Supplementary Materials on "Applications of Functional Dependence to Spatial Econometrics"

Zeqi Wu[a], Wen Jiang[b], and Xingbai Xu*[c]

[a]Institute of Statistics & Big Data, Renmin University of China

[b]New Huadu Business School, Minjiang University

[c]MOE Key Lab of Econometrics and Fujian Key Lab of Statistics, Wang Yanan Institute for Studies in Economics (WISE), Department of Statistics and Data Science, School of Economics, Xiamen University

This supplementary material contains an example of an SAR Tobit model, all the proofs for the main text except the LLN and the CLT, and more examples of spatial functional dependence. Throughout the proofs, we use $C, C_0, C_1, \ldots$ to represent some positive constants, which might be different from line to line.

S.1. Some Useful Lemmas

Lemma S.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{C}$ is a sub-σ-field of $\mathcal{F}$. For any random vector X and $p \geq 1$,*

$$\mathbb{E}_{\mathcal{C}} \left\| X - \frac{1}{2}\mathbb{E}_{\mathcal{C}} X \right\|^p \leq \left(\frac{3}{2}\right)^p \mathbb{E}_{\mathcal{C}} \left\| X \right\|^p .$$

*Corresponding author. E-mail address: xuxingbai@xmu.edu.cn. Address correspondence to: School of Economics, Xiamen University, Xiamen, China.

Proof. By the triangle inequality and conditional Lyapunov's inequality,

$$\left\| X - \frac{1}{2}\mathbb{E}_{\mathcal{C}}X \right\|_{L^p,\mathcal{C}} \leq \|X\|_{L^p,\mathcal{C}} + \frac{1}{2}\|\mathbb{E}_{\mathcal{C}}X\| \leq \|X\|_{L^p,\mathcal{C}} + \frac{1}{2}\|X\|_{L^p,\mathcal{C}} = \frac{3}{2}\|X\|_{L^p,\mathcal{C}}.$$

Taking both sides of the inequality to the pth power completes the proof. ■

Lemma S.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{C}$ is a sub-σ-field of $\mathcal{F}$. Suppose random vectors X and Y are independent conditional on $\mathcal{C}$ and $\mathbb{E}_{\mathcal{C}}Y = 0$ a.s. Then for any convex function f,*

$$\mathbb{E}_{\mathcal{C}}f(X) \leq \mathbb{E}_{\mathcal{C}}f(X+Y) \text{ a.s.}$$

Proof. It follows from the facts that X and Y are independent conditional on $\mathcal{C}$ and $\mathbb{E}_{\mathcal{C}}Y = 0$ that $\mathbb{E}_{\sigma(X)\vee\mathcal{C}}Y = 0$ a.s. Because $\mathbb{E}_{\sigma(X)\vee\mathcal{C}}X = X$, by conditional Jensen's inequality,

$$f(X) = f\left(\mathbb{E}_{\sigma(X)\vee\mathcal{C}}[X+Y]\right) \leq \mathbb{E}_{\sigma(X)\vee\mathcal{C}}f(X+Y) \text{ a.s.}$$

Thus,

$$\mathbb{E}_{\mathcal{C}}f(X) \leq \mathbb{E}_{\mathcal{C}}\mathbb{E}_{\sigma(X)\vee\mathcal{C}}f(X+Y) = \mathbb{E}_{\mathcal{C}}f(X+Y) \text{ a.s.}$$

■

Lemma S.3. *Consider the system (2.1). Let I_1 and I_2 be any two disjoint subsets of D_n. Then for any $i \in D_n$ and $p \geq 1$,*

$$\delta_p(i, I_1, n) \leq 3\delta_p\left(i, I_1 \bigcup I_2, n\right).$$

Proof. It suffices to consider the non-trivial case that $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$. Denote $\mathcal{G} \equiv \sigma(\epsilon_{j,n} : j \in D_n \backslash I_2) \vee \sigma\left(\epsilon^*_{j,n} : j \in I_1\right)$. Let $X = Y_{i,n} - Y_{i,n,I_1}$, $Y = Y_{i,n,I_2} - Y_{i,n,I_1 \bigcup I_2} - \mathbb{E}_{\mathcal{G}}\left[Y_{i,n,I_2} - Y_{i,n,I_1 \bigcup I_2}\right]$, and $f(x) = \|x\|^p$. Conditional on $\mathcal{G}$, X is a function of $\epsilon_{I_2,n}$ and Y is a function of $\epsilon^*_{I_2,n}$. So, X and Y

are independent conditional on $\mathcal{G}$. Lemma S.2 implies

$$\mathbb{E}_{\mathcal{G}} \|Y_{i,n} - Y_{i,n,I_1}\|^p \leq \mathbb{E}_{\mathcal{G}} \left\|Y_{i,n} - Y_{i,n,I_1} + Y_{i,n,I_2} - Y_{i,n,I_1 \bigcup I_2} - \mathbb{E}_{\mathcal{G}}\left[Y_{i,n,I_2} - Y_{i,n,I_1 \bigcup I_2}\right]\right\|^p. \quad \text{(S.1)}$$

Taking $X = Y_{i,n} - Y_{i,n,I_1} + Y_{i,n,I_2} - Y_{i,n,I_1 \bigcup I_2}$ in Lemma S.1 gives

$$\begin{aligned} &\mathbb{E}_{\mathcal{G}} \left\|Y_{i,n} - Y_{i,n,I_1} + Y_{i,n,I_2} - Y_{i,n,I_1 \bigcup I_2} - \mathbb{E}_{\mathcal{G}}\left[Y_{i,n,I_2} - Y_{i,n,I_1 \bigcup I_2}\right]\right\|^p \\ \leq &\left(\frac{3}{2}\right)^p \mathbb{E}_{\mathcal{G}} \left\|Y_{i,n} - Y_{i,n,I_1} + Y_{i,n,I_2} - Y_{i,n,I_1 \bigcup I_2}\right\|^p \\ \leq &\left(\frac{3}{2}\right)^p 2^{p-1}\left[\mathbb{E}_{\mathcal{G}} \left\|Y_{i,n} - Y_{i,n,I_1 \bigcup I_2}\right\|^p + \mathbb{E}_{\mathcal{G}} \|Y_{i,n,I_2} - Y_{i,n,I_1}\|^p\right], \end{aligned} \quad \text{(S.2)}$$

where the last inequality follows from the conditional Loève's c_r inequality. Combining (S.1) and (S.2) and taking the expectation yields

$$\mathbb{E} \|Y_{i,n} - Y_{i,n,I_1}\|^p \leq \frac{3^p}{2}\left[\mathbb{E} \left\|Y_{i,n} - Y_{i,n,I_1 \bigcup I_2}\right\|^p + \mathbb{E} \|Y_{i,n,I_2} - Y_{i,n,I_1}\|^p\right] = 3^p \mathbb{E} \left\|Y_{i,n} - Y_{i,n,I_1 \bigcup I_2}\right\|^p,$$

where the last equality is due to the fact that $Y_{i,n} - Y_{i,n,I_1 \bigcup I_2}$ and $Y_{i,n,I_2} - Y_{i,n,I_1}$ have the same distribution. Thus,

$$\delta_p(i, I_1, n) \leq 3\delta_p\left(i, I_1 \bigcup I_2, n\right).$$

■

Remark S.1. This lemma implies that the L^p-FDM $\delta_p(i, I, n)$ admits a similar property like monotonicity: $\delta_p(i, I, n) \leq 3\delta_p(i, J, n)$ for any $I \subset J$, which is useful in practice.

Lemma S.4. *Let $p \geq 1$ and $k \geq 1$. Consider the system (2.1). For a finite or infinite subset $J = \{j_1, j_2, \ldots\} \subset D_n$, we have $\delta_p(i, J, n) \leq \sum_{k=1}^{|J|} \delta_p(i, j_k, n)$.*

Proof. Denote $J_k = \{j_1, j_2, \ldots, j_k\}$ for $k \geq 1$ and $J_0 = \emptyset$. By the Minkowski inequality, we have

$$\delta_p(i, J, n) = \left\|Y_{i,n} - Y_{i,n,\{j_1,j_2,\ldots\}}\right\|_{L^p} = \left\|\sum_{k=1}^{|J|}\left(Y_{i,n,J_{k-1}} - Y_{i,n,J_k}\right)\right\|_{L^p}$$
$$\leq \sum_{k=1}^{|J|}\left\|Y_{i,n,J_{k-1}} - Y_{i,n,J_k}\right\|_{L^p} = \sum_{k=1}^{|J|}\delta_p(i, j_k, n).$$

■

Lemma S.5. *(Burkholder's inequality, Rio, 2009). Let $X_1, X_2, \ldots, X_n$ be a zero-mean martingale difference sequence and $p \geq 2$ is a constant. Then*

$$\left\|\sum_{i=1}^{n} X_i\right\|_{L^p} \leq \sqrt{p-1}\left(\sum_{i=1}^{n}\|X_i\|_{L^p}^2\right)^{1/2}. \tag{S.3}$$

Proof. When $p > 2$, the conclusion follows from Theorem 2.1 in Rio (2009). When $p = 2$, since $\mathrm{Cov}(X_i, X_j) = 0$ for all $i \neq j$, $\|\sum_{i=1}^n X_i\|_{L^2} = \left(\sum_{i=1}^n \|X_i\|_{L^2}^2\right)^{1/2}$. ■

Lemma S.6. *(Lemma A.1 in Jenish and Prucha, 2009). Suppose that Assumption 1 holds. Then, there exists a constant $C < \infty$ such that $\sup_{i\in\mathbb{R}^d}\left|\left\{j \in \mathbb{R}^d : m \leq d_{ij} < m+1\right\}\right| < Cm^{d-1}$.*

Lemma S.7. *(Generalization of Lemma 17.15 in Davidson, 1994). Let B and ρ be two nonnegative random variables and assume $\|\rho\|_{L^q} < \infty$, $\|B\|_{L^p} < \infty$, and $\|\rho B\|_{L^r} < \infty$, for $q^{-1} + p^{-1} = 1$, $q \geq 1$ and $r > t > 1$. Then $\|B\rho\|_{L^t} \leq 2\left(\|B\|_{L^p}^{r-t}\|\rho\|_{L^q}^{r-t}\|B\rho\|_{L^r}^{(t-1)r}\right)^{1/(tr-t)}$.*

Proof. Let $C = \left(\|B\|_{L^p}\|\rho\|_{L^q}\|B\rho\|_{L^r}^{-r}\right)^{1/(1-r)}$ and $B_1 = B \cdot 1(B\rho \leq C)$. By the Minkowski inequality,

$$\|B\rho\|_{L^t} \leq \|B_1\rho\|_{L^t} + \|(B - B_1)\rho\|_{L^t}. \tag{S.4}$$

We bound the two terms on the right hand side (r.h.s.) separately. The first part can be bounded as

$$\|B_1\rho\|_{L^t} = \left(\int_{B\rho\leq C} (B\rho)^t \,\mathrm{d}\mathbb{P}\right)^{1/t} \leq C^{(t-1)/t} \left(\int B\rho \mathrm{d}\mathbb{P}\right)^{1/t} \leq C^{(t-1)/t} \|\rho\|_{L^q}^{1/t} \|B\|_{L^p}^{1/t}, \tag{S.5}$$

where the first inequality is due to the fact that $B\rho \leq C$ and the second one follows from the Hölder inequality. The second part can be bounded as

$$\|(B - B_1)\,\rho\|_{L^t} = \left(\int_{B\rho>C} (B\rho)^t \,\mathrm{d}\mathbb{P}\right)^{1/t} \leq C^{(t-r)/t} \left(\int_{B\rho>C} (B\rho)^r \,\mathrm{d}\mathbb{P}\right)^{1/t} \leq C^{(t-r)/t} \|B\rho\|_{L^r}^{r/t}, \tag{S.6}$$

where the first inequality follows from the fact that $B\rho \leq C$ and $r > t$. Then the conclusion follows from (S.4)-(S.6). ■

Lemma S.8. *D_n is a countable lattice in a metric space. For any nonnegative matrix*[1] $M = (m_{ij})_{|D_n|\times|D_n|}$, *where the indexes of M belong to D_n, i.e., $i, j \in D_n$, denote* $\phi_M(s) = \sup_i \sum_{j:d_{ij}\geq s} m_{ij}$. *Let $K \geq 1$ be an integer. Then for any nonnegative matrices $M_1, \ldots, M_K$,*

$$\phi_{M_1\cdots M_K}(s) \leq \sum_{k=1}^{K} \phi_{M_k}\left(\frac{s}{K}\right) \prod_{j\in\{1,\ldots,K\}:j\neq k} \|M_j\|_\infty .$$

Proof. For all $k = 1, \ldots, K$, denote $M_k \equiv \left(m_{ij}^{(k)}\right)_{|D_n|\times|D_n|}$. Then

$$\phi_{M_1\cdots M_K}(s) = \sup_{i_0} \sum_{i_K:d_{i_0 i_K}\geq s} (M_1\cdots M_K)_{i_0 i_K} = \sup_{i_0} \sum_{i_K:d_{i_0 i_K}\geq s} \sum_{i_1,\ldots,i_{K-1}\in D_n} m_{i_0 i_1}^{(1)} \cdots m_{i_{K-1} i_K}^{(K)}.$$

Denote $I_k \equiv \left\{(i_1, \ldots, i_K) \in D_n^K : \text{there exists some } k \in \{1, 2, \ldots, K\} \text{ such that } d_{i_{k-1}i_k} \geq \frac{s}{K}\right\}$. By the triangle inequality,

$$\left\{(i_1, \ldots, i_K) \in D_n^K : d_{i_0 i_K} \geq s\right\} \subset \bigcup_{k=1}^{K} I_k.$$

[1] A matrix is nonnegative iff its elements are all nonnegative.

Consequently,

$$\phi_{M_1\cdots M_K}(s) \leq \sum_{k=1}^{K} \sup_{i_0} \sum_{(i_1,\ldots,i_K)\in I_k} m^{(1)}_{i_0 i_1}\cdots m^{(K)}_{i_{K-1}i_K}. \tag{S.7}$$

For any $k \in \{2,\ldots,K-1\}$,

$$\begin{aligned}
&\sup_{i_0} \sum_{(i_1,\ldots,i_K)\in I_k} m^{(1)}_{i_0 i_1}\cdots m^{(K)}_{i_{K-1}i_K} = \sup_{i_0} \sum_{i_1,\ldots,i_{k-1}\in D_n} \sum_{i_k: d_{i_{k-1}i_k}\geq s/K} \sum_{i_{k+1},\ldots,i_K\in D_n} m^{(1)}_{i_0 i_1}\cdots m^{(K)}_{i_{K-1}i_K} \\
=&\sup_{i_0} \sum_{i_1,\ldots,i_{k-1}\in D_n} m^{(1)}_{i_0 i_1}\cdots m^{(k-1)}_{i_{k-2}i_{k-1}} \sum_{i_k: d_{i_{k-1}i_k}\geq s/K} m^{(k)}_{i_{k-1}i_k} \left(\sum_{i_{k+1},\ldots,i_K\in D_n} m^{(k+1)}_{i_k i_{k+1}}\cdots m^{(K)}_{i_{K-1}i_K} \right) \\
\leq&\sup_{i_0} \sum_{i_1,\ldots,i_{k-1}\in D_n} m^{(1)}_{i_0 i_1}\cdots m^{(k-1)}_{i_{k-2}i_{k-1}} \left(\sup_{i_{k-1}} \sum_{i_k: d_{i_{k-1}i_k}\geq s/K} m^{(k)}_{i_{k-1}i_k} \right) \prod_{j=k+1}^{K} \|M_j\|_\infty \\
\leq&\phi_{M_k}\left(\frac{s}{K}\right) \prod_{j\in\{1,\ldots,K\}: j\neq k} \|M_j\|_\infty ,
\end{aligned} \tag{S.8}$$

where the first inequality is from

$$\sup_{i_k} \sum_{i_{k+1},\ldots,i_K\in D_n} m^{(k+1)}_{i_k i_{k+1}}\cdots m^{(K)}_{i_{K-1}i_K} = \sup_{i_k} \sum_{i_K\in D_n} (M_{k+1}\cdots M_K)_{i_k i_K} = \|M_{k+1}\cdots M_K\|_\infty \leq \prod_{j=k+1}^{K} \|M_j\|_\infty$$

and the last one follows from the definition of $\phi_{M_k}(s)$. (S.8) also holds for $k=1$ and $k=K$ by the same argument. Then the conclusion follows from (S.7)-(S.8). ■

Lemma S.9. *(Similar to Lemma A.7 in Xu and Lee, 2015). Let $0<\eta<1$ be a constant. Under Assumption 1 and $D_n=\{1,\ldots,n\}$, let $A_n=(a_{ij,n})$ be an $n\times n$ nonstochastic matrix satisfying $a_{ij,n}=0$ when $d_{ij}\geq \bar{d}_0$, where d_{ij} is the distance between individuals i and j. Suppose $\sup\|A_n\|_\infty \leq \eta^{\bar{d}_0}<1$, and, the random field $\{v_{i,n}\}$ satisfies $-1\leq v_{i,n}\leq 1$ and the L^p-FD coefficient of $\{v_{i,n}\}$ on an independent random field $\{u_{i,n}\}$ (denoted as $\Delta_{v,p}(s)$) satisfies $\Delta_{v,p}(s)\leq C\eta^s$, for some positive constants $C>0$, for all $s\geq 0$. Denote $G_n=\operatorname{diag}\{v_{1,n},\ldots,v_{n,n}\}$. Then, for any positive integer l,*

(i) The L^p-FD coefficient of $\left\{g^{(l)}_{i,n}\equiv (G_nA_nG_n)^l_{ii}\right\}$ (denoted as $\Delta^{(l)}_p(s)$) on $\{u_{i,n}\}$ satisfies $\Delta^{(l)}_p(s)\leq$

$C_1\eta^s$ *for some constant* $C_1 > 0$.

(ii) The L^p*-FD coefficient of* $\left\{h_{i,n} \equiv \left[(I_n - G_nA_nG_n)^{-1} G_nA_nG_n\right]_{ii}\right\}$ *(denoted as* $\Delta_{h,p}(s)$*) on* $\{u_{i,n}\}$ *satisfies* $\Delta_{h,p}(s) \leq C_3 s\eta^s$ *for some constant* $C_3 > 0$.

Proof. The proof is borrowed from Xu and Lee (2015).

(i) We note that $(G_nA_nG_n)^l_{ii} = \sum_{j_1} \cdots \sum_{j_{l-1}} a_{ij_1,n}a_{j_1j_2,n} \cdots a_{j_{l-1}i,n}v^2_{i,n}v^2_{j_1,n} \cdots v^2_{j_{l-1},n}$. When $a_{ij_1,n}a_{j_1j_2,n} \cdots a_{j_{l-1}i,n} \neq 0$, we have $d_{ij_1} < \bar{d}_0, d_{j_1j_2} < \bar{d}_0, \ldots, d_{j_{l-1}i} < \bar{d}_0$. Define $I_{k,s} \equiv \{\epsilon_{j,n} : d_{kj} \geq s\}$ for any $k \in D_n$, and for simplicity of the notation, let $j_0 = i$. Then $I_{i,s} \subset I_{j_h,s-h\bar{d}_0}$ for any $s \geq ld$ and $h \in \{0, \ldots, l-1\}$. Note that the absolute values of $v_{j,n}$'s are less than or equal to one and the product of $v_{j,n}$'s is a Lipschitz function. Then, when $s \geq ld$,

$$\left\|v^2_{i,n}v^2_{j_1,n} \cdots v^2_{j_{l-1},n} - v^2_{i,n,I_{i,s}}v^2_{j_1,n,I_{i,s}} \cdots v^2_{j_{l-1},n,I_{i,s}}\right\|_{L^p} \leq \sum_{h=0}^{l-1} \left\|v^2_{j_h,n} - v^2_{j_h,n,I_{i,s}}\right\|_{L^p}$$
$$\leq 2\sum_{h=0}^{l-1} \left\|v_{j_h,n} - v_{j_h,n,I_{i,s}}\right\|_{L^p} \leq 6\sum_{h=0}^{l-1} \left\|v_{j_h,n} - v_{j_h,n,I_{j_h,s-h\bar{d}_0}}\right\|_{L^p} \leq 6\sum_{h=0}^{l-1} \Delta_{v,p}(s-h\bar{d}_0) \leq 6\sum_{h=0}^{l-1} C\eta^{s-h\bar{d}_0},$$

where the second inequality follows from that v^2 is a Lipschitz function on $[-1,1]$ and the third one follows from Lemma S.3. When $0 \leq s < l\bar{d}_0$, $\left\|v^2_{j_h,n} - v^2_{j_h,n,I_{i,s}}\right\|_{L^p} \leq 2 \leq \max\{2, 6C\}\eta^{s-h\bar{d}_0}$, thus the above inequality still holds if we replace C by $\max\{2, 6C\}$. Thus, for any $s \geq 0$

$$\left\|g^{(l)}_{i,n} - g^{(l)}_{i,n,I_{i,s}}\right\|_{L^p} \leq \sum_{j_1} \cdots \sum_{j_{l-1}} |a_{ij_1,n}a_{j_1j_2,n} \cdots a_{j_{l-1}i,n}| \left\|v^2_{i,n}v^2_{j_1,n} \cdots v^2_{j_{l-1},n} - v^2_{i,n,I_{i,s}}v^2_{j_1,n,I_{i,s}} \cdots v^2_{j_{l-1},n,I_{i,s}}\right\|_{L^p}$$
$$\leq \|A_n\|^l_\infty \max\{2, 6C\} \sum_{h=0}^{l-1} \eta^{s-h\bar{d}_0} \leq \max\{2, 6C\}\eta^s\eta^{l\bar{d}_0}\frac{\eta^{-l\bar{d}_0} - 1}{\eta^{-\bar{d}_0} - 1} \leq \frac{\max\{2, 6C\}}{\eta^{-\bar{d}_0} - 1}\eta^s.$$

Thus,

$$\Delta^{(l)}_p(s) = \sup_{n,i} \left\|g^{(l)}_{i,n} - g^{(l)}_{i,n,I_{i,s}}\right\|_{L^p} \leq C_1\eta^s,$$

where $C_1 \equiv \frac{\max\{2,6C\}}{\eta^{-\bar{d}_0}-1}$.

