\right)\right)+\frac{1}{Tb_{1,T}b_{2,T}}\right).$$

Next, we need to find a bound for $\gamma=\max\left\{|a|,\,|b|\right\}$. For $\|\mathbf{t}\|\leq c_b\sqrt{Tb_{1,T}b_{2,T}}$ with $c_b>0$ we have

$$\begin{aligned}|b|&=\left|\left(Tb_{1,T}b_{1,T}\right)^{-1/2}\overline{B}_{2,T}\left(3,\,\mathbf{t}\right)\right|\leq\|\mathbf{t}\|^2\left\{\frac{1}{3!}\left(Tb_{1,T}b_{1,T}\right)^{-1/2}\left[\left|\Xi_{2,0}(0,\,3)\right|+3\left|\Xi_{2,0}(2,\,1)\right|\|\mathbf{t}\|\right]\right\} \qquad (S.78)\\&\leq\|\mathbf{t}\|^2\left\{\frac{c_b}{3!}\left(\left|\Xi_{2,0}(0,\,3)\right|+3\left|\Xi_{2,0}(2,\,1)\right|\right)\right\}\leq\|\mathbf{t}\|^2T_b,\end{aligned}$$

where $0<T_b<1/4$ by choosing $c_b$ sufficiently small. For a given $a$ we can choose a $c_a>0$ sufficiently small such that, for $\|\mathbf{t}\|\leq c_a\sqrt{Tb_{1,T}b_{1,T}}$,

$$\begin{aligned}|a|\leq&\|\mathbf{t}\|^2\left\{\frac{1}{3!}\left(Tb_{1,T}b_{1,T}\right)^{-1/2}\left[\left|\Xi_{2,0}(0,\,3)\right|+3\left|\Xi_{2,1}(2,\,1)\right|+O\left(b_{1,T}^2+\epsilon_{Tb_{2,T}}\left(3\right)\right)\right]\right. \qquad (S.79)\\&\left.\times\|\mathbf{t}\|+\left(Tb_{1,T}b_{1,T}\right)^{-1}\left[\left|R'_{0,4}\right|+\left|R'_{2,2}\right|\right]\|\mathbf{t}\|^2\right\}\\\leq&\|\mathbf{t}\|^2\left\{\frac{c_a}{3!}\left[\left|\Xi_{2,0}(0,\,3)\right|+3\left|\Xi_{2,0}(2,\,1)\right|+O\left(b_{1,T}^2+\epsilon_{Tb_{2,T}}\left(3\right)\right)\right]+c_a^2\left[\left|R'_{0,4}\right|+\left|R'_{2,2}\right|\right]\right\}\\\leq&\|\mathbf{t}\|^2\left\{\frac{1}{4}+O\left(b_{1,T}^2+\epsilon_{Tb_{2,T}}\left(3\right)\right)\right\}.\end{aligned}$$

From (S.78)-(S.79) we have for $\|\mathbf{t}\|\leq c_2\sqrt{Tb_{1,T}b_{1,T}}$ with $c_2=\min\left\{c_a,\,c_b\right\}$,

$$\exp\left(\gamma\right)\leq\exp\left\{\|\mathbf{t}\|^2\left[\frac{1}{4}+O\left(b_{1,T}^2+\epsilon_{Tb_{2,T}}\left(3\right)\right)\right]\right\},$$

or

$$\exp\left\{-\frac{1}{2}\mathbf{t}^2+\gamma\right\}\leq\exp\left\{\|\mathbf{t}\|^2\left[-\frac{1}{4}+O\left(b_{1,T}^2+\epsilon_{Tb_{2,T}}\left(3\right)\right)\right]\right\}\leq\exp\left\{-d_2\|\mathbf{t}\|^2\right\}, \qquad (S.80)$$

for some $d_2>0$. Note that $\psi\left(\mathbf{t}\right)=\exp\{\frac{1}{2}\|i\mathbf{t}\|^2+a\}$ and $\mathcal{A}_{2,T}\left(\mathbf{t},\,2\right)=\exp\{\frac{1}{2}\|i\mathbf{t}\|^2\}(1+b)$. Using (S.77)-(S.80) the result of the lemma follows. □

#### S.B.3.5 Proof of Theorem 6

Consider the following linear stochastic approximation to $U_T$,

$$\widetilde{U}_T\triangleq v_1\left(1-\frac{1}{2}\overline{c}_1b_{1,T}^{d_f}-\frac{1}{2}\sqrt{4\pi}\left\|K\right\|_2\left\|K_2\right\|_2v_2\left(Tb_{1,T}b_{2,T}\right)^{-1/2}-\frac{1}{2}\overline{c}_2b_{2,T}^2\right). \qquad (S.81)$$

