Online Supplementary Material: Inference in Mildly Explosive Autoregressions under Unconditional Heteroskedasticity

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This online supplement contains four Appendices: Appendix A contains a set of technical lemmas required for the proofs of the main results; Appendix B contains the proofs of the main results in Sections 3 and 4; Appendix C contains additional Monte Carlo results; Appendix D contains an empirical application to US house prices.

Appendix A: Technical Lemmas

This Appendix contains a set of technical lemmas that will be subsequently used in the proofs of the main results in Appendix B. As a matter of notation, we will use $\mathcal{C} = \mathcal{C}[0,1]$ to denote the space of continuous functions on $[0,1]$ and $\mathcal{D}$ the space of right continuous with left limit processes on $[0,1]$, '$\xrightarrow{p}$' to denote convergence in probability, '$\xrightarrow{w}$' to denote weak convergence in the space $\mathcal{D}$ endowed with the Skorohod metric, $\lfloor . \rfloor$ to denote the integer part of its argument, and $\mathbf{1}(\cdot)$ to denote the indicator function. For a random quantity δ, we write $\delta = \delta_0 + o_p(\delta_0)$ as $\delta = \delta_0 + s.o.$, where $s.o.$ represents a term of smaller order in probability. Further, we define $\dot{Y} = \frac{Y}{\nu} + \frac{1}{c}\mathbf{1}(\gamma = 0)$.

Lemma A.1 *For any real numbers a_1, a_2, a_3 that satisfy $0 < a_1 < \infty$, $-\infty < a_2, a_3 < \infty$, $a_1 \geq a_2$, we have $\lim_{T\to\infty} k_T^{-1} \sum_{t=1}^{T-|j|} \rho_T^{-a_1(T-t)+a_2|j|+a_3} < \infty$ holds uniformly in $j = 1, \ldots, T$.*

Lemma A.2 *[Guo et al., 2019] Under Assumptions 1-2, the following limiting results hold jointly:*

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(a) $(a_{k_T} k_T^{3/2} \rho_T^T)^{-1} \sum_{t=1}^{T} \sum_{j=t}^{T} \rho_T^{t-1-j} u_j = o_p(1);$

(b) $(a_{k_T} k_T^{3/2} \rho_T^{2T})^{-1} \sum_{t=1}^{T} \sum_{j=t}^{T} \rho_T^{2(t-1)-j} u_j = o_p(1);$

(c) $(a_{k_T}^2 k_T \rho_T^T)^{-1} \sum_{t=1}^{T} \sum_{j=t}^{T} \rho_T^{t-1-j} u_j u_t = o_p(1);$

(d) $(a_{k_T} k_T \rho_T^T)^{-2} \sum_{t=1}^{T} d_{t-1}^2 = Y_T^2/2c + o_p(1);$

(e) $(a_{k_T} k_T^{3/2} \rho_T^T)^{-1} \sum_{t=1}^{T} d_{t-1} = Y_T/c + o_p(1);$

(f) $(a_{k_T} a_T k_T \rho_T^T)^{-1} \sum_{t=1}^{T} d_{t-1} u_t = X_T Y_T + o_p(1).$

Lemma A.3 *Under Assumptions 1-2, the following limit results hold uniformly in j:*

(a) $(a_{k_T}^2 \mu_T k_T^{3/2} \rho_T^T)^{-2} \sum_{t=1}^{T-|j|} \Big[u_t u_{t+|j|} \big\{ \sum_{i_1=1}^{t-1} \rho_T^{t-1-i_1} u_{i_1} \big\} \big\{ \sum_{i_2=t+|j|}^{T} \rho_T^{t+|j|-1-i_2} u_{i_2} \big\} \Big] = o_p(k_T^{-1});$

(b) $(a_{k_T}^2 \mu_T k_T^{3/2} \rho_T^T)^{-2} \sum_{t=1}^{T-|j|} \Big[u_t u_{t+|j|} \big\{ \sum_{i_1=t}^{T} \rho_T^{t-1-i_1} u_{i_1} \big\} \big\{ \sum_{i_2=1}^{t+|j|-1} \rho_T^{t+|j|-1-i_2} u_{i_2} \big\} \Big] = o_p(k_T^{-1});$

(c) $(a_{k_T}^2 \mu_T k_T^{3/2} \rho_T^T)^{-2} \sum_{t=1}^{T-|j|} \Big[u_t u_{t+|j|} \big\{ \sum_{i_1=t}^{T} \rho_T^{t-1-i_1} u_{i_1} \big\} \big\{ \sum_{i_2=t+|j|}^{T} \rho_T^{t+|j|-1-i_2} u_{i_2} \big\} \Big] = o_p(k_T^{-1}).$

Lemma A.4 *Under Assumptions 1-3, the following limit results hold uniformly in j:*

(a) $\sum_{t=1}^{T-|j|} y_{t-1}^2 y_{t+|j|-1}^2 = O_p(a_{k_T}^4 \mu_T^4 k_T^5 \rho_T^{4T});$

(b) $\sum_{t=1}^{T-|j|} y_{t-1}^2 y_{t+|j|-1} = O_p(a_{k_T}^3 \mu_T^3 k_T^4 \rho_T^{3T}),\ \sum_{t=1}^{T-|j|} y_{t-1} y_{t+|j|-1}^2 = O_p(a_{k_T}^3 \mu_T^3 k_T^4 \rho_T^{3T});$

(c) $\sum_{t=1}^{T-|j|} y_{t-1} y_{t+|j|-1} = O_p(a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T}),\ \sum_{t=1}^{T-|j|} y_{t-1}^2 = O_p(a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T}),$

$\sum_{t=1}^{T-|j|} y_{t+|j|-1}^2 = O_p(a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T}).$

Lemma A.5 *Under Assumptions 1-3, the following limit results hold uniformly in j:*

(a) $\sum_{t=1}^{T-|j|} y_{t-1} y_{t+|j|-1}^2 u_t = O_p(a_{k_T}^3 a_T \mu_T^3 k_T^{7/2} \rho_T^{3T}),\ \sum_{t=1}^{T-|j|} y_{t-1}^2 y_{t+|j|-1} u_{t+|j|} = O_p(a_{k_T}^3 a_T \mu_T^3 k_T^{7/2} \rho_T^{3T});$

(b) $\sum_{t=1}^{T-|j|} y_{t-1} y_{t+|j|-1} u_t = O_p(a_{k_T}^2 a_T \mu_T^2 k_T^{5/2} \rho_T^{2T}),\ \sum_{t=1}^{T-|j|} y_{t-1} y_{t+|j|-1} u_{t+|j|} = O_p(a_{k_T}^2 a_T \mu_T^2 k_T^{5/2} \rho_T^{2T}),$

$\sum_{t=1}^{T-|j|} y_{t+|j|-1}^2 u_t = O_p(a_{k_T}^2 a_T \mu_T^2 k_T^{5/2} \rho_T^{2T}),\ \sum_{t=1}^{T-|j|} y_{t-1}^2 u_{t+|j|} = O_p(a_{k_T}^2 a_T \mu_T^2 k_T^{5/2} \rho_T^{2T});$

(c) $\sum_{t=1}^{T-|j|} y_{t-1} u_t = O_p(a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T),\ \sum_{t=1}^{T-|j|} y_{t-1} u_{t+|j|} = O_p(a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T),$

$$\sum_{t=1}^{T-|j|} y_{t+|j|-1}u_t = O_p(a_{k_T}a_T\mu_T k_T^{3/2}\rho_T^T),\ \sum_{t=1}^{T-|j|} y_{t+|j|-1}u_{t+|j|} = O_p(a_{k_T}a_T\mu_T k_T^{3/2}\rho_T^T).$$

Lemma A.6 *Under Assumptions 1-3, the following limit results hold uniformly in j:*

(a) $\hat{\rho}_T^{-2(T-t)+|j|-2} = \rho_T^{-2(T-t)+|j|-2} + o_p(1)$, $t = 1, ..., T-|j|$;

(b) $\sum_{t=1}^{T-|j|} \hat{\rho}_T^{-2(T-t)+|j|-2} y_{t-1}y_{t+|j|-1} = O_p(a_{k_T}^2\mu_T^2 k_T^3 \rho_T^{2T})$;

(c) $\sum_{t=1}^{T-|j|} \hat{\rho}_T^{-2(T-t)+|j|-2} y_{t-1} = O_p(a_{k_T}\mu_T k_T^2\rho_T^T)$, $\sum_{t=1}^{T-|j|} \hat{\rho}_T^{-2(T-t)+|j|-2} y_{t+|j|-1} = O_p(a_{k_T}\mu_T k_T^2\rho_T^T)$;

(d) $\sum_{t=1}^{T-|j|} \hat{\rho}_T^{-2(T-t)+|j|-2} y_{t-1}u_{t+|j|} = O_p(a_{k_T}a_T\mu_T k_T^{3/2}\rho_T^T)$, $\sum_{t=1}^{T-|j|} \hat{\rho}_T^{-2(T-t)+|j|-2} y_{t+|j|-1}u_t = O_p(a_{k_T}a_T\mu_T k_T^{3/2}\rho_T^T)$;

(e) $\sum_{t=1}^{T-|j|} \hat{\rho}_T^{-2(T-t)+|j|-2} u_t = O_p(a_T k_T^{1/2})$, $\sum_{t=1}^{T-|j|} \hat{\rho}_T^{-2(T-t)+|j|-2} u_{t+|j|} = O_p(a_T k_T^{1/2})$.

Lemma A.7 *Under Assumptions 1-3,* $\sum_{t=1}^{T} y_{t-1}\sigma_t^2 = O_p(a_{k_T}a_T^2\mu_T k_T^2\rho_T^T)$.

Proof of Lemma A.1: We have

$$
\begin{aligned}
& k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-a_1(T-t)+a_2|j|+a_3} = \left|k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-a_1(T-t)+a_2|j|+a_3}\right| \\
= \;& k_T^{-1}\frac{\left|\rho_T^{-(a_1-a_2)|j|+a_3-a_1} - \rho_T^{-a_1(T-1)+a_2|j|+a_3}\right|}{\rho_T^{a_1}-1} = \frac{\left|\rho_T^{-(a_1-a_2)|j|+a_3-a_1} - \rho_T^{-a_1(T-|j|)-(a_1-a_2)|j|+a_1+a_3}\right|}{a_1c+O(k_T^{-1})} \\
\leq \;& \frac{\left|\rho_T^{-(a_1-a_2)|j|+a_3-a_1}\right| + \left|\rho_T^{-a_1(T-|j|)-(a_1-a_2)|j|+a_1+a_3}\right|}{a_1c+O(k_T^{-1})}. \qquad \text{(A.1)}
\end{aligned}
$$

Taking the limit on both sides of $(A.1)$, we have

$$
\begin{aligned}
& \lim_{T\to\infty} k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-a_1(T-t)+a_2|j|+a_3} \leq \lim_{T\to\infty}\frac{\left|\rho_T^{-(a_1-a_2)|j|+a_3-a_1}\right| + \left|\rho_T^{-a_1(T-|j|)-(a_1-a_2)|j|+a_1+a_3}\right|}{a_1c+O(k_T^{-1})} \\
\leq \;& \lim_{T\to\infty}\frac{\left|\rho_T^{a_3-a_1}\right| + \left|\rho_T^{a_1+a_3}\right|}{a_1c+O(k_T^{-1})} = \frac{2}{a_1c} < \infty. \ \blacksquare \qquad \text{(A.2)}
\end{aligned}
$$

Proof of Lemma A.2: The proofs are essentially the same as Lemmas A.4-A.5 in Guo et al. (2019), with additional scaling factors a_{k_T} of certain order, either $a_{k_T}^{-1}$ or $a_{k_T}^{-2}$, added to ensure that the variance of e_t is well-behaved. We omit the proofs here but they are available upon request. ■

Proof of Lemma A.3: (a). Note that

$$\begin{aligned}
& \mathbb{E}\left|k_T(a_{k_T}^2\mu_T k_T^{3/2}\rho_T^T)^{-2}\sum_{t=1}^{T-|j|}\Big[u_t u_{t+|j|}\{\sum_{i_1=1}^{t-1}\rho_T^{t-1-i_1}u_{i_1}\}\{\sum_{i_2=t+|j|}^{T}\rho_T^{t+|j|-1-i_2}u_{i_2}\}\Big]\right| \\
\leq\ & k_T(\mu_T k_T^{3/2}\rho_T^T)^{-2}\sum_{t=1}^{T-|j|}\Big[\{\sum_{i_1=1}^{t-1}\rho_T^{t-1-i_1}\}\{\sum_{i_2=t+|j|}^{T}\rho_T^{t+|j|-1-i_2}\}\mathbb{E}\left|a_{k_T}^{-4}u_t u_{t+|j|}u_{i_1}u_{i_2}\right|\Big] \\
\leq\ & k_T(\mu_T k_T^{3/2}\rho_T^T)^{-2}\sum_{t=1}^{T-|j|}\Big[\{\sum_{i_1=1}^{t-1}\rho_T^{t-1-i_1}\}\{\sum_{i_2=t+|j|}^{T}\rho_T^{t+|j|-1-i_2}\}\Big] \\
& \times\{\mathbb{E}(a_{k_T}^{-1}u_t)^4\}^{1/4}\{\mathbb{E}(a_{k_T}^{-1}u_{t+|j|})^4\}^{1/4}\{\mathbb{E}(a_{k_T}^{-1}u_{i_1})^4\}^{1/4}\{\mathbb{E}(a_{k_T}^{-1}u_{i_2})^4\}^{1/4} \\
\leq\ & k_T(\mu_T k_T^{1/2})^{-2}\rho_T^{-2T}k_T^{-2}\left(\frac{(\rho_T^{T-|j|}-\rho_T^{|j|-T})\rho_T}{(\rho_T-1)^3(\rho_T+1)}-\frac{T-|j|}{(\rho_T-1)^2}\right)\times(\sum_{j=0}^{\infty}|c_j|)^4 K_2^{4/(4+\kappa_2)}K_1^{4/(4+\kappa_1)} \\
=\ & \nu^{-2}\rho_T^{-2T}k_T^{-1}\left(O(k_T^3\rho_T^T)+O(k_T^2 T)\right)\times O(1)\times K_2^{\frac{4}{4+\kappa_2}}K_1^{\frac{4}{4+\kappa_1}} = O(k_T^2\rho_T^{-T}) = o(1), \qquad \text{(A.3)}
\end{aligned}$$

which holds uniformly in j, where the second inequality holds due to the Cauchy-Schwarz inequality and the third inequality holds if $\sup_t\mathbb{E}[(a_{k_T}^{-1}u_t)^4]$ is bounded. This is indeed true as (see, Tanaka, 1996, pp. 501-502)

$$\begin{aligned}
|a_{k_T}^{-1}u_t| &= |\sum_{j=0}^{\infty}c_j a_{k_T}^{-1}\sigma_{t+|j|}\varepsilon_{t+|j|}| \leq \sum_{j=0}^{\infty}|c_j|^{3/4}a_{k_T}^{-1}\sigma_{t+|j|}(|c_j||\varepsilon_{t+|j|}|^4)^{1/4} \\
&\leq \sup_t(a_{k_T}^{-1}\sigma_t)\times(\sum_{j=0}^{\infty}|c_j|)^{3/4}(\sum_{j=0}^{\infty}|c_j||\varepsilon_{t+|j|}|^4)^{1/4}, \qquad \text{(A.4)}
\end{aligned}$$

so that

$$\begin{aligned}
& \sup_t\mathbb{E}(a_{k_T}^{-1}u_t)^4 \leq \sup_t\mathbb{E}(a_{k_T}^{-1}\sigma_t)^4\times(\sum_{j=0}^{\infty}|c_j|)^4\times\sup_t\mathbb{E}(\varepsilon_t^4) \\
\leq\ & (\sum_{j=0}^{\infty}|c_j|)^4 K_2^{4/(4+\kappa_2)}K_1^{4/(4+\kappa_1)} \leq (|c_0|+\sum_{j=0}^{\infty}j|c_j|)^4 K_2^{4/(4+\kappa_2)}K_1^{4/(4+\kappa_1)} < \infty. \qquad \text{(A.5)}
\end{aligned}$$

Finally, the $o(1)$ result in $(A.3)$ follows since $k_T^2\rho_T^{-T} = T^{2\alpha}\rho_T^{-T} = o(1)$, where $0 < \alpha < 1$ (Assumption 1). Then part (a) follows as we have shown its convergence in L^1, which implies convergence in probability. The proofs for (b) and (c) follow very similar steps as the proof for (a) and are hence omitted, but available upon request. ■

Proof of Lemma A.4: (a). Since $y_t = y_0\rho_T^t + \sum_{i=1}^t \rho_T^{t-i}u_i + \mu_T(\rho_T^t - 1)k_T/c$, we have

$$\begin{aligned}
&(a_{k_T}^4\mu_T^4k_T^5\rho_T^{4T})^{-1}\sum_{t=1}^{T-|j|} y_{t-1}^2 y_{t+|j|-1}^2 \\
= \;& (a_{k_T}^4\mu_T^4k_T^5\rho_T^{4T})^{-1}\sum_{t=1}^{T-|j|}\Bigg[\Big(y_0\rho_T^{t-1} + \sum_{i=1}^{t-1}\rho_T^{t-1-i}u_i + \mu_T(\rho_T^{t-1}-1)k_T/c\Big)^2 \\
&\times\Big(y_0\rho_T^{t+|j|-1} + \sum_{i=1}^{t+|j|-1}\rho_T^{t+|j|-1-i}u_i + \mu_T(\rho_T^{t+|j|-1}-1)k_T/c\Big)^2\Bigg] \qquad (A.6)\\
= \;& (a_{k_T}^4\mu_T^4k_T^5\rho_T^{4T})^{-1}\sum_{t=1}^{T-|j|}\Bigg[\Big(y_0\rho_T^{t-1} + \sum_{i=1}^{T}\rho_T^{t-1-i}u_i + \mu_T(\rho_T^{t-1}-1)k_T/c\Big)^2 \\
&\times\Big(y_0\rho_T^{t+|j|-1} + \sum_{i=1}^{T}\rho_T^{t+|j|-1-i}u_i + \mu_T(\rho_T^{t+|j|-1}-1)k_T/c\Big)^2\Bigg] + s.o. \qquad (A.7)\\
= \;& (a_{k_T}^4\mu_T^4k_T^5\rho_T^{4T})^{-1}\sum_{t=1}^{T-|j|}\rho_T^{2(t-1)}\rho_T^{2(t+|j|-1)}\Big(\sum_{i=1}^{T}\rho_T^{-i}u_i + \mu_T k_T/c\Big)^4 + s.o. \\
= \;& (\mu_T^4k_T^2)^{-1}\underbrace{\Big(k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2[T-(t-1)]}\rho_T^{-2[T-(t+|j|-1)]}\Big)}_{=O(1),\text{ by Lemma A.1}}\Big(a_{k_T}^{-1}k_T^{-1/2}\sum_{i=1}^{T}\rho_T^{-i}u_i + a_{k_T}^{-1}c^{-1}\nu\Big)^4 + s.o. \\
\stackrel{w}{\to}\;& \nu^{-5}\times O(1)\times\dot{Y}^4 + o_p(1) = O_p(1), \qquad (A.8)
\end{aligned}$$

where the approximation from $(A.6)$ to $(A.7)$ can be easily proved in a similar manner to $(A.3)$ in Lemma A.3. This proves (a).