(ii) Notice that

$$h_{i,n} = \left[(I_n - G_n A_n G_n)^{-1} G_n A_n G_n \right]_{ii} = \sum_{l=1}^{\infty} \left[(G_n A_n G_n)^l \right]_{ii} = \sum_{l=1}^{\infty} g_{i,n}^{(l)}$$

and

$$\left\| g_{i,n}^{(l)} - g_{i,n,I_{i,s}}^{(l)} \right\|_{L^p} = \sum_{j_1} \cdots \sum_{j_{l-1}} \left| a_{ij_1,n} a_{j_1 j_2,n} \cdots a_{j_{l-1}i,n} \right| \left\| v_{i,n}^2 v_{j_1,n}^2 \cdots v_{j_{l-1},n}^2 - v_{i,n,I_{i,s}}^2 v_{j_1,n,I_{i,s}}^2 \cdots v_{j_{l-1},n,I_{i,s}}^2 \right\|_{L^p}$$
$$\leq 2 \sum_{j_1} \cdots \sum_{j_{l-1}} \left| a_{ij_1,n} a_{j_1 j_2,n} \cdots a_{j_{l-1}i,n} \right| \leq 2 \left\| A_n \right\|_\infty^l$$

for any $i \in D_n$, $l \in \mathbb{N}$ and $s \geq 0$. Then, when $s \leq \bar{d}_0$,

$$\left\| h_{i,n} - h_{i,n,I_{i,s}} \right\|_{L^p} \leq \sum_{l=1}^{\infty} \left\| g_{i,n}^{(l)} - g_{i,n,I_{i,s}}^{(l)} \right\|_{L^p} \leq 2 \sum_{l=1}^{\infty} \left\| A_n \right\|_\infty^l \leq 2 \sum_{l=1}^{\infty} \eta^{l\bar{d}_0} = \frac{2\eta^{\bar{d}_0}}{1 - \eta^{\bar{d}_0}}.$$

When $s > \bar{d}_0$,

$$\left\| h_{i,n} - h_{i,n,I_{i,s}} \right\|_{L^p} \leq \sum_{l=1}^{\infty} \left\| g_{i,n}^{(l)} - g_{i,n,I_{i,s}}^{(l)} \right\|_{L^p} = \sum_{l \in \mathbb{N}: l\bar{d}_0 < s} \left\| g_{i,n}^{(l)} - g_{i,n,I_{i,s}}^{(l)} \right\|_{L^p} + \sum_{l \in \mathbb{N}: l\bar{d}_0 \geq s} \left\| g_{i,n}^{(l)} - g_{i,n,I_{i,s}}^{(l)} \right\|_{L^p}$$
$$\leq \sum_{l \in \mathbb{N}: l\bar{d}_0 < s} \frac{\max\{2, 6C\}}{\eta^{-\bar{d}_0} - 1} \eta^s + 2 \sum_{l \in \mathbb{N}: l\bar{d}_0 \geq s} \left\| A_n \right\|_\infty^l \leq \frac{\max\{2, 6C\}}{\eta^{-\bar{d}_0} - 1} \eta^s \left\lfloor \frac{s}{\bar{d}_0} \right\rfloor + 2 \sum_{l = \lfloor s/\bar{d}_0 \rfloor}^{\infty} \eta^{l\bar{d}_0}$$
$$= \frac{\max\{2, 6C\}}{\eta^{-\bar{d}_0} - 1} \eta^s \left\lfloor \frac{s}{\bar{d}_0} \right\rfloor + 2 \frac{\eta^{\bar{d}_0 \lfloor s/\bar{d}_0 \rfloor}}{1 - \eta^{\bar{d}_0}} \leq C_2 s \eta^s,$$

where $C_2 > 0$ is a constant. Taking $C_3 = \max\left\{ C_2, \frac{2\eta^{\bar{d}_0}}{1-\eta^{\bar{d}_0}} \right\}$, we have $\left\| h_{i,n} - h_{i,n,I_{i,s}} \right\|_{L^p} \leq C_3 s \eta^s$ for any $s \geq 0$. Thus,

$$\Delta_{h,p}(s) = \sup_{n,i} \left\| h_{i,n} - h_{i,n,I_{i,s}} \right\|_{L^p} \leq C_3 s \eta^s.$$

■

Lemma S.10. *Let* $2 < p_0 \leq q_0 \in \mathbb{R}$ *and* $2 \leq w_0 \in \mathbb{N}$ *satisfy* $\frac{1}{p_0} + \frac{w_0 - 1}{q_0} = \frac{1}{2}$, $r = \min\{d_{i_k j_l} : 1 \leq$

$k \leq u, 1 \leq l \leq v\}$, *and* $w = u + v \leq w_0$. *Denote* $\|Y\|_{L^p} \equiv \sup_{i,n} \|Y_{i,n}\|_{L^p}$ *for* $p > 1$. *If (i)* $\mathbb{E}Y_{i,n} = 0$ *for all* $i \in D_n$, *(ii)* $M \equiv \max(1, \|Y\|_{L^{q_0}}) < \infty$, *(iii)* $\{Y_{i,n}\}$ *is* L^2*-FD on an independent random field* $\{\epsilon_{i,n}\}$ *with the* L^2*-FD coefficient* $\Delta_2(s)$, *then, for any* $0 < s \leq r/2$,

$$|\mathrm{Cov}\left(Y_{i_1,n} \cdots Y_{i_u,n}, Y_{j_1,n} \cdots Y_{j_v,n}\right)| \leq 4wM^{w-1} \|Y\|_{L^{p_0}} \left[\Delta_2(s)\right]^{\frac{q_0 - 2w + 2}{2q_0 - 2w + 2}}.$$

Remark S.2. Lemma S.10 extends the covariance inequality of NED random fields (Lemma A.1, Xu and Lee, 2018) to FD random fields. Note that by Lyapunov's inequality ($p_0 \leq q_0$) and condition (ii) in this lemma, $\|Y\|_{L^{p_0}} < \infty$.

Proof. Let $I_{k,n}(s) = \{j : d(i_k, j) \geq s\}$ for $k = 1, \cdots, u$, $U \equiv \prod_{k=1}^{u} Y_{i_k,n,I_{k,n}(s)}$, $\Delta U = \prod_{k=1}^{u} Y_{i_k,n} - U$. Similarly, we define $J_{l,n}(s) = \{i : d(i, j_l) \geq s\}$ for $l = 1, \cdots, v$, $V \equiv \prod_{l=1}^{v} Y_{j_l,n,J_{l,n}(s)}$, $\Delta V = \prod_{l=1}^{v} Y_{j_l,n} - V$. When we construct $Y_{j_l,n,J_{l,n}(s)}$ for $l = 1, \cdots, v$, we choose the i.i.d. copies $\epsilon^*_{j,n}$ for $j \in J_{l,n}(s)$ to be independent of those $\epsilon^*_{i,n}$ for $i \in I_{k,n}(s)$, $k = 1, \cdots, u$. In this way, when $r \geq 2s$, U is independent of V, thus, $\mathrm{Cov}(U, V) = 0$. Let $t \equiv \frac{u-1}{w_0-1} q_0 \leq q_0$. By generalized Hölder's inequality and Lyapunov's inequality,

$$\|U + \Delta U\|_{L^2} = \|\prod_{k=1}^{u} Y_{i_k,n}\|_{L^2} \leq \|Y_{i_1,n}\|_{L^{p_0}} \|\prod_{k=2}^{u} Y_{i_k,n}\|_{L^t} \leq \|Y\|_{L^{p_0}} M^{u-1}. \tag{S.9}$$

Since $Y_{i_k,n,I_{k,n}(s)}$ and $Y_{i_k,n}$ are identically distributed for all $k = 1, \cdots, u$,

$$\|U\|_{L^2} \leq \|Y\|_{L^{p_0}} M^{u-1}. \tag{S.10}$$

Let $A \equiv \frac{q_0}{w-1} \geq \frac{q_0}{w_0-1} > 2$. In the following derivations, we use the convention that $\prod_{m=1}^{0} =$

$\prod_{m=u+1}^{u} = 1$.

$$\begin{aligned}
&\|\Delta U\|_{L^2} = \left\| \prod_{k=1}^{u} Y_{i_k,n} - \prod_{k=1}^{u} Y_{i_k,n,I_{k,n}(s)} \right\|_{L^2} \\
\le &\left\| \sum_{m=1}^{u} \left(\prod_{k=1}^{m-1} Y_{i_k,n} \right) \left(\prod_{k=m+1}^{u} Y_{i_k,n,I_{k,n}(s)} \right) \left(Y_{i_m,n} - Y_{i_m,n,I_{m,n}(s)} \right) \right\|_{L^2} \\
\le &\sum_{m=1}^{u} \left\| \left(\prod_{k=1}^{m-1} Y_{i_k,n} \right) \left(\prod_{k=m+1}^{u} Y_{i_k,n,I_{k,n}(s)} \right) \left(Y_{i_m,n} - Y_{i_m,n,I_{m,n}(s)} \right) \right\|_{L^2} \\
\le &2 \sum_{m=1}^{u} \left\| \left(\prod_{k=1}^{m-1} Y_{i_k,n} \right) \left(\prod_{k=m+1}^{u} Y_{i_k,n,I_{k,n}(s)} \right) \right\|_{L^2}^{\frac{A-2}{2A-2}} \cdot \left\| Y_{i_m,n} - Y_{i_m,n,I_{m,n}(s)} \right\|_{L^2}^{\frac{A-2}{2A-2}} \cdot \\
&\left\| \left(\prod_{k=1}^{m-1} Y_{i_k,n} \right) \left(\prod_{k=m+1}^{u} Y_{i_k,n,I_{k,n}(s)} \right) \left(Y_{i_m,n} - Y_{i_m,n,I_{m,n}(s)} \right) \right\|_{L^A}^{\frac{A}{2A-2}},
\end{aligned} \tag{S.11}$$

where the first inequality is by Lemma A.3 in Xu and Lee (2015), the second one is by Minkowski's inequality, and the last one is by $\|B\rho\|_{L^2} \le 2 \left(\|\rho\|_{L^2}^{A-2} \|B\|_{L^2}^{A-2} \|B\rho\|_{L^A}^{A} \right)^{1/(2A-2)}$ when $A > 2$ (Lemma 17.15, Davidson, 1994). Similar to (S.9),

$$\left\| \left(\prod_{k=1}^{m-1} Y_{i_k,n} \right) \left(\prod_{k=m+1}^{u} Y_{i_k,n,I_{k,n}(s)} \right) \right\|_{L^2} \le M^{u-1}. \tag{S.12}$$

Moreover, by generalized Hölder's inequality,

$$\begin{aligned}
&\left\| \left(\prod_{k=1}^{m-1} Y_{i_k,n} \right) \left(\prod_{k=m+1}^{u} Y_{i_k,n,I_{k,n}(s)} \right) \left(Y_{i_m,n} - Y_{i_m,n,I_{m,n}(s)} \right) \right\|_{L^A} \\
\le &\prod_{k=1}^{m-1} \|Y_{i_k,n}\|_{L^{Au}} \cdot \prod_{k=m+1}^{u} \left\| Y_{i_k,n,I_{k,n}(s)} \right\|_{L^{Au}} \cdot \left\| Y_{i_m,n} - Y_{i_m,n,I_{m,n}(s)} \right\|_{L^{Au}} \\
\le &\prod_{k=1}^{m-1} \|Y_{i_k,n}\|_{L^{q_0}} \cdot \prod_{k=m+1}^{u} \left\| Y_{i_k,n,I_{k,n}(s)} \right\|_{L^{q_0}} \cdot \left(\|Y_{i_m,n}\|_{L^{q_0}} + \left\| Y_{i_m,n,I_{m,n}(s)} \right\|_{L^{q_0}} \right) \\
\le &2M^u,
\end{aligned} \tag{S.13}$$

where the second inequality is by Lyapunov's inequality ($Au \leq q_0$) and Minkowski's inequality. Plugging (S.12) and (S.13) into (S.11), we have

$$\begin{aligned}&\|\Delta U\|_{L^2} \leq 2uM^{(u-1)\frac{A-2}{2A-2}}\Delta_2(s)^{\frac{A-2}{2A-2}}2^{\frac{A}{2A-2}}M^{\frac{Au}{2A-2}} \\ =&2^{\frac{3A-2}{2A-2}}uM^{\frac{2uA-2u-A+2}{2A-2}}\Delta_2(s)^{\frac{A-2}{2A-2}} \leq 4uM^u\Delta_2(s)^{\frac{A-2}{2A-2}},\end{aligned} \tag{S.14}$$

where the last inequality follows from the fact that $\frac{3A-2}{2A-2} < 2$, $M \geq 1$, and $\frac{2uA-2u-A+2}{2A-2} < u$. Similarly, we have $\|V + \Delta V\|_{L^2} \leq \|Y\|_{L^{p_0}}M^{v-1}$, $\|V\|_{L^2} \leq \|Y\|_{L^{p_0}}M^{v-1}$, and $\|\Delta V\|_{L^2} \leq 4vM^v\Delta_2(s)^{\frac{A-2}{2A-2}}$. Consequently, by $\mathrm{Cov}(U, V) = 0$, we have

$$\begin{aligned}&|\mathrm{Cov}\,(Y_{i_1,n}\cdots Y_{i_u,n}, Y_{j_1,n}\cdots Y_{j_v,n})| = |\mathrm{Cov}\,(U + \Delta U, V + \Delta V)| \\ \leq& |\mathrm{Cov}\,(U, V)| + |\mathrm{Cov}\,(U, \Delta V)| + |\mathrm{Cov}\,(\Delta U, V + \Delta V)| \\ \leq&\|U\|_{L^2}\|\Delta V\|_{L^2} + \|\Delta U\|_{L^2}\|V + \Delta V\|_{L^2} \\ \leq&\|Y\|_{L^{p_0}}M^{u-1} \cdot 4vM^v\Delta_2(s)^{\frac{A-2}{2A-2}} + 4uM^u\Delta_2(s)^{\frac{A-2}{2A-2}} \cdot \|Y\|_{L^{p_0}}M^{v-1} \\ =&4wM^{w-1}\|Y\|_{L^{p_0}}\left[\Delta_2(s)\right]^{\frac{q_0-2w+2}{2q_0-2w+2}},\end{aligned}$$

where the third inequality follows from the bounds for $\|V + \Delta V\|_{L^2}$ and $\|\Delta V\|_{L^2}$, (S.10), and (S.14), and the last step follows from $w = u + v$ and $A \equiv \frac{q_0}{w-1}$. ■

Lemma S.11. *(Corollary 1.8, Nagaev, 1979). When $X_1, X_2, \ldots, X_n$ are mean zero independent random variables, for any $p \geq 2$, $x > 0$,*

$$\mathbb{P}\left(|S_n| \geq x\right) \leq \left(1 + \frac{2}{p}\right)^p \frac{\mu_{n,p}}{x^p} + 2\exp\left(-\frac{2x^2}{e^p\left(p+2\right)^2\mu_{n,2}}\right), \tag{S.15}$$

where $S_n = \sum_{i=1}^n X_i$, $\mu_{n,p} = \sum_{i=1}^n \|X_i\|_{L^p}^p$.

Lemma S.12. *$\{x_n\}$ is a nonnegative sequence. If $x_n = O\left(n^\alpha\right)$ for some $\alpha < -1$, then $\sum_{n=1}^\infty x_n < \infty$ and $\sum_{m=n}^\infty x_m = O\left(n^{\alpha+1}\right)$ as $n \to \infty$.*

Proof. Since $x_n = O(n^{\alpha})$, there exists a constant C such that $x_n \leq Cn^{\alpha}$. Thus $\sum_{m=n}^{\infty} x_m \leq \sum_{m=n}^{\infty} Cn^{\alpha}$. Then the conclusion follows from $\sum_{m=n}^{\infty} Cm^{\alpha} \leq C\int_{n-1}^{\infty} x^{\alpha}\mathrm{d}x$ and

$$\lim_{n\to\infty} \frac{\int_{n-1}^{\infty} x^{\alpha}\mathrm{d}x}{n^{\alpha+1}} = \lim_{n\to\infty} \frac{-(n-1)^{\alpha}}{(\alpha+1)\,n^{\alpha}} = \frac{-1}{\alpha+1} > 0,$$

where the first equality follows from L'Hospital's rule. ■

S.2. An SAR Tobit Model

Here, we employ our new tools to establish the CLT for the score function of the SAR Tobit model studied in Xu and Lee (2015), which is a crucial step for establishing the asymptotic normality of the MLE (maximum likelihood estimator). The form of the SAR Tobit model is the same as (4.3) with $F(\cdot) \equiv \max\{0, \cdot\}$. We first state some assumptions.

Assumption S.1. *(1)* $\zeta = |\lambda| \sup_n \|W_n\|_{\infty} < 1$*;*

(2) $w_{ij,n}$ *can be nonzero only if* $d_{ij} < \bar{d}_0$*;*

(3) for each n, $\epsilon_{i,n}$*'s are i.i.d.* $N(0,\sigma^2)$ *random variables;* $X_{i,n}$*'s and* $\epsilon_{i,n}$*'s are independent;*

(4) for some $p \geq 6$, $\|X\|_{L^p} = \sup_{n,i} \left\| X_{i,n}'\beta \right\|_{L^p} < \infty$ *and* $\{X_{i,n}\}$ *is* L^p*-FD on an independent random field* $\{u_{i,n} : i \in D_n, n \geq 1\}$ *with the* L^p*-FD coefficient* $\Delta_{X,p}(s) = O\left(\zeta^{s/\bar{d}_0}\right)$ *satisfying* $\Delta_{X,p}(0) < \infty$*;* $\left(u_{i,n}', \epsilon_{i,n}\right)$*'s are independent over* i*;*

(5) $\Sigma = \lim_{n\to\infty} \Sigma_n$ *exists and is nonsingular, where* $\Sigma_n = \frac{1}{n}\mathrm{Var}(\sum_{i=1}^{n} q_{i,n})$ *and the expression of the score function* $q_{i,n}$ *can be found in Section 5 of Xu and Lee (2015, p.269).*[2]

Assumptions S.1(1)-(5) are similar to Assumptions 2, 3(1), 5, 10-11 in Xu and Lee (2015). We can also consider the case that $w_{ij,n}$ decreases as a power function of d_{ij} like Assumption 3(2) in

[2]Note that our $q_{i,n}$ corresponds to $q_{i,n}(\theta_0)$, the score function evaluated at the true model parameters, in Xu and Lee (2015).

Xu and Lee (2015), but we only consider the short distance connections for simplicity. We have the following CLT.

Proposition S.1. *Under Assumptions 1 and S.1,* $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} q_{i,n} \xrightarrow{d} N(0,\Sigma)$.