Consider the transformation $\mathbf{s} = (s_1,\, s_2)' = (\widetilde{U}_T\,(h_1,\, v_2)\,,\, v_2)' = \Delta_T\,(\mathbf{v})$ say, and its inverse $\mathbf{v} = \Delta_T^{-1}\,(\mathbf{s}) = (h_1^{\dagger}\,(s_1,\, s_2)\,,\, s_2)'$. Let $\gamma > 0$ be such that

$$\frac{T^{3\gamma}}{(Tb_{1,T}b_{2,T})^{3/2}} \to 0,$$

and define $\mathbf{L}_T = \{\mathbf{v} : |v_i| < l_i T^{\gamma},\, i = 1,\, 2\}$, where $l_i$ are some fixed constants. Using $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$ for $|x| < 1$, we have uniformly in the set $\mathbf{L}_T$,

$$h_1^{\dagger}\,(\mathbf{s}) = s_1 \left[1 + \frac{1}{2}\bar{c}_1 b_{1,T}^{d_f} + \frac{1}{2}\sqrt{4\pi}\,\|K_2\|\,\|K_2\|_2\, s_2\,(Tb_{1,T}b_{2,T})^{-1/2} + \frac{1}{2}\bar{c}_2 b_{2,T}^2\right] + o\left((Tb_{1,T}b_{2,T})^{-1/2}\right).$$

We have $\mathbb{P}(U_T \in \mathbf{C}) = \mathbb{P}(\mathbf{v} \in \Delta_T^{-1}\,(\mathbf{C} \times \mathbb{R}))$ and from Theorem 3,

$$\begin{aligned}&\sup_{\mathbf{C}} \left|\mathbb{P}\left(\mathbf{v} \in \Delta_T^{-1}\,(\mathbf{C} \times \mathbb{R})\right) - \mathbb{Q}_{2,T}^{(2)}\left(\Delta_T^{-1}\,(\mathbf{C} \times \mathbb{R})\right)\right| \\ &\quad = o\left((Tb_{1,T}b_{2,T})^{-1/2}\right) + \text{cost} \sup_{\mathbf{C}} \mathbb{Q}_{2,T}^{(2)}\left(\left(\partial \Delta_T^{-1}\,(\mathbf{C} \times \mathbb{R})\right)^{2\phi_T}\right), \end{aligned} \tag{S.82}$$

where $\phi_T = (Tb_{1,T}b_{2,T})^{-\rho}$, $1/2 < \rho < 1$. From the continuity of $\Delta_T$, we can obtain, for some $c > 0$,

$$Q_{2,T}^{(2)}\left(\left(\partial \Delta_T^{-1}\,(\mathbf{C} \times \mathbb{R})\right)^{2\phi_T}\right) \leq Q_{2,T}^{(2)}\left(\Delta_T^{-1}\,(\partial \mathbf{C})^{c\phi_T} \times \mathbb{R}\right), \tag{S.83}$$

and

$$\begin{aligned} Q_{2,T}^{(2)}\left(\Delta_T^{-1}\,(\mathbf{C} \times \mathbb{R})\right) &= \int_{\mathbf{L}_T \cap \Delta_T^{-1}(\mathbf{C} \times \mathbb{R})} \varphi_2\,(\mathbf{x})\, q_{2,T}^{(2)}\,(\mathbf{x})\, d\mathbf{x} + o\left((Tb_{1,T}b_{2,T})^{-1/2}\right) \\ &= \int_{\mathbf{L}_T^* \cap \{\mathbf{C} \times \mathbb{R}\}} \varphi_2\left(\Delta_T^{-1}\,(\mathbf{s})\right) q_{2,T}^{(2)}\left(\Delta_T^{-1}\,(\mathbf{s})\right) |\mathcal{J}|\, d\mathbf{s} + o\left((Tb_{1,T}b_{2,T})^{-1/2}\right), \end{aligned}$$

where $\varphi_2\,(\cdot)$ is the bivariate standard normal density, $\mathbf{L}_T^* = \Delta_T\,(\mathbf{L}_T)$, and $|\mathcal{J}|$ is the Jacobian of the transformation. Neglecting the terms that contribute $o((Tb_{1,T}b_{2,T})^{-1/2})$ to the integrals, we yield