(b). Similar to (a), with some algebra, we have both $(a_{k_T}^3\mu_T^3k_T^4\rho_T^{3T})^{-1}\sum_{t=1}^{T-|j|} y_{t-1}^2 y_{t+|j|-1}$ and $(a_{k_T}^3\mu_T^3k_T^4\rho_T^{3T})^{-1}\sum_{t=1}^{T-|j|} y_{t-1} y_{t+|j|-1}^2 \stackrel{w}{\to} \nu^{-4}\times O(1)\times\dot{Y}^3 + o_p(1) = O_p(1)$.

(c). Similar to (a) and (b), with some algebra, we have $(a_{k_T}^2\mu_T^2k_T^3\rho_T^{2T})^{-1}\sum_{t=1}^{T-|j|} y_{t-1}y_{t+|j|-1}$, $(a_{k_T}^2\mu_T^2k_T^3\rho_T^{2T})^{-1}\sum_{t=1}^{T-|j|} y_{t-1}^2$ and $(a_{k_T}^2\mu_T^2k_T^3\rho_T^{2T})^{-1}\sum_{t=1}^{T-|j|} y_{t+|j|-1}^2 \stackrel{w}{\to} \nu^{-3}\times O(1)\times\dot{Y}^2 + o_p(1) =$

$O_p(1)$. ■

Proof of Lemma A.5: (a). The first result is calculated by

$$\begin{aligned}
& (a_{k_T}^3 a_T \mu_T^3 k_T^{7/2} \rho_T^{3T})^{-1} \sum_{t=1}^{T-|j|} y_{t-1} y_{t+|j|-1}^2 u_t \\
= \;& (a_{k_T}^3 a_T \mu_T^4 k_T^5 \rho_T^{4T})^{-1} \sum_{t=1}^{T-|j|} \Bigg[\Big(y_0 \rho_T^{t-1} + \sum_{i=1}^{t} \rho_T^{t-1-i} u_i + \mu_T(\rho_T^{t-1} - 1) k_T / c \Big) \\
& \times \Big(y_0 \rho_T^{t+|j|-1} + \sum_{i=1}^{t+|j|-1} \rho_T^{t+|j|-1-i} u_i + \mu_T(\rho_T^{t+|j|-1} - 1) k_T / c \Big)^2 u_t \Bigg] \\
= \;& (a_{k_T}^3 a_T \mu_T^3 k_T^{7/2} \rho_T^{3T})^{-1} \sum_{t=1}^{T-|j|} \Bigg[\Big(y_0 \rho_T^{t-1} + \sum_{i=1}^{T} \rho_T^{t-1-i} u_i + \mu_T(\rho_T^{t-1} - 1) k_T / c \Big) \\
& \times \Big(y_0 \rho_T^{t+|j|-1} + \sum_{i=1}^{T} \rho_T^{t+|j|-1-i} u_i + \mu_T(\rho_T^{t+|j|-1} - 1) k_T / c \Big)^2 u_t \Bigg] + s.o. \\
= \;& (a_{k_T}^3 a_T \mu_T^3 k_T^{7/2} \rho_T^{3T})^{-1} \sum_{t=1}^{T-|j|} \rho_T^{2(t+|j|-1)} \rho_T^{t-1} u_t \Big(\sum_{i=1}^{T} \rho_T^{-i} u_i + \mu_T k_T / c \Big)^3 + s.o. \\
= \;& (\mu_T^3 k_T^{3/2})^{-1} \underbrace{\Big(a_T^{-1} k_T^{-1/2} \sum_{t=1}^{T-|j|} \rho_T^{-2[T-(t+|j|-1)]} \rho_T^{-[T-(t-1)]} u_t \Big)}_{=O_p(1)} \Big(a_{k_T}^{-1} k_T^{-1/2} \sum_{i=1}^{T} \rho_T^{-i} u_i + a_{k_T}^{-1} c^{-1} \nu \Big)^3 + s.o. \\
\overset{w}{\to} \;& \nu^{-4} \times O_p(1) \times \dot{Y}^3 + o_p(1) = O_p(1), \qquad \text{(A.9)}
\end{aligned}$$

where $\left(a_T^{-1} k_T^{-1/2} \sum_{t=1}^{T-|j|} \rho_T^{-2[T-(t+|j|-1)]} \rho_T^{-[T-(t-1)]} u_t \right) = O_p(1)$ is due to

$$\begin{aligned}
& \mathbb{E} \left(a_T^{-1} k_T^{-1/2} \sum_{t=1}^{T-|j|} \rho_T^{-2[T-(t+|j|-1)]} \rho_T^{-[T-(t-1)]} u_t \right)^2 \\
\leq \;& C(1)^2 \underbrace{\Big(k_T^{-1} \sum_{t=1}^{T-|j|} \rho_T^{-4[T-(t+|j|-1)]} \rho_T^{-2[T-(t-1)]} \Big)}_{=O(1), \text{ by Lemma A.1}} \times \{\sup_t \mathbb{E}(a_T^{-1} e_t)^4\}^{1/2} + s.o. \\
= \;& C(1)^2 \times O(1) \times O(1) = O(1), \qquad \text{(A.10)}
\end{aligned}$$

which uses the result that $\sup_t \mathbb{E}[(a_T^{-1} e_t)^4] \leq \sup_t \mathbb{E}[\varepsilon_t^4] \sup_t \mathbb{E}[(a_T^{-1} \sigma_t)^4] < \infty$. The second result is similar, i.e., $(a_{k_T}^3 a_T \mu_T^3 k_T^{7/2} \rho_T^{3T})^{-1} \sum_{t=1}^{T-|j|} y_{t-1}^2 y_{t+|j|-1} u_{t+|j|-1} =$

$$(\mu_T^3 k_T^{3/2})^{-1} \underbrace{\left(a_T^{-1} k_T^{-1/2} \sum_{t=1}^{T-|j|} \rho_T^{-[T-(t+|j|-1)]} \rho_T^{-2[T-(t-1)]} u_{t+|j|-1}\right)}_{=O_p(1)} \left(a_{k_T}^{-1} k_T^{-1/2} \textstyle\sum_{i=1}^{T} \rho_T^{-i} u_i + a_{k_T}^{-1} c^{-1} \nu\right)^3 +$$

$s.o. \xrightarrow{w} \nu^{-4} \times O_p(1) \times \dot{Y}^3 + o_p(1) = O_p(1)$.

(b). Similar to (a), with some algebra, we have $(a_{k_T}^2 a_T \mu_T^2 k_T^{5/2} \rho_T^{2T})^{-1} \sum_{t=1}^{T-|j|} y_{t-1} y_{t+|j|-1} u_t$, $(a_{k_T}^2 a_T \mu_T^2 k_T^{5/2} \rho_T^{2T})^{-1} \sum_{t=1}^{T-|j|} y_{t-1} y_{t+|j|-1} u_{t+|j|-1}$, $(a_{k_T}^2 a_T \mu_T^2 k_T^{5/2} \rho_T^{2T})^{-1} \sum_{t=1}^{T-|j|} y_{t+|j|-1}^2 u_t$ and $(a_{k_T}^2 a_T \mu_T^2 k_T^{5/2} \rho_T^{2T})^{-1} \sum_{t=1}^{T-|j|} y_{t-1}^2 u_{t+|j|-1} \xrightarrow{w} \nu^{-3} \times O_p(1) \times \dot{Y}^2 + o_p(1) = O_p(1)$.

(c). For a given j, we have

$$\begin{aligned}
& (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-1} \sum_{t=1}^{T-|j|} y_{t-1} u_t \\
= \; & (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-1} \sum_{t=1}^{T-|j|} \left(y_0 \rho_T^{t-1} + \sum_{i=1}^{t} \rho_T^{t-1-i} u_i + \mu_T(\rho_T^{t-1} - 1) k_T/c\right) u_t \\
= \; & (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-1} \sum_{t=1}^{T-|j|} \left(y_0 \rho_T^{t-1} + \sum_{i=1}^{T} \rho_T^{t-1-i} u_i + \mu_T(\rho_T^{t-1} - 1) k_T/c\right) u_t + s.o. \\
= \; & (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-1} \sum_{t=1}^{T-|j|} \rho_T^{t-1} u_t \left(\sum_{i=1}^{T} \rho_T^{-i} u_i + \mu_T k_T/c\right) + s.o. \\
= \; & (\mu_T k_T^{1/2})^{-1} \underbrace{\left(a_T^{-1} k_T^{-1/2} \sum_{t=1}^{T-|j|} \rho_T^{-[T-(t-1)]} u_t\right)}_{=O_p(1)} \left(a_{k_T}^{-1} k_T^{-1/2} \sum_{i=1}^{T} \rho_T^{-i} u_i + a_{k_T}^{-1} c^{-1} \nu\right) + s.o. \\
\xrightarrow{w} \; & \nu^{-2} \times O_p(1) \times \dot{Y} + o_p(1) = O_p(1),
\end{aligned} \tag{A.11}$$

where $a_T^{-1} k_T^{-1/2} \sum_{t=1}^{T-|j|} \rho_T^{-[T-(t-1)]} u_t = O_p(1)$ can be easily proved in a similar manner to $(A.10)$. Using the same approach, it is easy to check that the rest of the three cases (b), (c) and (d) are of the stated orders. ■

Proof of Lemma A.6: Using the fact that $\hat{\rho}_T = \rho_T + O_p(a_{k_T}^{-1} a_T (\mu_T k_T^{3/2} \rho_T^T)^{-1})$, a first order Taylor expansion yields

$$\begin{aligned}
\hat{\rho}_T^{-2(T-t)+|j|-2} - \rho_T^{-2(T-t)+|j|-2} &= (\hat{\rho}_T - \rho_T) \times \underbrace{(-2(T-t) + |j| - 2)}_{O(T) \text{ or smaller}} \underbrace{\rho_T^{-2(T-t)+|j|-3}}_{<\infty} + s.o. \\
&= O_p(a_{k_T}^{-1} a_T (\mu_T k_T^{3/2} \rho_T^T)^{-1}) \times O(T) \times O(1) + s.o. \\
&= \nu^{-1} O_p(a_{k_T}^{-1} a_T T k_T^{-1} \rho_T^{-T}) + s.o. = o_p(1).
\end{aligned} \tag{A.12}$$

For the remaining results (b)-(e), we only prove (b) as the rest of the cases can be proved in a similar manner to this case and cases in previous Lemmas A.4-A.5. We first show the results hold if $\hat{\rho}_T$ is replaced with ρ_T:

$$
\begin{aligned}
& (a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T})^{-1} \sum_{t=1}^{T-|j|} \rho_T^{-2(T-t)+|j|-2} y_{t-1} y_{t+|j|-1} \\
= \;& (a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T})^{-1} \sum_{t=1}^{T-|j|} \Bigg[\rho_T^{-2(T-t)+|j|-2} \Big(y_0 \rho_T^{t-1} + \sum_{i=1}^{t} \rho_T^{t-1-i} u_i + \mu_T (\rho_T^{t-1} - 1) k_T / c \Big) \\
& \times \Big(y_0 \rho_T^{t+|j|-1} + \sum_{i=1}^{t+|j|-1} \rho_T^{t-1-i} u_i + \mu_T (\rho_T^{t+|j|-1} - 1) k_T / c \Big) \Bigg] \\
= \;& (a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T})^{-1} \sum_{t=1}^{T-|j|} \Bigg[\rho_T^{-2(T-t)+|j|-2} \Big(y_0 \rho_T^{t-1} + \sum_{i=1}^{T} \rho_T^{t-1-i} u_i + \mu_T (\rho_T^{t-1} - 1) k_T / c \Big) \\
& \times \Big(y_0 \rho_T^{t+|j|-1} + \sum_{i=1}^{T} \rho_T^{t-1-i} u_i + \mu_T (\rho_T^{t+|j|-1} - 1) k_T / c \Big) \Bigg] + s.o. \\
= \;& (a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T})^{-1} \sum_{t=1}^{T-|j|} \rho_T^{(t+|j|-1)} \rho_T^{t-1} \rho_T^{-2(T-t)+|j|-2} \Big(\sum_{i=1}^{T} \rho_T^{-i} u_i + \mu_T k_T / c \Big)^2 + s.o. \\
= \;& (\mu_T^2 k_T)^{-1} \underbrace{\Big(k_T^{-1} \sum_{t=1}^{T-|j|} \rho_T^{-[T-(t+|j|-1)]} \rho_T^{-[T-(t-1)]} \rho_T^{-2(T-t)+|j|-2} \Big)}_{=O(1), \text{ by Lemma A.1}} \Big(a_{k_T}^{-1} k_T^{-1/2} \sum_{i=1}^{T} \rho_T^{-i} u_i + a_{k_T}^{-1} c^{-1} \nu \Big)^2 + s.o. \\
\stackrel{w}{\to} \;& \nu^{-3} \times O(1) \times \dot{Y}^2 + o_p(1) = O_p(1). \qquad \text{(A.13)}
\end{aligned}
$$

Next, using the results in (a), we have

$$
\begin{aligned}
& (a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T})^{-1} \sum_{t=1}^{T-|j|} \hat{\rho}_T^{-2(T-t)+|j|-2} y_{t-1} y_{t+|j|-1} \\
= \;& (a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T})^{-1} \sum_{t=1}^{T-|j|} \rho_T^{-2(T-t)+|j|-2} y_{t-1} y_{t+|j|-1} + \underbrace{(a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T})^{-1} \sum_{t=1}^{T-|j|} y_{t-1} y_{t+|j|-1}}_{=O_p(1), \text{ by Lemma A.4(c)}} \times o_p(1) \\
= \;& O_p(1). \quad \blacksquare \qquad \text{(A.14)}
\end{aligned}
$$

Proof of Lemma A.7: Since $y_{t-1} = y_0\rho_T^{t-1} + \sum_{i=1}^{t-1}\rho_T^{t-1-i}u_i + \mu_T(\rho_T^{t-1}-1)k_T/c$, we have

$$
\begin{aligned}
& (a_{k_T}a_T^2\mu_T k_T^2\rho_T^T)^{-1}\sum_{t=1}^T y_{t-1}\sigma_t^2 \\
= \quad & (a_{k_T}a_T^2\mu_T k_T^2\rho_T^T)^{-1}\sum_{t=1}^T \Big(y_0\rho_T^{t-1} + \sum_{i=1}^{t-1}\rho_T^{t-1-i}u_i + \mu_T(\rho_T^{t-1}-1)k_T/c\Big)\sigma_t^2 \\
= \quad & (a_{k_T}a_T^2\mu_T k_T^2\rho_T^T)^{-1}\sum_{t=1}^T \Big(\sum_{i=1}^{T}\rho_T^{t-1-i}u_i\Big)\sigma_t^2 + \sum_{t=1}^T \mu_T\rho_T^{t-1}k_T\sigma_t^2/c + s.o. \\
= \quad & (\mu_T k_T^{1/2})^{-1}\Big(a_{k_T}^{-1}k_T^{-1/2}\sum_{i=1}^T\rho_T^{-i}u_i\Big)\Big(a_T^{-2}k_T^{-1}\sum_{t=1}^T\rho_T^{-(T-t)-1}\sigma_t^2\Big) + (a_{k_T}c)^{-1}a_T^{-2}k_T^{-1}\sum_{t=1}^T\rho_T^{-(T-t)-1}\sigma_t^2 + s.o. \\
\overset{w}{\to} \quad & \dot{Y}\times\int_0^\infty \mathrm{e}^{-cr}g(1)^2dr = \frac{g(1)^2\dot{Y}}{c}. \quad \blacksquare
\end{aligned}
\tag{A.15}
$$

References

[1] Guo, G., Sun, Y., Wang, S. (2019), "Testing for moderate explosiveness," *Econometrics Journal* 22, 73-95.

[2] Tanaka, K. (1996), *Time Series Analysis*. Wiley, New York.

Appendix B: Proofs of Main Results

This appendix contains proofs of the main results in the paper. Let P^* denote the bootstrap probability measure and E^* the expectation with respect to P^*. We will use $\mathcal{C} = \mathcal{C}[0,1]$ to denote the space of continuous functions on $[0,1]$ and $\mathcal{D}$ the space of right continuous with left limit processes on $[0,1]$, '$\xrightarrow{p}$' to denote convergence in probability, '$\xrightarrow{w}$' to denote weak convergence in the space $\mathcal{D}$ endowed with the Skorohod metric, '$\xrightarrow{w}_p$' to denote weak convergence in probability under the bootstrap measure (Giné and Zinn, 1990), $\lfloor . \rfloor$ to denote the integer part of its argument, and $\mathbf{1}(\cdot)$ to denote the indicator function. We will write $Z_T^* = o_{p^*}(1)$ if, for any $\epsilon_1 > 0, \epsilon_2 > 0$, $\lim_{T\to\infty} P[P^*(|Z_T^*| > \epsilon_1) > \epsilon_2] = 0$. Similarly, we write $Z_T^* = O_{p^*}(1)$ if, for all $\epsilon > 0$, there exists an $M_\epsilon < \infty$ such that $\lim_{T\to\infty} P[P^*(|Z_T^*| > M_\epsilon) > \epsilon] = 0$. Finally, for a random quantity δ, we write $\delta = \delta_0 + o_p(\delta_0)$ as $\delta = \delta_0 + s.o.$, where $s.o.$ represents a term of smaller order in probability.

Proof of Theorem 1: Let $\widetilde{X}_T = a_T^{-1} k_T^{-1/2} \sum_{t=1}^{T} \rho_T^{-(T-t)-1} e_t$. We first show that $X_T = C(1)\widetilde{X}_T + o_p(1)$. Under Assumption 2, the standard Beveridge-Nelson decomposition (see, e.g., Phillips and Solo, 1992) states that

$$u_t = C(1)e_t + \widetilde{e}_{t-1} - \widetilde{e}_t, \tag{B.1}$$

where $\widetilde{e}_t = \widetilde{C}(L)e_t = \sum_{j=0}^{\infty} \widetilde{c}_j e_{t-|j|}$, $\widetilde{c}_j = \sum_{k=j+1}^{\infty} c_k$. Furthermore, the assumption $\sum_{j=0}^{\infty} j|c_j| < \infty$ ensures $\sum_{j=0}^{\infty} \widetilde{c}_j^2 < \infty$, which implies $\mathbb{E}(a_T^{-2}\widetilde{e}_t^2) \le \sum_{j=0}^{\infty} \widetilde{c}_j^2 \sup_t \mathbb{E}(a_T^{-2}\sigma_t^2) \sup_t \mathbb{E}(\varepsilon_t^2) \le K_2^{2/(4+\kappa_2)} K_1^{2/(4+\kappa_1)} \sum_{j=0}^{\infty} \widetilde{c}_j^2 < \infty$, $\forall t$. We first prove the result for X_T. We have

$$\begin{aligned}
X_T &= a_T^{-1} k_T^{-1/2} \sum_{t=1}^{T} \rho_T^{-(T-t)-1} u_t \\
&= a_T^{-1} k_T^{-1/2} \sum_{t=1}^{T} \rho_T^{-(T-t)-1} C(1) e_t + a_T^{-1} k_T^{-1/2} \sum_{t=1}^{T} \rho_T^{-(T-t)-1} (\widetilde{e}_{t-1} - \widetilde{e}_t) \\
&= C(1)\widetilde{X}_T + a_T^{-1} k_T^{-1/2} \sum_{t=1}^{T} \rho_T^{-(T-t)-1} (\widetilde{e}_{t-1} - \widetilde{e}_t),
\end{aligned} \tag{B.2}$$

where we need to show that the second term $a_T^{-1} k_T^{-1/2} \sum_{t=1}^{T} \rho_T^{-(T-t)-1} (\widetilde{e}_{t-1} - \widetilde{e}_t)$ is $o_p(1)$.