Proof. Let $z_{i,n} \equiv \frac{Y_{i,n}-\lambda W_{i\cdot,n}Y_n - X_{i,n}'\beta}{\sigma} = \frac{\epsilon_{i,n}}{\sigma}$, $r_{i,n} \equiv \left[\left(I_n - \lambda\tilde{W}_n\right)^{-1}\tilde{W}_n\right]_{ii}$, where $\tilde{W}_n = G(Y_n)WG(Y_n)$ and $G(Y_n) = \mathrm{diag}\left\{1(Y_{1,n}>0),\ldots,1(Y_{n,n}>0)\right\}$, and $\phi(\cdot)$, $\Phi(\cdot)$ be the probability density function and cumulative distribution function of the standard normal distribution, respectively. From Proposition 1(1) and Lemma A.9 in Xu and Lee (2015), $\{Y_{i,n}\}$, $\{W_{i\cdot,n}Y_n\}$, $\{z_{i,n}\}$, $\left\{z_{i,n}^2\right\}$ and $\left\{\frac{\phi(z_{i,n})}{\Phi(z_{i,n})}\right\}$ are uniformly (in i and n) L^p, L^p, L^p, $L^{p/2}$ and L^p bounded, respectively. And from the proof of Proposition 5 in Xu and Lee (2015), $|r_{i,n}| \le \lambda\frac{\zeta^2}{1-\zeta}$. Then, by Hölder's and Minkowski's inequality, $\{q_{i,n}\}$ is uniformly $L^{p/2}$ bounded. To apply Theorem 3.5, we show that every term in $q_{i,n}$ is L^2-FD on $\left\{\left(u_{i,n}', \epsilon_{i,n}\right)'\right\}$. By Proposition 4.3(5)(ii) and Assumptions S.1(1)-(4), the L^p-FD coefficient of $Y_{i,n}$ is $O\left(\zeta^{s/\left(2\bar{d}_0\right)}\right)$. For $\{W_{i\cdot,n}Y_n\}$, denote $I_{i,s} = \{j : d_{ij} \ge s\}$. When $s \ge \bar{d}_0$,

$$
\begin{aligned}
&\left\|W_{i\cdot,n}Y_n - W_{i\cdot,n}Y_{n,I_{i,s}}\right\|_{L^p} \le \sum_{k=1}^{n}|w_{ik,n}|\left\|Y_{k,n}-Y_{k,n,I_{i,s}}\right\|_{L^p} = \sum_{k:d_{ik}<\bar{d}_0}|w_{ik,n}|\left\|Y_{k,n}-Y_{k,n,I_{i,s}}\right\|_{L^p} \\
\le& 3\sum_{k:d_{ik}<\bar{d}_0}|w_{ik,n}|\left\|Y_{k,n}-Y_{k,n,I_{k,s-\bar{d}_0}}\right\|_{L^p} \le \sum_{k:d_{ik}<\bar{d}_0}|w_{ik,n}|\,O\left(\zeta^{\left(s-\bar{d}_0\right)/\left(2\bar{d}_0\right)}\right) = O\left(\zeta^{s/\left(2\bar{d}_0\right)}\right)
\end{aligned}
$$

as $s \to \infty$, where the second inequality follows from Lemma S.3 and $I_{i,s} \subset I_{k,s-\bar{d}_0}$ for any k satisfying $d_{ik} < \bar{d}_0$ and the last step follows from $\sup_n \|W_n\|_\infty < \infty$. When $0 \le s < \bar{d}_0$, $\left\|W_{i\cdot,n}Y_n - W_{i\cdot,n}Y_{n,I_{i,s}}\right\|_{L^p} \le 2\sup_{n,i}\|W_{i\cdot,n}Y_n\|_{L^p} < \infty$. Thus, the L^p-FD coefficient of $\{W_{i\cdot,n}Y_n\}$ is $O\left(\zeta^{s/\left(2\bar{d}_0\right)}\right)$. Since $z_{i,n} = \frac{\epsilon_{i,n}}{\sigma}$, all of $\{z_{i,n}\}$, $\left\{z_{i,n}^2\right\}$ and $\left\{\frac{\phi(z_{i,n})}{\Phi(z_{i,n})}\right\}$ are independent random fields. By Proposition 5.4, the L^p-FD coefficients of $\{1(Y_{i,n})>0\}$ and $\{1(Y_{i,n}=0)\}$ on $\left\{\left(u_{i,n}', \epsilon_{i,n}\right)'\right\}$ are both $O\left(\zeta^{\frac{s}{2(p+1)\bar{d}_0}}\right)$. Since $|\lambda|\sup_n\|W_n\|_\infty = \zeta < \left(\zeta^{\frac{1}{6\bar{d}_0}}\right)^{\bar{d}_0}$, by Lemma S.9, the L^2-FD coefficient of $\{r_{i,n}\}$ is $O\left(s\zeta^{\frac{s}{6\bar{d}_0}}\right)$. So, by Proposition 5.6, all terms except $r_{i,n}$ in $q_{i,n}$ are $L^{p/3}$-FD on $\left\{\left(u_{i,n}', \epsilon_{i,n}\right)'\right\}$ with the $L^{p/3}$-FD coefficient $O\left(\zeta^{\frac{s}{2(p+1)\bar{d}_0}}\right)$. We illustrate this for $\left\{1(Y_{i,n}=0)\frac{\phi(z_{i,n})}{\Phi(z_{i,n})}W_{i\cdot,n}Y_n\right\}$

as an example. By Proposition 5.6, the $L^{p/2}$-FD coefficient of $\left\{\frac{\phi(z_{i,n})}{\Phi(z_{i,n})} W_{i\cdot,n} Y_n\right\}$ is $O\left(\zeta^{s/\left(2\bar{d}_0\right)}\right)$; by Hölder's inequality, $\left\{\frac{\phi(z_{i,n})}{\Phi(z_{i,n})} W_{i\cdot,n} Y_n\right\}$ is uniformly (in i and n) $L^{p/2}$ bounded. Now, by Proposition 5.6 again, the $L^{p/3}$-FD coefficient of $\left\{1(Y_{i,n}=0)\frac{\phi(z_{i,n})}{\Phi(z_{i,n})} W_{i\cdot,n} Y_n\right\}$ is $O\left(\zeta^{\frac{s}{2(p+1)\bar{d}_0}}\right)$. Other terms can be calculated similarly. Since $\frac{p}{3} \geq 2$ by Assumption S.1(4), we conclude that the L^2-FD coefficient of $\{q_{i,n}\}$ is $\max\left\{O\left(s\zeta^{\frac{s}{6\bar{d}_0}}\right), O\left(\zeta^{\frac{s}{2(p+1)\bar{d}_0}}\right)\right\} = O\left(\zeta^{\frac{s}{2(p+1)\bar{d}_0}}\right)$. Hence, by Assumption S.1(5), Theorem 3.5 and Slutsky's theorem, we have the conclusion. ■

S.3. Proofs for Appendix B

Proof of Lemma B.1. Recall $I_{i,m,\iota} = \{j \in D_n : d_{ij} \in [\iota_{m-1}, \iota_m)\}$. The conclusion follows from

$$
\begin{aligned}
&\|V_{i,n,\iota}(m)\|_{L^p} = \|\mathbb{E}\left(Y_{i,n}|\mathcal{F}_{i,n}\left(\iota_m\right)\right) - \mathbb{E}\left(Y_{i,n}|\mathcal{F}_{i,n}\left(\iota_{m-1}\right)\right)\|_{L^p} \\
&= \left\|\mathbb{E}\left(Y_{i,n}|\mathcal{F}_{i,n}\left(\iota_m\right)\right) - \mathbb{E}\left(Y_{i,n,I_{i,m,\iota}}|\mathcal{F}_{i,n}\left(\iota_{m-1}\right)\right)\right\|_{L^p} \\
&= \left\|\mathbb{E}\left(Y_{i,n}|\mathcal{F}_{i,n}\left(\iota_m\right)\right) - \mathbb{E}\left(Y_{i,n,I_{i,m,\iota}}|\mathcal{F}_{i,n}\left(\iota_m\right)\right)\right\|_{L^p} = \left\|\mathbb{E}\left(Y_{i,n} - Y_{i,n,I_{i,m,\iota}}|\mathcal{F}_{i,n}\left(\iota_m\right)\right)\right\|_{L^p} \\
&\leq \left\|Y_{i,n} - Y_{i,n,I_{i,m,\iota}}\right\|_{L^p} \leq \theta_{m,p,\iota},
\end{aligned}
$$

where the second and third equalities follow from the independence of $\epsilon_{j,n}$'s and $\epsilon^*_{j,n}$'s, and the first inequality follows from the conditional Jensen's inequality. ■

Proof of Theorem B.1. Assume $\mathbb{E}Y_{j,n} = 0$ for all j and n w.l.o.g. to shorten formulas in the proof. Recall that $V_{j,n,\iota}(m) \equiv \mathbb{E}\left(Y_{j,n}|\mathcal{F}_{j,n}\left(\iota_m\right)\right) - \mathbb{E}\left(Y_{j,n}|\mathcal{F}_{j,n}\left(\iota_{m-1}\right)\right)$ in Lemma B.1. Since $\epsilon_{i,n}$'s are independent, $V_{i,n,\iota}(m)$ and $V_{j,n,\iota}(m)$ are independent if $d_{ij} \geq 2\iota_m$. The idea of the proof is to divide $\{V_{j,n,\iota}(m)\}_{j\in T_n}$ into several subsequences such that the random variables in each subsequence are independent with mean zero. Thus, every subsequence is a martingale difference sequence and we can apply Burkholder's inequality (Lemma S.5). Hence, it suffices to group the spatial units such that the distance of any two spatial units in the same group is greater than or equal to $2\iota_m$.

First, we partition $\mathbb{R}^d$ using big cubes (a cube is a left closed and right open interval in $\mathbb{R}^d$ in this proof) with length of sides $2\iota_m$: for any $m \in \mathbb{N}$,

$$\mathbb{R}^d = \bigcup_{(k_1,\cdots,k_d)\in\mathbb{Z}^d} S\left(k_1, \ldots, k_d\right),$$

where $S\left(k_1, \cdots, k_d\right) = \left[2k_1\iota_m, 2\left(k_1+1\right)\iota_m\right) \times \left[2k_2\iota_m, 2\left(k_2+1\right)\iota_m\right) \times \cdots \times \left[2k_d\iota_m, 2\left(k_d+1\right)\iota_m\right)$. To shorten the notation, denote $\vec{k} = (k_1, \ldots, k_d)$. Then $S(\vec{k}) \equiv S\left(k_1, \ldots, k_d\right) \equiv [2\iota_m\vec{k}, 2\iota_m(\vec{k}+1))$, where $(a_1, \ldots, a_d) + b \equiv (a_1 + b, \ldots, a_d + b)$ for any vector $(a_1, \ldots, a_d)$ and scalar b. So, the above partition can be written as $\mathbb{R}^d = \bigcup_{\vec{k}\in\mathbb{Z}^d} S(\vec{k})$.

Second, we classify these big cubes $S(\vec{k})$'s into 2^d groups such that each cube will not be in the same group as its adjacent cube. Two cubes $S\left(\vec{k}_1\right)$ and $S\left(\vec{k}_2\right)$ belong to the same group iff $\vec{k}_1 \equiv \vec{k}_2 \pmod 2$, which is defined as $k_{1i} \equiv k_{2i} \pmod 2$ for all $i \in \{1, \cdots, d\}$, i.e., k_{1i} and k_{2i} share the same parity for all i. Let $A = \{(a_1, \ldots, a_d) : a_i = 0 \text{ or } 1 \text{ for all } i\}$ and notice that $|A| = 2^d$. So,

$$\mathbb{R}^d = \bigcup_{\vec{a}\in A}\left[\bigcup_{\vec{k}\in\mathbb{Z}^d:\vec{k}\equiv\vec{a} \pmod 2} S(\vec{k})\right].$$

Consequently, each group corresponds to each $\vec{a} \in A$, and within every group $\{S(\vec{k}) : \vec{k} \in \mathbb{Z}^d, \vec{k} \equiv \vec{a} \pmod 2\}$, the distance of any two big cubes is greater than or equal to $2\iota_m$.

Third, we partition each cube $S(\vec{k})$ into $(2\iota_m)^d$ disjoint unit cubes. Denote $I(\iota_m) \equiv \{(i_1, \ldots, i_d) : i_j \in \{0, 1, \ldots, 2\iota_m - 1\} \text{ for all } j\}$, and notice that $|I(\iota_m)| = (2\iota_m)^d$. Then

$$S\left(\vec{k}\right) = \bigcup_{\vec{i}\in I(\iota_m)} [2\iota_m\vec{k} + \vec{i}, 2\iota_m\vec{k} + \vec{i} + 1) \equiv \bigcup_{\vec{i}\in I(\iota_m)} S\left(\vec{k}, \vec{i}\right).$$

So,

$$\mathbb{R}^d = \bigcup_{\vec{a}\in A}\bigcup_{\vec{i}\in I(\iota_m)}\left[\bigcup_{\vec{k}\in\mathbb{Z}^d:\vec{k}\equiv\vec{a} \pmod 2} S(\vec{k}, \vec{i})\right].$$

Under Assumption 1, there is at most one spatial unit in each unit cube $S(\vec{k},\vec{i})$.

Finally, for each $\vec{a} \in A$ and $\vec{i} \in I(\iota_m)$, denote $U(\vec{a},\vec{i}) \equiv T_n \cap \left[\bigcup_{\vec{k}\in\mathbb{Z}^d:\vec{k}\equiv\vec{a}\ (\text{mod } 2)} S(\vec{k},\vec{i})\right]$ and $U(\vec{a}) \equiv \bigcup_{\vec{i}\in I(\iota_m)} U(\vec{a},\vec{i})$. Then $T_n = \bigcup_{\vec{a}\in A} U(\vec{a}) = \bigcup_{\vec{a}\in A}\bigcup_{\vec{i}\in I(\iota_m)} U(\vec{a},\vec{i})$. Figure S.1 shows an example of the above partition.

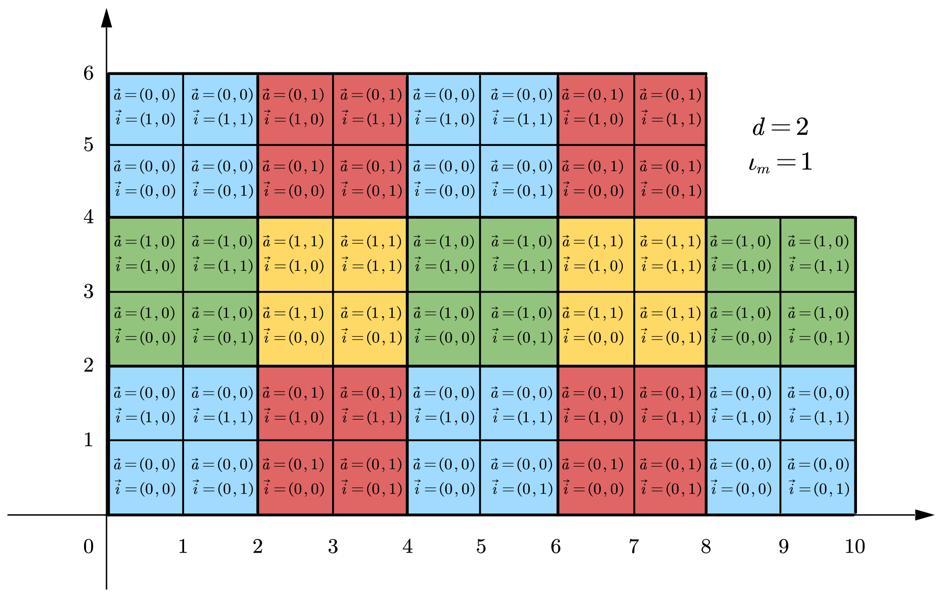

Figure S.1: An example of the partition (The squares with the same color belong to the same big group ($U(\vec{a})$), and every big group is divided into four smaller groups ($U(\vec{a},\vec{i})$).)

For each $\vec{a} \in A$ and $\vec{i} \in I(\iota_m)$, the random variables in $\left\{V_{j,n,\iota}(m) : j \in U(\vec{a},\vec{i})\right\}$ are independent. From the definition of $\mathscr{I}$, $\lim_{m\to\infty}\iota_m = \infty$. Since $Y_{j,n} = \sum_{m=1}^{\infty} V_{j,n,\iota}(m)$ for all $j \in U$,

$$\left\|\sum_{j\in T_n} Y_{j,n}\right\|_{L^p} = \left\|\sum_{j\in T_n}\sum_{m=1}^{\infty} V_{j,n,\iota}(m)\right\|_{L^p} = \left\|\sum_{\vec{a}\in A}\sum_{\vec{i}\in I(\iota_m)}\sum_{j\in U(\vec{a},\vec{i})}\sum_{m=1}^{\infty} V_{j,n,\iota}(m)\right\|_{L^p}$$
$$\leq \sum_{m=1}^{\infty}\sum_{\vec{a}\in A}\sum_{\vec{i}\in I(\iota_m)}\left\|\sum_{j\in U(\vec{a},\vec{i})} V_{j,n,\iota}(m)\right\|_{L^p} \leq \sum_{m=1}^{\infty}\sum_{\vec{a}\in A}\sum_{\vec{i}\in I(\iota_m)}\sqrt{p-1}\left(\sum_{j\in U(\vec{a},\vec{i})}\|V_{j,n,\iota}(m)\|_{L^p}^2\right)^{1/2}$$
$$\leq \sum_{m=1}^{\infty}\sum_{\vec{a}\in A}\sum_{\vec{i}\in I(\iota_m)}\sqrt{p-1}\left|U(\vec{a},\vec{i})\right|^{1/2}\theta_{m,p,\iota} \leq \sqrt{p-1}\sum_{m=1}^{\infty} 2^d\iota_m^{d/2}\left|T_n\right|^{1/2}\theta_{m,p,\iota} = 2^d\sqrt{p-1}\Theta_{p,\iota}\left|T_n\right|^{1/2},$$

where the second inequality follows from (S.3) (Since $V_{j,n}(m)$'s are independent with mean zero for all $j \in U(\vec{a},\vec{i})$, we can regard $\{V_{j,n}(m)\}_{j\in U_{s,t}}$ as a martingale difference sequence), the third inequality follows from (B.1) and the last inequality follows from the power-mean inequality

$$\left(\frac{\sum_{\vec{a}\in A}\sum_{\vec{i}\in I(\iota_m)}\left|U(\vec{a},\vec{i})\right|^{1/2}}{2^d(2\iota_m)^d}\right)^2 \le \frac{\sum_{\vec{a}\in A}\sum_{\vec{i}\in I(\iota_m)}\left|U(\vec{a},\vec{i})\right|}{2^d(2\iota_m)^d} = \frac{|T_n|}{2^d(2\iota_m)^d}.$$

We obtain the desired result. ■

Proof of Theorem B.2. The idea of the proof is borrowed from that for Theorem 3 in Wu and Wu (2016). From Theorem B.1, for any $p \ge 2$, we have

$$\|S_n\|_{L^p} = \left\|\sum_{i\in T_n} Y_{i,n}\right\|_{L^p} \le 2^d\sqrt{p-1}\Theta_{p,\iota^{(p)}}|T_n|^{1/2}.$$

Consequently, $\|Z_n\|_{L^p} \le 2^d\sqrt{p-1}\Theta_{p,\iota^{(p)}}$ for $p\ge 2$. Recall a Taylor's formula: $(1-s)^{-1/2} = 1+\sum_{k=1}^{\infty}a_k s^k$, where $|s|<1$ and $a_k = (2k)!/\left(2^{2k}(k!)^2\right)$ for $k \ge 0$. By Stirling's formula, $a_k \sim (k\pi)^{-1/2}$. Hence, $k! \sim \sqrt{2}(k/e)^k a_k^{-1}$ and $a_k/a_{k-1}\to 1$. Thus, there exist constants $c_1, c_2 > 0$ such that $c_1(k/e)^k a_k^{-1} \le k!$ and $a_k \le c_2 a_{k-1}$ hold for all $k\ge 1$. By (B.3), when $\alpha k \ge 2$, we have $\Theta_{\alpha k,\iota^{(\alpha k)}} \le \gamma_0(\alpha k)^{\nu}$. As a result, when $\alpha k\ge 2$,

$$\begin{aligned}\frac{t^k\|Z_n\|_{L^{\alpha k}}^{\alpha k}}{k!} &\le \frac{t^k\left[2^d\sqrt{\alpha k-1}\Theta_{\alpha k,\iota^{(\alpha k)}}\right]^{\alpha k}}{c_1(k/e)^k a_k^{-1}} \le \frac{2^{d\alpha k}t^k(\alpha k-1)^{\alpha k/2}\gamma_0^{\alpha k}(\alpha k)^{\alpha k\nu}}{c_1(k/e)^k a_k^{-1}}\\ &= \frac{a_k t^k}{c_1 t_0^k}\frac{(\alpha k-1)^{\alpha k/2}}{(\alpha k)^{\alpha k/2}} \le \frac{a_k t^k}{c_1 t_0^k\sqrt{e}},\end{aligned}$$

where the equality is by $t_0 = \left(2^{\alpha d}e\alpha\gamma_0^{\alpha}\right)^{-1}$ and $\nu = \frac{1}{\alpha}-\frac{1}{2}$, and the last step is due to $(x-1)^{x/2}/x^{x/2} \le e^{-1/2}$ for all $x\ge 2$. When $0<\alpha k\le 2$ and $k\ge 1$, we have $\|Z_n\|_{L^{\alpha k}} \le \|Z_n\|_{L^2} \le 2^d\Theta_{2,\iota^{(2)}} \le 2^d 2^{\nu}\gamma_0$, and so

$$\frac{t^k \left\|Z_n\right\|_{L^{\alpha k}}^{\alpha k}}{k!} \leq \frac{t^k \left(2^d 2^\nu \gamma_0\right)^{\alpha k}}{k!} \leq \frac{t^k 2^{d\alpha k} 2^{\nu\alpha k} \gamma_0^{\alpha k}}{c_1 \left(k/e\right)^k a_k^{-1}} = \frac{a_k t^k}{c_1 t_0^k} \frac{2^{\nu\alpha k}}{\left(\alpha k\right)^k} \leq \frac{a_k t^k}{c_1 t_0^k} \frac{2^{2/\alpha - 1}}{\min\left\{\alpha, \alpha^{2/\alpha}\right\}},$$

where the equality is by $t_0 = \left(2^{\alpha d} e\alpha\gamma_0^\alpha\right)^{-1}$, and the last inequality follows from the following facts: $2^{\nu\alpha k} \leq 2^{2\nu}$, $\nu = \frac{1}{\alpha} - \frac{1}{2}$, and

$$\left(\alpha k\right)^k \geq \begin{cases} \alpha & \text{if } \alpha \geq 1, \\ \alpha^{2/\alpha} & \text{if } \alpha < 1. \end{cases}$$