$$\varphi_2\left(\Delta_T^{-1}\,(\mathbf{s})\right) = \varphi\,(s_1)\,\varphi\,(s_2)\left(1 - \frac{1}{2}s_1^2\left[\bar{c}_1 b_{1,T}^{d_f} + \frac{1}{2}\sqrt{4\pi}\,\|K\|_2\,\|K_2\|_2\, s_2\,(Tb_{1,T}b_{2,T})^{-1/2} + \frac{1}{2}\bar{c}_2 b_{2,T}^2\right]\right), \tag{S.84}$$

and

$$q_{2,T}^{(2)}\,(\mathbf{v}) = 1 + \frac{1}{3!}\,(Tb_{1,T}b_{2,T})^{-1/2}\left(\Xi_{2,0}\,(0,\, 3)\,\mathcal{H}_3\,(v_2) + \Xi_{2,0}\,(2,\, 1)\,\mathcal{H}_2\,(h_1)\,\mathcal{H}_1\,(v_2)\right), \tag{S.85}$$

where

$$|\mathcal{J}| = 1 + \frac{1}{2}\bar{c}_1 b_{1,T}^{d_f} + \frac{1}{2}\sqrt{4\pi}\,\|K_2\|\,\|K_2\|_2\, s_2\,(Tb_{1,T}b_{2,T})^{-1/2} + \frac{1}{2}\bar{c}_2 b_{2,T}^2.$$

For $j=1,\,2,\,3$ let $p_j\left(\mathbf{s}\right)$ denote polynomials not depending on $T$. We have

$$Q^{(2)}_{2,T}\left(\Delta^{-1}_T\left(\mathbf{C}\times\mathbb{R}\right)\right)=\int_{\mathbf{C}}\varphi\left(s_1\right)\left\{\int_{\mathbb{R}}\left[1+p_1\left(\mathbf{s}\right)\left(Tb_{1,T}b_{2,T}\right)^{-1/2}+p_2\left(\mathbf{s}\right)b^{d_f}_{1,T}+p_3\left(\mathbf{s}\right)b^2_{2,T}\right]\varphi\left(s_2\right)ds_2\right\}ds_1 \tag{S.86}$$

$$+\,o\left(\left(Tb_{1,T}b_{2,T}\right)^{-1/2}\right)$$

$$=\int_{\mathbf{C}}\varphi\left(s_1\right)\left[1+r_1\left(s_1\right)\left(Tb_{1,T}b_{2,T}\right)^{-1/2}+r_2\left(s_1\right)b^{d_f}_{1,T}+r_3\left(s_1\right)b^2_{2,T}\right]ds_1$$

$$+\,o\left(\left(Tb_{1,T}b_{2,T}\right)^{-1/2}\right),$$

where $r_j\left(s_1\right)$ are polynomials in $s_1$ for $j=1,\,2,\,3$ with bounded coefficients. Integration with respect to $s_2$ in $\mathbb{R}$ yields $r_1\left(x\right)=0$, $r_2\left(x\right)=-2^{-1}\overline{c}_1\left(x^2-1\right)$ and $r_3\left(x\right)=-2^{-1}\overline{c}_2\left(x^2-1\right)$. Using (S.82)-(S.86) provides the second-order Edgeworth expansion for the linear stochastic approximation $\widetilde{U}_T$. Since Lemma S.21 below shows that $\widetilde{U}_T$ and $U_T$ have the same Edgeworth expansion, the proof is concluded. □

**Lemma S.21.** *Let Assumptions 4, 5 ($p>1$) and 6-8, 11-13 hold. Then, $U_T$ has the same Edgeworth expansion as $\widetilde{U}_T$ uniformly for convex Borel sets up to the order $O((Tb_{1,T}b_{2,T})^{-1/2})$.*

*Proof of Lemma S.21.* We first expand $U_T\left(\mathbf{v}\right)$ around $\mathbf{0}$ in $\mathbf{L}_T$ with $|\eta_2|\leq 1$,

$$U_T=d_Th_1-\frac{1}{2}d^3_T\mathsf{V}_{2,T}h_1v_2\left(Tb_{1,T}b_{2,T}\right)^{-1/2}+U^*_{1,T}\left(Tb_{1,T}b_{2,T}\right)^{-1}, \tag{S.87}$$

where $d_T=\left(1+\mathsf{B}_{2,T}\right)^{-1/2}$ and

$$U^*_{1,T}=\frac{3}{8}\left(1+\mathsf{B}_{2,T}+\eta_2\mathsf{V}_{2,T}v_2\left(Tb_{1,T}b_{2,T}\right)^{-1/2}\right)^{-5/2}\mathsf{V}^2_{2,T}h_1v^2_2.$$