Following standard calculations,

$$
\begin{aligned}
a_T^{-1}\sum_{t=1}^{T}\rho_T^{-(T-t)-1}(\widetilde{e}_{t-1}-\widetilde{e}_t) &= a_T^{-1}\sum_{t=1}^{T}\rho_T^{-(T-t)-1}\widetilde{e}_{t-1} - a_T^{-1}\sum_{t=1}^{T}\rho_T^{-(T-t)-1}\widetilde{e}_t \\
&= a_T^{-1}\sum_{t=0}^{T-1}\rho_T^{-(T-t)}\widetilde{e}_t - a_T^{-1}\sum_{t=1}^{T}\rho_T^{-(T-t)-1}\widetilde{e}_t \\
&= a_T^{-1}\rho_T^{-T}\widetilde{e}_0 - a_T^{-1}\rho_T^{-1}\widetilde{e}_T + a_T^{-1}\sum_{t=1}^{T-1}(\rho_T^{-(T-t)} - \rho_T^{-(T-t)-1})\widetilde{e}_t \\
&= a_T^{-1}\rho_T^{-T}\widetilde{e}_0 - a_T^{-1}\rho_T^{-1}\widetilde{e}_T + ca_T^{-1}k_T^{-1}\sum_{t=1}^{T-1}\rho_T^{-(T-t)-1}\widetilde{e}_t. \qquad \text{(B.3)}
\end{aligned}
$$

Since $\mathbb{E}(a_T^{-2}\widetilde{e}_t^2) < \infty$, $\forall t$, we have $k_T^{-1/2}a_T^{-1}\rho_T^{-T}\widetilde{e}_0 = o_p(1)$ and $k_T^{-1/2}a_T^{-1}\rho_T^{-1}\widetilde{e}_T = o_p(1)$. Further,

$$
\begin{aligned}
\mathbb{E}\left[(ca_T^{-1}k_T^{-1}\sum_{t=1}^{T-1}\rho_T^{-(T-t)-1}\widetilde{e}_t)^2\right] &= c^2k_T^{-2}\sum_{t=1}^{T-1}\sum_{s=1}^{T-1}\left(\rho_T^{-(T-t)-1}\rho_T^{-(T-s)-1}\mathbb{E}(a_T^{-2}\widetilde{e}_t\widetilde{e}_s)\right) \\
&\leq c^2k_T^{-2}O\left((\sum_{t=1}^{T-1}\rho_T^{-(T-t)-1})^2\right)\times O(1) \\
&= c^2k_T^{-2}O\left(k_T^2\right) = O(1), \qquad \text{(B.4)}
\end{aligned}
$$

where the second inequality is due to the fact that, for any $t,s = 1,\ldots,T$, $\mathbb{E}(a_T^{-2}\widetilde{e}_t\widetilde{e}_s) = \sum_{j=0}^{\infty}\widetilde{c}_j\widetilde{c}_{|t-s|+j}\mathbb{E}(a_T^{-2}e_{t-|j|}^2) \leq K_2^{2/(4+\kappa_2)}K_1^{2/(4+\kappa_1)}(\sum_{j=0}^{\infty}\widetilde{c}_j)^2 < \infty$. Thus we have $a_T^{-1}k_T^{-1/2}\sum_{t=1}^{T}\rho_T^{-(T-t)-1}(\widetilde{e}_{t-1} - \widetilde{e}_t) = k_T^{-1/2}O_p(1) = o_p(1)$. Similarly, letting $\widetilde{Y}_T = a_T^{-1}k_T^{-1/2}\sum_{t=1}^{T}\rho_T^{-t}e_t$, and using entirely analogous arguments, we can show that $Y_T = C(1)\widetilde{Y}_T + o_p(1)$.

Next, we establish that as $T\to\infty$, $[\widetilde{X}_T, \widetilde{Y}_T] \xrightarrow{w} [\widetilde{X}, \widetilde{Y}]$, where $\widetilde{X} \sim MN(0, V_{\widetilde{x}})$, $V_{\widetilde{x}} = \frac{g(1)^2}{2c}$, and $\widetilde{Y} \sim MN(0, V_{\widetilde{y}})$, $V_{\widetilde{y}} = \int_0^\infty \mathbb{E}^{-2cr}g(r)^2dr$. We write:

$$
[\widetilde{X}_T, \widetilde{Y}_T] = \sum_{t=1}^{T}\xi_{T,t}\varepsilon_t + o_p(1), \quad \xi_{T,t} = k_T^{-1/2}\left[a_T^{-1}\rho_T^{-(T-t)-1}, a_{k_T}^{-1}\rho_T^{-t}\right]\sigma_t. \qquad \text{(B.5)}
$$

Defining $\mathcal{H} = \sigma\text{-}field\{\{\sigma_s\}_1^T\}$, $\mathcal{F}_{t-1} = \sigma\text{-}field\{\{\sigma_s\}_1^T, \{\varepsilon_s\}_1^{t-1}\}$, and using the law of iterated expectations, the conditional variance can be calculated as:

$$
\begin{aligned}
\mathbb{E}\left[(\widetilde{X}_T,\widetilde{Y}_T)(\widetilde{X}_T,\widetilde{Y}_T)'|\mathcal{H}\right] &= \sum_{t=1}^{T}\mathbb{E}\left[\left\{\mathbb{E}(\xi_{T,t}\xi_{T,t}'\varepsilon_t^2|\mathcal{F}_{t-1})\right\}|\mathcal{H}\right] + o_p(1) \\
&= \sum_{t=1}^{T}\xi_{T,t}\xi_{T,t}' + o_p(1) \\
&= k_T^{-1}\mathrm{diag}\left(a_T^{-2}\sum_{t=1}^{T}\rho_T^{-2(T-t+1)}\sigma_t^2, a_{k_T}^{-2}\sum_{t=1}^{T}\rho_T^{-2t}\sigma_t^2\right) + o_p(1), \quad \text{(B.6)}
\end{aligned}
$$

since the off-diagonal term satisfies

$$
\begin{aligned}
\mathbb{E}\left|a_T^{-1}a_{k_T}^{-1}k_T^{-1}\rho_T^{-(T+1)}\sum_{t=1}^{T}\sigma_t^2\right| &\leq \sup_t\mathbb{E}(a_T^{-2}\sigma_t^2)\times\frac{a_T}{a_{k_T}}\frac{T}{k_T}\rho_T^{-(T+1)} \\
&\leq \{\sup_t\mathbb{E}(a_T^{-1}\sigma_t)^4\}^{1/2}\times\left(\frac{T}{k_T}\right)^{\gamma+1}\rho_T^{-(T+1)}\to 0, \qquad \text{(B.7)}
\end{aligned}
$$

by the second part of Assumption 2(c), Assumption 3, and the fact that for any fixed $\kappa \geq 1$, $\rho_T^{-T}(T/k_T)^\kappa \to 0$ (Proposition A.1 of Phillips and Magdalinos, 2007). Now the first term in $(B.6)$ is

$$
\begin{aligned}
a_T^{-2}k_T^{-1}\sum_{t=1}^{T}\rho_T^{-2(T-t+1)}\sigma_t^2 &= a_T^{-2}k_T^{-1}\sum_{t=1}^{T}\rho_T^{-2t}\sigma_{T-t+1}^2 = a_T^{-2}k_T^{-1}\int_2^{T+1}\rho_T^{-2\lfloor t\rfloor}\sigma_{T-\lfloor t\rfloor+1}^2dt \\
&= \int_{2/k_T}^{(T+1)/k_T}\rho_T^{-2\lfloor k_Tr\rfloor}\left(\frac{\sigma_{T-\lfloor k_Tr\rfloor+1}}{a_T}\right)^2dr \\
&= \int_{2/k_T}^{(T+1)/k_T}\left[\left(1+\frac{c}{k_T}\right)^{k_T}\right]^{-2\frac{\lfloor k_Tr\rfloor}{k_T}}\left(\frac{\sigma_{T-\lfloor k_Tr\rfloor+1}}{a_T}\right)^2dr \\
&\overset{w}{\to}\int_0^\infty \mathrm{e}^{-2cr}g(1)^2dr = \frac{g(1)^2}{2c}, \qquad \text{(B.8)}
\end{aligned}
$$

by the continuous mapping theorem and Assumption 2(c). For the second term,

$$
\begin{aligned}
a_{k_T}^{-2}k_T^{-1}\sum_{t=1}^{T}\rho_T^{-2t}\sigma_t^2 &= a_{k_T}^{-2}k_T^{-1}\int_2^{T+1}\rho_T^{-2\lfloor t\rfloor}\sigma_{\lfloor t\rfloor}^2dt = \int_{2/k_T}^{(T+1)/k_T}\rho_T^{-2\lfloor k_Tr\rfloor}\left(\frac{\sigma_{\lfloor k_Tr\rfloor}}{a_{k_T}}\right)^2dr \\
&= \int_{2/k_T}^{(T+1)/k_T}\left[\left(1+\frac{c}{k_T}\right)^{k_T}\right]^{-2\frac{\lfloor k_Tr\rfloor}{k_T}}\left(\frac{\sigma_{\lfloor k_Tr\rfloor}}{a_{k_T}}\right)^2dr \\
&\overset{w}{\to}\int_0^\infty \mathrm{e}^{-2cr}g(r)^2dr. \qquad \text{(B.9)}
\end{aligned}
$$

The (conditional) Lindeberg condition is implied by the (conditional) Lyapounov condition in view of the bounded $4+\kappa_1$ moments for $\{\varepsilon_t\}$ assumed by Assumption 2(b). Finally, since σ_t is independent of ε_s for any s and t [Assumption 2(d)], condition (2.3) of Wang (2014) is satisfied ensuring asymptotic mixed Gaussianity despite the fact that convergence of the conditional variance obtains in distribution. Further, by diagonality of (B.6), $\widetilde{X}_T$ and $\widetilde{Y}_T$ are asymptotically independent. Hence, X_T and Y_T are also asymptotically independent. This completes our proof. ■

Proof of Theorem 2: The proofs of (a)-(c) are similar to the proofs of the results in Theorem 3.1 in Guo et al. (2019), with some differences, due to the additional scaling factor a_T or a_{k_T}. Specifically, we take (c) as an example to show how to deal with this. To prove (c), by Lemma A.1,

$$
\begin{aligned}
& (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-1} \sum_{t=1}^{T} y_{t-1} u_t = (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-1} \sum_{t=1}^{T} (d_{t-1} + \mu_T(\rho_T^{t-1} - 1) k_T / c) u_t \\
= \ & (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-1} \sum_{t=1}^{T} d_{t-1} u_t + (a_{k_T} a_T k_T^{1/2} \rho_T^T)^{-1} c^{-1} \sum_{t=1}^{T} \rho_T^{t-1} u_t - (a_{k_T} a_T k_T^{1/2} \rho_T^T)^{-1} c^{-1} \sum_{t=1}^{T} u_t \\
= \ & X_T Y_T / (\mu_T k_T^{1/2}) + a_{k_T}^{-1} c^{-1} (a_T^{-1} k_T^{-1/2} \sum_{t=1}^{T} \rho_T^{-(T-t)-1} u_t) + O_p(T^{1/2} (k_T^{1/2} \rho_T^T)^{-1} a_{k_T}^{-1}) \qquad \text{(B.10)} \\
= \ & X_T Y_T / (\mu_T k_T^{1/2}) + X_T / (a_{k_T} c) + o_p(1) \overset{w}{\to} \begin{cases} X(\frac{Y}{\nu} + \frac{1}{c}) & \gamma = 0 \\ \frac{XY}{\nu} & \gamma > 0 \end{cases} := X(\frac{Y}{\nu} + \frac{1}{c} \mathbf{1}(\gamma = 0)).
\end{aligned}
$$

The proofs for (a) and (b) follow Guo et al. (2019) and the steps used in the proof of (c) and are hence omitted. For (d), we have

$$
\begin{aligned}
& a_T^{-1} T^{-1/2} \sum_{t=1}^{T} u_t = a_T^{-1} T^{-1/2} \sum_{t=1}^{T} C(1) e_t + a_T^{-1} T^{-1/2} \sum_{t=1}^{T} (\widetilde{e}_{t-1} - \widetilde{e}_t) \\
= \ & C(1) \times a_T^{-1} T^{-1/2} \sum_{t=1}^{T} e_t + \underbrace{a_T^{-1} T^{-1/2} \widetilde{e}_0}_{=o_p(1)} - \underbrace{a_T^{-1} T^{-1/2} \widetilde{e}_T}_{=o_p(1)} = C(1) \times a_T^{-1} T^{-1/2} \sum_{t=1}^{T} e_t + o_p(1),
\end{aligned}
$$

with $\mathbb{E}[(a_T^{-1} T^{-1/2} \sum_{t=1}^T e_t)^2 | \{\sigma_t\}_1^T] = a_T^{-2} T^{-1} \sum_{t=1}^T \sigma_t^2 \overset{w}{\to} \int_0^1 g(r)^2 dr$. To show the independence of U with X and Y, similar to $(B.7)$, we can show that for the key elements of the cross product of $a_T^{-1} T^{-1/2} \sum_{t=1}^T u_t$ and X_T, it follows

$$
\begin{aligned}
\mathbb{E} \left| T^{-1/2} a_T^{-2} k_T^{-1/2} \sum_{t=1}^{T} \rho_T^{-(T-t+1)} \sigma_t^2 \right| & \le \sup_t \mathbb{E}(a_T^{-2} \sigma_t^2) \times T^{-1/2} c^{-1} k_T^{1/2} (1 - \rho_T^{-T}) \\
& \le \{\sup_t \mathbb{E}(a_T^{-1} \sigma_t)^4\}^{1/2} \times c^{-1} T^{-1/2} k_T^{1/2} \to 0, \qquad \text{(B.11)}
\end{aligned}
$$

while for the cross product of $a_T^{-1}T^{-1/2}\sum_{t=1}^{T}u_t$ and Y_T, it follows

$$\mathbb{E}\left|T^{-1/2}a_T^{-1}a_{k_T}^{-1}k_T^{-1/2}\sum_{t=1}^{T}\rho_T^{-t}\sigma_t^2\right| \leq \sup_t\mathbb{E}(a_{k_T}^{-2}\sigma_t^2)\times\frac{a_{k_T}}{a_T}T^{-1/2}c^{-1}k_T^{1/2}(1-\rho_T^{-T})$$
$$\leq \{\sup_t\mathbb{E}(a_{k_T}^{-1}\sigma_t)^4\}^{1/2}\times\left(\frac{k_T}{T}\right)^{\gamma}c^{-1}T^{-1/2}k_T^{1/2}\to 0. \quad \text{(B.12)}$$

■

Proof of Theorem 3: The proofs are similar to the proofs of Theorem 2, so to save space, we only give a proof of (c). We have

$$\begin{aligned}
&(a_{k_T}a_Tk_T\rho_T^T)^{-1}\sum_{t=1}^{T}y_{t-1}u_t = (a_{k_T}a_Tk_T\rho_T^T)^{-1}\sum_{t=1}^{T}(d_{t-1}+\mu_T(\rho_T^{t-1}-1)k_T/c)u_t \\
=\;& (a_{k_T}a_Tk_T\rho_T^T)^{-1}\sum_{t=1}^{T}d_{t-1}u_t+(\mu_Tk_T^{1/2})(a_{k_T}a_Tk_T^{1/2}\rho_T^T)^{-1}c^{-1}\sum_{t=1}^{T}\rho_T^{t-1}u_t \\
&-(\mu_Tk_T^{1/2})(a_{k_T}a_Tk_T^{1/2}\rho_T^T)^{-1}c^{-1}\sum_{t=1}^{T}u_t \\
=\;& X_TY_T+(\mu_Tk_T^{1/2})a_{k_T}^{-1}c^{-1}(a_T^{-1}k_T^{-1/2}\sum_{t=1}^{T}\rho_T^{-(T-t)-1}u_t)+O_p(T^{1/2}(k_T^{1/2}\rho_T^T)^{-1}a_T^{-1}(\mu_Tk_T^{1/2})) \\
=\;& X_TY_T+\nu X_T/(a_{k_T}c)+o_p(1) = X_TY_T+o_p(1)\xrightarrow{w} XY, \qquad \text{(B.13)}
\end{aligned}$$

where the last equality holds due to the fact that $a_T = T^{\gamma}$ is at least $O(1)$ under $\gamma \geq 0$. ■

Proof of Lemma 1: Let $\widetilde{\Phi}_T(j) = a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)-2}\sum_{k=0}^{\infty}c_kc_{k+|j|}\sigma_{t-k}^2$. We first establish

$\sum_{j=-(T-1)}^{T-1} w(j/b_T)\Phi_T(j) = \sum_{j=-(T-1)}^{T-1} w(j/b_T)\widetilde{\Phi}_T(j) + o_p(1)$. By definition,

$$\begin{aligned}
\Phi_T(j) - \widetilde{\Phi}_T(j) &= a_T^{-2}k_T^{-1}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty} c_k c_i \sum_{t=1}^{T-|j|} \rho_T^{-2(T-t)+|j|-2}\sigma_{t-k}\sigma_{t+|j|-i}\varepsilon_{t-k}\varepsilon_{t+|j|-i} - \widetilde{\Phi}_T(j) \\
&= \underbrace{a_T^{-2}k_T^{-1}\sum_{k=0}^{\infty} c_k c_{k+|j|} \sum_{t=1}^{T-|j|} \rho_T^{-2(T-t)+|j|-2}\sigma_{t-k}^2(\varepsilon_{t-k}^2 - 1)}_{H_{1T}(j)} \\
&\quad + \underbrace{a_T^{-2}k_T^{-1}\sum_{k=0}^{\infty} c_k c_{k+|j|} \sum_{t=1}^{T-|j|} \rho_T^{-2(T-t)+|j|-2}\sigma_{t-k}^2(1 - \rho_T^{-|j|})}_{H_{2T}(j)} \qquad \text{(B.14)} \\
&\quad + \underbrace{a_T^{-2}k_T^{-1}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty} c_k c_i \sum_{t=1}^{T-|j|} \mathbf{1}(i \neq k+|j|)\rho_T^{-2(T-t)+|j|-2}\sigma_{t-k}\sigma_{t+|j|-i}\varepsilon_{t-k}\varepsilon_{t+|j|-i}}_{H_{3T}(j)}.
\end{aligned}$$