Because $e^x = 1 + \sum_{k=1}^\infty x^k/k!$,

$$\begin{aligned} m\left(t\right) = 1 + \sum_{k=1}^\infty \frac{t^k \mathbb{E}\left|Z_n\right|^{\alpha k}}{k!} &= 1 + \sum_{1\leq k<2/\alpha} \frac{t^k \left\|Z_n\right\|_{L^{\alpha k}}^{\alpha k}}{k!} + \sum_{k\geq 2/\alpha} \frac{t^k \left\|Z_n\right\|_{L^{\alpha k}}^{\alpha k}}{k!} \\ \leq 1 + \sum_{1\leq k<2/\alpha} \frac{a_k t^k}{c_1 t_0^k} \frac{2^{2/\alpha-1}}{\min\left\{\alpha, \alpha^{2/\alpha}\right\}} &+ \sum_{k\geq 2/\alpha} \frac{a_k t^k}{c_1 t_0^k \sqrt{e}} \leq 1 + c'_\alpha \sum_{k=1}^\infty a_k \frac{t^k}{t_0^k} \\ \leq 1 + c'_\alpha \sum_{k=1}^\infty c_2 a_{k-1} \frac{t^k}{t_0^k} &= 1 + c_\alpha \frac{t}{t_0} \sum_{k=0}^\infty a_k \frac{t^k}{t_0^k} = 1 + c_\alpha \frac{t/t_0}{\left(1 - t/t_0\right)^{1/2}}, \end{aligned}$$

where $c'_\alpha, c_\alpha \geq 0$ are constants depending only on α, and the last step follows from the formula $\left(1-s\right)^{-1/2} = 1 + \sum_{k=1}^\infty a_k s^k$. Using Markov's inequality and letting $t = \frac{t_0}{2}$ and $x = \sqrt{\left|T_n\right|}\epsilon$, we obtain

$$\begin{aligned} &\mathbb{P}\left(\left|S_n\right| \geq \left|T_n\right|\epsilon\right) = \mathbb{P}\left(\left|Z_n\right| \geq x\right) = \mathbb{P}\left[\exp\left(t\left|Z_n\right|^\alpha\right) \geq \exp\left(tx^\alpha\right)\right] \leq \exp\left(-tx^\alpha\right) m(t) \\ &\leq \left(1 + \frac{\sqrt{2}c_\alpha}{2}\right) \exp\left(-\frac{x^\alpha}{2^{\alpha d+1} e\alpha\gamma_0^\alpha}\right) = \left(1 + \frac{\sqrt{2}c_\alpha}{2}\right) \exp\left(-\frac{\left|T_n\right|^{1/(1+2\nu)} \epsilon^{2/(1+2\nu)}}{2^{\alpha d+1} e\alpha\gamma_0^\alpha}\right), \end{aligned}$$

where the last step is due to $\alpha = 2/(1+2\nu)$. ■

Proof of Lemma B.2. Since $I_{i,m,\iota} \subset \{j : d_{ij} \geq \iota_{m-1}\}$, by Lemma S.3,

$$\theta_{m,p,\iota} = \sup_{n,i\in D_n} \delta_p\left(i, I_{i,m,\iota}, n\right) \leq \sup_{n,i\in D_n} 3\delta_p\left(i, \{j : d_{ij} \geq \iota_{m-1}\}, n\right) = 3\Delta_p(\iota_{m-1}).$$

■

Proof of Lemma B.3. Since $\lim_{s\to\infty} \Delta_1(s) = 0$, for all $m \geq 1$, there exists ι_m such that $\Delta_1(\iota_m) \leq m^{-2}$. Let $\iota_0 = 0$ and w.l.o.g., we suppose that $\iota_m > \iota_{m-1}$ for all $m \geq 1$. For this sequence $\iota = (\iota_0, \iota_1, \cdots)$, by Lemma B.2,

$$\sum_{m=s}^{\infty} \theta_{m,1,\iota} \leq 3\sum_{m=s}^{\infty} \Delta_1(\iota_{m-1}) = 3\sum_{m=s-1}^{\infty} \Delta_1(\iota_m) \leq 3\sum_{m=s-1}^{\infty} m^{-2} \to 0 \text{ as } s \to \infty.$$

The desired result follows. ■

Proof of Lemma B.4. We select ι satisfying $\iota_m = m^{\lfloor 3/(\kappa-d/2)\rfloor+1}$. Notice that $\iota_m < \iota_{m+1}$ for all $m \geq 0$. Then, for sufficiently large s, by Lemma B.2,

$$\begin{aligned}\Theta_{s,p,\iota} &= \sum_{m=s}^{\infty} \iota_m^{d/2}\theta_{m,p,\iota} \leq 3\sum_{m=s}^{\infty} \iota_m^{d/2}\Delta_p(\iota_{m-1}) = 3\sum_{m=s}^{\infty} \iota_m^{d/2} O\left(\iota_{m-1}^{-\kappa}\right)\\ &\leq C_1 \sum_{m=s}^{\infty} m^{-3} = o\left(s^{-1}\right) \text{ as } s \to \infty.\end{aligned}$$

To show $\Theta_{p,\iota} < \infty$, we only need to show that $\iota_m^{d/2}\theta_{m,p,\iota} < \infty$ for every $m \in \mathbb{N}$. This directly follows from $\iota_m^{d/2} < \infty$ and $\theta_{m,p,\iota} \leq 3\Delta_p(0) < \infty$ by Lemma S.3. ■

Proof of Lemma B.5. Let ι be a sequence satisfying $\iota_m = m^{\lfloor 3/(\kappa-d/2)\rfloor+1}$. Then $\iota \in \mathscr{I}$. By the conditions in this lemma, for all $p \geq 2$,

$$\begin{aligned}\Theta_{p,\iota} &= \sum_{m=1}^{\infty} (\iota_m)^{d/2}\theta_{m,p,\iota} \leq 3\sum_{m=1}^{\infty} (\iota_m)^{d/2}\Delta_p(\iota_{m-1}) \leq 3O(p^\nu)\sum_{m=1}^{\infty} (\iota_m)^{d/2} O\left((\iota_{m-1})^{-\kappa}\right)\\ &\leq 3O(p^\nu)\sum_{m=1}^{\infty} m^{-3} = O(p^\nu) \text{ as } p \to \infty,\end{aligned}$$

where the first inequality follows from Lemma B.2, the second inequality follows from $\Delta_p(s) \leq O(s^{-\kappa})O(p^{\nu})$, and the third equality follows from the fact that $O\left((\iota_{m-1})^{-\kappa}\right)$ does not depend on p. Thus, we have $\gamma_0 = \sup_{p\geq 2} p^{-\nu}\Theta_{p,\iota} < \infty$. ■

S.4. Some Proofs for Section 3

The proofs in this section rely heavily on the theory of the second-type L^p-FD coefficient in Appendix B. Recall $\mathscr{I} \equiv \{\iota = (\iota_0, \iota_1, \ldots) : \iota_0 = 0, \iota_m > \iota_{m-1}, \iota_m \in \mathbb{N} \text{ for all } m \geq 1\}$.

Proof of Theorem 3.1. By the condition in this theorem and Lemma B.4, there exists a sequence $\iota \in \mathscr{I}$ such that $\Theta_{p,\iota} < \infty$. Applying Theorem B.1 and letting $C \equiv 2^d\sqrt{p-1}\Theta_{p,\iota}$ yields the result.■

Proof of Theorem 3.2. By Condition (ii) in this theorem and Lemma B.5, there exists a sequence $\iota \in \mathscr{I}$ such that $\gamma_0 \equiv \sup_{p\geq 2} p^{-\nu}\Theta_{p,\iota} < \infty$. Taking $\iota^{(p)} = \iota$ for all real numbers $p \geq 2$ in Theorem B.2 yields the result. ■

Proof of Theorem 3.3. This proof is inspired by Wu and Wu (2016). We only need to consider the case that $\frac{x}{\|Y_\cdot\|_{2,\omega}|T_n|^{1/2}} \geq 1$, since otherwise we can select C_2 and C_3 satisfying $C_2 \exp(-C_3) \geq 1$ and then the result holds trivially. Recall that $V_{j,n,\iota}(m) \equiv \mathbb{E}\left(Y_{j,n}|\mathcal{F}_{j,n}(\iota_m)\right) - \mathbb{E}\left(Y_{j,n}|\mathcal{F}_{j,n}(\iota_{m-1})\right)$ in Lemma B.1. In the following proof, we take $\iota = (\iota_0, \iota_1, \cdots)$ as $\iota_0 = 0$ and $\iota_m = \lfloor m^\kappa \rfloor + 1$ for $m \geq 1$, where κ can be any number such that $\kappa \geq 1$ and $\kappa > \frac{3}{2(\omega-d)} > 0$. Thus, $\iota_m > \iota_{m-1}$ and $m^\kappa \leq \iota_m \leq 2m^\kappa$ for any $m \geq 1$ and there exists a constant $C_\kappa > 0$ such that $C_\kappa\iota_m \leq \iota_{m-1} + 1$ for any $m \geq 1$. As in the proof of Theorem B.1, we decompose $S_n = \sum_{j\in T_n} Y_{j,n}$ as

$$S_n = \sum_{m=1}^{\infty}\sum_{\vec{a}\in A}\sum_{\vec{i}\in I(\iota_m)}\sum_{j\in U(\vec{a},\vec{i})} V_{j,n,\iota}(m),$$

where A, $I(\iota_m)$, $U(\vec{a},\vec{i})$ are all defined in the proof of Theorem B.1. To make the presentation clearer, we denote

$$W(\vec{a},\vec{i},m) \equiv \sum_{j\in U(\vec{a},\vec{i})} V_{j,n,\iota}(m). \tag{S.16}$$

Then

$$S_n = \underbrace{\sum_{m=|T_n|+1}^{\infty} \sum_{\vec{a}\in A} \sum_{\vec{i}\in I(\iota_m)} W(\vec{a},\vec{i},m)}_{S_{n1}} + \underbrace{\sum_{m=1}^{|T_n|} \sum_{\vec{a}\in A} \sum_{\vec{i}\in I(\iota_m)} W(\vec{a},\vec{i},m)}_{S_{n2}}.$$

By our construction of $U(\vec{a},\vec{i})$, the $V_{j,n,\iota}(m)$'s in (S.16) are independent. Thus, by Burkholder's inequality (S.3) and Lemma B.1, for any $\vec{a}$, $\vec{i}$, m and p,

$$\left\|W(\vec{a},\vec{i},m)\right\|_{L^p} \le \sqrt{p-1}\left(\sum_{j\in U(\vec{a},\vec{i})} \|V_{j,n,\iota}(m)\|_{L^p}^2\right)^{1/2} \le \sqrt{p-1}\left|U(\vec{a},\vec{i})\right|^{1/2}\theta_{m,p,\iota}. \tag{S.17}$$

Now we will handle S_{n1} and S_{n2} respectively.

$\underline{S_{n1}}$: Following the proof of Theorem B.1, we have

$$\begin{aligned} \|S_{n1}\|_{L^p} &= \left\|\sum_{m=|T_n|+1}^{\infty} \sum_{\vec{a}\in A} \sum_{\vec{i}\in I(\iota_m)} W(\vec{a},\vec{i},m)\right\|_{L^p} \le \sum_{m=|T_n|+1}^{\infty} \sum_{\vec{a}\in A} \sum_{\vec{i}\in I(\iota_m)} \left\|W(\vec{a},\vec{i},m)\right\|_{L^p} \\ &\le \sum_{m=|T_n|+1}^{\infty} \sum_{\vec{a}\in A} \sum_{\vec{i}\in I(\iota_m)} \sqrt{p-1}\left|U(\vec{a},\vec{i})\right|^{1/2}\theta_{m,p,\iota} \le 2^d\sqrt{p-1}\Theta_{|T_n|+1,p,\iota}|T_n|^{1/2}, \end{aligned} \tag{S.18}$$

where the second inequality follows from (S.17) and the last one follows from the power-mean inequality

$$\left(\frac{\sum_{\vec{a}\in A}\sum_{\vec{i}\in I(\iota_m)}\left|U(\vec{a},\vec{i})\right|^{1/2}}{2^d\,(2\iota_m)^d}\right)^2 \le \frac{\sum_{\vec{a}\in A}\sum_{\vec{i}\in I(\iota_m)}\left|U(\vec{a},\vec{i})\right|}{2^d\,(2\iota_m)^d} = \frac{|T_n|}{2^d\,(2\iota_m)^d}.$$

By Markov's inequality and (S.18), for any $x>0$,

$$\mathbb{P}\left(|S_{n1}| \ge x\right) \le \frac{\|S_{n1}\|_{L^p}^p}{x^p} \le \frac{2^{pd}\,(p-1)^{p/2}\,\Theta_{|T_n|+1,p,\iota}^p|T_n|^{p/2}}{x^p}.$$

Note that, by Lemma B.2, the definition of $\|Y.\|_{p,\omega}$ and $C_\kappa \iota_m \le \iota_{m-1}+1$,

$$\theta_{m,p,\iota} \le 3\Delta_p(\iota_{m-1}) \le 3\|Y.\|_{p,\omega}(\iota_{m-1}+1)^{-\omega} \le 3C_\kappa^{-\omega}\|Y.\|_{p,\omega}\iota_m^{-\omega}. \tag{S.19}$$

Hence,

$$\begin{aligned}\Theta_{|T_n|+1,p,\iota} &= \sum_{m=|T_n|+1}^{\infty} \iota_m^{d/2}\theta_{m,p,\iota} \le 3C_\kappa^{-\omega}\|Y.\|_{p,\omega}\sum_{m=|T_n|+1}^{\infty}\iota_m^{d/2-\omega} \\ &\le 3C_\kappa^{-\omega}\|Y.\|_{p,\omega}\sum_{m=|T_n|+1}^{\infty} m^{\kappa(d/2-\omega)} \le C_1\|Y.\|_{p,\omega}|T_n|^{\kappa(d/2-\omega)+1} \le C_1\|Y.\|_{p,\omega}|T_n|^{1/p-1/2},\end{aligned}$$

where C_1 is a constant not depending on n, the second inequality follows from $\iota_m \ge m^\kappa$, the third one follows from Lemma S.12, and the last one follows from $\omega > d$ and

$$\kappa(d/2-\omega)+1 \le \frac{3(d/2-\omega)}{2(\omega-d)}+1 = \frac{-(\omega-d)-3/2d}{2(\omega-d)} \le -\frac{1}{2} \le \frac{1}{p}-\frac{1}{2}.$$

Thus

$$\mathbb{P}(|S_{n1}| \ge x) \le \frac{C\|Y.\|_{p,\omega}^p|T_n|}{x^p}, \tag{S.20}$$

where the constant C does not depend on n nor x.

$\underline{S_{n2}}$: Denote $R_m \equiv \sum_{\vec{a}\in A}\sum_{\vec{i}\in I(\iota_m)} W(\vec{a},\vec{i},m)$. Then $S_{n2} = \sum_{m=1}^{|T_n|} R_m$. Recall $W(\vec{a},\vec{i},m) \equiv \sum_{j\in U(\vec{a},\vec{i})} V_{j,n,\iota}(m)$ and the $V_{j,n,\iota}(m)$'s in $U(\vec{a},\vec{i})$ are independent. For any $x>0$,

$$\begin{aligned}\mathbb{P}\left(\left|W(\vec{a},\vec{i},m)\right| \ge x\right) &\le C_{p1}\frac{\sum_{j\in U(\vec{a},\vec{i})}\|V_{j,n,\iota}(m)\|_{L^p}^p}{x^p} + 2\exp\left(-\frac{x^2}{C_{p2}\sum_{j\in U(\vec{a},\vec{i})}\|V_{j,n,\iota}(m)\|_{L^2}^2}\right) \\ &\le C_{p1}\frac{\left|U(\vec{a},\vec{i})\right|\theta_{m,p,\iota}^p}{x^p} + 2\exp\left(-\frac{x^2}{C_{p2}\left|U(\vec{a},\vec{i})\right|\theta_{m,2,\iota}^2}\right),\end{aligned}$$

where $C_{p1} = \left(1+\frac{2}{p}\right)^p$, $C_{p2} = \frac{2}{e^p(p+2)^2}$, the first inequality follows from Lemma S.11 and the last one follows from (B.1) in Lemma B.1. Thus, we have

$$
\begin{aligned}
&\mathbb{P}\left(|R_m| \geq 2^d \left(2\iota_m\right)^d x\right) \leq \sum_{\vec{a}\in A}\sum_{\vec{i}\in I(\iota_m)} \mathbb{P}\left(\left|W(\vec{a},\vec{i},m)\right| \geq x\right) \\
\leq &\sum_{\vec{a}\in A}\sum_{\vec{i}\in I(\iota_m)} C_{p1}\frac{\left|U(\vec{a},\vec{i})\right| \theta^p_{m,p,\iota}}{x^p} + \sum_{\vec{a}\in A}\sum_{\vec{i}\in I(\iota_m)} 2\exp\left(-\frac{x^2}{C_{p2}\left|U(\vec{a},\vec{i})\right|\theta^2_{m,2,\iota}}\right) \\
\leq &C_{p1}\frac{|T_n|\theta^p_{m,p,\iota}}{x^p} + 2^{2d+1}\iota_m^d \exp\left(-\frac{x^2}{C_{p2}|T_n|\theta^2_{m,2,\iota}}\right),
\end{aligned}
$$

where the last inequality follows from $\sum_{\vec{a}\in A}\sum_{\vec{i}\in I(\iota_m)}\left|U(\vec{a},\vec{i})\right| = |T_n|$, $|I(\iota_m)| = (2\iota_m)^d$ and $|A| = 2^d$. By letting $y = 2^d \left(2\iota_m\right)^d x$, we obtain

$$
\begin{aligned}
&\mathbb{P}\left(|R_m| \geq y\right) \leq C_{p1}\frac{2^{2pd}\iota_m^{pd}|T_n|\theta^p_{m,p,\iota}}{y^p} + 2^{2d+1}\iota_m^d \exp\left(-\frac{y^2}{C_{p2}2^{4d}\iota_m^{2d}|T_n|\theta^2_{m,2,\iota}}\right) \\
\leq &C_2\frac{\|Y.\|^p_{p,\omega}\,\iota_m^{pd-p\omega}|T_n|}{y^p} + 2^{2d+1}\iota_m^d \exp\left(-\frac{y^2}{C_3\|Y.\|^2_{2,\omega}|T_n|\iota_m^{2d-2\omega}}\right),
\end{aligned}
$$

where the last step follows from (S.19). Therefore, by letting $\lambda_m = C_\lambda m^{1/2-\kappa(\omega-d)}$, where $C_\lambda^{-1} = \sum_{m=1}^{\infty} m^{1/2-\kappa(\omega-d)} < \infty$ by $\kappa > \frac{3}{2(\omega-d)}$, we have $\sum_{m=1}^{|T_n|}\lambda_m \leq 1$ and for any $x > 0$,

$$
\begin{aligned}
&\mathbb{P}\left(|S_{n2}| \geq x\right) \leq \sum_{m=1}^{|T_n|}\mathbb{P}\left(|R_m| \geq \lambda_m x\right) \\
\leq &\frac{C_2\|Y.\|^p_{p,\omega}|T_n|}{x^p}\sum_{m=1}^{|T_n|}\frac{\iota_m^{pd-p\omega}}{\lambda_m^p} + 2^{3d+1}|T_n|^{\kappa d}\sum_{m=1}^{|T_n|}\exp\left(-\frac{\lambda_m^2 x^2}{C_3\|Y.\|^2_{2,\omega}|T_n|\iota_m^{2d-2\omega}}\right),
\end{aligned}
\tag{S.21}
$$

where the last step follows from $\iota_m \leq 2m^\kappa \leq 2|T_n|^\kappa$. We analyze the two terms on the r.h.s. of (S.21) respectively. Note that

$$
\frac{\iota_m^{pd-p\omega}}{\lambda_m^p} = O\left(m^{\kappa p(d-\omega)}m^{p\kappa(\omega-d)-p/2}\right) = O\left(m^{-p/2}\right) \text{ as } m\to\infty,
$$

thus

$$\sum_{m=1}^{|T_n|} \frac{\iota_m^{pd-p\omega}}{\lambda_m^p} \leq \sum_{m=1}^{\infty} \frac{\iota_m^{pd-p\omega}}{\lambda_m^p} < C_4 \tag{S.22}$$

for some constant $C_4 > 0$. For the second term, consider $h(u) = \sum_{m=1}^{|T_n|} \exp\left(-\frac{\lambda_m^2 u^2}{C_3 \iota_m^{2d-2\omega}}\right)$ for $u \geq 1$. Denote $C_5 \equiv C_3^{-1} C_\lambda^2$. Then

$$\frac{\lambda_m^2}{C_3 \iota_m^{2d-2\omega}} m^{-1} \geq C_3^{-1} C_\lambda^2 m^{1-2\kappa(\omega-d)} m^{2\kappa(\omega-d)} m^{-1} = C_5.$$

Thus,

$$h(u) \leq \sum_{m=1}^{|T_n|} \exp\left(-C_5 m u^2\right) \leq \sum_{m=1}^{\infty} \exp\left(-C_5 m u^2\right) = \frac{\exp\left(-C_5 u^2\right)}{1 - \exp\left(-C_5 u^2\right)} \leq \frac{\exp\left(-C_5 u^2\right)}{1 - \exp\left(-C_5\right)}.$$

By letting $u = \frac{x}{\|Y_\cdot\|_{2,\omega} |T_n|^{1/2}} \geq 1$ (as mentioned at the beginning of the proof, we only consider $\frac{x}{\|Y_\cdot\|_{2,\omega} |T_n|^{1/2}} \geq 1$), we obtain that

$$\sum_{m=1}^{|T_n|} \exp\left(-\frac{\lambda_m^2 x^2}{C_3 \|Y_\cdot\|_{2,\omega}^2 |T_n| \iota_m^{2d-2\omega}}\right) \leq C_6 \exp\left(-\frac{C_5 x^2}{\|Y_\cdot\|_{2,\omega}^2 |T_n|}\right). \tag{S.23}$$

Hence, the result follows from (S.20)-(S.23) and $\mathbb{P}\left(|S_n| \geq 2x\right) \leq \mathbb{P}\left(|S_{n1}| \geq x\right) + \mathbb{P}\left(|S_{n2}| \geq x\right)$.