We now express $U_T$ in terms of $\widetilde{U}_T$ where the latter is defined in (S.81). Substituting for $\mathsf{B}_{2,T}$ and $\mathsf{V}_{2,T}$ in (S.87), we yield $U_T=\widetilde{U}_T+U^*_T\left(Tb_{1,T}b_{2,T}\right)^{-1}$ where $U^*_T=\sum^3_{i=1}U^*_{i,T}$,

$$U^*_{2,T}=h_1\left(O\left(\left(b_{1,T}b_{2,T}\right)^{-1}\log T+Tb_{2,T}b^{1+d_f+\varrho}_{1,T}\right)+o\left(Tb^3_{2,T}b_{1,T}\right)\right)$$

and

$$U^*_{3,T}=h_1v_2O\left(\left(Tb_{1,T}b_{2,T}\right)^{1/2}\left(b^2_{1,T}+\epsilon_T\left(2\right)\right)\right).$$

We now show that $U^*_T(Tb_{1,T}b_{2,T})^{-1}$ can be neglected with error $o((Tb_{1,T}b_{2,T})^{1/2})$. This follows from Theorem 2 in Chibisov (1972) provided that the following condition holds,

$$\mathbb{P}\left(\left|U^*_T\right|>\gamma_T\sqrt{Tb_{1,T}b_{2,T}}\right)\leq\sum^3_{i=1}\mathbb{P}\left(\left|U^*_{i,T}\right|>\frac{1}{3}\gamma_T\sqrt{Tb_{1,T}b_{2,T}}\right)=o\left(\left(Tb_{1,T}b_{2,T}\right)^{-1/2}\right), \tag{S.88}$$

for some positive sequence $\{\gamma_T\}$ such that $\gamma_T\to 0$ and $\gamma_T\sqrt{Tb_{1,T}b_{2,T}}\to\infty$. Note that

$$\left(Tb_{1,T}b_{2,T}\right)^{-1/2}U^*_{2,T}=h_1O\left(\left(Tb_{2,T}\right)^{1/2}b^{-3/2}_{1,T}\left(Tb_{2,T}\right)^{-1}\log T+\left(Tb_{2,T}b_{1,T}\right)^{1/2}b^{d_f+\varrho}_{1,T}\right).$$

By Assumption 13 the right-hand side above is $O((Tb_{2,T}b_{1,T})^{-\upsilon})$ for some $\upsilon > 0$. Further,

$$(Tb_{1,T}b_{2,T})^{-1/2}\, U_{3,T}^{*} = h_1 v_2 O\left(b_{1,T}^{2} + \epsilon_T(2)\right) = O((Tb_{2,T}b_{1,T})^{-\upsilon}),$$

for some $\upsilon > 0$. Since $h_1$ and $v_2$ have finite moments of all orders, we can take $\gamma_T = 1/\log T$ and apply Chebyshev's inequality to establish $\mathbb{P}(|U_{i,T}^{*}| > 3^{-1}\gamma_T\sqrt{Tb_{1,T}b_{2,T}}) = o((Tb_{1,T}b_{2,T})^{-1/2})$ for $i = 2,\, 3$.

It remains to show $\mathbb{P}(|U_{1,T}^{*}| > 3^{-1}\gamma_T\sqrt{Tb_{1,T}b_{2,T}}) = o((Tb_{1,T}b_{2,T})^{-1/2})$. We have

$$\begin{aligned}
&\mathbb{P}\left(\left|U_{1,T}^{*}\right| > \frac{1}{3}\gamma_T\sqrt{Tb_{1,T}b_{2,T}}\right)\\
&< \mathbb{P}\left(\left|\frac{3}{8}\mathsf{V}_{2,T}^{2}h_1v_2^{2}\right|(Tb_{1,T}b_{2,T})^{-1/4} > \gamma_T^{1/2}\right)\\
&\quad + \mathbb{P}\left(\left|1 + \mathsf{B}_{2,T} + \eta_2\mathsf{V}_{2,T}v_2\,(Tb_{1,T}b_{2,T})^{-1/2}\right|(Tb_{1,T}b_{2,T})^{-1/4} > \gamma_T^{1/2}\right).\\
&\triangleq A_1 + A_2.
\end{aligned}$$