To that end, we establish some bounds for $H_{1T}(j)$, $H_{2T}(j)$ and $H_{3T}(j)$. Regarding $H_{1T}(j)$, let $\zeta_{T,j} = a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)+|j|-2}\sigma_{t-k}^2(\varepsilon_{t-k}^2 - 1)$, we have

$$\begin{aligned}
\mathbb{E}[\zeta_{T,j}^2] &= \mathbb{E}\left(a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)+|j|-2}\sigma_{t-k}^2(\varepsilon_{t-k}^2 - 1)\right)^2 \\
&= k_T^{-2}\sum_{t=1}^{T-|j|}\rho_T^{-4(T-t)+2|j|-4}\left\{a_T^{-4}\mathbb{E}\sigma_{t-k}^4\right\}\left\{\mathbb{E}(\varepsilon_{t-k}^2 - 1)^2\right\} \\
&\quad + 2k_T^{-2}\sum_{s<t,s,t=1}^{T-|j|}\rho_T^{-4T+2(t+s)+2|j|-4}\left\{\mathbb{E}(a_T^{-4}\sigma_{t-k}^2\sigma_{s-k}^2)\right\}\underbrace{\left\{\mathbb{E}(\varepsilon_{s-k}^2 - 1)\right\}}_{=0}\underbrace{\left\{\mathbb{E}(\varepsilon_{t-k}^2 - 1)\right\}}_{=0} \\
&= k_T^{-2}\sum_{t=1}^{T-|j|}\rho_T^{-4(T-t)+2|j|-4}\left\{a_T^{-4}\mathbb{E}\sigma_{t-k}^4\right\}\left\{\mathbb{E}\varepsilon_{t-k}^4 - 2\mathbb{E}\varepsilon_{t-k}^2 + 1\right\} \\
&= k_T^{-2}\frac{\rho_T^{-2|j|} - \rho_T^{-4T+2|j|}}{\rho_T^4 - 1}\left\{a_T^{-4}\mathbb{E}\sigma_{t-k}^4\right\}\left\{\mathbb{E}\varepsilon_{t-k}^4 - 1\right\} \\
&= C_1 k_T^{-1}\rho_T^{-2|j|} \leq C_1 k_T^{-1}, \text{ uniformly in } j, \qquad \text{(B.15)}
\end{aligned}$$

where C_1 is a finite positive constant which does not depend on j and T. Then (B.15) implies $\mathbb{E}|\zeta_{T,j}| \leq (\mathbb{E}[\zeta_{T,j}^2])^{1/2} \leq O(k_T^{-1/2})$, uniformly in j. Thus, $\mathbb{E}\Big|H_{1T}(j)\Big| = \mathbb{E}\Big|\sum_{k=0}^{\infty} c_k c_{k+|j|}\zeta_{T,j}\Big| \leq$

$\sum_{k=0}^{\infty}\left|c_k c_{k+|j|}\right|\mathbb{E}\left|\zeta_{T,j}\right| = k_T^{-1/2}C_1^{1/2}\sum_{k=0}^{\infty}\left|c_k c_{k+|j|}\right| = O(k_T^{-1/2})$. Regarding $H_{2T}(j)$,

$$
\begin{aligned}
& \mathbb{E}\Big|H_{2T}(j)\Big| = \mathbb{E}\Big|a_T^{-2}k_T^{-1}\sum_{k=0}^{\infty}c_k c_{k+|j|}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)+|j|-2}\sigma_{t-k}^2(1-\rho_T^{-|j|})\Big| \\
\leq\ & \{\sup_t\mathbb{E}(a_T^{-1}\sigma_{t-k})^4\}^{1/2}\times\sum_{k=0}^{\infty}|c_k c_{k+|j|}|\times k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)+|j|-2}(1-\rho_T^{-|j|}) \\
=\ & C_2\sum_{k=0}^{\infty}|c_k c_{k+|j|}|(\rho_T^{-|j|}-\rho_T^{-2|j|}), \qquad \text{(B.16)}
\end{aligned}
$$

where C_2 is a finite positive constant which does not depend on j and T. Regarding $H_{3T}(j)$,

$$
\begin{aligned}
& \mathbb{E}\Big|H_{3T}(j)\Big| = \mathbb{E}\Big|a_T^{-2}k_T^{-1}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}c_k c_i\sum_{t=1}^{T-|j|}\mathbf{1}(i\neq k+|j|)\rho_T^{-2(T-t)+|j|-2}\sigma_{t-k}\sigma_{t+|j|-i}\varepsilon_{t-k}\varepsilon_{t+|j|-i}\Big| \\
\leq\ & \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}|c_k c_i|\left(\mathbb{E}\left[\mathbf{1}(i\neq k+|j|)\sum_{t=1}^{T-|j|}a_T^{-2}k_T^{-1}\rho_T^{-2(T-t)+|j|-2}\sigma_{t-k}\sigma_{t+|j|-i}\varepsilon_{t-k}\varepsilon_{t+|j|-i}\right]^2\right)^{1/2} \\
=\ & \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}|c_k c_i|\mathbf{1}(i\neq k+|j|)\left(\mathbb{E}\left[\sum_{t=1}^{T-|j|}a_T^{-4}k_T^{-2}\rho_T^{-4(T-t)+2|j|-4}\sigma_{t-k}^2\sigma_{t+|j|-i}^2\varepsilon_{t-k}^2\varepsilon_{t+|j|-i}^2\right]\right. \\
& \left. +2\sum_{s<t,s,t=1}^{T-|j|}a_T^{-4}k_T^{-2}\rho_T^{-4T+2(t+s)+2|j|-4}\{\mathbb{E}(\sigma_{t-k}\sigma_{s-k}\sigma_{t+|j|-i}\sigma_{s+|j|-i})\}\underbrace{\{\mathbb{E}(\varepsilon_{t-k}\varepsilon_{s-k}\varepsilon_{t+|j|-i}\varepsilon_{s+|j|-i})\}}_{=0}\right)^{1/2} \\
\leq\ & \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}|c_k c_i|\mathbf{1}(i\neq k+|j|)\left(k_T^{-2}\sum_{t=1}^{T-|j|}\rho_T^{-4(T-t)+2|j|-4}\right. \\
& \left.\times\{\sup_t\mathbb{E}(a_T^{-1}\sigma_{t-k})^4\}^{1/2}\{\sup_t\mathbb{E}(a_T^{-1}\sigma_{t+|j|-i})^4\}^{1/2}\{\sup_t\mathbb{E}\varepsilon_{t-k}^4\}^{1/2}\{\sup_t\mathbb{E}\varepsilon_{t+|j|-i}^4\}^{1/2}\right)^{1/2} \\
=\ & k_T^{-1/2}C_3\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}|c_k c_i|\mathbf{1}(i\neq k+|j|)\rho_T^{-|j|}\leq k_T^{-1/2}C_3(\sum_{k=0}^{\infty}|c_k|)^2, \text{ uniformly in } j, \qquad \text{(B.17)}
\end{aligned}
$$

where C_3 is a finite positive constant which does not depend on j and T. Now combining the above three bounds, we claim

$$\begin{aligned}
&\mathbb{E}\left|\sum_{j=-(T-1)}^{T-1} w(j/b_T)[\Phi_T(j)-\widetilde{\Phi}_T(j)]\right| \\
\leq\ & \mathbb{E}\left|\sum_{j=-(T-1)}^{T-1} w(j/b_T)H_{1T}(j)\right| + \mathbb{E}\left|\sum_{j=-(T-1)}^{T-1} w(j/b_T)H_{2T}(j)\right| + \mathbb{E}\left|\sum_{j=-(T-1)}^{T-1} w(j/b_T)H_{3T}(j)\right| \\
=\ & o(1)+o(1)+o(1)=o(1). \qquad \text{(B.18)}
\end{aligned}$$

Define $H_{iT} = \sum_{j=-(T-1)}^{T-1} w(j/b_T)H_{iT}(j),\ i = 1,2,3$. The claim (B.18) holds if $\mathbb{E}|H_{iT}| = o(1),\ i=1,2,3$. We consider each of these terms in turn.

For $\mathbb{E}|H_{1T}|$,

$$\begin{aligned}
&\mathbb{E}\left|\sum_{j=-(T-1)}^{T-1} w(j/b_T)H_{1T}(j)\right| \leq k_T^{-1/2}\mathrm{sup}_x|w(x)|C_1^{1/2}\sum_{j=-(T-1)}^{T-1}\sum_{k=0}^{\infty}|c_kc_{k+|j|}| \\
\leq\ & k_T^{-1/2}\times 1\times C_1^{1/2}\sum_{k=0}^{\infty}|c_k|\sum_{j=-(T-1)}^{T-1}|c_{k+|j|}| \leq k_T^{-1/2}C_1^{1/2}(\sum_{k=0}^{\infty}|c_k|)^2 = O(k_T^{-1/2}). \text{(B.19)}
\end{aligned}$$

For $\mathbb{E}|H_{2T}|$,

$$\begin{aligned}
&\mathbb{E}\left|\sum_{j=-(T-1)}^{T-1} w(j/b_T)H_{2T}(j)\right| \leq C_2\sum_{j=-(T-1)}^{T-1}|w(j/b_T)|\sum_{k=0}^{\infty}|c_kc_{k+|j|}|(\rho_T^{-|j|}-\rho_T^{-2|j|}) \\
=\ & 2C_2\left(\sum_{j=0}^{b_T}|w(j/b_T)|\sum_{k=0}^{\infty}|c_kc_{k+j}|(\rho_T^{-j}-\rho_T^{-2j}) + \sum_{j=b_T+1}^{T-1}|w(j/b_T)|\sum_{k=0}^{\infty}|c_kc_{k+j}|\underbrace{(\rho_T^{-j}-\rho_T^{-2j})}_{\leq 1}\right) \\
\leq\ & 2C_2\left(\mathrm{sup}_x|w(x)|(\sum_{k=0}^{\infty}|c_k|)^2\sum_{j=0}^{b_T}(\rho_T^{-j}-\rho_T^{-2j}) + \mathrm{sup}_x|w(x)|\sum_{j=b_T+1}^{T-1}\sum_{k=0}^{\infty}|c_kc_{k+j}|\right) \\
\leq\ & 2C_2\left(\mathrm{sup}_x|w(x)|(\sum_{k=0}^{\infty}|c_k|)^2\times O(k_T^{-1}b_T^2) + \mathrm{sup}_x|w(x)|\sum_{k=0}^{\infty}|c_k|\sum_{j=b_T+1}^{T-1}|c_{k+j}|\right) \\
\leq\ & O(k_T^{-1}b_T^2) + \mathrm{sup}_x|w(x)|\sum_{k=0}^{\infty}|c_k|\sum_{j=b_T+1}^{T-1}|c_j| = o(1)+o(1)=o(1), \qquad \text{(B.20)}
\end{aligned}$$

which uses the fact that, by binomial expansion,

$$\begin{aligned}
&\sum_{j=0}^{b_T}(\rho_T^{-j}-\rho_T^{-2j}) = \sum_{j=0}^{b_T}\rho_T^{-j} - \sum_{j=0}^{b_T}\rho_T^{-2j} = \frac{1-\rho_T^{-b_T-1}}{1-\rho_T^{-1}} - \frac{1-\rho_T^{-2b_T-2}}{1-\rho_T^{-2}} \\
= \quad & \frac{\rho_T+\rho_T^{-2b_T}-\rho_T^{-b_T+1}-\rho_T^{-b_T}}{\rho_T^2-1} = \frac{[\rho_T^{2b_T+1}+1]-[\rho_T^{b_T}(1+\rho_T)]}{\rho_T^{2b_T}(\rho_T^2-1)} \\
= \quad & \frac{[2+c(2b_T+1)k_T^{-1}+O(b_T^2k_T^{-2})]-[(1+cb_Tk_T^{-1}+O(b_T^2k_T^{-2}))(2+ck_T^{-1})]}{O(ck_T^{-1})} \\
= \quad & \frac{O(b_T^2k_T^{-2})}{O(ck_T^{-1})} = O(k_T^{-1}b_T^2),
\end{aligned} \tag{B.21}$$

and $\sum_{j=b_T+1}^{T-1}|c_j| = o(b_T^{-1})$, (see Chang and Park, 2002, p.434).

For $\mathbb{E}|H_{3T}|$,

$$\begin{aligned}
&\mathbb{E}\left|\sum_{j=-(T-1)}^{T-1} w(j/b_T)H_{3T}(j)\right| \leq \sum_{j=-(T-1)}^{T-1}|w(j/b_T)|k_T^{-1/2}C_3(\sum_{k=0}^{\infty}|c_k|)^2 \\
= \quad & \left(k_T^{-1/2}\sum_{j=-(T-1)}^{T-1}|w(j/b_T)|\right)\times C_3(\sum_{k=0}^{\infty}|c_k|)^2 = o(1),
\end{aligned} \tag{B.22}$$

since $k_T^{-1/2}\sum_{j=-(T-1)}^{T-1}|w(j/b_T)| = \left(k_T^{-1/2}b_T\right)\times\left(b_T^{-1}\sum_{j=-(T-1)}^{T-1}|w(j/b_T)|\right) = O(k_T^{-1/2}b_T) = o(1)$ using the fact that $b_T^{-1}\sum_{j=-(T-1)}^{T-1}|w(j/b_T)| \to \int_{-\infty}^{+\infty}|w(x)|dx < +\infty$. Therefore, to prove the lemma, it suffices to show that $\sum_{j=-(T-1)}^{T-1} w(j/b_T)\widetilde{\Phi}_T(j) \xrightarrow{w} V_x$. We can write

$$\begin{aligned}
&\sum_{j=-(T-1)}^{T-1} w(j/b_T)\widetilde{\Phi}_T(j) = \sum_{j=-(T-1)}^{T-1} w(j/b_T)a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)-2}\sum_{k=0}^{\infty}c_kc_{k+|j|}\sigma_{t-k}^2 \\
= \quad & \sum_{j=-(T-1)}^{T-1} w(j/b_T)a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)-2}\sum_{k=0}^{\infty}c_kc_{k+|j|}\sigma_t^2 \\
& + \sum_{j=-(T-1)}^{T-1} w(j/b_T)a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)-2}\sum_{k=0}^{\infty}c_kc_{k+|j|}(\sigma_{t-k}^2-\sigma_t^2) := G_{1T}+G_{2T}.
\end{aligned} \tag{B.23}$$

The proof is then completed if we prove $G_{1T} \xrightarrow{w} V_x$, and $G_{2T} = o_p(1)$. Regarding G_{1T}, we have

$$
\begin{aligned}
&\sum_{j=-(T-1)}^{T-1} w(j/b_T)a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)-2}\sum_{k=0}^{\infty}c_kc_{k+|j|}\sigma_t^2 \\
= &\sum_{j=-(T-1)}^{T-1} w(j/b_T)a_T^{-2}k_T^{-1}\sum_{t=1}^{T}\rho_T^{-2(T-t)-2}\sum_{k=0}^{\infty}c_kc_{k+|j|}\sigma_t^2 \\
&-\sum_{j=-(T-1)}^{T-1} w(j/b_T)a_T^{-2}k_T^{-1}\sum_{t=T-|j|+1}^{T}\rho_T^{-2(T-t)-2}\sum_{k=0}^{\infty}c_kc_{k+|j|}\sigma_t^2. \qquad \text{(B.24)}
\end{aligned}
$$

By Lemma 6 of Jansson (2002), i.e., $\sum_{j=-(T-1)}^{T-1} w(j/b_T)\sum_{k=0}^{\infty}c_kc_{k+|j|}\to C(1)^2$, we can write the first term in (B.24) as

$$
\begin{aligned}
&\sum_{j=-(T-1)}^{T-1} w(j/b_T)a_T^{-2}k_T^{-1}\sum_{t=1}^{T}\rho_T^{-2(T-t)-2}\sum_{k=0}^{\infty}c_kc_{k+|j|}\sigma_t^2 \\
= &\left(\sum_{j=-(T-1)}^{T-1} w(j/b_T)\sum_{k=0}^{\infty}c_kc_{k+|j|}\right)\left(a_T^{-2}k_T^{-1}\sum_{t=1}^{T}\rho_T^{-2(T-t)-2}\sigma_t^2\right) \\
\xrightarrow{w} &\ C(1)^2\lim_{T\to\infty}\int_0^{\infty}\mathrm{e}^{-2cr}g(1)^2dr=\frac{C(1)^2g(1)^2}{2c}=V_x. \qquad \text{(B.25)}
\end{aligned}
$$

The second term in (B.24) converges in probability to zero since

$$
\begin{aligned}
&\mathbb{E}\left|\sum_{j=-(T-1)}^{T-1} w(j/b_T)a_T^{-2}k_T^{-1}\sum_{t=T-|j|+1}^{T}\rho_T^{-2(T-t)-2}\sum_{k=0}^{\infty}c_kc_{k+|j|}\sigma_t^2\right| \qquad \text{(B.26)}\\
\leq &\ k_T^{-1}\mathrm{sup}_x|w(x)|\{\mathrm{sup}_t\mathbb{E}(a_T^{-1}\sigma_t)^4\}^{1/2}\sum_{j=-(T-1)}^{T-1}\sum_{t=T-|j|+1}^{T}\rho_T^{-2(T-t)-2}\sum_{k=0}^{\infty}|c_kc_{k+|j|}| \\
\leq &\ k_T^{-1}K_2^{2/(4+\kappa_2)}\sum_{j=-(T-1)}^{T-1}|j|\sum_{k=0}^{\infty}|c_kc_{k+|j|}|\leq k_T^{-1}K_2^{2/(4+\kappa_2)}\sum_{k=0}^{\infty}|c_k|\sum_{j=-(T-1)}^{T-1}|j||c_{k+|j|}|=O(k_T^{-1}),
\end{aligned}
$$

where the second inequality in (B.26) uses the fact that $\rho_T^{-2(T-t)-2}\leq 1$ for $t=T-|j|+1,\ldots,T$, and the last equality in (B.26) holds since by Assumption 2(a), $\sum_{j=0}^{\infty}j|c_j|<\infty$, we have for any $k\geq 0$, $\sum_{j=0}^{\infty}j|c_{k+j}|\leq\sum_{j=0}^{\infty}(k+j)|c_{k+j}|\leq\sum_{j=0}^{\infty}j|c_j|<\infty$.