Proof of Theorem 3.6. Denote $\Lambda \equiv \{\lambda \in \mathbb{R}^{p_Y} : \|\lambda\| = 1\}$. It suffices to show that $\lambda'\left(\widehat{V}_n - V_n\right)\lambda = o_p(1)$ as $n \to \infty$. Let $y_{i,n} = \lambda' Y_{i,n}$ so that $\{y_{i,n}\}$ is L^2-FD on $\{\epsilon_{i,n}\}$ with the L^2-FD coefficient

$\Delta_2(s)$. Then,

$$\begin{aligned}
&\lambda'\left(\widehat{V}_n - V_n\right)\lambda \\
=&\sum_{s\geq 0} k_n(s)|T_n|^{-1}\sum_{i\in T_n}\sum_{j\in T_n:d_{ij}\in[s,s+1)} y_{i,n}y_{j,n} - \sum_{s\geq 0}|T_n|^{-1}\sum_{i\in T_n}\sum_{j\in T_n:d_{ij}\in[s,s+1)}\mathbb{E}(y_{i,n}y_{j,n}) \\
=&|T_n|^{-1}\sum_{i\in T_n}(y_{i,n}^2 - Ey_{i,n}^2) + \sum_{s\geq 1} k_n(s)|T_n|^{-1}\sum_{i\in T_n}\sum_{j\in T_n:d_{ij}\in[s,s+1)}[y_{i,n}y_{j,n} - \mathbb{E}(y_{i,n}y_{j,n})] + \qquad (S.24)\\
&\sum_{s\geq 1}[k_n(s)-1]\,|T_n|^{-1}\sum_{i\in T_n}\sum_{j\in T_n:d_{ij}\in[s,s+1)}\mathbb{E}(y_{i,n}y_{j,n}) \\
\equiv& P_{n,1} + P_{n,2} + P_{n,3}.
\end{aligned}$$

In the following proof, we will show that as $n\to\infty$, the three terms in the last line of (S.24) are all $o_p(1)$.

$\underline{P_{n,3}}$:

$$\begin{aligned}
&|P_{n,3}| \leq \sum_{s\geq 1}|k_n(s)-1|\,|T_n|^{-1}\sum_{i\in T_n}\sum_{j\in T_n:d_{ij}\in[s,s+1)}|\mathbb{E}(y_{i,n}y_{j,n})| \\
\leq& 2\|Y\|_{L^2}\sum_{s\geq 1}|k_n(s)-1|\,\Delta_2(\frac{s}{2})|T_n|^{-1}\sum_{i\in T_n}|\{j\in T_n : d_{ij}\in[s,s+1)\}| \\
\leq& 2C\|Y\|_{L^2}\sum_{s\geq 1}|k_n(s)-1|\,\Delta_2(\frac{s}{2})s^{d-1} = 2^{c_\Delta+1}CC_kC_\Delta\|Y\|_{L^2}b_n^{-c_k-1}\sum_{s\geq 1}s^{c_k-c_\Delta+d} \\
=& o(1),
\end{aligned}$$

where the second inequality follows from Corollary 6.1, the third one is by Lemma S.6, the first equality is from conditions (ii) and (iii) in this theorem, and the last step follows from conditions (iii) and (iv) in this theorem ($b_n\to\infty$ and $c_k - c_\Delta + d < -1$).

$\underline{P_{n,2}}$: Recall $P_{n,2} = \sum_{s\geq 1} k_n(s)|T_n|^{-1}\sum_{i\in T_n}\sum_{j\in T_n:d_{ij}\in[s,s+1)}[y_{i,n}y_{j,n} - \mathbb{E}(y_{i,n}y_{j,n})]$, $|k_n(\cdot)|\leq 1$, and $k_n(u) = 0$ when $u > b_n$. Thus,

$$\begin{aligned}\mathbb{E}P_{n,2}^2 \leq & |T_n|^{-2} \sum_{i,j\in T_n: d_{ij}\in[1,b_n]} \sum_{k,l\in T_n: d_{kl}\in[1,b_n]} |\mathrm{Cov}(y_{i,n}y_{j,n}, y_{k,n}y_{l,n})| \\ \leq & |T_n|^{-2} \sum_{s\geq 0} \sum_{i,j,k,l\in T_n: d_{ij}\in[1,b_n], d_{kl}\in[1,b_n], d_{ij;kl}\in[s,s+1)} |\mathrm{Cov}(y_{i,n}y_{j,n}, y_{k,n}y_{l,n})|,\end{aligned} \tag{S.25}$$

where $d_{ij;kl} \equiv \min\{d_{ik}, d_{il}, d_{jk}, d_{jl}\}$. When $s = 0$, we have $i = k$, $i = l$, $j = k$, or $j = l$. Thus,

$$\begin{aligned}& |T_n|^{-2} \sum_{i,j,k,l\in T_n: d_{ij}\in[1,b_n], d_{kl}\in[1,b_n], d_{ij;kl}=0} |\mathrm{Cov}(y_{i,n}y_{j,n}, y_{k,n}y_{l,n})| \\ \leq & 4|T_n|^{-2} \sum_{i\in T_n} \sum_{j\in T_n: d_{ij}\in[1,b_n]} \sum_{l\in T_n: d_{il}\in[1,b_n]} |\mathrm{Cov}(y_{i,n}y_{j,n}, y_{i,n}y_{l,n})| \\ \leq & 4|T_n|^{-2} \sum_{i\in T_n} \sum_{j\in T_n: d_{ij}\in[1,b_n]} \sum_{l\in T_n: d_{il}\in[1,b_n]} \left(\left|\mathbb{E}y_{i,n}^2 y_{j,n}y_{l,n}\right| + |\mathbb{E}y_{i,n}y_{j,n}|\,|\mathbb{E}y_{i,n}y_{l,n}|\right) \\ \leq & 4|T_n|^{-2} \sum_{i\in T_n} \sum_{r=1}^{\lfloor b_n \rfloor} \sum_{j\in T_n: d_{ij}\in[r,r+1)} \sum_{l\in T_n: d_{il}\in[r,r+1)} 2\|Y\|_{L^4}^4 \\ \leq & 8C^2\|Y\|_{L^4}^4 |T_n|^{-1} \sum_{r=0}^{\lfloor b_n \rfloor} r^{2(d-1)} \leq 8C^2\|Y\|_{L^4}^4 |T_n|^{-1} \int_0^{b_n+1} x^{2(d-1)} \mathrm{d}x \\ = & \frac{8C^2\|Y\|_{L^4}^4}{2d-1} |T_n|^{-1}(b_n+1)^{2d-1} = o(1),\end{aligned}$$

where the second inequality follows from Minkowski's inequality, the third one is by generalized Hölder's inequality and Lyapunov's inequality, the fourth one is by Lemma S.6, and the last equality holds under condition (iv) in this theorem.

When $s \geq 1$, we apply Lemma S.10 to bound the covariance of the product terms. Specifically, we take $w_0 = w = 4$ so that $|\mathrm{Cov}\left(y_{i,n}y_{j,n}, y_{k,n}y_{l,n}\right)| \leq 16M^3\|Y\|_{L^{p_0}} \left[\Delta_2(x)\right]^{\frac{q_0-6}{2q_0-6}}$ when $0 < x \leq$

$d_{ij;kl}/2$, where $M = \max\{1, \|Y\|_{L^{q_0}}\}$. Then,

$$\mathbb{E}P_{n,2}^2 \le 16M^3\|Y\|_{L^{p_0}}|T_n|^{-2}\sum_{s\ge1}\sum_{i,j,k,l\in T_n:d_{ij}\in[1,b_n],d_{kl}\in[1,b_n],d_{ij;kl}\in[s,s+1)}[\Delta_2(s/2)]^{\frac{q_0-6}{2q_0-6}} + o(1)$$

$$\le 16M^3\|Y\|_{L^{p_0}}|T_n|^{-2}\sum_{s\ge1}4\sum_{i,k\in T_n:d_{ik}\in[s,s+1)}\sup_{n,i}|\{j\in T_n : d_{ij}\le b_n\}|^2[\Delta_2(s/2)]^{\frac{q_0-6}{2q_0-6}} + o(1)$$

$$\le 64C^3d^{-2}M^3\|Y\|_{L^{p_0}}|T_n|^{-1}\sum_{s\ge1}\left(2^d b_n^d\right)^2[\Delta_2(s/2)]^{\frac{q_0-6}{2q_0-6}}s^{d-1} + o(1)$$

$$= 2^{\frac{(q_0-6)c_\Delta}{2q_0-6}+6+2d}C^3d^{-2}M^3\|Y\|_{L^{p_0}}C_b^{2d}C_\Delta^{\frac{q_0-6}{2q_0-6}}|T_n|^{-1+2dc_b}\sum_{s\ge1}s^{-\frac{(q_0-6)c_\Delta}{2q_0-6}+d-1} + o(1) = o(1),$$

where the third inequality is by Lemma S.6 and

$$\sup_{n,i}|\{j\in T_n : d_{ij}\le b_n\}| \le \sum_{s=0}^{\lfloor b_n\rfloor}\sup_{n,i}|\{j\in T_n : d_{ij}\in[s,s+1)\}|$$

$$\le C\sum_{s=0}^{\lfloor b_n\rfloor}s^{d-1} \le C\int_0^{b_n+1}x^{d-1}\mathrm{d}x \le Cd^{-1}(b_n+1)^d \le 2^dCd^{-1}b_n^d,$$

the first equality is from conditions (iii) and (iv) in this theorem, and the last equality holds under conditions (iii) and (iv) in this theorem ($-1+2dc_b<0$ and $-\frac{(q_0-6)c_\Delta}{2q_0-6}+d-1<-1$).

$\underline{P_{n,1}}$: Recall $P_{n,1} = |T_n|^{-1}\sum_{i\in T_n}(y_{i,n}^2 - \mathbb{E}y_{i,n}^2)$. So

$$\mathbb{E}P_{n,1}^2 \le |T_n|^{-2}\sum_{s\ge0}\sum_{i\in T_n}\sum_{j\in T_n:d_{ij}\in[s,s+1)}\left|\mathrm{Cov}(y_{i,n}^2, y_{j,n}^2)\right|.$$

When $s=0$, because $\left|\mathrm{Var}(y_{i,n}^2)\right| \le \left|\mathbb{E}y_{i,n}^4\right|$,

$$|T_n|^{-2}\sum_{i\in T_n}\left|\mathrm{Var}(y_{i,n}^2)\right| \le |T_n|^{-2}\sum_{i\in T_n}\left|\mathbb{E}y_{i,n}^4\right| = O(|T_n|^{-1}).$$

Thus,

$$\begin{aligned}
\mathbb{E}P_{n,1}^2 \leq & |T_n|^{-2} \sum_{s\geq 1} \sum_{i\in T_n} \sum_{j\in T_n: d_{ij}\in[s,s+1)} \left|\mathrm{Cov}(y_{i,n}^2, y_{j,n}^2)\right| + o(1) \\
\leq & 16M^3 \|Y\|_{L^{p_0}} |T_n|^{-2} \sum_{s\geq 1} \sum_{i\in T_n} \sum_{j\in T_n: d_{ij}\in[s,s+1)} \left[\Delta_2(s/2)\right]^{\frac{q_0-6}{2q_0-6}} + o(1) \\
\leq & 16M^3 \|Y\|_{L^{p_0}} |T_n|^{-1} \sum_{s\geq 1} C s^{d-1} \left[\Delta_2(s/2)\right]^{\frac{q_0-6}{2q_0-6}} + o(1) \\
= & 2^{\frac{(q_0-6)c_\Delta}{2q_0-6}+4} C M^3 \|Y\|_{L^{p_0}} |T_n|^{-1} \sum_{s\geq 1} s^{-\frac{(q_0-6)c_\Delta}{2q_0-6}+d-1} + o(1) = o(1),
\end{aligned}$$

where the second inequality follows from Lemma S.10, the third one is by Lemma S.6, the first equality is from condition (iii) in this theorem, and the last equality holds under condition (iii) in this theorem that $-\frac{(q_0-6)c_\Delta}{2q_0-6} + d - 1 < -1$. ■

S.5. Proofs for Section 4

Proof of Proposition 4.1. For any $I \subset D_n$, denote $X_{n,I} = \left(\left(X'_{i,n,I}\right)_{i\in D_n}\right)'$. Then

$$\delta_p(i,j,n) = \left\|h_{i,n}(X_n) - h_{i,n}\left(X_{n,\{j\}}\right)\right\|_{L^p} \leq m_{ij,n}\left\|X_{j,n} - X_{j,n}^*\right\|_{L^p} \leq 2\|X\|_{L^p} m_{ij,n},$$

where the first inequality is from (4.2) and the second one follows from the Minkowski inequality. Therefore, for any $s \in [0,\infty)$, by Lemma S.4,

$$\Delta_p(s) \leq \sup_{n\geq 1} \sup_{i\in D_n} \sum_{j\in D_n: d_{ij}\geq s} \delta_p(i,j,n) \leq 2\|X\|_{L^p} \sup_{n\geq 1} \sup_{i\in D_n} \sum_{j\in D_n: d_{ij}\geq s} m_{ij,n} = 2\|X\|_{L^p}\phi(s) \to 0 \text{ as } s\to\infty.$$

So, the conclusion holds. ■

Proof of Proposition 4.2. **(i)** For all $i, k \in D_n$, by (4.2),

$$\delta_p(i,k,n) = \left\|Y_{i,n} - Y_{i,n,\{k\}}\right\|_{L^p} \leq \sum_{j\in D_n} m_{ij,n}\left\|X_{j,n} - X_{j,n,\{k\}}\right\|_{L^p} = \sum_{j\in D_n} m_{ij,n}\delta_{X,p}(j,k,n).$$

(ii) For any $i \in D_n$, $s \in [0, \infty)$ and $\tilde{s} \in [0, s]$, by (4.2),

$$\left\|Y_{i,n} - Y_{i,n,\{k:d_{ik}\geq s\}}\right\|_{L^p} \leq \sum_{j\in D_n} m_{ij,n} \left\|X_{j,n} - X_{j,n,\{k:d_{ik}\geq s\}}\right\|_{L^p} = \sum_{j\in D_n} m_{ij,n}\delta_{X,p}\left(j, \{k : d_{ik} \geq s\}, n\right)$$

$$= \sum_{j:d_{ij}\geq\tilde{s}} m_{ij,n}\delta_{X,p}\left(j, \{k : d_{ik} \geq s\}, n\right) + \sum_{j:d_{ij}<\tilde{s}} m_{ij,n}\delta_{X,p}\left(j, \{k : d_{ik} \geq s\}, n\right)$$

$$\leq 3 \sum_{j:d_{ij}\geq\tilde{s}} m_{ij,n}\delta_{X,p}\left(j, D_n, n\right) + 3 \sum_{j:d_{ij}<\tilde{s}} m_{ij,n}\delta_{X,p}\left(j, \{k : d_{jk} \geq s - \tilde{s}\}, n\right)$$

$$\leq 3\Delta_{X,p}(0) \sum_{j:d_{ij}\geq\tilde{s}} m_{ij,n} + 3\left(\sup_{n,i\in D_n} \sum_{j:d_{ij}\geq 0} m_{ij,n}\right)\left(\sup_{n,j\in D_n} \delta_{X,p}\left(j, \{k : d_{jk} \geq s - \tilde{s}\}, n\right)\right)$$

$$\leq 3\Delta_{X,p}(0)\phi\left(\tilde{s}\right) + 3\phi\left(0\right)\Delta_{X,p}(s - \tilde{s}),$$

where the second inequality follows from Lemma S.3 and the fact that for all $i, j \in D_n$ satisfying $d_{ij} < \tilde{s}$, $\{k : d_{ik} \geq s\} \subset \{k : d_{jk} \geq s - \tilde{s}\}$. Thus, taking the supremum on both sides of the above inequality yields

$$\Delta_p\left(s\right) = \sup_{n,i\in D_n} \left\|Y_{i,n} - Y_{i,n,\{k:d_{ik}\geq s\}}\right\|_{L^p} \leq 3\Delta_{X,p}(0)\phi\left(\tilde{s}\right) + 3\phi\left(0\right)\Delta_{X,p}(s - \tilde{s}).$$

So, the conclusion holds. ■

Proof of Proposition 4.3. **(1)** In this proof, for any vector or matrix $A = (a_{ij})_{n\times m}$, we denote $|A| \equiv (|a_{ij}|)_{n\times m}$. Direct calculations show that $|A+B| \leq^* |A|+|B|$ and $|AB| \leq^* |A||B|$, where $A = (a_{ij})_{m\times n} \leq^* B = (b_{ij})_{m\times n}$ means $\forall i, j : a_{ij} \leq b_{ij}$. To shorten formulas, denote $v_{i,n} \equiv X'_{i,n}\beta + \epsilon_{i,n}$, $V_n \equiv X_n\beta + \epsilon_n$, and the solution of (4.3) as $Y_n\left(V_n\right)$. Then $Y_n\left(V_n\right) = F\left(\lambda W_n Y_n\left(V_n\right) + V_n\right)$. Consider $Y_n^{(1)} = Y_n\left(V_n^{(1)}\right)$ and $Y_n^{(2)} = Y_n\left(V_n^{(2)}\right)$. So, for any $1 \leq i \leq n$,

$$\left|Y_{i,n}^{(1)} - Y_{i,n}^{(2)}\right| = \left|F\left(\lambda w_{i\cdot,n}Y_n^{(1)} + v_{i,n}^{(1)}\right) - F\left(\lambda w_{i\cdot,n}Y_n^{(2)} + v_{i,n}^{(2)}\right)\right|$$

$$\leq L\left|\lambda\right| \sum_{j=1}^{n} \left|w_{ij,n}\right| \left|Y_{j,n}^{(1)} - Y_{j,n}^{(2)}\right| + L\left|v_{i,n}^{(1)} - v_{i,n}^{(2)}\right|.$$

The above inequality can be written in a matrix form:

$$(I_n - L|\lambda||W_n|)\left|Y_n^{(1)} - Y_n^{(2)}\right| \leq^* L\left|V_n^{(1)} - V_n^{(2)}\right|. \tag{S.26}$$

Since $M_n \equiv L(I_n - L|\lambda W_n|)^{-1} = L\sum_{l=0}^{\infty}(L|\lambda W_n|)^l$, all entries of M_n are nonnegative. As a result, we can multiply M_n/L on both sides of (S.26): $\left|Y_n^{(1)} - Y_n^{(2)}\right| \leq^* M_n\left|V_n^{(1)} - V_n^{(2)}\right|$. So, for any $1 \leq i \leq n$,

$$\left|Y_{i,n}^{(1)} - Y_{i,n}^{(2)}\right| \leq \sum_{j=1}^{n} m_{ij,n}\left|v_{i,n}^{(1)} - v_{i,n}^{(2)}\right|. \tag{S.27}$$

We take $v_{i,n}^{(1)} = v_{i,n}$ and $v_{i,n}^{(2)} \equiv 0$. Next, we will show that $Y_{i,n}^{(2)}$'s are uniformly bounded.

$$\left|Y_{i,n}^{(2)}\right| = \left|F\left(\lambda w_{i\cdot,n}Y_n^{(2)}\right)\right| \leq \left|F\left(\lambda w_{i\cdot,n}Y_n^{(2)}\right) - F(0)\right| + |F(0)| \leq \left|L\lambda w_{i\cdot,n}Y_n^{(2)}\right| + |F(0)|.$$

Denote the n-dimensional vector $l_n \equiv (1, \ldots, 1)'$. Then the above inequality can be written as $\left|Y_n^{(2)}\right| \leq^* \left|L\lambda W_n Y_n^{(2)}\right| + F(0)l_n$. Consequently, $\left|Y_n^{(2)}\right| \leq^* F(0)M_n l_n/L$. So, it follows from

$$\|M_n\|_\infty \leq L\sum_{l=0}^{\infty}\left\|(L|\lambda W_n|)^l\right\|_\infty \leq L\sum_{l=0}^{\infty}\zeta^l = \frac{L}{1-\zeta} \tag{S.28}$$

that $\sup_{i,n}\left|Y_{i,n}^{(2)}\right| \leq \frac{F(0)}{1-\zeta}$, i.e., $Y_{i,n}^{(2)}$'s are uniformly bounded. Hence, by the Minkowski inequality, $\sup_{i,n}\left|Y_{i,n}^{(2)}\right| \leq \frac{F(0)}{1-\zeta}$, and (S.27)-(S.28), we have

$$\|Y_{i,n}\|_{L^p} \leq \left\|Y_{i,n} - Y_{i,n}^{(2)}\right\|_{L^p} + \left\|Y_{i,n}^{(2)}\right\|_{L^p} \leq \sum_{j=1}^{n} m_{ij,n}\|v_{i,n}\|_{L^p} + \frac{F(0)}{1-\zeta} \leq \frac{LC_{x\epsilon,p} + F(0)}{1-\zeta}.$$