Using Chebyshev's inequality $A_1 = o((Tb_{1,T}b_{2,T})^{-1/2})$. Using $(Tb_{1,T}b_{2,T})^{-1/10}\,\gamma_T^{-1/5} \to 0$ we yield

$$A_2 < C_2\mathbb{P}\left(\left|v_2\,(Tb_{1,T}b_{2,T})^{-1/2}\right| > c_2\right) = o\left((Tb_{1,T}b_{2,T})^{-1/2}\right),$$

where $C_2$ and $c_2$ are some positive constants and we have used Chebyshev's inequality. □

## S.B.4 Proof of the Results of Section 5

### S.B.4.1 Proof of Theorem 7

Consider first the numerator of $t_{\mathrm{DM},i}$. We have

$$\begin{aligned}
T_n^{1/2}\overline{d}_L &= \delta^2 O_{\mathbb{P}}\left(T_n^{1/2}T_n^{-1}n_\delta\right) + O_{\mathbb{P}}\left(T_n^{1/2}T_n^{-1}\,(T_n - n_\delta)^{1/2}\right)\mathscr{N}(0,\, J_{\mathrm{DM}})\\
&= \delta^2 O_{\mathbb{P}}\left(T_n^{-1/2}n_\delta\right) + O_{\mathbb{P}}(1)\,,
\end{aligned}$$

for some $J_{\mathrm{DM}} \in (0,\, \infty)$ where $n_\delta$ depends on the length of the segment where the mean of $x_t^{(2)}$ shifts by $\delta$. The factor $\delta^2$ follows from the quadratic loss.

Next, we focus on the expansion of the denominator of $t_{\mathrm{DM},i}$ which hinges on which LRV estimator is used. We begin with part (i). Under Assumption 9 $b_{1,T} \to 0$ as $T \to \infty$. Using Theorem S.1,

$$\begin{aligned}
\widehat{J}_{d_L,\mathrm{NW87},T} &= \sum_{k=-\lfloor b_T^{-1}\rfloor}^{\lfloor b_T^{-1}\rfloor}(1 - |b_{1,T}k|)\,\widehat{\Gamma}(k)\\
&= \sum_{k=-\lfloor b_{1,T}^{-1}\rfloor}^{\lfloor b_{1,T}^{-1}\rfloor}(1 - |b_{1,T}k|)\int_0^1 c(u,\, k)\,du
\end{aligned}$$

$$+\sum_{k=-\lfloor b_{1,T}^{-1}\rfloor}^{\lfloor b_{1,T}^{-1}\rfloor}(1-|b_{1,T}k|)\left(2^{-1}\left(\frac{T_b-T_m-1}{T_n}\right)\left(\frac{T_n-T_b-2}{T_n}\right)\delta^4+o_{\mathbb{P}}(1)\right)$$
$$=CJ_{\mathrm{DM}}+\sum_{k=-\lfloor b_{1,T}^{-1}\rfloor}^{\lfloor b_{1,T}^{-1}\rfloor}(1-|b_{1,T}k|)\left(2^{-1}\left(\frac{T_b-T_m-1}{T_n}\right)\left(\frac{T_n-T_b-2}{T_n}\right)\delta^4+o_{\mathbb{P}}(1)\right),$$

for some $C>0$ such that $C<\infty$. By Exercise 1.7.12 in Brillinger (1975),

$$\sum_{k=-\lfloor b_{1,T}^{-1}\rfloor}^{\lfloor b_{1,T}^{-1}\rfloor}(1-|b_{1,T}k|)\exp(-i\omega k)=b_{1,T}\left(\frac{\sin\frac{\lfloor b_{1,T}^{-1}\rfloor\omega}{2}}{\sin\frac{\omega}{2}}\right)^2.$$

Evaluating the expression above at $\omega=0$ and applying L'Hôpital's rule we yield,

$$\sum_{k=-\lfloor b_{1,T}^{-1}\rfloor}^{\lfloor b_{1,T}^{-1}\rfloor}(1-|b_{1,T}k|)=b_{1,T}\left(\frac{\frac{\lfloor b_{1,T}^{-1}\rfloor}{2}}{\frac{1}{2}}\right)^2=\left\lfloor b_{1,T}^{-1}\right\rfloor.$$

Therefore, $\widehat{J}_{d_L,\mathrm{NW87},T}=CJ_{\mathrm{DM}}+\delta^4O_{\mathbb{P}}\left(b_{1,T}^{-1}\right)$ and

$$|t_{\mathrm{DM,NW87}}|\leq\frac{\delta^2O_{\mathbb{P}}\left(T_n^{-1/2}n_\delta\right)+O_{\mathbb{P}}(1)}{\left(\delta^4O\left(b_{1,T}^{-1}\right)\right)^{1/2}}=\frac{\delta^2O\left(T_n^{\zeta}\right)}{\delta^2O\left(b_{1,T}^{-1/2}\right)}=O\left(T_n^{\zeta}b_{1,T}^{1/2}\right),\tag{S.89}$$

which implies $\mathbb{P}_\delta(|t_{\mathrm{DM,NW87}}|>z_\alpha)\to0$.