Regarding G_{2T}, let $m_T=K_3T^{\omega}$, $0<\omega<1$ and K_3 is a constant which does not depend on T. It follows

$$
\begin{aligned}
& \mathbb{E}\left(\sum_{j=-(T-1)}^{T-1}|w(j/b_T)|a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)-2}\sum_{k=0}^{\infty}|c_kc_{k+|j|}|\left|\sigma_{t-k}^2-\sigma_t^2\right|\right)\\
= \quad & \mathbb{E}\left(\sum_{j=-(T-1)}^{T-1}|w(j/b_T)|\sum_{k=0}^{\infty}|c_kc_{k+|j|}|a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)-2}\left|\sigma_{t-k}^2-\sigma_t^2\right|\right)\\
= \quad & \mathbb{E}\left(\sum_{j=-(T-1)}^{T-1}|w(j/b_T)|\sum_{k=0}^{m_T}|c_kc_{k+|j|}|a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)-2}\left|\sigma_{t-k}^2-\sigma_t^2\right|\right)\\
& +\mathbb{E}\left(\sum_{j=-(T-1)}^{T-1}|w(j/b_T)|\sum_{k=m_T+1}^{\infty}|c_kc_{k+|j|}|a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)-2}\left|\sigma_{t-k}^2-\sigma_t^2\right|\right)\\
\leq \quad & \sum_{j=-(T-1)}^{T-1}|w(j/b_T)|\sum_{k=0}^{m_T}|c_kc_{k+|j|}|\mathbb{E}\left(\sup_{j<m_T}a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)-2}\left|\sigma_{t-k}^2-\sigma_t^2\right|\right)\\
& +\left(\sum_{j=-(T-1)}^{T-1}|w(j/b_T)|\sum_{k=m_T+1}^{\infty}|c_kc_{k+|j|}|k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)-2}\right)\times 2\{\sup_t\mathbb{E}(a_T^{-1}\sigma_t)^4\}^{1/2}\\
\leq \quad & \sup_x|w(x)|(\sum_{k=0}^{\infty}|c_k|)^2\mathbb{E}\left(\sup_{j<m_T,k\leq m_T}a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)-2}\left|\sigma_{t-k}^2-\sigma_t^2\right|\right)\\
& +2\{\sup_t\mathbb{E}(a_T^{-1}\sigma_t)^4\}^{1/2}\sup_x|w(x)|\sum_{k=m_T+1}^{\infty}|c_k|\sum_{j=-(T-1)}^{T-1}|c_{k+|j|}|\times C_4 \qquad \text{(B.27)}\\
= \quad & o(1)+o(1)=o(1),
\end{aligned}
$$

since for the first term in (B.27), following Lemma A.1 in Cavaliere and Taylor (2009), $\sup_{j<m_T,k\leq m_T}a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)-2}\left|\sigma_{t-k}^2-\sigma_t^2\right| \leq k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)-2}\times\sup_{t\leq T-|j|,j<m_T,k\leq m_T}a_T^{-2}|\sigma_{t-k}^2-\sigma_t^2\Big| = k_T^{-1}\times O(k_T)\times o_p(1)=o_p(1)$.[1] For the second term in (B.27), it follows $k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)-2} = C_4\times\rho_T^{-2|j|}\leq C_4$, where C_4 is a finite constant independent of T and j, and $\sum_{k=m_T+1}^{\infty}|c_k|\sum_{j=-(T-1)}^{T-1}|c_{k+|j|}|\leq$

[1]This directly follows from the fact that if $\sigma^2(\cdot)$ has continuous sample paths almost surely, then $\sup_{t\leq T-|j|,j<m_T,k\leq m_T}a_T^{-2}|\sigma_{t-k}^2-\sigma_t^2\Big|=o_p(1)$; if $\sigma^2(\cdot)$ does not have continuous sample paths almost surely, a similar proof of Lemma A.1 in Cavaliere and Taylor (2009) can be given to establish this result.

$(\sum_{k=m_T+1}^{\infty}|c_k|)(\sum_{k=0}^{\infty}|c_k|) \to 0$ as $T \to \infty$. ■

Proof of Theorem 4: The proof of this theorem builds extensively on Lemmas A.3-A.4. By the definition of $\hat{\Omega}$, we have

$$T(a_{k_T}a_T\mu_T k_T^{3/2}\rho_T^T)^{-2}\hat{\Omega} \;=\; \sum_{j=-(T-1)}^{T-1} w(j/b_T)(a_{k_T}a_T\mu_T k_T^{3/2}\rho_T^T)^{-2}T\hat{\Gamma}(j). \tag{B.28}$$

We first work out the details of $(a_{k_T}a_T\mu_T k_T^{3/2}\rho_T^T)^{-2}T\hat{\Gamma}(j)$, using the fact that $\hat{u}_t = \dot{u}_t - \dot{y}_{t-1}(\hat{\rho}_T - \rho_T)$. We have

$$\begin{aligned}
& (a_{k_T}a_T\mu_T k_T^{3/2}\rho_T^T)^{-2}T\hat{\Gamma}(j) \\
= \;& (a_{k_T}a_T\mu_T k_T^{3/2}\rho_T^T)^{-2}\sum_{t=1}^{T-|j|}\dot{y}_{t-1}\hat{u}_t\dot{y}_{t+|j|-1}\hat{u}_{t+|j|} \\
= \;& (a_{k_T}a_T\mu_T k_T^{3/2}\rho_T^T)^{-2}\sum_{t=1}^{T-|j|}\Big[\dot{y}_{t-1}\dot{u}_t\dot{y}_{t+|j|-1}\dot{u}_{t+|j|} - (\hat{\rho}_T - \rho_T)\dot{y}_{t-1}^2\dot{y}_{t+|j|-1}\dot{u}_{t+|j|} \\
& -(\hat{\rho}_T - \rho_T)\dot{y}_{t-1}\dot{y}_{t+|j|-1}^2\dot{u}_t + (\hat{\rho}_T - \rho_T)^2\dot{y}_{t-1}^2\dot{y}_{t+|j|-1}^2\Big] \\
:= \;& A_j - B_j - C_j + D_j.
\end{aligned} \tag{B.29}$$

Specifically, by Lemma A.3 and $\bar{y}_{-1} = O_p(T^{-1}a_{k_T}\mu_T k_T^2\rho_T^T)$ which is implied by Theorem 2(b), we have
B_j:

$$\begin{aligned}
& (a_{k_T}a_T\mu_T k_T^{3/2}\rho_T^T)^{-2}\sum_{t=1}^{T-|j|}(\hat{\rho}_T - \rho_T)\dot{y}_{t-1}^2\dot{y}_{t+|j|-1}\dot{u}_{t+|j|} \\
= \;& (a_{k_T}a_T\mu_T k_T^{3/2}\rho_T^T)^{-2}\sum_{t=1}^{T-|j|}(\hat{\rho}_T - \rho_T)(y_{t-1} - \bar{y}_{-1})^2(y_{t+|j|-1} - \bar{y}_{-1})u_{t+|j|} \\
& -(a_{k_T}a_T\mu_T k_T^{3/2}\rho_T^T)^{-2}\sum_{t=1}^{T-|j|}(\hat{\rho}_T - \rho_T)(y_{t-1} - \bar{y}_{-1})^2(y_{t+|j|-1} - \bar{y}_{-1})\bar{u} := B_{1j} - B_{2j}.
\end{aligned} \tag{B.30}$$

B_{1j}:

$$
\begin{aligned}
& (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \sum_{t=1}^{T-|j|} (\hat{\rho}_T - \rho_T) \dot{y}_{t-1}^2 \dot{y}_{t+|j|-1} u_{t+|j|} \\
= & (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} (\hat{\rho}_T - \rho_T) \sum_{t=1}^{T-|j|} (y_{t-1} - \bar{y}_{-1})^2 (y_{t+|j|-1} - \bar{y}_{-1}) u_{t+|j|} \\
= & (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} (\hat{\rho}_T - \rho_T) \sum_{t=1}^{T-|j|} \Big[y_{t-1}^2 y_{t+|j|-1} u_{t+|j|} - y_{t-1}^2 u_{t+|j|} \bar{y}_{-1} - 2 y_{t-1} y_{t+|j|-1} u_{t+|j|} \bar{y}_{-1} \\
& + 2 y_{t-1} u_{t+|j|} \bar{y}_{-1}^2 + y_{t+|j|-1} u_{t+|j|} \bar{y}_{-1}^2 - u_{t+|j|} \bar{y}_{-1}^3 \Big] \\
= & (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \times O_p((a_{k_T} a_T^{-1} \mu_T k_T^{3/2} \rho_T^T)^{-1}) \times \Big[O_p(a_{k_T}^3 a_T \mu_T^3 k_T^{7/2} \rho_T^{3T}) \\
& + O_p(a_{k_T}^2 a_T \mu_T^2 k_T^{5/2} \rho_T^{2T}) O_p(T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T) + O_p(a_{k_T}^2 a_T \mu_T^2 k_T^{5/2} \rho_T^{2T}) O_p(T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T) \\
& + O_p(a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T) O_p((T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T)^2) + O_p(a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T) O_p((T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T)^2) \\
& + O_p(a_{k_T} T^{1/2}) O_p((T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T)^3) \Big] \\
= & O_p(k_T^{-1}) + O_p(T^{-1}) + O_p(T^{-2} k_T) + O_p(a_{k_T} a_T^{-1} T^{-5/2} k_T^{3/2}) = O_p(k_T^{-1}).
\end{aligned}
$$

B_{2j}:

$$
\begin{aligned}
& (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \sum_{t=1}^{T-|j|} (\hat{\rho}_T - \rho_T) \dot{y}_{t-1}^2 \dot{y}_{t+|j|-1} \bar{u} \\
= & (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} (\hat{\rho}_T - \rho_T) \bar{u} \sum_{t=1}^{T-|j|} (y_{t-1} - \bar{y}_{-1})^2 (y_{t+|j|-1} - \bar{y}_{-1}) \\
= & (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} (\hat{\rho}_T - \rho_T) \bar{u} \sum_{t=1}^{T-|j|} \Big[y_{t-1}^2 y_{t+|j|-1} - y_{t-1}^2 \bar{y}_{-1} - 2 y_{t-1} y_{t+|j|-1} \bar{y}_{-1} \\
& + 2 y_{t-1} \bar{y}_{-1}^2 + y_{t+|j|-1} \bar{y}_{-1}^2 - \bar{y}_{-1}^3 \Big] \\
= & (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \times O_p((a_{k_T} a_T^{-1} \mu_T k_T^{3/2} \rho_T^T)^{-1}) \times O_p(a_T T^{-1/2}) \times \Big[O_p(a_{k_T}^3 \mu_T^3 k_T^4 \rho_T^{3T}) \\
& + O_p(a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T}) O_p(T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T) + O_p(a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T}) O_p(T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T) \\
& + O_p(a_{k_T} \mu_T k_T^2 \rho_T^T) O_p((T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T)^2) + O_p(a_{k_T} \mu_T k_T^2 \rho_T^T) O_p((T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T)^2) \\
& + O_p((T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T)^3) \Big] \\
= & O_p(T^{-1/2} k_T^{-1/2}) + O_p(T^{-3/2} k_T^{1/2}) + O_p(T^{-5/2} k_T^{3/2}) = o_p(k_T^{-1}),
\end{aligned}
$$

which gives $B_j = O_p(k_T^{-1})$. $C_j = O_p(k_T^{-1})$ follows from the same proof as that for B_j.

D_j:

$$
\begin{aligned}
& (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \sum_{t=1}^{T-|j|} (\hat{\rho}_T - \rho_T)^2 \dot{y}_{t-1}^2 \dot{y}_{t+|j|-1}^2 \\
= & (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \sum_{t=1}^{T-|j|} (\hat{\rho}_T - \rho_T)^2 \Big[y_{t-1}^2 y_{t+|j|-1}^2 - 2 y_{t-1}^2 y_{t+|j|-1} \bar{y}_{-1} + y_{t-1}^2 \bar{y}_{-1}^2 - 2 y_{t-1} y_{t+|j|-1}^2 \bar{y}_{-1} \\
& + 4 y_{t-1} y_{t+|j|-1} \bar{y}_{-1}^2 - 2 y_{t-1} \bar{y}_{-1}^3 + y_{t+|j|-1}^2 \bar{y}_{-1}^2 - 2 y_{t+|j|-1} \bar{y}_{-1}^3 + \bar{y}_{-1}^4 \Big] \\
= & (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \times O_p((a_{k_T} a_T^{-1} \mu_T k_T^{3/2} \rho_T^T)^{-2}) \times \Big[O_p(a_{k_T}^4 \mu_T^4 k_T^5 \rho_T^{4T}) \\
& + O_p(a_{k_T}^3 \mu_T^3 k_T^4 \rho_T^{3T}) O_p(T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T) + O_p(a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T}) O_p((T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T)^2) \\
& + O_p(a_{k_T}^3 \mu_T^3 k_T^4 \rho_T^{3T}) O_p(T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T) + O_p(a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T}) O_p((T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T)^2) \\
& + O_p(a_{k_T} \mu_T k_T^2 \rho_T^T) O_p((T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T)^3) + O_p(a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T}) O_p((T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T)^2) \\
& + O_p(a_{k_T} \mu_T k_T^2 \rho_T^T) O_p((T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T)^3) + O_p((T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T)^4) \Big] \\
= & O_p(k_T^{-1}) + O_p(T^{-1}) + O_p(T^{-2} k_T) + O_p(T^{-3} k_T^2) + O_p(T^{-4} k_T^2) = O_p(k_T^{-1}).
\end{aligned}
$$

Thus B_j, C_j and D_j are all $O_p(k_T^{-1})$. We next calculate A_j.

A_j:

$$
\begin{aligned}
& (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \sum_{t=1}^{T-|j|} \dot{y}_{t-1} \dot{u}_t \dot{y}_{t+|j|-1} \dot{u}_{t+|j|} \\
= & (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \sum_{t=1}^{T-|j|} \dot{y}_{t-1} (u_t - \bar{u}) \dot{y}_{t+|j|-1} (u_{t+|j|} - \bar{u}) \\
= & (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \sum_{t=1}^{T-|j|} \dot{y}_{t-1} u_t \dot{y}_{t+|j|-1} u_{t+|j|} + (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \bar{u}^2 \sum_{t=1}^{T-|j|} \dot{y}_{t-1} \dot{y}_{t+|j|-1} \\
& - (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \bar{u} \sum_{t=1}^{T-|j|} \Big[\dot{y}_{t-1} \dot{y}_{t+|j|-1} u_t + \dot{y}_{t-1} \dot{y}_{t+|j|-1} u_{t+|j|} \Big] \\
:= & A_{1j} + A_{2j} - A_{3j}. \qquad \text{(B.31)}
\end{aligned}
$$

In what follows, we derive the orders for A_{2j}, A_{3j}, and claim they are asymptotically negligible. Similar to the foregoing analysis, we have

$$
\begin{aligned}
A_{2j} &= (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \bar{u}^2 \sum_{t=1}^{T-|j|} \dot{y}_{t-1} \dot{y}_{t+|j|-1} && \text{(B.32)} \\
&= (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \times O_p(a_T^2 T^{-1}) \times O_p(a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T}) + s.o. = O_p(T^{-1}) = o_p(k_T^{-1}), \\
A_{3j} &= (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \bar{u} \sum_{t=1}^{T-|j|} \left[\dot{y}_{t-1} \dot{y}_{t+|j|-1} u_t + \dot{y}_{t-1} \dot{y}_{t+|j|-1} u_{t+|j|} \right] && \text{(B.33)} \\
&= (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \times O_p(a_T T^{-1/2}) \times O_p(a_{k_T}^2 a_T \mu_T^2 k_T^{5/2} \rho_T^{2T}) + s.o. \\
&= O_p(T^{-1/2} k_T^{-1/2}) = o_p(k_T^{-1}).
\end{aligned}
$$

Then we analyze A_{1j}:

$$
\begin{aligned}
& (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \sum_{t=1}^{T-|j|} \dot{y}_{t-1} u_t \dot{y}_{t+|j|-1} u_{t+|j|} \\
= \;& (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \sum_{t=1}^{T-|j|} y_{t-1} u_t y_{t+|j|-1} u_{t+|j|} + (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \sum_{t=1}^{T-|j|} u_t u_{t+|j|} \bar{y}_{-1}^2 \\
& - (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \sum_{t=1}^{T-|j|} y_{t-1} u_t u_{t+|j|} \bar{y}_{-1} - (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \sum_{t=1}^{T-|j|} u_t y_{t+|j|-1} u_{t+|j|} \bar{y}_{-1} \\
:= \;& A_{1j,1} + A_{1j,2} - A_{1j,3} - A_{1j,4}, && \text{(B.34)}
\end{aligned}
$$

where $A_{1j,2}$:

$$
\begin{aligned}
& (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \sum_{t=1}^{T-|j|} u_t u_{t+|j|} \bar{y}_{-1}^2 = T^{-1} k_T (T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T)^{-2} \bar{y}_{-1}^2 \times a_T^{-2} T^{-1} \sum_{t=1}^{T-|j|} u_t u_{t+|j|} \\
= \;& O(T^{-1} k_T) O_p(1) \times a_T^{-2} T^{-1} \sum_{t=1}^{T-|j|} u_t u_{t+|j|} = O_p(T^{-1} k_T) \times a_T^{-2} T^{-1} \sum_{t=1}^{T-|j|} u_t u_{t+|j|}. && \text{(B.35)}
\end{aligned}
$$

$A_{1j,3}$:

$$
\begin{aligned}
& (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \sum_{t=1}^{T-|j|} y_{t-1} u_t u_{t+|j|} \bar{y}_{-1} \\
= \;& T^{-1/2} k_T^{1/2} \underbrace{(T^{-1} a_{k_T} \mu_T k_T^2 \rho_T^T)^{-1} \bar{y}_{-1}}_{=O_p(1)} \times (T^{1/2} a_{k_T} a_T^2 \mu_T k_T^{3/2} \rho_T^T)^{-1} \sum_{t=1}^{T-|j|} y_{t-1} u_t u_{t+|j|} \\
= \;& O_p(T^{-1/2} k_T^{1/2}) \times (T^{1/2} a_{k_T} a_T^2 \mu_T k_T^{3/2} \rho_T^T)^{-1} \sum_{t=1}^{T-|j|} y_{t-1} u_t u_{t+|j|}, && \text{(B.36)}
\end{aligned}
$$

$A_{1j,4} = O_p(T^{-1/2}k_T^{1/2}) \times (T^{1/2}a_{k_T}a_T^2\mu_T k_T^{3/2}\rho_T^T)^{-1}\sum_{t=1}^{T-|j|} u_t y_{t+|j|-1}u_{t+|j|}$ follows the same way as $A_{1j,3}$.