So, $Y_{i,n}$'s are uniformly L^p-bounded. ■

(2) With (S.27), conclusion (i) follows from Propositions 4.1, and conclusion (ii) follows from 4.2 and that the FDM $\delta_{X\beta+\epsilon}(i,j,n)$ of $\{X_{i,n}'\beta + \epsilon_{i,n} : i \in D_n, n \geq 1\}$ on $\{u_{i,n} : i \in D_n, n \geq 1\}$ is bounded by $(\|\beta\| + 1)\delta_{X\epsilon}(i,j,n)$. ■

(3) We prove the conclusion using Proposition 4.1. First, condition (4.2) follows from (S.27) and $C_{x\epsilon,p} \equiv \sup_{n,i\in D_n} \|v_{i,n}\|_{L^p} < \infty$. Under Assumption 3(1), $\left(|W_n|^l\right)_{ij} = 0$ if $d_{ij} > 0$ and $l \leq d_{ij}/\bar{d}_0$. Thus, when $s > 0$, by the Neumann's expansion $m_{ij,n} = \sum_{l=0}^{\infty} L^{l+1} |\lambda|^l \left(|W_n|^l\right)_{ij}$,

$$\begin{aligned}
&\phi(s) = \sup_{n,i} \sum_{j\in D_n:d_{ij}\geq s} m_{ij,n} = \sup_{n,i} \sum_{j\in D_n:d_{ij}\geq s} \sum_{l=0}^{\infty} L^{l+1} \left(|\lambda W_n|^l\right)_{ij} \\
&\leq \sup_{n,i} \sum_{j\in D_n:d_{ij}\geq s} \sum_{l=\lfloor s/\bar{d}_0\rfloor+1}^{\infty} L^{l+1} \left(|\lambda W_n|^l\right)_{ij} = \sum_{l=\lfloor s/\bar{d}_0\rfloor+1}^{\infty} \sum_{j\in D_n:d_{ij}\geq s} L^{l+1} \left(|\lambda W_n|^l\right)_{ij} \\
&\leq \sum_{l=\lfloor s/\bar{d}_0\rfloor+1}^{\infty} L^{l+1} \|\lambda W_n\|_{\infty}^l \leq \sum_{l=\lfloor s/\bar{d}_0\rfloor+1}^{\infty} L\zeta^l \leq \frac{L}{1-\zeta}\zeta^{s/\bar{d}_0} \to 0 \quad \text{as } s \to \infty.
\end{aligned} \tag{S.29}$$

From (S.28), $\phi(0) = \sup_n \|M_n\|_{\infty} \leq \frac{L}{1-\zeta} < \infty$. In sum, (S.29) holds for all $s \in [0,\infty)$. Therefore, by Proposition 4.1, $\{Y_{i,n}\}$ is L^p-FD on the random field $\left\{\left(X_{i,n}', \epsilon_{i,n}\right)'\right\}$ with the L^p-FD coefficient $\Delta_p(s) \leq 2C_{x\epsilon,p}\phi(s) \leq \frac{2LC_{x\epsilon,p}}{1-\zeta}\zeta^{s/\bar{d}_0}$. ■

(4) To apply Proposition 4.1, we only need to calculate $\phi(s) \equiv \sup_{n,i} \sum_{j\in D_n:d_{ij}\geq s} m_{ij,n}$ for $s > 0$ and $s = 0$. From (S.28), $\phi(0) = \sup_n \|M_n\|_{\infty} \leq \frac{L}{1-\zeta} < \infty$. Next, consider $s > 0$. We fix a constant $\tilde{s} > 0$, whose value depends on s and will be determined later. Define $\tilde{W}_n = (\tilde{w}_{ij,n})_{n\times n}$, where

$$\begin{cases} \tilde{w}_{ij,n} = w_{ij,n}, & d_{ij} < \tilde{s}, \\ \tilde{w}_{ij,n} = 0, & d_{ij} \geq \tilde{s}. \end{cases}$$

Define $\tilde{M}_n \equiv (\tilde{m}_{ij,n})_{n\times n} \equiv L\left(I_n - L\left|\lambda\tilde{W}_n\right|\right)^{-1}$ and $\tilde{\phi}(s) \equiv \sup_{n,i} \sum_{j\in D_n:d_{ij}\geq s} \tilde{m}_{ij,n}$. Then

$$\begin{aligned}
&\phi(s) = \sup_{n,i} \sum_{j\in D_n:d_{ij}\geq s} m_{ij,n} \leq \sup_{n,i} \sum_{j\in D_n:d_{ij}\geq s} \{\tilde{m}_{ij,n} + |m_{ij,n} - \tilde{m}_{ij,n}|\} \\
&\leq \sup_{n,i} \sum_{j\in D_n:d_{ij}\geq s} \tilde{m}_{ij,n} + \sup_{n,i} \sum_{j\in D_n:d_{ij}\geq s} |m_{ij,n} - \tilde{m}_{ij,n}| \leq \frac{L}{1-\zeta}\zeta^{s/\tilde{s}} + \sup_n \left\|M_n - \tilde{M}_n\right\|_{\infty},
\end{aligned} \tag{S.30}$$

where the last step follows from (S.29). In order to bound $\sup_n \left\|M_n - \tilde{M}_n\right\|_{\infty}$, we denote $\psi(s) \equiv$

$\sup_{n,i} \sum_{j\in D_n:d_{ij}\geq s} |w_{ij,n}|$. Then, when s is large enough,

$$\psi(s) = \sup_{n,i} \sum_{j\in D_n:d_{ij}\geq s} |w_{ij,n}| \leq \sup_{n,i} \sum_{m=\lfloor s\rfloor}^{\infty} \sum_{j\in D_n:d_{ij}\in[m,m+1)} cd_{ij}^{-\alpha} \leq Cc \sum_{m=\lfloor s\rfloor}^{\infty} m^{d-1}m^{-\alpha} \leq C_1 s^{d-\alpha}, \tag{S.31}$$

for some constants $C, C_1 > 0$, where the second inequality follows from Lemma S.6 and the last one follows from Lemma S.12. Therefore,

$$\begin{aligned}
&\sup_n \left\| M_n - \tilde{M}_n \right\|_\infty = \sup_n \left\| \sum_{l=0}^{\infty} L^{l+1} |\lambda|^l \left(W_n^l - \tilde{W}_n^l \right) \right\|_\infty \\
=& \sup_n \left\| \sum_{l=1}^{\infty} L^{l+1} |\lambda|^l \sum_{h=0}^{l-1} \tilde{W}_n^h \left(W_n - \tilde{W}_N \right) W_n^{l-1-h} \right\|_\infty \\
\leq& \sup_n \sum_{l=1}^{\infty} L^{l+1} |\lambda|^l \sum_{h=0}^{l-1} \left\| \tilde{W}_n^h \right\|_\infty \left\| W_n - \tilde{W}_n \right\|_\infty \left\| W_n^{l-1-h} \right\|_\infty \\
\leq& \sum_{l=1}^{\infty} L^{l+1} |\lambda|^l l \left(\sup_n \|W_n\|_\infty \right)^{l-1} \psi(\tilde{s}) \leq \psi(\tilde{s}) L^2 |\lambda| \sum_{l=1}^{\infty} l\zeta^{l-1} = \frac{\psi(\tilde{s}) L^2 |\lambda|}{(1-\zeta)^2},
\end{aligned} \tag{S.32}$$

where the first equality follows from Neumann's expansion, the second equality follows from the fact that $A^l - B^l = \sum_{h=0}^{l-1} B^h (A-B) A^{l-1-h}$ for all square matrices A and B, and the second inequality follows from $\left\| W_n - \tilde{W}_n \right\|_\infty \leq \psi(\tilde{s})$ and $\sup_n \left\| \tilde{W}_n \right\|_\infty \leq \sup_n \|W_n\|_\infty$.

Combining (S.30)-(S.32), when $s \geq \tilde{s}$,

$$\phi(s) \leq L(1-\zeta)^{-1} \zeta^{s/\tilde{s}} + \frac{C_1 L^2 |\lambda|}{(1-\zeta)^2} \tilde{s}^{d-\alpha}.$$

When s is large enough, taking $\tilde{s} = \frac{s}{(d-\alpha)\log s/\log\zeta} < s$ yields

$$\phi(s) \leq \frac{L}{1-\zeta} s^{d-\alpha} + \frac{C_1 L^2 |\lambda|}{(1-\zeta)^2} \left(\frac{\log\zeta}{d-\alpha} \right)^{d-\alpha} s^{d-\alpha} (\log s)^{\alpha-d} = O\left(s^{-(\alpha-d)} (\log s)^{\alpha-d} \right). \tag{S.33}$$

Then the conclusion follows from Proposition 4.1. ■

(5) We apply Proposition 4.2 to prove the conclusion. First, condition (4.2) follows from (S.27) and $\phi(s) \le \frac{L}{1-\zeta}\zeta^{s/\bar{d}_0}$. By Assumption 4(2) and the definition of FD coefficient, $\left\{X_{i,n}'\beta + \epsilon_{i,n}\right\}$ is L^p-FD on $\{u_{i,n}\}$ with the L^p-FD coefficient less than $(\|\beta\|+1)\,\Delta_{X\epsilon,p}(s)$. Then the conclusion follows from Proposition 4.2. ■

(6) We still apply Proposition 4.2. Condition (4.2) follows from (S.27) and $\phi(s) = O\left(s^{-(\alpha-d)}(\log s)^{\alpha-d}\right)$. From the proof of conclusion (5), $\left\{X_{i,n}'\beta + \epsilon_{i,n}\right\}$ is L^p-FD on $\{u_{i,n}\}$ with the L^p-FD coefficient less than $(\|\beta\|+1)\,\Delta_{X\epsilon,p}(s)$. Then the conclusion follows from Proposition 4.2. ■

Proof of Proposition 4.4. By Assumptions 5(2) and (4)-(5), it follows from Proposition 4.3(1) and (4) that $\{\epsilon_{i,n}\}$ is uniformly L^p-bounded and L^p-FD on $\{v_{i,n}\}$ with L^p-FD coefficient $\Delta_{\epsilon,p}(s) = O\left(s^{-(\alpha-d)}(\log s)^{\alpha-d}\right)$, which satisfies $\Delta_{\epsilon,p}(0) < \infty$. Together with Assumption 5(3), $\left\{\left(X_{i,n}', \epsilon_{i,n}\right)'\right\}$ is L^p-FD on $\left\{\left(u_{i,n}', v_{i,n}\right)'\right\}$ with the L^p-FD coefficient $\Delta_{X\epsilon,p}(s) \le \Delta_{\epsilon,p}(s)+\Delta_{X,p}(s) = O\left(s^{-(\alpha-d)}(\log s)^{\alpha-d}\right)$. Thus, Assumption 4(2) is satisfied. Therefore, by Assumption 5(1)-(2), the desired result follows from Proposition 4.3(6). ■

Proof of Proposition 4.5. Since $\epsilon_n = v_n - \rho M_n v_n$, the FDM $\delta_{\epsilon,p}(i,j,n)$ of $\{\epsilon_{i.n}\}$ on $\{v_{i,n}\}$ satisfies $\delta_{\epsilon,p}(i,j,n) \le 2\left[\rho\left|m_{ij,n}\right| + 1(i=j)\right]\|v\|_{L^p}$ for any $i,j \in D_n$. Thus, $\Delta_{\epsilon,p}(0) < \infty$ and

$$\Delta_{\epsilon,p}(s) \le \sup_{i,n} \sum_{j:d_{ij}\ge s} \delta_{\epsilon,p}(i,j,n) \le 2\rho\|v\|_{L^p} \sum_{j:d_{ij}\ge s} \left|m_{ij,n}\right| = O\left(s^{-(\alpha-d)}\right)$$

as $s \to \infty$, where the first step follows from Lemma S.4 and the last step follows from Assumption 6(2) and (S.31). Together with Assumption 6(3), $\left\{\left(X_{i,n}', \epsilon_{i,n}\right)'\right\}$ is L^p-FD on $\left\{\left(u_{i,n}', v_{i,n}\right)'\right\}$ with the L^p-FD coefficient $\Delta_{X\epsilon,p}(s) \le \Delta_{\epsilon,p}(s) + \Delta_{X,p}(s) = O\left(s^{-(\alpha-d)}(\log s)^{\alpha-d}\right)$. Thus, Assumption 4(2) is satisfied. Therefore, with Assumption 6(1)-(2), the desired result follows from Proposition 4.3(6). ■

Proof of Proposition 4.6. In this proof, for any vector or matrix $A = (a_{ij})_{n\times m}$, we denote $|A| \equiv (|a_{ij}|)_{n\times m}$. Direct calculations show that $|A+B| \le^* |A|+|B|$ and $|AB| \le^* |A||B|$, where $A = (a_{ij})_{m\times n} \le^* B = (b_{ij})_{m\times n}$ means $\forall i,j : a_{ij} \le b_{ij}$. We regard the underlying independent random field as $\left\{\left(u_{i,n}', q_{i,n}\right)'\right\}$ and denote the FDM of $\{X_{i,n}\}$ and $\{\epsilon_{i,n}\}$ with respect to $\left\{\left(u_{i,n}', q_{i,n}\right)'\right\}$

as $\delta_{X,p}$ and $\delta_{\epsilon,p}$ respectively. Then $\delta_{X,p} \leq \delta_{X\epsilon,p}$ and $\delta_{\epsilon,p} \leq \delta_{X\epsilon,p}$. For any $i \in D_n$, denote $\lambda(q_{i,n}) \equiv \lambda_1 1(q_{i,n} \leq \gamma) + \lambda_2 1(q_{i,n} > \gamma)$ and $\beta(q_{i,n}) \equiv \beta_1 1(q_{i,n} \leq \gamma) + \beta_2 1(q_{i,n} > \gamma)$. Then $|\lambda(q_{i,n})| \leq \lambda$ and $\|\beta(q_{i,n})\| \leq \|\beta\| \equiv \max\{\|\beta_1\|, \|\beta_2\|\}$. We first establish the uniform L^p-boundedness of $\{Y_{i,n}\}$. It follows from $|Y_{i,n}| \leq \lambda \sum_{j=1}^n |w_{ij,n}| |Y_{j,n}| + \|X_{i,n}\| \|\beta\| + |\epsilon_{i,n}|$ that

$$\sup_{i \in D_n} \|Y_{i,n}\|_{L^p} \leq \lambda \sup_n \|W_n\|_\infty \sup_{i \in D_n} \|Y_{i,n}\|_{L^p} + \|\beta\| \|X\|_{L^p} + \|\epsilon\|_{L^p}.$$

Consequently,

$$\|Y\|_{L^p} \equiv \sup_{n,i} \|Y_{i,n}\|_{L^p} \leq \frac{\|\beta\| \|X\|_{L^p} + \|\epsilon\|_{L^p}}{1 - \lambda \sup_n \|W_n\|_\infty} < \infty.$$

Now, for any fixed unit $i \in D_n$ and any $s > 0$, denote $I_s \equiv \{j \in D_n : d_{ij} \geq s\}$. Notice that $I_s \neq \emptyset$, as $I_s \ni i$. For any $k \in D_n \backslash I_s$, define

$$Y_{k,n,I_s} = \lambda(q_{k,n}) W_{k\cdot,n} Y_{n,I_s} + X'_{k,n,I_s} \beta(q_{k,n}) + \epsilon_{k,n,I_s}.$$

Thus,

$$\begin{aligned} |Y_{k,n} - Y_{k,n,I_s}| &= \left|\lambda(q_{k,n}) W_{k\cdot,n} (Y_n - Y_{n,I_s}) + \left(X'_{k,n} - X'_{k,n,I_s}\right) \beta(q_{k,n}) + \epsilon_{k,n} - \epsilon_{k,n,I_s}\right| \\ &\leq \lambda |W_{k\cdot,n}| |Y_n - Y_{n,I_s}| + \|\beta\| \|X_{k,n} - X_{k,n,I_s}\| + |\epsilon_{k,n} - \epsilon_{k,n,I_s}|. \end{aligned}$$

For any matrix $A \in \mathbb{R}^{m \times n}$, $A^{\cdot I} \in \mathbb{R}^{m \times (n - |I|)}$ denotes the sub-matrix of A after deleting the columns whose indexes belong to I, $A^{I\cdot} \in \mathbb{R}^{(m-|I|) \times n}$ denotes the sub-matrix of A after deleting the rows whose indexes belong to I, and $A^{II} \in \mathbb{R}^{(m-|I|)\times(n-|I|)}$ denotes the sub-matrix A after deleting both the rows and the columns whose indexes belong to I. Denote $\|X_n - X_{n,I_s}\| \equiv (\|X_{1,n} - X_{1,n,I_s}\|, \ldots, \|X_{n,n} - X_{n,n,I_s}\|)'$ in this proof (and only in this proof). Writing the last equation in a vector form, we have

$$\left|Y_n^{I_s\cdot} - Y_{n,I_s}^{I_s\cdot}\right| \leq^* \lambda \left|W_n^{I_s\cdot}\right| |Y_n - Y_{n,I_s}| + \|\beta\| \|X_n - X_{n,I_s}\|^{I_s\cdot} + \left|\epsilon_n^{I_s\cdot} - \epsilon_{n,I_s}^{I_s\cdot}\right|$$

$$=\lambda \left|W_n^{I_sI_s}\right| \left|Y_n^{I_s\cdot} - Y_{n,I_s}^{I_s\cdot}\right| + \lambda \sum_{r\in I_s} \left|Y_{r,n} - Y_{r,n,I_s}\right| \left|W_{\cdot r,n}^{I_s\cdot}\right| + \|\beta\| \left\|X_n - X_{n,I_s}\right\|^{I_s\cdot} + \left|\epsilon_n^{I_s\cdot} - \epsilon_{n,I_s}^{I_s\cdot}\right|,$$

where $W_{\cdot r,n}$ is the rth column of W_n and $\epsilon_{n,I_s} = (\epsilon_{1,n,I_s}, \ldots, \epsilon_{n,n,I_s})'$. Denote $M_n^{I_s} \equiv \left(m_{ab,n}^{I_s}\right)_{a,b\in D_n\backslash I_s} \equiv \left(I - \lambda\left|W_n^{I_sI_s}\right|\right)^{-1}$, where the indexes of $M_n^{I_s}$ and $W_n^{I_sI_s}$ are the same, i.e., $a, b \in D_n\backslash I_s$. By Neumann's expansion, all entries of $M_n^{I_s}$ are nonnegative. Thus, by the above inequality,

$$\left|Y_n^{I_s\cdot} - Y_{n,I_s}^{I_s\cdot}\right| \leq^* M_n^{I_s} \left\{\lambda \sum_{r\in I_s} \left|Y_{r,n} - Y_{r,n,I_s}\right| \left|W_{\cdot r,n}^{I_s\cdot}\right| + \|\beta\| \left\|X_n - X_{n,I_s}\right\|^{I_s\cdot} + \left|\epsilon_n^{I_s\cdot} - \epsilon_{n,I_s}^{I_s\cdot}\right|\right\}. \tag{S.34}$$

Note that $W_n^{I_sI_s}$ is the spatial weights matrix of units $D_n\backslash I_s$. By the same argument as in the proof of Proposition 4.3(4) and (S.28), we have

$$\sup_{s,n,a\in D_n\backslash I_s} \sum_{b\in D_n\backslash I_s:d_{ab}\geq m} m_{ab,n}^{I_s} = O\left(m^{-(\alpha-d)} (\log m)^{\alpha-d}\right) \tag{S.35}$$

as $m \to \infty$ and $\sup_{n,s} \left\|M_n^{I_s}\right\|_\infty < \infty$. We extend $M_n^{I_s}$ to an $n \times n$ matrix $\tilde{M}_n^{I_s} \equiv \left(\tilde{m}_{ab,n}^{I_s}\right)_{a,b\in D_n}$ by filling 0's, i.e.,

$$\begin{cases} \tilde{m}_{ab,n}^{I_s} \equiv m_{ab,n}^{I_s}, & a \in D_n\backslash I_s \text{ and } b \in D_n\backslash I_s, \\ \tilde{m}_{ab,n}^{I_s} \equiv 0, & \text{otherwise.} \end{cases}$$

Then (S.35) becomes

$$\phi_{\tilde{M}}(m) \equiv \sup_{s,n,a\in D_n} \sum_{b\in:d_{ab}\geq m} \tilde{m}_{ab,n}^{I_s} = O\left(m^{-(\alpha-d)} (\log m)^{\alpha-d}\right)$$

as $m \to \infty$ and $\sup_{n,s} \left\|\tilde{M}_n^{I_s}\right\|_\infty \leq \phi_{\tilde{M}}(0) < \infty$. As $i \in D_n\backslash I_s$, by taking the L^p-norm on both sides of the ith row of inequality (S.34), we have

$$\delta_{Y,p}(i,I_s,n) \leq 2\lambda \left\|Y\right\|_{L^p} \sum_{r\in I_s}\sum_{k=1}^{n} \tilde{m}_{ik,n}^{I_s} w_{kr,n} + \|\beta\| \sum_{k=1}^{n} \tilde{m}_{ik,n}^{I_s}\delta_{X,p}(k,I_s,n) + \sum_{k=1}^{n} \tilde{m}_{ik,n}^{I_s}\delta_{\epsilon,p}(k,I_s,n)$$
$$\leq 2\lambda \left\|Y\right\|_{L^p} \sum_{r\in I_s}\sum_{k=1}^{n} \tilde{m}_{ik,n}^{I_s} w_{kr,n} + (\|\beta\|+1) \sum_{k=1}^{n} \tilde{m}_{ik,n}^{I_s}\delta_{X\epsilon,p}(k,I_s,n), \tag{S.36}$$

where the first inequality follows from $\left\|Y_{r,n}-Y_{r,n,I_s}\right\|_{L^p} \leq 2\left\|Y\right\|_{L^p}$ and the last one follows from $\delta_{X,p} \leq \delta_{X\epsilon,p}$ and $\delta_{\epsilon,p} \leq \delta_{X\epsilon,p}$. We analyze the last two terms in (S.36) respectively. For the first term,

$$\sum_{r\in I_s}\sum_{k=1}^{n} \tilde{m}_{ik,n}^{I_s} w_{kr,n} \leq \sup_{n,i\in D_n} \sum_{r\in D_n:d_{ir}\geq s}\sum_{k=1}^{n} \tilde{m}_{ik,n}^{I_s} w_{kr,n}.$$