Under Assumption 10 with $q=1/3$, similar derivations yield $|t_{\mathrm{DM,NW87}}|=O(T_n^{\zeta-1/6})$ and $\mathbb{P}_\delta(|t_{\mathrm{DM,NW87}}|>z_\alpha)\to0$.

In part (ii), $b_{1,T}=T^{-1}$. Proceeding as in (S.89) we have $|t_{\mathrm{DM,KVB}}|=O(T_n^{\zeta-1})$ and $\mathbb{P}_\delta(|t_{\mathrm{DM,KVB}}|>z_\alpha)\to0$ since $T_n^{\zeta-1}\to0$.

Finally, we consider part (iii). Using Theorem 1, we have

$$\widehat{J}_{d_L,\mathrm{DK},T}=\sum_{k=-T_n+1}^{T_n-1}K_1\left(\widehat{b}_{1,T}k\right)\frac{n_T}{T_n}\sum_{r=1}^{\lfloor T_n/n_T\rfloor}\widehat{c}_{\mathrm{DK},T}(rn_T/T,\,k)$$
$$=\sum_{k=-T_n+1}^{T_n-1}K_1\left(\widehat{b}_{1,T}k\right)\frac{n_T}{T_n}\sum_{r=1}^{\lfloor T_n/n_T\rfloor}\Big(c(rn_T/T,\,k)$$
$$+\delta^2\mathbf{1}\left\{\left(|rn_T+k/2+n_{2,T}/2+1)-T_j^0|/n_{2,T}\right)\in(0,\,1)\right\}\Big)+o_{\mathbb{P}}(1)$$

$$= J_{\mathrm{DM}} + \delta^2 O_{\mathbb{P}} \left( \widehat{b}_{1,T}^{-1} \frac{T \widehat{\overline{b}}_{2,T}}{n_T} \frac{n_T}{T_n} \right) + o_{\mathbb{P}} (1) .$$

It follows that

$$|t_{\mathrm{DM,DK}}| = \frac{\delta^2 O_{\mathbb{P}} \left( T_n^{-1/2} n_\delta \right) + O_{\mathbb{P}} (1)}{\left( J_{\mathrm{DM}} + \delta^2 O_{\mathbb{P}} \left( b_{1,T}^{-1} \widehat{\overline{b}}_{2,T} \right) \right)^{1/2}}$$
$$= \delta^2 O \left( T_n^{\zeta} \right),$$

and so $\mathbb{P}_\delta(|t_{\mathrm{DM,DK}}| > z_\alpha) \rightarrow 1$ since $T_n^{\zeta} \rightarrow \infty$. □