To calculate $A_{1j,1}$, let $\dot{Y}_T := (\mu_T k_T^{1/2})^{-1}Y_T + 1/(a_{k_T}c)$. It then follows $\dot{Y}_T \overset{w}{\to} Y/\nu + 1/(c)\mathbf{1}(\gamma = 0)$. By Lemma A.2 and the assumption $a_{k_T}^{-1}y_0 = o_p(k_T^{1/2})$, with some algebra, we have
$A_{1j,1}$:

$$
\begin{aligned}
& (a_{k_T}a_T\mu_T k_T^{3/2}\rho_T^T)^{-2}\sum_{t=1}^{T-|j|} y_{t-1}u_t y_{t+|j|-1}u_{t+|j|} \\
= \; & (a_{k_T}a_T\mu_T k_T^{3/2}\rho_T^T)^{-2}\sum_{t=1}^{T-|j|}\Big[\big\{y_0\rho_T^{t-1} + \sum_{i_1=1}^{t-1}\rho_T^{t-1-i_1}u_{i_1} + \mu_T(\rho_T^{t-1}-1)k_T/c\big\}u_t \times \\
& \big\{y_0\rho_T^{t+|j|-1} + \sum_{i_2=1}^{t+|j|-1}\rho_T^{t+|j|-1-i_2}u_{i_2} + \mu_T(\rho_T^{t+|j|-1}-1)k_T/c\big\}u_{t+|j|}\Big] \\
= \; & (a_{k_T}a_T\mu_T k_T^{3/2}\rho_T^T)^{-2}\sum_{t=1}^{T-|j|}\Big[\big\{\sum_{i_1=1}^{T}\rho_T^{t-1-i_1}u_{i_1} + \mu_T\rho_T^{t-1}k_T/c\big\}u_t \times \\
& \big\{\sum_{i_2=1}^{T}\rho_T^{t+|j|-1-i_2}u_{i_2} + \mu_T\rho_T^{t+|j|-1}k_T/c\big\}u_{t+|j|}\Big] + o_p(k_T^{-1}) \\
= \; & \left(a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\rho_T^{-2(T-t)+|j|-2}u_t u_{t+|j|}\right)\left((\mu_T k_T^{1/2})^{-1}a_{k_T}^{-1}k_T^{-1/2}\sum_{t=1}^{T}\rho_T^{-t}u_t + a_{k_T}^{-1}c^{-1}\right)^2 + o_p(k_T^{-1}) \\
= \; & \Phi_T(j)\dot{Y}_T^2 + o_p(k_T^{-1}). \qquad \text{(B.37)}
\end{aligned}
$$

Now, we combine A_j-D_j and use Lemma 1 to obtain

$$\sum_{j=-(T-1)}^{T-1} w(j/b_T)(A_j - B_j - C_j + D_j) \tag{B.38}$$
$$= \sum_{j=-(T-1)}^{T-1} w(j/b_T)(A_{1j,1} + A_{1j,2} - A_{1j,3} - A_{1j,4} \underbrace{+A_{2j} - A_{3j}}_{o_p(k_T^{-1})} \underbrace{-B_j - C_j + D_j}_{O_p(k_T^{-1})})$$
$$= \sum_{j=-(T-1)}^{T-1} w(j/b_T)(A_{1j,1} + A_{1j,2} - A_{1j,3} - A_{1j,4}) + o_p(1)$$
$$= \sum_{j=-(T-1)}^{T-1} w(j/b_T)\{\underbrace{\Phi_T(j)\dot{Y}_T^2 + o_p(k_T^{-1})}_{A_{1j,1}}\} + \underbrace{\sum_{j=-(T-1)}^{T-1} w(j/b_T)A_{1j,2}}_{\xrightarrow{p}0} - \underbrace{\sum_{j=-(T-1)}^{T-1} w(j/b_T)A_{1j,3}}_{\xrightarrow{p}0}$$
$$- \underbrace{\sum_{j=-(T-1)}^{T-1} w(j/b_T)A_{1j,4}}_{\xrightarrow{p}0} + o_p(1) = \dot{Y}_T^2 \sum_{j=-(T-1)}^{T-1} w(j/b_T)\Phi_T(j) + o_p(1) \xrightarrow{w} V_x(\frac{Y}{\nu} + \frac{1}{c}\mathbf{1}(\gamma = 0))^2.$$

Note in the above $\sum_{j=-(T-1)}^{T-1} w(j/b_T)A_{1j,2} \xrightarrow{p} 0$ holds due to

$$\sum_{j=-(T-1)}^{T-1} w(j/b_T)A_{1j,2} = O_p(T^{-1}k_T) \times \underbrace{a_T^{-2}T^{-1} \sum_{j=-(T-1)}^{T-1} w(j/b_T) \sum_{t=1}^{T-|j|} u_t u_{t+|j|}}_{\xrightarrow{w}\sigma_u^2} \xrightarrow{p} 0,$$

where the result $a_T^{-2}T^{-1} \sum_{j=-(T-1)}^{T-1} w(j/b_T) \sum_{t=1}^{T-|j|} u_t u_{t+|j|} \xrightarrow{w} \sigma_u^2$ can be proved analogously to Lemma 1. Next, $\sum_{j=-(T-1)}^{T-1} w(j/b_T)A_{1j,3} \xrightarrow{p} 0$ holds due to

$$\sum_{j=-(T-1)}^{T-1} w(j/b_T)A_{1j,3} = O_p(T^{-1/2}k_T^{1/2}) \times \underbrace{(T^{1/2}a_{k_T}a_T^2\mu_T k_T^{3/2}\rho_T^T)^{-1} \sum_{j=-(T-1)}^{T-1} w(j/b_T) \sum_{t=1}^{T-|j|} y_{t-1}u_t u_{t+|j|}}_{=O_p(T^{-1/2}k_T^{1/2})} \xrightarrow{p} 0,$$

since

$$
\begin{aligned}
& (T^{1/2}a_{k_T}a_T^2\mu_T k_T^{3/2}\rho_T^T)^{-1}\sum_{j=-(T-1)}^{T-1} w(j/b_T)\sum_{t=1}^{T-|j|} y_{t-1}u_t u_{t+|j|} \\
= \; & (T^{1/2}a_{k_T}a_T^2\mu_T k_T^{3/2}\rho_T^T)^{-1}\sum_{j=-(T-1)}^{T-1} w(j/b_T)\sum_{t=1}^{T-|j|} y_{t-1}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty} c_k c_i \sigma_{t-k}\sigma_{t+|j|-i}\varepsilon_{t-k}\varepsilon_{t+|j|-i} \\
= \; & (T^{1/2}a_{k_T}a_T^2\mu_T k_T^{3/2}\rho_T^T)^{-1}\sum_{j=-(T-1)}^{T-1} w(j/b_T)\sum_{t=1}^{T} y_{t-1}\sum_{k=0}^{\infty} c_k c_{k+|j|}\sigma_{t-k}^2 + o_p(1) \qquad \text{(B.39)} \\
= \; & (T^{1/2}a_{k_T}a_T^2\mu_T k_T^{3/2}\rho_T^T)^{-1}\sum_{j=-(T-1)}^{T-1} w(j/b_T)\sum_{t=1}^{T} y_{t-1}\sum_{k=0}^{\infty} c_k c_{k+|j|}\sigma_{t}^2 + o_p(1) \qquad \text{(B.40)} \\
& \overset{w}{\to} \sum_{j=-(T-1)}^{T-1} w(j/b_T)\sum_{k=0}^{\infty} c_k c_{k+|j|} \times (T^{1/2}a_{k_T}a_T^2\mu_T k_T^{3/2}\rho_T^T)^{-1}\sum_{t=1}^{T} y_{t-1}\sigma_t^2 \\
& \overset{w}{\to} C(1)^2 \times (T^{1/2}a_{k_T}a_T^2\mu_T k_T^{3/2}\rho_T^T)^{-1} O_p(a_{k_T}a_T^2\mu_T k_T^2\rho_T^T) = O_p(T^{-1/2}k_T^{1/2}), \qquad \text{(B.41)}
\end{aligned}
$$

where $(B.39)$ and $(B.40)$ can be established in a similar manner to the proof of Lemma 1, $(B.41)$ holds because of Lemma A.6. Finally, $\sum_{j=-(T-1)}^{T-1} w(j/b_T)A_{1j,4} \overset{p}{\to} 0$ follows from the same arguments used for $A_{1j,3}$. The result of Theorem 4 follows. ■

Proof of Theorem 5: This is proved in the main text, see (18)-(19). ■

Proof of Theorem 6: We will prove the result for X_T^*. First we note that conditionally on $\hat{u}_t$, $\hat{u}_t^*$, $t = 1, \ldots, T$ is normally distributed over time with zero mean, thus the partial sum process is normally distributed with zero mean. In what follows, we focus on deriving its variance. Using the fact that $\mathrm{Cov}(\eta_s, \eta_t) = \mathbb{E}(\eta_s \eta_t) = K(\frac{s-t}{l_T})$, it follows

$$\begin{aligned}
\mathbb{E}^*\left[X_T^{*2}\right] &= \mathbb{E}^*\left[a_T^{-1} k_T^{-1/2} \sum_{t=1}^{T} \hat{\rho}_T^{-(T-t)-1} u_t^*\right]^2 = \mathbb{E}^*\left[a_T^{-1} k_T^{-1/2} \sum_{t=1}^{T} \hat{\rho}_T^{-(T-t)-1} \eta_t \hat{u}_t\right]^2 \\
&= a_T^{-2} k_T^{-1} \sum_{t=1}^{T} \sum_{s=1}^{T} \hat{\rho}_T^{-[2T-(t+s)]-2} \hat{u}_t \hat{u}_s \mathbb{E}(\eta_s \eta_t) \\
&= a_T^{-2} k_T^{-1} \sum_{t=1}^{T} \sum_{s=1}^{T} K(\frac{s-t}{l_T}) \hat{\rho}_T^{-[2T-(t+s)]-2} \hat{u}_t \hat{u}_s \\
&= \sum_{j=-(T-1)}^{T-1} K(\frac{j}{l_T}) \left[a_T^{-2} k_T^{-1} \sum_{t=1}^{T-|j|} \hat{\rho}_T^{-2(T-t)+|j|-2} \hat{u}_t \hat{u}_{t+|j|}\right] \\
&= \sum_{j=-(T-1)}^{T-1} K(\frac{j}{l_T}) \left[a_T^{-2} k_T^{-1} \sum_{t=1}^{T-|j|} \hat{\rho}_T^{-2(T-t)+|j|-2} u_t u_{t+|j|}\right] + o_p(1) \qquad \text{(B.42)} \\
&= \sum_{j=-(T-1)}^{T-1} K(\frac{j}{l_T}) \left[a_T^{-2} k_T^{-1} \sum_{t=1}^{T-|j|} \rho_T^{-2(T-t)+|j|-2} u_t u_{t+|j|}\right] \\
&\quad + o_p(1) \times \underbrace{\sum_{j=-(T-1)}^{T-1} K(\frac{j}{l_T}) \left[a_T^{-2} k_T^{-1} \sum_{t=1}^{T-|j|} u_t u_{t+|j|}\right]}_{\xrightarrow{w} \sigma_u^2} + o_p(1) \qquad \text{(B.43)} \\
&\xrightarrow{w} V_x, \qquad \text{(B.44)}
\end{aligned}$$

where the $o_p(1)$ in $(B.42)$ is not surprising as, under $\hat{u}_t = \dot{u}_t - \dot{y}_{t-1}(\hat{\rho}_T - \rho_T)$,

$$\begin{aligned}
& a_T^{-2} k_T^{-1} \sum_{t=1}^{T-|j|} \hat{\rho}_T^{-2(T-t)+|j|-2} (\hat{u}_t \hat{u}_{t+|j|} - u_t u_{t+|j|}) \\
=\ & a_T^{-2} k_T^{-1} \sum_{t=1}^{T-|j|} \hat{\rho}_T^{-2(T-t)+|j|-2} \Big[(\dot{u}_t \dot{u}_{t+|j|} - u_t u_{t+|j|}) - (\hat{\rho}_T - \rho_T) \dot{y}_{t-1} \dot{u}_{t+|j|} \\
& - (\hat{\rho}_T - \rho_T) \dot{y}_{t+|j|-1} \dot{u}_t + (\hat{\rho}_T - \rho_T)^2 \dot{y}_{t-1} \dot{y}_{t+|j|-1}\Big] \\
:=\ & \widetilde{A}_j - \widetilde{B}_j - \widetilde{C}_j + \widetilde{D}_j, \qquad \text{(B.45)}
\end{aligned}$$

where, using the results from Lemma A.6, we can easily derive

$\widetilde{B}_j$:

$$\begin{aligned} & a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\hat{\rho}_T^{-2(T-t)+|j|-2}(\hat{\rho}_T-\rho_T)\dot{y}_{t-1}\dot{u}_{t+|j|} \\ = \;& a_T^{-2}k_T^{-1}(\hat{\rho}_T-\rho_T)\sum_{t=1}^{T-|j|}\hat{\rho}_T^{-2(T-t)+|j|-2}\Big[y_{t-1}u_{t+|j|}+\bar{y}_{-1}\bar{u}-y_{t-1}\bar{u}-\bar{y}_{-1}u_{t+|j|}\Big] \\ = \;& a_T^{-2}k_T^{-1}\times O_p((\mu_T k_T^{3/2}\rho_T^T)^{-1})\times O_p(a_{k_T}a_T\mu_T k_T^{3/2}\rho_T^T)+s.o.=O_p(a_{k_T}a_T^{-1}k_T^{-1}). \end{aligned} \tag{B.46}$$

$\widetilde{C}_j=O_p(a_{k_T}a_T^{-1}k_T^{-1})$, which follows the same proof as $\widetilde{B}_j$.

$\widetilde{D}_j$:

$$\begin{aligned} & a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\hat{\rho}_T^{-2(T-t)+|j|-2}(\hat{\rho}_T-\rho_T)^2\dot{y}_{t-1}\dot{y}_{t+|j|-1} \\ = \;& a_T^{-2}k_T^{-1}(\hat{\rho}_T-\rho_T)^2\sum_{t=1}^{T-|j|}\hat{\rho}_T^{-2(T-t)+|j|-2}\Big[y_{t-1}y_{t+|j|-1}-y_{t-1}\bar{y}_{-1}-y_{t+|j|-1}\bar{y}_{-1}+\bar{y}_{-1}^2\Big] \\ = \;& a_T^{-2}k_T^{-1}\times O_p((\mu_T k_T^{3/2}\rho_T^T)^{-2})\times O_p(a_{k_T}^2\mu_T^2k_T^3\rho_T^{2T})+s.o.=O_p(a_{k_T}^2a_T^{-2}k_T^{-1}). \end{aligned} \tag{B.47}$$

$\widetilde{A}_j$:

$$\begin{aligned} & a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\hat{\rho}_T^{-2(T-t)+|j|-2}(\dot{u}_t\dot{u}_{t+|j|}-u_tu_{t+|j|}) \\ = \;& a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\hat{\rho}_T^{-2(T-t)+|j|-2}\bar{u}^2-a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\hat{\rho}_T^{-2(T-t)+|j|-2}u_{t+|j|}\bar{u}-a_T^{-2}k_T^{-1}\sum_{t=1}^{T-|j|}\hat{\rho}_T^{-2(T-t)+|j|-2}u_t\bar{u} \\ = \;& (a_T^{-2}k_T^{-1})O_p(a_T^2T^{-1})O(k_T)+O_p(a_TT^{-1/2})O_p(a_T^{-1}k_T^{-1/2})+O_p(a_TT^{-1/2})O_p(a_T^{-1}k_T^{-1/2}) \\ = \;& O_p(T^{-1})+O_p(T^{-1/2}k_T^{-1/2})=o_p(k_T^{-1}). \end{aligned} \tag{B.48}$$

Therefore, the sum of the four terms in $(B.45)$ is at most $O_p(k_T^{-1})$. Under Assumption 6, $k_T^{-1/2}l_T\to 0$ as $T\to\infty$, the equality in $(B.42)$ holds. Moreover, $(B.43)$ follows from Lemma A.6(a), the final limit result $(B.44)$ follows from Lemma 1 since l_T and $K(\cdot)$ also satisfy the conditions needed in the proof of Lemma 1 (Assumption 5). ■

Proof of Theorem 7: To prove this theorem, it suffices to show

(i) $a_T^{-1}a_{k_T}\mu_Tk_T^{3/2}\hat{\rho}_T^T(\hat{\rho}_T^*-\hat{\rho}_T)\xrightarrow{w}_p 2cX/\Big(\frac{Y}{\nu}+\frac{1}{c}\mathbf{1}(\gamma=0)\Big)$;

(ii) $(a_T^{-1}a_{k_T}\mu_Tk_T^{3/2}\hat{\rho}_T^T)^2\hat{\Lambda}^*\xrightarrow{w}_p 4c^2V_x/\Big(\frac{Y}{\nu}+\frac{1}{c}\mathbf{1}(\gamma=0)\Big)^2$.

First, consider (i). In light of the OLS formula for $(\hat{\rho}_T^* - \hat{\rho}_T)$, we need to work out the asymptotics for the bootstrap data - we first prove a similar result as Theorem 2 for the bootstrap data. Denoting $1/\infty = 0$, we have the following joint convergence results:

$$
\begin{aligned}
&(a') \quad (a_{k_T}^2 \mu_T^2 k_T^3 \hat{\rho}_T^{2T})^{-1} \sum_{t=1}^T y_{t-1}^{*2} \overset{w}{\to}_p \frac{1}{2c}\Big[\frac{Y}{\nu} + \frac{1}{c}\mathbf{1}(\gamma=0)\Big]^2, \\
&(b') \quad (a_{k_T} \mu_T k_T^2 \hat{\rho}_T^{T})^{-1} \sum_{t=1}^T y_{t-1}^{*} \overset{w}{\to}_p \frac{1}{c}\Big[\frac{Y}{\nu} + \frac{1}{c}\mathbf{1}(\gamma=0)\Big], \\
&(c') \quad (a_{k_T} a_T \mu_T k_T^{3/2} \hat{\rho}_T^{T})^{-1} \sum_{t=1}^T y_{t-1}^{*} u_t^* \overset{w}{\to}_p X\Big[\frac{Y}{\nu} + \frac{1}{c}\mathbf{1}(\gamma=0)\Big], \\
&(d') \quad a_T^{-1} T^{-1/2} \sum_{t=1}^T u_t^* \overset{w}{\to} U \sim MN(0, \sigma_u^2), \quad \sigma_u^2 = C(1)^2 \int_0^1 g(r)^2 dr,
\end{aligned}
\tag{B.49}
$$

where $y_t^* = \hat{\mu}_T + \hat{\rho}_T y_{t-1}^* + u_t^*$, $t = 1, \ldots, T$, $y_0^* = y_0$. Note that we can define

$$
d_t^* = \hat{\rho}_T d_{t-1}^* + u_t^*, \quad d_0^* = y_0^*, \tag{B.50}
$$

so that $y_t^* = d_t^* + \hat{\mu}_T(\hat{\rho}_T^t - 1)k_T/\hat{c}$. The results in $(B.49)$ are the bootstrap counterparts to those stated in Theorem 2, which build upon the following results that are entirely analogous to Lemma A.1,

(a'') $(a_{k_T} k_T^{3/2} \hat{\rho}_T^T)^{-1} \sum_{t=1}^T \sum_{j=t}^T \hat{\rho}_T^{t-1-j} u_j^* = o_{p^*}(1)$; (b'') $(a_{k_T} k_T^{3/2} \hat{\rho}_T^{2T})^{-1} \sum_{t=1}^T \sum_{j=t}^T \hat{\rho}_T^{2(t-1)-j} u_j^* = o_{p^*}(1)$;
(c'') $(a_{k_T}^2 k_T \hat{\rho}_T^T)^{-1} \sum_{t=1}^T \sum_{j=t}^T \hat{\rho}_T^{t-1-j} u_j^* u_t^* = o_{p^*}(1)$; (d'') $(a_{k_T} k_T \hat{\rho}_T^T)^{-2} \sum_{t=1}^T d_{t-1}^{*2} = Y_T^{*2}/2c + o_{p^*}(1)$;
(e'') $(a_{k_T} k_T^{3/2} \hat{\rho}_T^T)^{-1} \sum_{t=1}^T d_{t-1}^* = Y_T^*/c + o_{p^*}(1)$; (f'') $(a_{k_T} a_T k_T \hat{\rho}_T^T)^{-1} \sum_{t=1}^T d_{t-1}^* u_t^* = X_T^* Y_T^* + o_{p^*}(1)$.