Denote $\phi_{\tilde{M}^{I_s}W}(s) \equiv \sup_{n,i\in D_n}\sum_{r\in D_n:d_{ir}\geq s}\sum_{k=1}^{n} \tilde{m}_{ik,n}^{I_s} w_{kr,n}$. By (S.31) and Lemma S.8, as $s\to\infty$,

$$\phi_{\tilde{M}^{I_s}W}(s)$$
$$\leq \sup_n \left\|W_n\right\|_\infty O\left(\left(\frac{s}{2}\right)^{-(\alpha-d)}\left(\log\frac{s}{2}\right)^{\alpha-d}\right) + \sup_{n,\tilde{s}}\left\|\tilde{M}_n^{I_{\tilde{s}}}\right\|_\infty O\left(\left(\frac{s}{2}\right)^{-(\alpha-d)}\right) = O\left(s^{-(\alpha-d)}\left(\log s\right)^{\alpha-d}\right).$$

For the second term, we use the same argument as that in the proof of Proposition 4.2. We have

$$\sum_{k=1}^{n} \tilde{m}_{ik,n}^{I_s}\delta_{X\epsilon,p}(k,I_s,n) = \sum_{k=1}^{n} \tilde{m}_{ik,n}^{I_s}\delta_{X\epsilon,p}\left(k,\{j:d_{ij}\geq s\},n\right)$$
$$= \sum_{k:d_{ik}\geq s/2} \tilde{m}_{ik,n}^{I_s}\delta_{X\epsilon,p}\left(k,\{j:d_{ij}\geq s\},n\right) + \sum_{k:d_{ik}< s/2} \tilde{m}_{ik,n}^{I_s}\delta_{X\epsilon,p}\left(k,\{j:d_{ij}\geq s\},n\right)$$
$$\leq 3 \sum_{k:d_{ik}\geq s/2} \tilde{m}_{ik,n}^{I_s}\delta_{X\epsilon,p}\left(k,D_n,n\right) + 3\sum_{k:d_{ik}< s/2} \tilde{m}_{ik,n}^{I_s}\delta_{X\epsilon,p}\left(k,\left\{j:d_{kj}\geq\frac{s}{2}\right\},n\right)$$
$$\leq 3\Delta_{X\epsilon,p}(0)\sum_{k:d_{ik}\geq s/2} \tilde{m}_{ik,n}^{I_s} + 3\left(\sup_{n,i\in D_n}\sum_{k=1}^{n}\tilde{m}_{ik,n}^{I_s}\right)\left(\sup_{n,k\in D_n}\delta_{X\epsilon,p}\left(k,\left\{j:d_{kj}\geq\frac{s}{2}\right\},n\right)\right)$$
$$\leq 3\Delta_{X\epsilon,p}(0)\phi_{\tilde{M}_n^{I_s}}\left(\frac{s}{2}\right) + 3\phi_{\tilde{M}_n^{I_s}}(0)\,\Delta_{X\epsilon,p}\left(\frac{s}{2}\right) = O\left(s^{-(\alpha-d)}\left(\log s\right)^{\alpha-d}\right),$$

where the first inequality follows from Lemma S.3 and the fact that $\{j:d_{ij}\geq s\}\subset\left\{j:d_{kj}\geq\frac{s}{2}\right\}$

holds for any $k \in D_n$ satisfying $d_{ik} < \frac{s}{2}$, and the last inequality follows from Assumption 7(3). Thus, the two terms on the r.h.s. of (S.36) are both $O\left(s^{-(\alpha-d)}(\log s)^{\alpha-d}\right)$ as $s \to \infty$. Hence, as $s \to \infty$,

$$\Delta_{Y,p}(s) = \sup_{n,i} \delta_{Y,p}(i, I_s, n) = O\left(s^{-(\alpha-d)}(\log s)^{\alpha-d}\right).$$

Finally, as $\Delta_{Y,p}(0) = \sup_{n,i} \delta_{Y,p}(i, D_n, n) \leq 2\|Y\|_{L^p}$, the conclusion follows. ■

Proof of Proposition 4.7. (1) To simplify the notation, we assume $\lambda, \gamma, \rho \geq 0$ and all entries of W_n are nonnegative in the proof.[3] Note that all expectations and L^p-norms are taken conditional on $\mathcal{C}$, but we omit the subscript $\mathcal{C}$ to simplify the notation. It follows from $\sup_N \left\|S_N^{-1}\right\|_\infty = \sup_N \left\|\sum_{l=0}^\infty (\lambda W_N)^l\right\|_\infty \leq \sum_{l=0}^\infty \lambda^l = \frac{1}{1-\lambda}$ that $\sup_N \|A_N\|_\infty = \sup_N \left\|S_N^{-1}(\gamma I_N + \rho W_N)\right\|_\infty \leq \frac{\gamma+\rho}{1-\lambda} = \zeta < 1$. Besides, S_N^{-1}, A_N and their products are all nonnegative, so Lemma S.8 is applicable. Since $y_{it} = \sum_{h=0}^\infty \sum_{j=1}^N \left(A_N^h S_N^{-1}\right)_{ij} \epsilon_{j(t-h)}$ and ϵ_{jt}'s are independent conditional on $\mathcal{C}$ by Assumption 11, for any pair $((i_1, t_1), (i_2, t_2)) \in D_{NT}^2$,

$$\delta_p^{\mathcal{C}}(i_1t_1, i_2t_2) = \left\|\left(A_N^{t_1-t_2} S_N^{-1}\right)_{i_1i_2}\left(\epsilon_{i_2t_2} - \epsilon_{i_2t_2}^*\right)\right\|_{L^p} \leq 2\|\epsilon\|_{L^p}\left(A_N^{t_1-t_2} S_N^{-1}\right)_{i_1i_2}$$

if $t_1 \geq t_2$; $\delta_p^{\mathcal{C}}(i_1t_1, i_2t_2) = 0$ otherwise. Thus, by Lemma S.4,

$$\begin{aligned}
\Delta_p^{\mathcal{C}}(s) \leq & \sup_{(N,T)} \sup_{(i_1,t_1)\in D_{NT}} \sum_{(i_2,t_2)\in D_{NT}: d_{i_1t_1;i_2t_2}\geq s} \delta_p^{\mathcal{C}}(i_1t_1, i_2t_2) \\
\leq & 2\|\epsilon\|_{L^p} \sup_{(N,T)} \sup_{(i_1,t_1)\in D_{NT}} \sum_{(i_2,t_2)\in D_{NT}: d_{i_1t_1;i_2t_2}\geq s, t_2\leq t_1} \left(A_N^{t_1-t_2} S_N^{-1}\right)_{i_1i_2} \\
\leq & 2\|\epsilon\|_{L^p} \sup_{(N,T)} \sup_{(i_1,t_1)\in D_{NT}} \left\{\sum_{(i_2,t_2): t_1-t_2\geq s} \left(A_N^{t_1-t_2} S_N^{-1}\right)_{i_1i_2} + \sum_{(i_2,t_2): d_{i_1i_2}\geq s, 0\leq t_1-t_2<s} \left(A_N^{t_1-t_2} S_N^{-1}\right)_{i_1i_2}\right\}
\end{aligned} \tag{S.37}$$

a.s. We now bound the above two terms separately.

<u>**Term 1**</u> For any $(i_1, t_1) \in D_{NT}$ and $s \in [0, \infty)$, because $\sup_N \left\|S_N^{-1}\right\|_\infty \leq \frac{1}{1-\lambda}$ that $\sup_N \|A_N\|_\infty \leq$

[3] This simplicity of notation does not change the essence of the proof; without it, we need to add many absolute value signs in the proof.

ζ,

$$\begin{aligned}\sum_{(i_2,t_2):t_1-t_2\geq s}\left(A_N^{t_1-t_2}S_N^{-1}\right)_{i_1i_2} &\leq \sum_{h=\lfloor s\rfloor}^{\infty}\sum_{i_2=1}^{N}\left(A_N^{h}S_N^{-1}\right)_{i_1i_2}\leq \sum_{h=\lfloor s\rfloor}^{\infty}\sup_N\left\|A_N^{h}S_N^{-1}\right\|_{\infty}\\ \leq \sum_{h=\lfloor s\rfloor}^{\infty}\|A_N\|_{\infty}^{h}\left\|S_N^{-1}\right\|_{\infty} &\leq \sum_{h=\lfloor s\rfloor}^{\infty}\frac{\zeta^h}{1-\lambda}=\frac{\zeta^{\lfloor s\rfloor}}{(1-\lambda)(1-\zeta)}=\frac{\zeta^{\lfloor s\rfloor}}{1-\lambda-\gamma-\rho}.\end{aligned} \tag{S.38}$$

Term 2 For any $(i_1,t_1)\in D_{NT}$,

$$\sum_{(i_2,t_2):d_{i_1i_2}\geq s,0\leq t_1-t_2<s}\left(A_N^{t_1-t_2}S_N^{-1}\right)_{i_1i_2}\leq \sum_{h=0}^{\lfloor s\rfloor}\sum_{i_2:d_{i_1i_2}\geq s}\left(A_N^{h}S_N^{-1}\right)_{i_1i_2}. \tag{S.39}$$

Recall the definition of $\phi_M(s)$ in Lemma S.8. For any $s\in[0,\infty)$,

$$\phi_{\gamma I_N+\rho W_N}(s)=\gamma 1(s=0)+\sup_i\sum_{j:d_{ij}\geq s}\rho w_{ij,N}\leq C_1\rho\left(s+1\right)^{-(\alpha-d)},$$

for some constant C_1 that does not depend on s, where the inequality follows from (S.31). From (S.33), there exists a constant $C_2>0$, such that for any $s\in[0,\infty)$,

$$\phi_{S_N^{-1}}(s)\leq C_2\left(s+1\right)^{-(\alpha-d)}\left(\log\left(s+2\right)\right)^{\alpha-d}.$$

Because $A_N^hS_N^{-1}=S_N^{-h}\left(\gamma I_N+\rho W_N\right)^hS_N^{-1}$ for any $h\in\{0,1,2,\ldots\}$, $\left\|S_N^{-1}\right\|_{\infty}\leq\frac{1}{1-\lambda}$, and $\|\gamma I_N+\rho W_N\|_{\infty}\leq\gamma+\rho$, by Lemma S.8, for any $s\in[0,\infty)$, we have

$$\begin{aligned}&\phi_{A_N^hS_N^{-1}}(s)\leq(h+1)\left(\frac{\gamma+\rho}{1-\lambda}\right)^h\phi_{S_N^{-1}}\left(\frac{s}{2h+1}\right)+\frac{h}{(1-\lambda)^2}\left(\frac{\gamma+\rho}{1-\lambda}\right)^{h-1}\phi_{\gamma I_N+\rho W_N}\left(\frac{s}{2h+1}\right)\\ \leq&C_3\zeta^h\left(h+1\right)\left(\frac{s}{2h+1}+1\right)^{-(\alpha-d)}\left[\log\left(\frac{s}{2h+1}+2\right)\right]^{\alpha-d}\\ =&C_3\zeta^h\left(h+1\right)\left(2h+1\right)^{\alpha-d}\left(s+2h+1\right)^{-(\alpha-d)}\left[\log\left(\frac{s}{2h+1}+2\right)\right]^{\alpha-d}\\ \leq&C_4\zeta^h\left(h+1\right)^{\alpha-d+1}\left(s+1\right)^{-(\alpha-d)}\left(\log\left(s+2\right)\right)^{\alpha-d},\end{aligned}$$

where $C_3, C_4 > 0$ are constants depending neither on s nor h, and the last inequality follows from the fact that $(s+2h+1)^{-(\alpha-d)}$ and $\left[\log\left(\frac{s}{2h+1}+2\right)\right]^{\alpha-d}$ both are decreasing in h and $h \geq 0$. Thus, by (S.39),

$$\sup_{(i_1,t_1)\in D_{NT}} \sum_{(i_2,t_2):d_{i_1 i_2}\geq s, 0\leq t_1-t_2<s} \left(A_N^{t_1-t_2} S_N^{-1}\right)_{i_1 i_2} \leq \sup_{(i_1,t_1)\in D_{NT}} \sum_{h=0}^{\lfloor s\rfloor} \phi_{A_N^h S_N^{-1}}(s)$$
$$\leq \sum_{h=0}^{\lfloor s\rfloor} C_4 \zeta^h (h+1)^{\alpha-d+1} (s+1)^{-(\alpha-d)} (\log(s+2))^{\alpha-d} \leq C_5 (s+1)^{-(\alpha-d)} (\log(s+2))^{\alpha-d}, \tag{S.40}$$

where $C_5 > 0$ is a constant not depending on s, and the last step follows from $\sum_{h=0}^{\infty} \zeta^h (h+1)^{\alpha-d+1} < \infty$. Combining (S.37), (S.38) and (S.40), as $s \to \infty$, we have

$$\Delta_p^{\mathcal{C}}(s) \leq 2\|\epsilon\|_{L^p} \left[\frac{\zeta^{\lfloor s\rfloor}}{1-\lambda-\gamma-\rho} + C_5 (s+1)^{-(\alpha-d)} (\log(s+2))^{\alpha-d}\right] = \|\epsilon\|_{L^p} O\left(s^{-(\alpha-d)} (\log s)^{\alpha-d}\right).$$

■

(2) Multiplying both sides of (4.8) by W_N, we obtain $W_N Y_{N,t} = \sum_{h=0}^{\infty} W_N A_N^h S_N^{-1} \varepsilon_{N,t-h}$. Then the proof for the L^p-FD property of $\{W_{i\cdot,N} Y_{Nt}\}$ is similar to that for $\{y_{it}\}$, and thus we omit it here. ■

Proof of Proposition 4.8. In this proof, all the statements are conditional on $\mathcal{C}$. We first show the uniform L^p-boundedness of $\{y_{it}\}$. For any $(i,t) \in D_{NT}$, denote $\xi_{it} = e_i'(\gamma_{0\tau} l_N + Z_{Nt}\alpha_\tau + l_N B_\tau' F_t + V_{Nt})$. By Assumption 12(2), we have $\|\xi\|_{L^p,\mathcal{C}} \equiv \sup_{N,T,i,t} \|\xi_{it}\|_{L^p,\mathcal{C}} < \infty$. Recall (4.8): $y_{it} = \sum_{h=0}^{\infty} \sum_{j=1}^{N} \left(A_N^h S_N^{-1}\right)_{ij} \xi_{j,t-h}$. By $\sup_N \left\|S_N^{-1}\right\|_\infty \leq \frac{1}{1-|\gamma_{\tau 1}|}$ and $\sup_N \|A_N\|_\infty \leq \zeta$, we have

$$\|Y\|_{L^p,\mathcal{C}} = \sup_{N,T,i,t} \|y_{it}\|_{L^p,\mathcal{C}} \leq \sum_{h=0}^{\infty} \sup_N \left\|A_N^h S_N^{-1}\right\|_\infty \|\xi\|_{L^p,\mathcal{C}}$$
$$\leq \sum_{h=0}^{\infty} \frac{\zeta^h \|\xi\|_{L^p,\mathcal{C}}}{1-|\gamma_{\tau 1}|} = \frac{\|\xi\|_{L^p,\mathcal{C}}}{1-|\gamma_{\tau 1}|-|\gamma_{\tau 2}|-|\gamma_{\tau 3}|} < \infty.$$

Since $\sup_N \|W_N\| = 1$, $\sup_{N,T,i,t} \left\|\overline{Y}_{i,t-1}\right\|_{L^p,\mathcal{C}} \leq \sum_{j=1}^{N} |w_{ij,N}| \|y_{j,t-1}\|_{L^p,\mathcal{C}} \leq \|Y\|_{L^p,\mathcal{C}} < \infty$. Similarly, $\sup_{N,T,i,t}\left\|e_i'W_N^2Y_{N,t-1}\right\|_{L^p,\mathcal{C}}<\infty$ and $\sup_{N,T,i,t}\left\|e_i'W_N^3Y_{N,t-1}\right\|_{L^p,\mathcal{C}}<\infty$. Thus, $\|\Psi\|_{L^p,\mathcal{C}}\equiv\sup_{N,T,i,t}\|\Psi_{it}\|_{L^p,\mathcal{C}}<\infty$ as each component of Ψ_{it} has been shown to be uniformly L^p-bounded. By Proposition 4.7, $\{y_{it}\}$ and $\left\{\overline{Y}_{it}\right\}$ are both L^p-FD on $\{v_{it}\}$ with the L^p-FD coefficient $\Delta_p^{(1)}(s)=O\left(s^{-(\alpha-d)}(\log s)^{\alpha-d}\right)$ as $s\to\infty$. Next, we will show that $\{\Psi_{it}\}$ is also L^p-FD on $\{v_{it}\}$, and it suffices to show each component of Ψ_{it} is L^p-FD. Denote the L^p-FD coefficient of $\{y_{i,t-1}\}$ as $\Delta_p^{(2)}(s)$. Note that $\Delta_p^{(1)}(0)\le 2\|Y\|_{L^p,\mathcal{C}}$ and when $s\ge 1$

$$\{(i_1,t_1)\in D_{NT}:d_{it;i_1t_1}\ge s\}\subset\{(i_1,t_1)\in D_{NT}:d_{i,t-1;i_1t_1}\ge s-1\}.$$

Thus, by Lemma S.3, $\Delta_p^{(2)}(s)\le 3\Delta_p^{(1)}(s-1)=O\left(s^{-(\alpha-d)}(\log s)^{\alpha-d}\right)$. Denote the L^p-FD coefficient of $\left\{\overline{Y}_{i,t-1}\right\}$ as $\Delta_p^{(3)}(s)$. Define $W_{NT}=(w_{i_1t_1;i_2t_2})_{D_{NT}\times D_{NT}}$ as

$$w_{i_1t_1;i_2t_2}=\begin{cases}w_{i_1i_2,N}, & t_1=t_2,\\ 0, & \text{otherwise},\end{cases}$$

i.e.,

$$W_{NT}=\begin{pmatrix}W_N & & \\ & W_N & \\ & & \ddots\end{pmatrix}.$$

Denote $\mathbb{Q}_{NT}=\left(Y_{N,T-1}',Y_{N,T-2}',\ldots\right)'$ and $\mathbb{P}_{NT}=\left(\overline{Y}_{1,T-1},\ldots,\overline{Y}_{N,T-1},\overline{Y}_{1,T-2},\ldots,\overline{Y}_{N,T-2},\ldots\right)'$. Then

$$\mathbb{P}_{NT}=W_{NT}\mathbb{Q}_{NT}.$$

Because

$$\phi_W(s)\equiv\sup_{(N,T)}\sup_{(i_1,t_1)\in D_{NT}}\sum_{(i_2,t_2)\in D_{NT}:d_{i_1t_1;i_2t_2}\ge s}w_{i_1t_1;i_2t_2}=\sup_{N,i\in D_N}\sum_{j\in D_N:d_{ij}\ge s}w_{ij,N}=O(s^{d-\alpha})$$

by (S.31), we have $\Delta_p^{(3)}(s) = O\left(s^{-(\alpha-d)}(\log s)^{\alpha-d}\right)$ by Proposition 4.2. By similar arguments, we conclude that $\left\{e_i' W_N^2 Y_{N,t-1}\right\}$ and $\left\{e_i' W_N^3 Y_{N,t-1}\right\}$ are both L^p-FD on $\{v_{it}\}$ with the L^p-FD coefficient $O\left(s^{-(\alpha-d)}(\log s)^{\alpha-d}\right)$. Conditioning on $\mathcal{C}$, $\{1\}, \{z_{i,t}\}, \{F_t\}$ are all nonstochastic. So far, we have shown that the L^p-FD coefficient of each component of $\Psi_{it} = (x_{it}', r_{it}')'$ is either $O\left(s^{-(\alpha-d)}(\log s)^{\alpha-d}\right)$ or 0. Therefore, the L^p-FD coefficient of $\{\Psi_{it}\}$ is $O\left(s^{-(\alpha-d)}(\log s)^{\alpha-d}\right)$. Since $\gamma_{1\tau}$ and ϕ_τ are true parameters, $u_{it} \equiv y_{it} - \gamma_{1\tau}\overline{Y}_{it} - x_{it}'\phi_\tau = v_{it}$. Thus, $\psi_\tau(u_{it})$'s are independent over i and t, i.e., L^q-FD coefficient of $\{\psi_\tau(u_{it})\}$ on $\{v_{it}\}$ is zero for any $q \geq 1$. By Proposition 5.6, the L^2-FD coefficient of $\{s_{it} \equiv \psi_\tau(u_{it}) \cdot \Psi_{it}\}$ on $\{v_{it}\}$ (denoted as $\Delta_2(s)$) satisfies

$$\Delta_2(s) \leq \|\Psi\|_{L^p,\mathcal{C}} \times 0 + \sup_{N,T,i,t} \|\psi_\tau(u_{it})\|_{L^{\frac{2p}{p-2}},\mathcal{C}} O\left(s^{-(\alpha-d)}(\log s)^{\alpha-d}\right) = O\left(s^{-(\alpha-d)}(\log s)^{\alpha-d}\right).$$