# S.C Figures

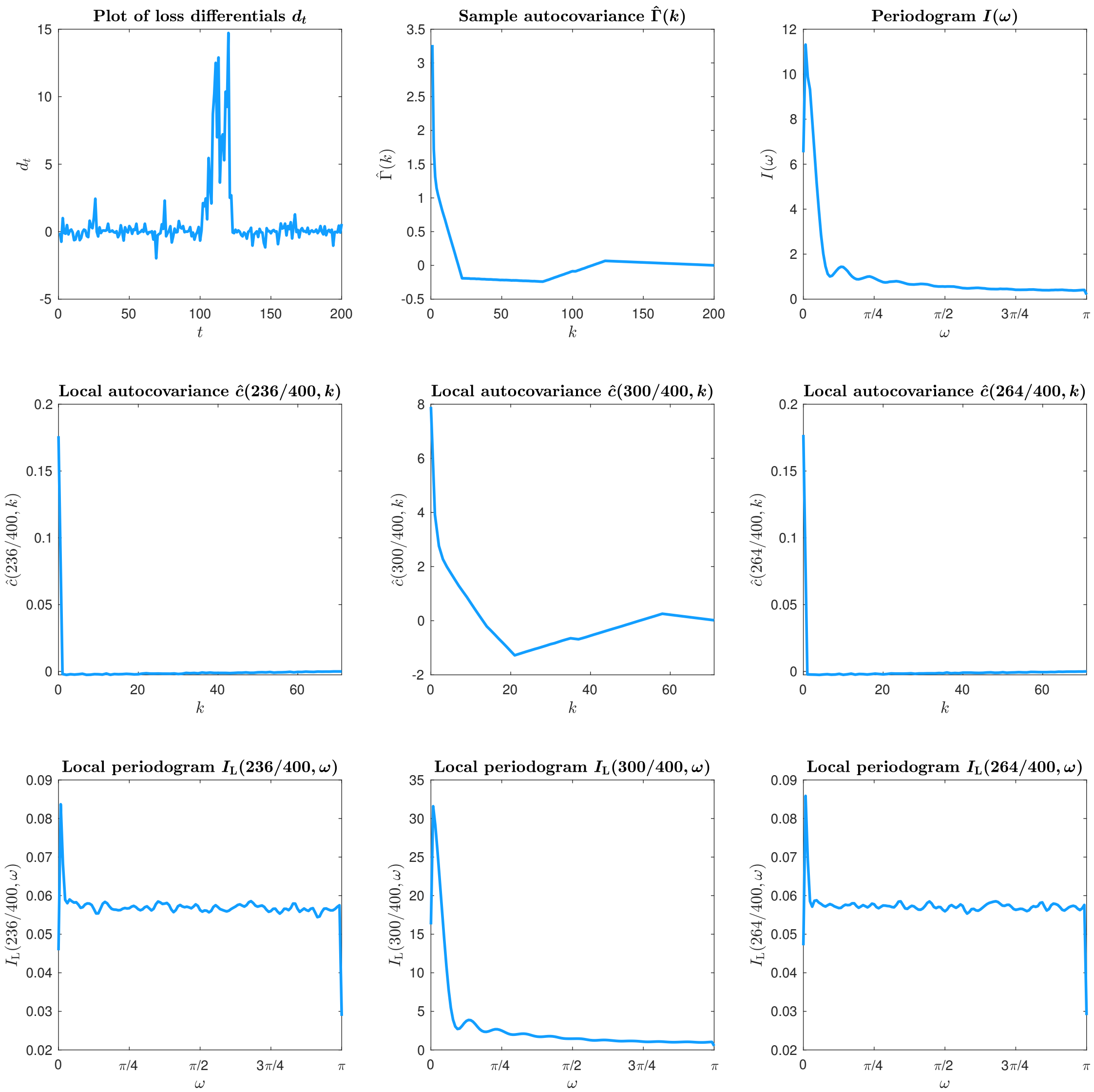


Figure S.1: Plots of loss differentials $d_t$, sample autocovariance $\widehat{\Gamma}(k)$, periodogram $I(\omega)$, sample local autocovariance $\widehat{c}(u, k)$ and local periodogram $I_{\mathrm{L}}(u, \omega)$. In all panels $\delta = 2$.