The proofs of (a'')-(f'') are entirely the same as the proofs of (a)-(f) in Lemma A.1, with an additional result $\hat{c} - c = k_T(\hat{\rho}_T - \rho_T) = a_T a_{k_T}^{-1} (\mu_T k_T^{1/2} \rho_T^T)^{-1} a_T^{-1} a_{k_T} \mu_T k_T^{3/2} \rho_T^T (\hat{\rho}_T - \rho_T) = a_T a_{k_T}^{-1} \rho_T^{-T} \left(\frac{2cX}{Y + \nu/c\mathbf{1}(\gamma=0)} + o_p(1)\right) = O_p(a_T a_{k_T}^{-1} \rho_T^{-T}) = o_p(1)$, so we omit them for brevity. Given (a'')-(f''), to see how the proofs for the (a')-(d') are unchanged or follow with minor modifications to that of Theorem 2, we take (c') as an example. Using the fact that

$\frac{\hat{\mu}_T k_T^{1/2}}{\mu_T k_T^{1/2}} = 1 + \nu^{-1} O_p(a_T T^{-1/2} k_T^{1/2})$,

$$(a_{k_T} a_T \mu_T k_T^{3/2} \hat{\rho}_T^T)^{-1} \sum_{t=1}^T y_{t-1}^* u_t^* = (a_{k_T} a_T \mu_T k_T^{3/2} \hat{\rho}_T^T)^{-1} \sum_{t=1}^T (d_{t-1}^* + \hat{\mu}_T(\hat{\rho}_T^{t-1} - 1) k_T/\hat{c}) u_t^*$$

$$= (a_{k_T} a_T \mu_T k_T^{3/2} \hat{\rho}_T^T)^{-1} \sum_{t=1}^T d_{t-1}^* u_t^* + \frac{\hat{\mu}_T k_T^{1/2}}{\mu_T k_T^{1/2}} (a_{k_T} a_T k_T^{1/2} \hat{\rho}_T^T)^{-1} \hat{c}^{-1} (\sum_{t=1}^T \hat{\rho}_T^{t-1} u_t^* - \sum_{t=1}^T u_t^*)$$

$$= X_T^* Y_T^* / (\mu_T k_T^{1/2}) + (1 + \nu^{-1} O_p(a_T T^{-1/2} k_T^{1/2})) a_{k_T}^{-1} \hat{c}^{-1} a_T^{-1} k_T^{-1/2} \sum_{t=1}^T \hat{\rho}_T^{-(T-t)-1} u_t^*$$

$$-(1 + \nu^{-1} O_p(a_T T^{-1/2} k_T^{1/2})) \times O_{p*}(T^{1/2} (k_T^{1/2} \hat{\rho}_T^T)^{-1} a_{k_T}^{-1})$$

$$= X_T^* Y_T^* / (\mu_T k_T^{1/2}) + X_T^* / (a_{k_T} c) + \nu^{-1} O_{p*}(a_T a_{k_T}^{-1} T^{-1/2} k_T^{1/2}) + o_{p*}(1) \overset{w}{\to}_p \frac{1}{2c} [\frac{Y}{\nu} + \frac{1}{c} \mathbf{1}(\gamma = 0)]^2,$$

where $O_{p*}(a_T a_{k_T}^{-1} T^{-1/2} k_T^{1/2}) = O_{p*}(T^{-(1/2-\gamma)} k_T^{1/2-\gamma}) = o_{p*}(1)$ by Assumption $3'$. With the above results (a')-(d') at hand, we thus have,

$$(a_{k_T} a_T \mu_T k_T^{3/2} \hat{\rho}_T^T)^{-1} \sum_{t=1}^T (y_{t-1}^* - \bar{y}_{-1}^*) u_t^* = (a_{k_T} a_T \mu_T k_T^{3/2} \hat{\rho}_T^T)^{-1} \left(\sum_{t=1}^T y_{t-1}^* u_t^* - T^{-1} \sum_{t=1}^T y_{t-1}^* \sum_{t=1}^T u_t^* \right)$$

$$= (a_{k_T} a_T \mu_T k_T^{3/2} \hat{\rho}_T^T)^{-1} \sum_{t=1}^T y_{t-1}^* u_t^* - \underbrace{T^{-1/2} k_T^{1/2} (a_{k_T} \mu_T k_T^{3/2} \hat{\rho}_T^T)^{-1} \sum_{t=1}^T y_{t-1}^* \times a_T^{-1} T^{-1/2} \sum_{t=1}^T u_t^*}_{O_{p*}(T^{-1/2} k_T^{1/2})}$$

$$\overset{w}{\to}_p X \Big[\frac{Y}{\nu} + \frac{1}{c} \mathbf{1}(\gamma = 0) \Big], \tag{B.51}$$

$$a_{k_T}^{-2} (\mu_T k_T^{3/2} \hat{\rho}_T^T)^{-2} \sum_{t=1}^T (y_{t-1}^* - \bar{y}_{-1}^*)^2 = a_{k_T}^{-2} (\mu_T k_T^{3/2} \hat{\rho}_T^T)^{-2} \left(\sum_{t=1}^T y_{t-1}^{*2} - T^{-1} (\sum_{t=1}^T y_{t-1}^*)^2 \right)$$

$$= a_{k_T}^{-2} (\mu_T k_T^{3/2} \hat{\rho}_T^T)^{-2} \sum_{t=1}^T y_{t-1}^{*2} - \underbrace{T^{-1} k_T [(a_{k_T} \mu_T k_T^{3/2} \hat{\rho}_T^T)^{-1} \sum_{t=1}^T y_{t-1}^*]^2}_{O_{p*}(T^{-1} k_T)}$$

$$\overset{w}{\to}_p \frac{1}{2c} \Big[\frac{Y}{\nu} + \frac{1}{c} \mathbf{1}(\gamma = 0) \Big]^2, \tag{B.52}$$

which gives

$$a_T^{-1} a_{k_T} \mu_T k_T^{3/2} \hat{\rho}_T^T (\hat{\rho}_T^* - \hat{\rho}_T) = \frac{(a_{k_T} a_T \mu_T k_T^{3/2} \hat{\rho}_T^T)^{-1} \sum_{t=1}^T (y_{t-1}^* - \bar{y}_{-1}^*) u_t^*}{a_{k_T}^{-2} (\mu_T k_T^{3/2} \hat{\rho}_T^T)^{-2} \sum_{t=1}^T (y_{t-1}^* - \bar{y}_{-1}^*)^2} \overset{w}{\to}_p 2cX / \Big(\frac{Y}{\nu} + \frac{1}{c} \mathbf{1}(\gamma = 0) \Big).$$

The proof of (ii) is similar in spirit to the proof of (i) in terms of showing that the limit result based on the original data is transferable to the bootstrap data. We outline the main steps of the argument. In particular, we need to show a similar result as Theorem 4 holds for the bootstrap data, i.e.,

$$T(a_{k_T}a_T\mu_T k_T^{3/2}\hat{\rho}_T^T)^{-2}\hat{\Omega}^* \xrightarrow{w}_p V_x\left[\frac{Y}{\nu}+\frac{1}{c}\mathbf{1}(\gamma=0)\right]^2. \tag{B.53}$$

The proof shares entirely the same logic as the proof of Theorem 4, and it builds on several preliminary results which are analogous to Lemmas A.2-A.4 and A.6 with corresponding quantities replaced by their bootstrap analogues. Next, in view of $(B.52)$,

$$T(a_{k_T}^2\mu_T^2 k_T^3\hat{\rho}_T^{2T})^{-1}Q_T^* = a_{k_T}^{-2}(\mu_T k_T^{3/2}\hat{\rho}_T^T)^{-2}\sum_{t=1}^T \dot{y}_{t-1}^{*2} \xrightarrow{w}_p \frac{1}{2c}\left[\frac{Y}{\nu}+\frac{1}{c}\mathbf{1}(\gamma=0)\right]^2. \tag{B.54}$$

Finally, combining $(B.53)$ and $(B.54)$,

$$\begin{aligned} &(a_T^{-1}a_{k_T}\mu_T k_T^{3/2}\hat{\rho}_T^T)^2\hat{\Lambda}^* = T^{-1}(a_T^{-1}a_{k_T}\mu_T k_T^{3/2}\hat{\rho}_T^T)^2 Q_T^{*-2}\hat{\Omega}^* \qquad (B.55)\\ = \; &\left[T(a_{k_T}^2\mu_T^2 k_T^3\hat{\rho}_T^{2T})^{-1}Q_T^*\right]^{-2}\left[T(a_{k_T}a_T\mu_T k_T^{3/2}\hat{\rho}_T^T)^{-2}\hat{\Omega}^*\right] \xrightarrow{w}_p 4c^2V_x/\Big(\frac{Y}{\nu}+\frac{1}{c}\mathbf{1}(\gamma=0)\Big)^2, \end{aligned}$$

which gives (ii). The theorem then follows. ■

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Appendix C: Additional Monte Carlo Results

Notes to Tables

1. Table C.1 reports empirical coverage rate of various inferential methods for $\rho = 1 + c/T^{\alpha}$, 95% nominal rate, $T = 50, 100, 200$, $c = 0.5$, $\alpha = 0.8$, $\mu_T = 0$.
2. Table C.2 reports empirical coverage rate of various inferential methods for $\rho = 1 + c/T^{\alpha}$, 95% nominal rate, $T = 50, 100, 200$, $c = 0.5$, $\alpha = 0.8$, $\mu_T = T^{-\alpha/4}$.
3. Table C.3 reports effective interval length ratio (benchmark: t_{hac}) of various inferential methods for $\rho = 1 + c/T^{\alpha}$, 95% nominal rate, $T = 50, 100, 200$, $c = 0.5$, $\alpha = 0.8$, $\mu_T = 0$.
4. Table C.4 reports effective interval length ratio (benchmark: t_{hac}) of various inferential methods for $\rho = 1 + c/T^{\alpha}$, 95% nominal rate, $T = 50, 100, 200$, $c = 0.5$, $\alpha = 0.8$, $\mu_T = T^{-\alpha/4}$.

Table C.1: Empirical coverage rate of various inferential methods for $\rho = 1 + c/T^{\alpha}$, 95% nominal rate, $T = 50, 100, 200$, $c = 0.5$, $\alpha = 0.8$, $\mu_T = 0$.

DGP	0			1			2			3			4			5			6		
T	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200
Panel A: no serial correlation																					
t_{hac}	57.6	61.3	63.0	50.9	55.6	61.8	66.3	67.4	65.9	72.0	74.5	77.2	67.9	70.6	70.0	64.2	69.4	73.3	67.4	72.1	77.1
PM	98.6	98.5	98.7	99.0	99.1	99.2	98.9	99.1	98.9	92.9	92.7	93.1	95.9	96.2	97.0	93.6	93.9	94.1	90.3	89.6	90.3
GSW	67.9	66.1	65.9	54.8	56.8	59.2	78.5	77.2	76.5	82.8	77.9	74.5	76.6	74.1	71.1	64.8	62.5	62.2	67.5	67.2	69.0
DWB_3	83.4	83.3	84.0	80.5	81.3	83.5	89.5	87.2	86.7	89.5	88.3	89.3	89.3	87.6	87.1	88.3	87.6	88.3	89.5	89.9	90.8
DWB_5	81.1	81.8	82.7	77.3	78.7	81.6	88.4	85.9	86.6	88.0	87.2	88.7	87.9	87.0	86.5	85.8	85.7	87.2	87.7	88.6	89.9
DWB_{10}	78.8	79.3	80.6	73.2	74.8	78.5	86.4	84.3	85.4	86.1	85.1	86.9	86.2	85.6	85.9	82.2	82.4	84.5	85.3	86.8	88.7
DWB_r	85.0	83.5	84.0	82.5	81.2	83.5	90.6	87.2	87.1	90.7	88.6	89.6	90.3	87.3	87.2	89.9	87.5	88.2	90.6	89.8	90.8
Panel B: AR case, $\phi = 0.5$																					
t_{hac}	82.9	87.7	90.0	71.7	80.1	86.5	91.8	94.6	95.2	91.2	94.9	96.8	90.8	94.2	95.2	80.2	87.5	91.6	77.7	86.3	90.7
PM	97.9	97.9	97.8	99.4	99.3	99.5	99.4	99.3	99.3	87.8	87.8	88.3	92.6	93.8	94.5	90.6	90.9	90.9	87.5	87.5	87.7
GSW	89.3	90.7	91.2	76.3	80.5	84.3	97.5	98.2	98.8	95.8	95.9	96.1	96.0	96.8	96.3	77.9	80.2	83.5	76.0	78.7	82.1
DWB_3	88.8	89.2	88.8	85.0	87.3	88.3	94.7	95.0	94.1	92.5	93.4	93.3	93.5	93.4	91.9	90.1	91.7	92.1	89.0	91.4	92.9
DWB_5	86.2	86.2	85.5	82.3	84.6	86.4	93.6	92.8	91.0	90.4	90.1	89.1	91.6	90.7	88.0	87.8	89.7	90.3	87.0	89.5	91.9
DWB_{10}	82.0	82.0	81.5	77.3	80.1	82.6	91.5	89.8	88.1	86.9	86.0	83.7	88.1	87.3	84.5	84.1	85.7	86.9	83.2	86.4	89.6
DWB_r	90.2	89.1	88.9	86.4	87.0	88.3	95.6	94.9	94.2	93.8	93.4	93.3	94.6	93.5	92.1	91.6	92.0	92.2	90.4	91.2	93.0
Panel C: MA case, $\theta = 0.5$																					
t_{hac}	83.9	90.2	92.1	74.6	85.4	90.2	90.2	94.6	95.4	91.7	95.9	97.6	91.6	95.5	96.1	82.1	89.9	93.8	81.3	88.9	93.7
PM	98.4	98.2	98.4	99.2	99.3	99.3	99.2	99.1	99.3	90.3	89.4	89.4	93.7	94.5	95.6	91.4	92.1	92.2	88.0	88.4	88.8
GSW	89.7	89.0	87.8	78.8	80.8	83.0	95.0	95.0	95.2	95.0	94.2	94.0	95.6	95.7	94.4	78.4	79.5	80.9	77.2	78.3	81.0
DWB_3	90.3	90.9	90.5	87.1	90.2	90.8	94.7	94.6	93.9	93.7	94.2	94.0	94.1	94.1	92.6	91.8	93.1	93.6	91.5	93.7	94.9
DWB_5	87.5	87.7	86.4	83.8	86.2	87.4	93.1	92.9	91.3	91.5	91.5	91.1	92.2	92.0	89.6	89.1	90.7	91.3	89.3	92.0	93.3
DWB_{10}	83.6	83.8	82.3	79.3	81.2	82.9	90.9	90.0	88.6	88.5	88.4	87.3	89.6	89.1	86.7	85.7	86.8	88.0	86.3	88.9	91.1
DWB_r	92.5	90.9	90.1	89.7	89.9	90.8	95.7	94.7	94.1	95.0	93.9	94.1	95.2	94.1	92.6	93.5	93.2	93.7	93.3	93.7	95.0

Table C.2: Empirical coverage rate of various inferential methods for $\rho = 1 + c/T^{\alpha}$, 95% nominal rate, $T = 50, 100, 200$, $c = 0.5$, $\alpha = 0.8$, $\mu_T = T^{-\alpha/4}$.

DGP	0			1			2			3			4			5			6		
T	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200
Panel A: no serial correlation																					
t_{hac}	89.2	92.5	93.9	91.6	93.8	94.0	81.6	86.4	90.1	74.8	78.7	83.8	87.2	91.7	93.4	83.6	88.4	90.2	78.1	83.3	86.5
PM	99.8	99.9	100	100	100	100	99.7	99.8	100	95.3	96.2	97.2	99.9	100	100	96.5	97.0	97.2	93.1	94.3	94.3
GSW	93.8	94.5	94.9	96.2	96.6	96.7	92.2	95.5	98.3	84.0	82.4	82.4	92.6	93.3	94.0	88.3	87.8	87.4	82.5	83.1	83.3
DWB_3	90.1	91.7	92.7	91.9	93.5	94.3	87.4	88.3	92.8	86.8	89.7	91.6	89.3	91.1	93.1	91.6	91.8	92.8	91.8	92.0	93.9
DWB_5	88.1	90.3	91.6	90.5	92.6	93.8	86.8	87.3	92.2	84.8	87.4	90.3	87.3	89.3	92.0	90.7	90.9	92.1	91.2	91.6	93.3
DWB_{10}	84.0	87.1	88.9	87.5	90.4	92.6	85.7	85.3	90.6	80.9	84.0	86.9	83.6	86.0	89.6	88.6	89.5	91.1	89.7	90.3	92.6
DWB_r	91.0	91.7	92.6	92.3	93.3	94.2	87.8	88.2	92.8	88.2	89.7	91.6	90.5	91.2	93.4	91.7	91.7	92.8	92.0	92.0	93.9
Panel B: AR case, $\phi = 0.5$																					
t_{hac}	86.7	91.9	93.8	88.7	92.7	94.7	91.8	94.0	95.1	90.1	93.4	94.8	86.7	90.7	92.9	87.4	91.8	94.7	84.2	90.4	93.4
PM	98.2	98.6	99.1	99.5	99.6	99.7	99.4	99.5	99.6	89.8	90.6	91.6	98.2	99.1	99.6	93.2	93.9	94.5	89.4	90.6	91.0
GSW	90.5	91.9	93.2	90.9	94.1	95.3	97.9	98.6	99.2	94.6	94.7	94.6	89.9	91.7	92.5	87.2	88.8	89.5	83.6	86.5	87.8
DWB_3	86.5	88.7	90.3	88.7	91.0	92.5	87.5	85.8	89.6	84.2	86.5	89.0	86.0	87.7	90.7	90.3	89.9	91.2	89.8	90.7	92.5
DWB_5	84.9	87.7	89.8	87.7	90.7	92.6	86.9	84.8	89.3	81.9	84.2	87.8	84.0	86.5	90.2	89.1	89.3	91.0	88.5	89.8	92.3
DWB_{10}	81.1	84.9	87.9	84.9	89.2	91.8	85.7	83.5	88.3	77.5	80.9	84.9	80.1	83.4	87.9	86.7	87.9	89.8	86.2	88.1	91.3
DWB_r	87.4	88.3	90.7	89.0	91.1	92.7	87.9	85.8	89.7	86.1	86.3	88.9	86.9	87.8	91.1	90.6	90.1	91.2	90.0	90.7	92.6
Panel C: MA case, $\theta = 0.5$																					
t_{hac}	92.9	95.9	96.9	94.4	96.7	97.4	93.3	95.7	96.7	91.6	94.8	96.1	92.0	95.3	96.8	91.2	95.4	96.8	87.7	93.5	95.8
PM	99.0	99.4	99.6	99.8	99.7	99.9	99.6	99.8	99.8	92.0	92.9	94.4	99.4	99.8	99.9	94.5	95.1	95.6	91.1	92.0	92.2
GSW	93.6	94.5	94.7	95.4	96.1	97.0	97.0	98.0	99.0	94.1	93.0	92.4	92.6	93.5	93.8	89.3	89.9	91.2	86.1	87.0	88.2
DWB_3	91.2	93.1	94.4	92.9	95.1	96.0	90.5	89.5	93.7	88.8	91.2	92.8	90.5	92.6	94.6	93.3	93.5	94.7	93.2	93.9	95.7
DWB_5	88.7	90.9	92.3	91.4	93.7	94.9	89.1	87.6	92.3	85.8	88.1	90.3	87.8	90.2	92.6	91.9	92.1	93.1	91.9	92.6	94.3
DWB_{10}	84.5	87.2	89.2	88.4	91.2	93.2	87.3	85.4	90.1	81.5	84.2	86.7	83.8	86.3	89.6	89.6	90.1	91.4	89.6	90.7	92.8
DWB_r	93.0	93.1	94.4	94.1	95.1	96.1	91.4	89.8	93.7	91.0	91.2	92.8	92.5	92.6	94.5	94.4	93.8	94.7	94.1	94.0	95.7

Table C.3: Effective interval length ratio (benchmark: t_{hac}) of various inferential methods for $\rho = 1 + c/T^{\alpha}$, 95% nominal rate, $T = 50, 100, 200$, $c = 0.5$, $\alpha = 0.8$, $\mu_T = 0$.

DGP	0			1			2			3			4			5			6		
T	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200
Panel A: no serial correlation																					
PM	2.01	2.08	2.22	1.97	2.06	2.29	2.20	2.20	2.28	1.32	1.29	1.34	1.56	1.62	1.73	1.06	1.18	1.15	1.18	1.28	1.25
GSW	1.10	1.03	1.01	0.92	0.85	0.86	1.16	1.04	0.99	1.11	0.93	0.87	1.01	0.94	0.90	0.79	0.79	0.68	0.71	0.69	0.61
DWB_3	1.20	1.10	1.09	1.20	1.06	1.01	1.07	0.94	0.91	1.18	1.05	1.03	1.14	1.04	1.02	1.38	1.21	0.99	1.63	1.22	1.05
DWB_5	1.17	1.08	1.07	1.18	1.03	0.99	1.06	0.91	0.90	1.16	1.03	1.01	1.12	1.02	1.00	1.32	1.17	1.01	1.47	1.14	1.00
DWB_{10}	1.13	1.04	1.04	1.15	1.00	0.97	1.08	0.90	0.87	1.13	0.99	0.98	1.09	1.00	0.98	1.28	1.11	0.99	1.41	1.09	0.94
DWB_r	1.22	1.11	1.09	1.23	1.06	1.02	1.08	0.94	0.92	1.20	1.05	1.03	1.16	1.04	1.02	1.41	1.20	1.05	1.64	1.19	1.02
Panel B: AR case, $\phi = 0.5$																					
PM	1.79	1.74	1.75	1.96	1.84	1.84	2.12	1.97	1.89	1.10	1.15	1.17	1.21	1.39	1.47	1.02	1.11	1.12	1.23	0.96	1.22
GSW	1.14	1.04	0.99	1.01	0.93	0.88	1.39	1.19	1.09	1.09	1.04	0.95	0.89	0.94	0.94	0.76	0.79	0.75	0.73	0.54	0.67
DWB_3	1.05	0.91	0.84	1.03	0.87	0.78	0.86	0.75	0.68	1.15	0.97	0.90	1.07	0.91	0.83	1.42	1.14	0.94	1.70	1.26	0.96
DWB_5	1.02	0.88	0.81	1.01	0.84	0.75	0.85	0.71	0.64	1.13	0.94	0.88	1.05	0.88	0.80	1.42	1.14	0.90	1.69	1.27	0.93
DWB_{10}	0.97	0.84	0.77	0.97	0.81	0.71	0.86	0.69	0.61	1.07	0.89	0.84	1.00	0.85	0.76	1.41	1.10	0.87	1.48	1.22	0.89
DWB_r	1.07	0.91	0.84	1.05	0.87	0.78	0.88	0.75	0.68	1.16	0.97	0.90	1.09	0.91	0.84	1.44	1.15	0.93	1.54	1.26	0.96
Panel C: MA case, $\theta = 0.5$																					
PM	1.59	1.55	1.54	1.72	1.65	1.63	1.89	1.72	1.68	0.99	1.03	1.02	1.25	1.21	1.26	1.09	0.95	0.91	0.98	1.04	1.05
GSW	1.06	0.96	0.89	0.97	0.86	0.80	1.23	1.01	0.94	1.00	0.93	0.82	0.96	0.85	0.82	0.85	0.69	0.63	0.62	0.60	0.58
DWB_3	1.04	0.90	0.82	1.05	0.88	0.78	0.87	0.76	0.68	1.12	0.94	0.87	1.06	0.89	0.81	1.57	1.70	0.96	3.83	1.33	0.99
DWB_5	0.99	0.85	0.76	1.00	0.83	0.73	0.85	0.71	0.63	1.09	0.89	0.81	1.01	0.84	0.76	1.52	1.60	0.90	3.73	1.22	0.93
DWB_{10}	0.91	0.78	0.71	0.95	0.78	0.68	0.83	0.67	0.59	1.04	0.82	0.76	0.94	0.78	0.71	1.51	1.43	0.85	3.31	1.14	0.87
DWB_r	1.08	0.90	0.82	1.08	0.88	0.78	0.90	0.76	0.68	1.17	0.95	0.87	1.09	0.89	0.81	1.61	1.63	0.95	3.83	1.27	0.92

Table C.4: Effective interval length ratio (benchmark: t_{hac}) of various inferential methods for $\rho = 1 + c/T^{\alpha}$, 95% nominal rate, $T = 50, 100, 200$, $c = 0.5$, $\alpha = 0.8$, $\mu_T = T^{-\alpha/4}$.

DGP	0			1			2			3			4			5			6		
T	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200
Panel A: no serial correlation																					
PM	5.65	7.27	9.48	8.10	10.5	13.9	3.37	4.33	6.06	1.67	1.83	2.15	5.44	7.26	9.55	2.01	2.24	2.51	1.81	1.76	1.76
GSW	1.11	1.05	1.02	1.17	1.12	1.11	1.30	1.32	1.48	1.09	0.94	0.87	1.03	1.02	1.01	0.96	0.89	0.82	0.98	0.79	0.67
DWB_3	1.14	1.08	1.06	1.04	1.02	1.01	0.99	0.95	1.01	1.39	1.19	1.11	1.22	1.12	1.08	1.68	1.33	1.14	1.60	1.28	1.20
DWB_5	1.07	1.03	1.02	0.99	0.99	0.99	0.98	0.92	0.98	1.38	1.14	1.07	1.13	1.06	1.03	1.62	1.26	1.11	1.51	1.21	1.14
DWB_{10}	0.95	0.95	0.96	0.92	0.95	0.95	1.00	0.87	0.94	1.37	1.05	0.98	1.00	0.96	0.96	1.51	1.18	1.04	1.44	1.15	1.10
DWB_r	1.18	1.08	1.06	1.07	1.02	1.01	1.01	0.95	1.01	1.38	1.19	1.11	1.27	1.12	1.08	1.71	1.32	1.15	1.66	1.26	1.20
Panel B: AR case, $\phi = 0.5$																					
PM	3.05	3.68	4.84	4.34	5.46	7.30	2.46	2.53	2.87	1.05	1.21	1.39	2.49	3.67	5.05	1.66	1.45	1.06	1.39	1.47	1.48
GSW	1.08	0.99	0.98	1.12	1.06	1.05	1.43	1.26	1.26	0.95	0.97	0.92	0.87	0.93	0.95	0.96	0.77	0.49	0.85	0.82	0.77
DWB_3	1.16	1.06	1.02	1.07	0.99	0.96	0.95	0.90	0.92	1.25	1.10	1.09	1.30	1.10	1.05	1.52	1.47	1.06	1.59	1.29	1.11
DWB_5	1.12	1.05	1.02	1.06	0.98	0.98	0.95	0.88	0.93	1.22	1.08	1.06	1.24	1.08	1.04	1.40	1.48	1.02	1.55	1.23	1.05
DWB_{10}	1.04	1.00	0.98	1.04	0.98	0.96	0.99	0.86	0.92	1.12	1.03	1.01	1.33	1.00	0.99	1.29	1.34	0.94	1.39	1.15	1.00
DWB_r	1.17	1.07	1.02	1.10	0.99	0.96	0.95	0.90	0.92	1.26	1.11	1.08	1.32	1.10	1.05	1.54	1.49	1.05	1.65	1.28	1.10
Panel C: MA case, $\theta = 0.5$																					
PM	3.30	4.16	5.59	4.83	6.18	8.27	2.37	2.58	3.11	1.06	1.10	1.36	2.87	4.13	5.75	1.22	1.55	1.73	1.06	1.28	1.27
GSW	0.99	0.89	0.88	1.03	0.98	0.95	1.29	1.14	1.13	0.94	0.81	0.81	0.81	0.84	0.86	0.65	0.74	0.75	0.63	0.70	0.62
DWB_3	1.14	1.05	1.00	1.03	0.98	0.96	0.92	0.88	0.91	1.29	1.12	1.04	1.21	1.08	1.02	1.67	1.18	1.12	1.95	1.29	1.07
DWB_5	1.06	0.97	0.95	0.96	0.92	0.92	0.90	0.83	0.84	1.24	1.05	0.98	1.11	1.02	0.96	1.55	1.11	1.02	1.80	1.19	0.99
DWB_{10}	0.96	0.88	0.87	0.89	0.84	0.88	0.90	0.78	0.80	1.12	0.93	0.89	0.96	0.90	0.87	1.40	1.01	0.97	1.73	1.10	0.90
DWB_r	1.16	1.05	1.00	1.07	0.99	0.96	0.96	0.88	0.89	1.31	1.11	1.05	1.26	1.08	1.02	1.68	1.17	1.08	1.99	1.21	0.99

Appendix D: Application to US house prices

The U.S. experienced a sharp increase in home prices between the years 1997 and 2006. While the increase was larger in some areas and smaller in others, real home prices went up by about 85% over this period for the country as a whole (Shiller, 2015). Figure D.1 plots three housing price indices - the national home price index, the 20-city composite index, the 10-city composite index, all of which nearly doubled from 2002 to 2006.[2] As Shiller (2015) notes, there was a "rocket taking off" that eventually crashed in 2006 and caused the 2008 financial crisis, the most severe of its kind since the Great Depression of the 1930s. A variety of potential explanations has been advanced for this rapid escalation in home prices including lax lending standards, the Federal Reserve's low interest rate policy, promotion in the media, and excessively optimistic investor beliefs from a behavioral perspective (see Glaeser, 2013; Shiller, 2015; Mian and Sufi, 2015; Griffin, Kruger and Maturana, 2021; and the references therein). In our study, while not investigating the causes of the house price boom during 2002-2006, we conduct an econometric analysis aimed at measuring the extent of explosive behavior in U.S. housing prices to help practitioners and policymakers gauge the intensity of these dramatic price accelerations. We thus provide a simple yet robust and effective tool to identify the degree of house price exuberance in the presence of potential serial correlation and heteroskedasticity, the latter emphasized by Case and Shiller (2003).

Similar to our foregoing analysis of the stock indices, we collect $T = 50$ observations of the three monthly Case-Shiller indices displayed in Figure D.1 by first pinning down the peak point and then gathering data backward until the T-th observation. The peak for the national index is at March 2006, while the peaks for the 10/20-city composite indices are at April 2006, which makes the starting month to be February or March 2002.[3] The middle and lower panels in Figure D.1 present, respectively, the variance profiles and estimated volatilities for the three series after fitting an autoregressive model using the same specifications as the stock indices in the previous section. These plots reveal considerable instability in the sample volatility paths for all three indices with a trending volatility specification appearing to provide a suitable characterization of the nature of the volatility process. Table D.1 presents the results from the diagnostic tests for stationary volatility (Panel A) and the bubble detection tests (Panel B) as described in Section 6.1. The $\mathcal{H}_{AD}$ and $\mathcal{H}_{CVM}$ tests turn out to be significant at least at the 10% level for all series, thereby formally corroborating the nonstationarity of the volatility paths and its type observed in Figure D.1. All of the bubble

[2]The three indices are the main indices from the well known Case-Shiller Index, which was developed in the 1980s by three economists - Allan Weiss, Karl Case and Robert Shiller. Specifically, the national home price index records the value of residential housing by tracking the purchase and resale price of single-family homes, which covers nine major census divisions in the U.S. The 10-city composite index covers Boston, Chicago, Denver, Las Vegas, Los Angeles, Miami, New York, San Diego, San Francisco, and Washington, D.C.. The 20-city composite index, further includes Atlanta, Charlotte, Cleveland, Dallas, Detroit, Minneapolis, Phoenix, Portland, Seattle, and Tampa. The data can be downloaded from the economic research data website of Federal Reserve Bank of St. Louis, https://fred.stlouisfed.org/release/tables?rid=199&eid=243552.

[3]See Phillips and Yu (2011) and Fabozzi and Xiao (2018) for empirical evidence on the bubble's timeline.

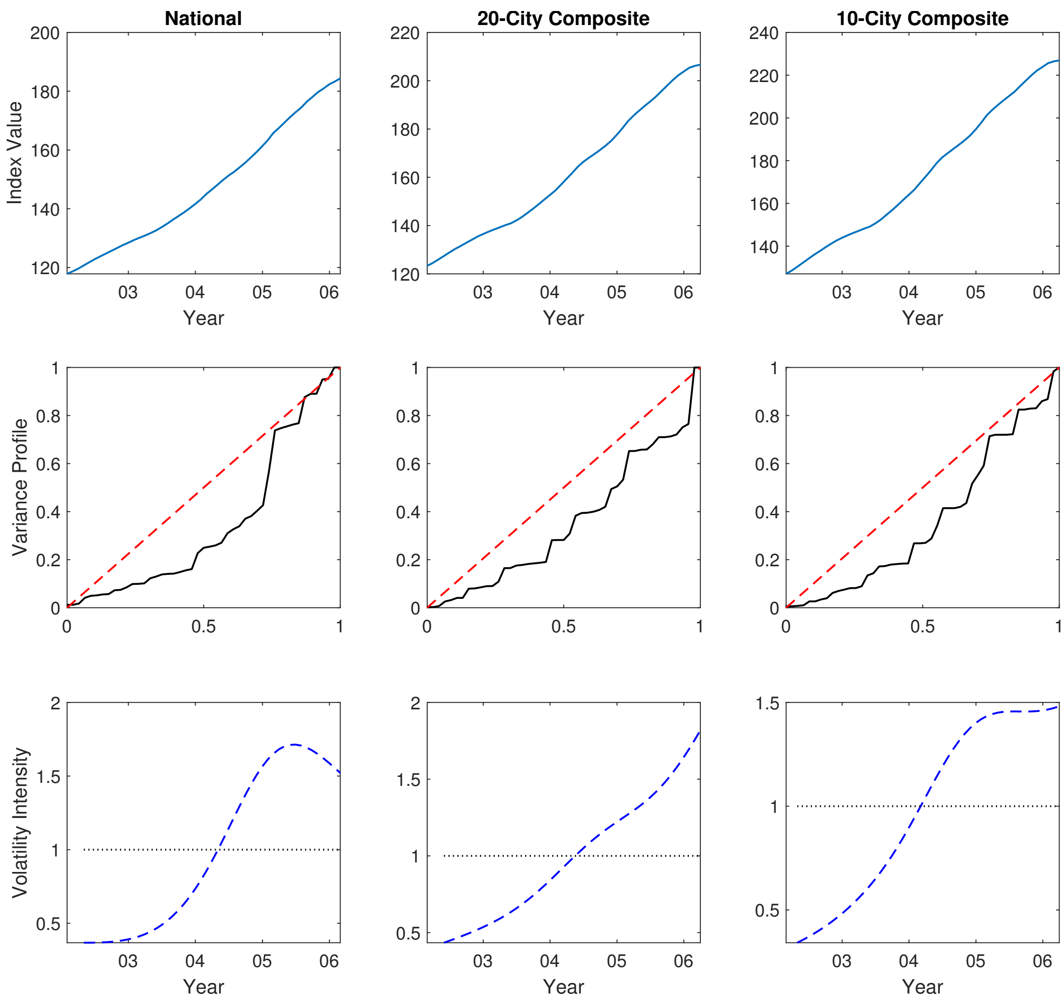

Figure D.1: Plots of three U.S. housing price indices and their variance profile, nonparametric volatility estimates during 2002-2006.

Table D.1: Nonstationary volatility tests (Cavaliere and Taylor, 2007b) and bubble tests (Harvey et al. 2019, 2020) results for U.S. housing price series.

Panel A: Tests for stationary volatility.

	National	20-City Composite	10-City Composite
$\mathcal{H}_{KS}$	1.177	1.093	1.314
$\mathcal{H}_{R}$	1.101	1.010	1.294*
$\mathcal{H}_{CVM}$	0.478**	0.424*	0.567**
$\mathcal{H}_{AD}$	2.165*	2.620**	2.822**

Panel B: Explosiveness estimates and p-values from bubble tests.

	National	20-City Composite	10-City Composite
$\hat{\rho}_T$	1.012	1.009	1.005
$\mathcal{U}$	0.002	0.000	0.000
supBZ	0.000	0.000	0.000
supDF	0.002	0.000	0.002
uPSY	0.000	0.002	0.002
PSY	0.018	0.010	0.016
sPSY	0.000	0.000	0.000

Note: *denotes 10%, **denotes 5%, and ***denotes 1% significance level for the above tests.

tests are significant at the 1% level except PSY (although the p-values for PSY are all below 2%) for each of the three series. Therefore, we move all three series to the second stage of constructing confidence intervals for the degree of explosiveness. The results are presented in Table D.2. As with our analysis of the stock indices, GSW provides a tighter interval and PM provides a much wider interval than our DWB-based method, both of which are consistent with our earlier discussions on the impact of ignoring possible nonstationarities in the second moments of the time series.

In conclusion, our analysis shows that the three housing price indices, all of which appear to possess trending (increasing) volatility, exhibit mildly explosive behavior as opposed to a severe explosion during the 2002-2006 housing market boom. Existing methods that do not allow for nonstationary volatility tend to understate/overstate the sampling variability around the point estimates. More recently, U.S. housing prices have been increasing at a record pace despite high unemployment during the COVID-19 pandemic, which is believed to be induced by the Federal Reserve's unlimited quantitative easing approach in response to

Table D.2: AR(1) estimates and 95% confidence intervals of various methods for U.S. housing price series.

	National	20-City Composite	10-City Composite
$\hat{\rho}_T$	1.012	1.009	1.005
t_{hac}	[1.001,1.038]	[1.001,1.029]	[1.001,1.032]
PM	[1.001,1.184]	[1.001,1.151]	[1.001,1.108]
GSW	[1.001,1.027]	[1.001,1.025]	[1.001,1.024]
DWB_3	[1.001,1.042]	[1.001,1.026]	[1.001,1.027]
DWB_5	[1.001,1.040]	[1.001,1.023]	[1.001,1.026]
DWB_{10}	[1.001,1.044]	[1.001,1.023]	[1.001,1.026]
DWB_r	[1.001,1.043]	[1.001,1.026]	[1.001,1.032]

the pandemic. In principle, our method can also be applied to analyze the intensity of this recent surge in home prices. We leave such an investigation as a potential topic for future research.

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