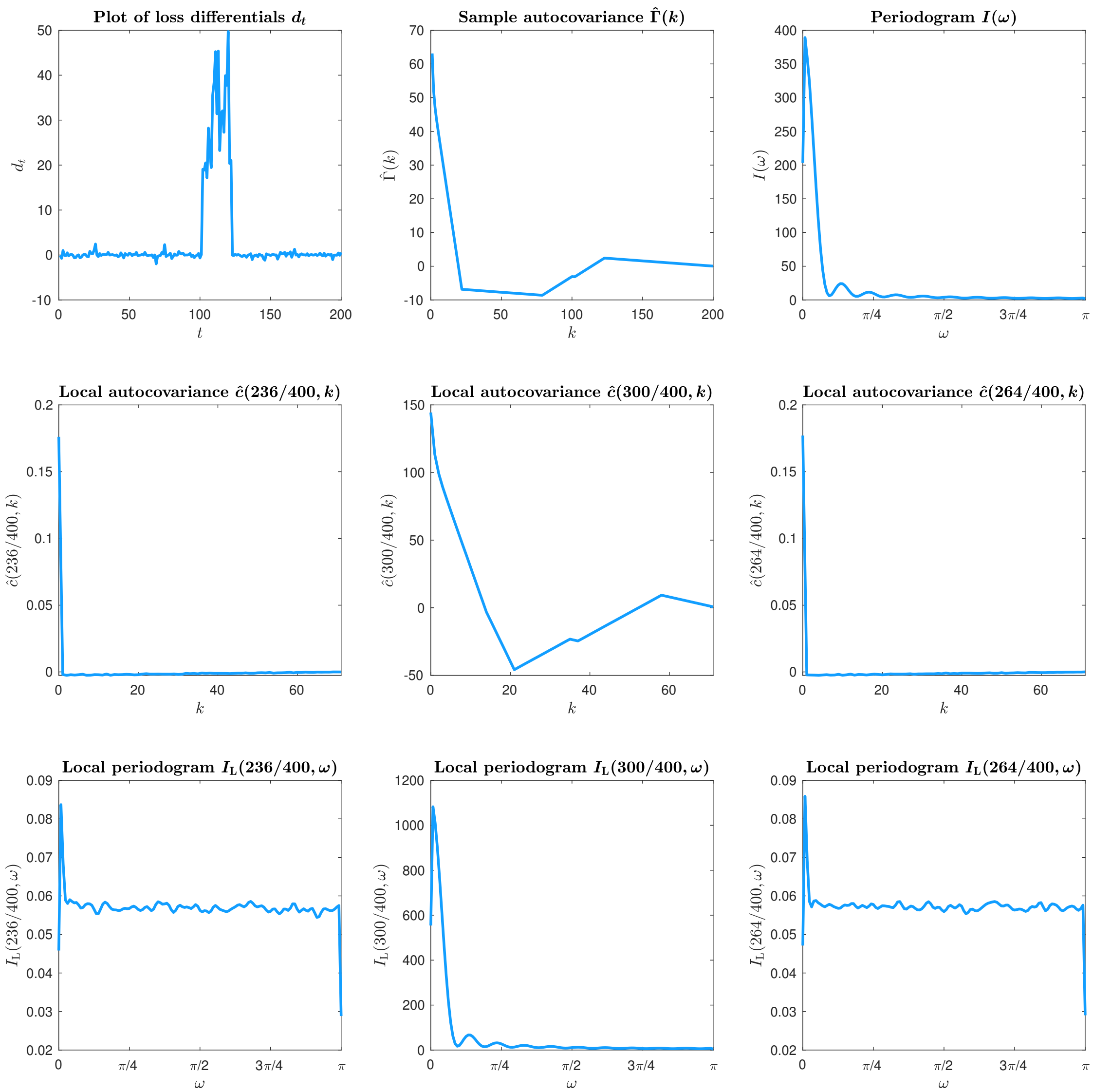


Figure S.2: Plots of loss differentials $d_t$, sample autocovariance $\widehat{\Gamma}(k)$, periodogram $I(\omega)$, sample local autocovariance $\widehat{c}(u, k)$ and local periodogram $I_{\mathrm{L}}(u, \omega)$. In all panels $\delta = 5$.

# References


Anderson, T.W., 1958. An Introduction to Multivariate Statistical Analysis. New York: Wiley.

Andrews, D.W.K., 1991. Heteroskedasticity and autocorrelation consistent covariance matrix estimation. Econometrica 59, 817–858.

Bentkus, R.Y., Rudzkis, R.A., 1982. On the distribution of some statistical estimates of spectral density. Theory of Probability and Its Applications 27, 795–814.

Brillinger, D., 1975. Time Series Data Analysis and Theory. New York: Holt, Rinehart and Winston.

Casini, A., 2022. Comment on Andrews (1991) "Heteroskedasticity and autocorrelation consistent covariance matrix estimation". Econometrica 90, 1–2.

Casini, A., 2023. Theory of evolutionary spectra for heteroskedasticity and autocorrelation robust inference in possibly misspecified and nonstationary models. Journal of Econometrics, 235, 372–392.

Casini, A., Perron, P., 2024. Change-point analysis of time series with evolutionary spectra. Journal of Econometrics 242, 105811.

Chibisov, D.M., 1972. An asymptotic expansion for the distribution of a statistic admitting an asymptotic expansion. Theory of Probability and Its Applications 17, 621–630.

Feller, W., 1971. An Introduction to Probability Theory and Its Applications. New York: Wiley.

Mikosch, T., Stărica, C., 2004. Nonstationarities in financial time series, the long-range dependence, and the IGARCH effects. The Review of Economics and Statistics 86, 378–390.

Roueff, F., von Sachs, R., 2011. Locally stationary long memory estimation. Stochastic Processes and their Applications 121, 813–844.

Taniguchi, M., 1987. Validity of edgeworth expansions of minimum contrast estimators for gaussian ARMA processes. Journal of Multivariate Analysis 21, 1–28.

Taniguchi, M., Puri, M.L., 1996. Valid Edgeworth expansions of M-estimators in regression models with weakly dependent residuals. Econometric Theory 12, 331–346.

Velasco, C., Robinson, P.M., 2001. Edgeworth expansions for spectral density estimates and studentized sample mean. Econometric Theory 17, 497–539.