Finally, since $\alpha - d > \frac{1}{2}(d+1)$ (as $\alpha > \frac{3d}{2} + \frac{1}{2}$ from Assumption 12(3)) and $\sup_{N,T,i,t}\|s_{it}\|_{L^p,\mathcal{C}} \leq \|\Psi\|_{L^p,\mathcal{C}} < \infty$ (as $|\psi_\tau(\cdot)| < 1$), by Corollary 3.1, $\Sigma_{NT}^{-1/2}(G_{NT} - \mathbb{E}G_{NT}) \xrightarrow{d} N(0, I)$, where

$$\Sigma_{NT} = \text{Var}\left(\sum_{t=1}^{T}\sum_{i=1}^{N} s_{it} \mid \mathcal{C}\right) = \tau(1-\tau)\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}\left(\Psi_{it}\Psi_{it}' \mid \mathcal{C}\right)$$

by Assumption 12(6)[4]. As $\Omega = \lim_{N,T\to\infty}(NT)^{-1}\Sigma_{NT}$ (Assumption 12(5)), by Slutsky's theorem,

$$\Omega^{-1/2}\frac{G_{NT} - \mathbb{E}G_{NT}}{\sqrt{NT}} \xrightarrow{d} N(0, I).$$

■

Proof of Proposition 4.9. In this proof, all the statements are conditional on $\mathcal{C}$. From the proof of Proposition 4.8, the L^p-FD coefficient of $\{\Psi_{it}\}$ is $O\left(s^{-(\alpha-d)}(\log s)^{\alpha-d}\right)$ and $\|\Psi\|_{L^p,\mathcal{C}} \equiv \sup_{N,T,i,t}\|\Psi_{it}\|_{L^p,\mathcal{C}} < \infty$. By Proposition 5.2, the $L^{p/2}$-FD coefficient (denoted as $\Delta_{p/2}(s)$) of each

[4] Refer to page 6 in Xu, Wang, Shin and Zheng (2022).

entry of $\{\Psi_{it}\Psi_{it}'\}$ on $\{v_{it}\}$ satisfies[5]

$$\Delta_{p/2}(s) \leq C_1 \left(2\|\Psi\|_{L^p,\mathcal{C}} + 1\right) O\left(s^{-(\alpha-d)} (\log s)^{\alpha-d}\right) = O\left(s^{-(\alpha-d)} (\log s)^{\alpha-d}\right) \to 0$$

as $s \to \infty$. By Hölder's inequality and $\|\Psi\|_{L^p,\mathcal{C}} < \infty$, each entry of $\{\Psi_{it}\Psi_{it}'\}$ is uniformly $L^{p/2}$-bounded. Then the result follows from Theorem D.1 and

$$(NT)^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}\left(\Psi_{it}\Psi_{it}' \mid \mathcal{C}\right) \xrightarrow{p} \lim_{N,T\to\infty}(NT)^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}\left(\Psi_{it}\Psi_{it}' \mid \mathcal{C}\right).$$

■.

S.6. Proofs for Section 5

Proof of Proposition 5.1. Since $\|H_{i,n}(y) - H_{i,n}(y^\bullet)\| \leq C\|y - y^\bullet\|$,

$$\delta_{Z,p}(i,I,n) = \|H_{i,n}(Y_{i,n}) - H_{i,n}(Y_{i,n,I})\|_{L^p} \leq C\|Y_{i,n} - Y_{i,n,I}\|_{L^p} = C\delta_{Y,p}(i,I,n). \tag{S.41}$$

We obtain the desired result. ■

Proof of Proposition 5.2. By (5.1) and $B_{i,n}(y,y^\bullet) \leq C_1(\|y\|^a + \|y^\bullet\|^a + 1)$,

$$\begin{aligned}
&\delta_{Z,p}(i,I,n) = \|H_{i,n}(Y_{i,n}) - H_{i,n}(Y_{i,n,I})\|_{L^p} \\
\leq& C_1\|(\|Y_{i,n}\|^a + \|Y_{i,n,I}\|^a + 1)\cdot\|Y_{i,n} - Y_{i,n,I}\|\|_{L^p} \\
\leq& C_1\|\|Y_{i,n}\|^a + \|Y_{i,n,I}\|^a + 1\|_{L^r}\cdot\|Y_{i,n} - Y_{i,n,I}\|_{L^q} \leq C_1(2\|Y\|_{L^{ar}}^a + 1)\delta_{Y,q}(i,I,n),
\end{aligned} \tag{S.42}$$

where the second inequality follows from the generalized Hölder inequality, and the last one follows from the Minkowski inequality. ■

Proof of Proposition 5.3. Denote $B \equiv \|Y_{i,n}\|^a + \|Y_{i,n,I}\|^a + 1$, $\rho \equiv \|Y_{i,n} - Y_{i,n,I}\|$, and $r \equiv$

[5] Note that $\Psi_{it}\Psi_{it}'$ is a matrix.

$\frac{q}{a+1} > p$. Because $q \geq \frac{ap}{p-1}$, by the Lyapunov inequality and the Minkowski inequality, we have $\|\rho\|_{L^p} \leq 2\|Y_{i,n}\|_{L^p} \leq 2\|Y\|_{L^q} < \infty$ and

$$\|B\|_{L^{p/(p-1)}} \leq \|B\|_{L^{q/a}} \leq \|\|Y_{i,n}\|^a\|_{L^{q/a}} + \|\|Y_{i,n,I}\|^a\|_{L^{q/a}} + 1 \leq 2\|Y\|_{L^q}^a + 1 < \infty.$$

Because $\frac{1}{r} = \frac{a}{q} + \frac{1}{q}$ (as $r \equiv \frac{q}{a+1}$), by the generalized Hölder inequality,

$$\|B\rho\|_{L^r} \leq \|\rho\|_{L^q}\|B\|_{L^{q/a}} \leq 2\|Y\|_{L^q}(2\|Y\|_{L^q}^a + 1) < \infty.$$

Then, by Lemma S.7, for all $n \geq 1$, $i \in D_n$, and $j \in D_n$, there exists a constant $C_2 > 0$ such that

$$\begin{aligned}
&\delta_{Z,p}(i,I,n) = \|Z_{i,n} - Z_{i,n,I}\|_{L^p} \leq C_1\|(\|Y_{i,n}\|^a + \|Y_{i,n,I}\|^a + 1)\cdot\|Y_{i,n} - Y_{i,n,I}\|\|_{L^p} = C_1\|B\rho\|_{L^p} \\
\leq& 2C_1\left(\|\rho\|_{L^p}^{r-p}\|B\|_{L^{p/(p-1)}}^{r-p}\|B\rho\|_{L^r}^r\right)^{1/(pr-p)} \leq C_2\left(\|Y_{i,n} - Y_{i,n,I}\|_{L^p}\right)^{(q-ap-p)/(pq-ap-p)} \\
=& C_2\{\delta_{Y,p}(i,I,n)\}^{(q-ap-p)/(pq-ap-p)}.
\end{aligned}$$

Thus the desired result follows. ■

Proof of Proposition 5.4. For any $\epsilon > 0$, let $B \equiv \{|Y_{i,n}| < \epsilon, |Y_{i,n,I}| < \epsilon\}$. It follows from the inequality $|1(x_1 > 0) - 1(x_2 > 0)| \leq \frac{|x_1 - x_2|}{\epsilon}1(|x_1| \geq \epsilon \text{ or } |x_2| \geq \epsilon) + 1(|x_1| < \epsilon, |x_2| < \epsilon)$ that

$$\begin{aligned}
&\|Z_{i,n} - Z_{i,n,I}\|_{L^p} = \|1(Y_{i,n} > 0) - 1(Y_{i,n,I} > 0)\|_{L^p} \leq \left\{\frac{1}{\epsilon^p}\int_{B^c}|Y_{i,n} - Y_{i,n,I}|^p \, \mathrm{d}\mathbb{P} + \mathbb{P}(B)\right\}^{1/p} \\
\leq& \frac{\|Y_{i,n} - Y_{i,n,I}\|_{L^p}}{\epsilon} + \mathbb{P}(|Y_{i,n}| < \epsilon)^{1/p} \leq \frac{\delta_{Y,p}(i,I,n)}{\epsilon} + (C_1\epsilon)^{1/p},
\end{aligned}$$

for some constant $C_1 > 0$, where the second inequality follows from the fact that $(a^p + b^p)^{1/p} \leq a + b$ for any a, $b \geq 0$ and $p \geq 1$, and the last one comes from the uniform boundedness of the density function of $Y_{i,n}$. By letting $\epsilon = \{\delta_{Y,p}(i,I,n)\}^{p/(p+1)}$, we have

$$\delta_{Z,p}(i,I,n) = \|Z_{i,n} - Z_{i,n,I}\|_{L^p} \leq \left(1 + C_1^{1/p}\right)\{\delta_{Y,p}(i,I,n)\}^{1/(p+1)} \leq C_2\{\delta_{Y,p}(i,I,n)\}^{1/(p+1)},$$

for some constant $C_2 > 0$, which gives the conclusion. ■

Proof of Proposition 5.6. The result follows from

$$\begin{aligned}
&\delta_{X,p}(i,I,n) = \|Y_{i,n}Z_{i,n} - Y_{i,n,I}Z_{i,n,I}\|_{L^p} = \|(Y_{i,n} - Y_{i,n,I})Z_{i,n} + Y_{i,n,I}(Z_{i,n} - Z_{i,n,I})\|_{L^p} \\
&\leq \|(Y_{i,n} - Y_{i,n,I})Z_{i,n}\|_{L^p} + \|Y_{i,n,I}(Z_{i,n} - Z_{i,n,I})\|_{L^p} \\
&\leq \|Y_{i,n} - Y_{i,n,I}\|_{L^{q_1}} \|Z\|_{L^{r_1}} + \|Z_{i,n} - Z_{i,n,I}\|_{L^{q_2}} \|Y\|_{L^{r_2}} \\
&= \|Z\|_{L^{r_1}} \delta_{Y,q_1}(i,I,n) + \|Y\|_{L^{r_2}} \delta_{Z,q_2}(i,I,n),
\end{aligned} \tag{S.43}$$

where the second inequality follows from the generalized Hölder inequality (as $p^{-1} = q_1^{-1} + r_1^{-1} = q_2^{-1} + r_2^{-1}$). ■

Proof of Proposition 5.7. The proof is similar to that of Proposition 5.3. It follows from $X_{i,n} = Y_{i,n}Z_{i,n}$ that $X_{i,n,I} = Y_{i,n,I}Z_{i,n,I}$. Denote $\rho \equiv |Y_{i,n} - Y_{i,n,I}|$ and $r = \frac{q}{2} > p$. By the Lyapunov inequality and the Minkowski inequality, $\|\rho\|_{L^p} = \|Y_{i,n} - Y_{i,n,I}\|_{L^p} \leq 2\|Y\|_{L^q} < \infty$. And by the generalized Hölder inequality ($q = 2r$),

$$\|Z_{i,n}\rho\|_{L^r} \leq \|\rho\|_{L^q} \|Z_{i,n}\|_{L^q} \leq 2\|Y\|_{L^q} \|Z\|_{L^q} < \infty.$$

By the fact that $\frac{p}{p-1} \leq q$,

$$\|Z_{i,n}\|_{L^{p/(p-1)}} \leq \|Z_{i,n}\|_{L^q} \leq \|Z\|_{L^q} < \infty.$$

Then, by Lemma S.7,

$$\begin{aligned}
&\|(Y_{i,n} - Y_{i,n,I})Z_{i,n}\|_{L^p} = \|Z_{i,n}\rho\|_{L^p} \leq 2\left(\|\rho\|_{L^p}^{r-p} \|Z_{i,n}\|_{L^{p/(p-1)}}^{r-p} \|Z_{i,n}\rho\|_{L^r}^{r}\right)^{1/(pr-p)} \\
&\leq C_1 (\|\rho\|_{L^p})^{(q-2p)/(pq-2p)} = C_1 \|Y_{i,n} - Y_{i,n,I}\|_{L^p}^{(q-2p)/(pq-2p)} = C_1 \{\delta_{Y,p}(i,I,n)\}^{(q-2p)/(pq-2p)}
\end{aligned}$$

for some constant $C_1 > 0$ that does not depend on i, I, n. Similarly,

$$\|Y_{i,n,I}\left(Z_{i,n} - Z_{i,n,I}\right)\|_{L^p} \leq C_2 \|Z_{i,n} - Z_{i,n,I}\|_{L^p}^{(q-2p)/(pq-2p)} = C_2 \left\{\delta_{Z,p}(i, I, n)\right\}^{(q-2p)/(pq-2p)}.$$

By the above two results,

$$\begin{aligned}
&\delta_{X,p}(i, I, n) = \|Y_{i,n}Z_{i,n} - Y_{i,n,I}Z_{i,n,I}\|_{L^p} = \|\left(Y_{i,n} - Y_{i,n,I}\right) Z_{i,n} + Y_{i,n,I}\left(Z_{i,n} - Z_{i,n,I}\right)\|_{L^p} \\
\leq &\|\left(Y_{i,n} - Y_{i,n,I}\right) Z_{i,n}\|_{L^p} + \|Y_{i,n,I}\left(Z_{i,n} - Z_{i,n,I}\right)\|_{L^p} \\
\leq &C_1 \left\{\delta_{Y,p}(i, I, n)\right\}^{(q-2p)/(pq-2p)} + C_2 \left\{\delta_{Z,p}(i, I, n)\right\}^{(q-2p)/(pq-2p)}.
\end{aligned}$$

We obtain the desired result. ■

S.7. Proofs for Section 6

Proof of Theorem 6.1. (1) By the independence of $\epsilon_{i,n}$ and $\epsilon^*_{i,n}$,

$$\begin{aligned}
&Y_{i,n} - \mathbb{E}\left(Y_{i,n} | \mathcal{F}_{i,n}(s)\right) = Y_{i,n} - \mathbb{E}\left[Y_{i,n,\{j \in D_n : d_{ij} \geq s\}} \middle| \mathcal{F}_{i,n}(s)\right] \\
=&\mathbb{E}\left[Y_{i,n} - Y_{i,n,\{j \in D_n : d_{ij} \geq s\}} \middle| \epsilon_{k,n}, k \in D_n\right].
\end{aligned}$$

Thus, the desired conclusion follows from

$$\begin{aligned}
&\sup_{n, i \in D_n} \|Y_{i,n} - \mathbb{E}\left(Y_{i,n} | \mathcal{F}_{i,n}(s)\right)\|_{L^p} = \sup_{n, i \in D_n} \left\|\mathbb{E}\left[Y_{i,n} - Y_{i,n,\{j \in D_n : d_{ij} \geq s\}} \middle| \epsilon_{k,n}, k \in D_n\right]\right\|_{L^p} \\
\leq &\sup_{n, i \in D_n} \left\|Y_{i,n} - Y_{i,n,\{j \in D_n : d_{ij} \geq s\}}\right\|_{L^p} = \sup_{n, i \in D_n} \delta_p\left(i, \{j \in D_n : d_{ij} \geq s\}, n\right) = \Delta_p(s) \to 0 \quad \text{as } s \to \infty,
\end{aligned}$$

where the inequality follows from the conditional Jensen inequality. ■

(2) From $Y_{i,n} = \sum_{j \in D_n} w_{ij,n}\epsilon_{j,n}$ and the independence of $\epsilon_{j,n}$'s,

$$\Delta_p(s) = \sup_{n,i \in D_n} \left\| Y_{i,n} - Y_{i,n,\{j \in D_n : d_{ij} \geq s\}} \right\|_{L^p} = \sup_{n,i \in D_n} \left\| \sum_{j \in D_n : d_{ij} \geq s} w_{ij,n}\left(\epsilon_{j,n} - \epsilon^*_{j,n}\right) \right\|_{L^p}$$

$$\leq 2 \sup_{n,i \in D_n} \left\| \sum_{j \in D_n : d_{ij} \geq s} w_{ij,n}\epsilon_{j,n} \right\|_{L^p} = 2 \sup_{n,i \in D_n} \left\| Y_{i,n} - \mathbb{E}\left[Y_{i,n} \mid \mathcal{F}_{i,n}(s)\right] \right\|_{L^p} \leq 2\psi(s).$$

Thus the result follows. ■

Proof of Corollary 6.1. For any $i \in D_n$, $m > 0$, denote $\xi^m_{i,n} = \mathbb{E}\left(Y_{i,n} | \mathcal{F}_{i,n}(m)\right) - \mathbb{E}Y_{i,n}$ and $\eta^m_{i,n} = Y_{i,n} - \mathbb{E}(Y_{i,n} | \mathcal{F}_{i,n}(m))$. From Theorem 6.1, $\left\|\eta^m_{i,n}\right\|_{L^2} \leq \Delta_2(m)$. Because the conditional expectation minimizes the L^2-distance, we have $\left\|\xi^m_{i,n}\right\|_{L^2} \leq \left\|\mathbb{E}\left(Y_{i,n} | \mathcal{F}_{i,n}(m)\right)\right\|_{L^2} \leq \left\|Y_{i,n}\right\|_{L^2} \leq \|Y\|_{L^2}$ and $\left\|\eta^m_{i,n}\right\|_{L^2} \leq \left\|Y_{i,n}\right\|_{L^2} \leq \|Y\|_{L^2}$. For any $i \neq j$ and $0 < s \leq \frac{d_{ij}}{2}$, since $Y_{i,n} - \mathbb{E}Y_{i,n} = \xi^s_{i,n} + \eta^s_{i,n}$ and $Y_{j,n} - \mathbb{E}Y_{j,n} = \xi^s_{j,n} + \eta^s_{j,n}$, we have

$$\left|\mathrm{Cov}\left(Y_{i,n}, Y_{j,n}\right)\right| \leq \left|\mathbb{E}\left(\xi^s_{i,n}\xi^s_{j,n}\right)\right| + \left|\mathbb{E}\left(\xi^s_{i,n}\eta^s_{j,n}\right)\right| + \left|\mathbb{E}\left(\eta^s_{i,n}Y_{j,n}\right)\right|.$$

We bound each term on the r.h.s. of the above inequality respectively. First, because $\mathcal{F}_{i,n}(s)$ and $\mathcal{F}_{j,n}(s)$ are independent, $\left|\mathbb{E}\left(\xi^s_{i,n}\xi^s_{j,n}\right)\right| = 0$. Second, by the Cauchy-Schwartz inequality,

$$\left|\mathbb{E}\left(\xi^s_{i,n}\eta^s_{j,n}\right)\right| \leq \left\|\xi^s_{i,n}\right\|_{L^2} \left\|\eta^s_{j,n}\right\|_{L^2} \leq \|Y\|_{L^2} \Delta_2(s),$$

$$\left|\mathbb{E}\left(\eta^s_{i,n}Y_{j,n}\right)\right| \leq \left\|\eta^s_{i,n}\right\|_{L^2} \left\|Y_{j,n}\right\|_{L^2} \leq \|Y\|_{L^2} \Delta_2(s).$$

In sum, we have $\left|\mathrm{Cov}\left(Y_{i,n}, Y_{j,n}\right)\right| \leq 2\|Y\|_{L^2} \Delta_2(s)$ for any $0 < s \leq \frac{d_{ij}}{2}$. ■

S.8. More Examples of Spatial Functional Dependence

S.8.1. The Semiparametric SAR model in Su and Jin (2010) and Su (2012)

In this section, we discuss the application of our FD theory to the semiparametric SAR models considered in Su and Jin (2010) and Su (2012). Since the model in Su (2012) is a special case of the one in Su and Jin (2010), we focus on the model considered in Su and Jin (2010):

$$Y_n = F\left(\lambda W_n Y_n + X_n \beta + \boldsymbol{m}(Z_n) + \epsilon_n\right) \tag{S.44}$$

where $W_n = (w_{ij,n})_{n \times n}$ is a nonstochastic and nonzero spatial weights matrix, $F : \mathbb{R} \to \mathbb{R}$ is a Borel-measurable function, $F(a) \equiv (F(a_1), \ldots, F(a_n))'$ for any column vector $a = (a_1, \ldots, a_n)' \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $\beta \in \mathbb{R}^K$ are true model parameters, $X_n = (X_{1,n}, X_{2,n}, \ldots, X_{n,n})' \in \mathbb{R}^{n \times K_1}$ and $Z_n = (Z_{1,n}, Z_{2,n}, \ldots, Z_{n,n})' \in \mathbb{R}^{n \times K_2}$ are the exogenous variable matrices, $\epsilon_n = (\epsilon_{1,n}, \epsilon_{2,n}, \ldots, \epsilon_{n,n})' \in \mathbb{R}^n$ is the disturbance term, $\boldsymbol{m}(Z_n) = (m(Z_{1,n}), \ldots, m(Z_{n,n}))'$, and $m(\cdot) : \mathbb{R}^{K_2} \to \mathbb{R}$ is an unknown function.

To establish the FD properties of Y_n generated by model (S.44), we state some assumptions.

Assumption S.2. *(1) The Lipschitz constant of $F : \mathbb{R} \to \mathbb{R}$ is L, and $\zeta \equiv L\,|\lambda| \sup_n \|W_n\|_\infty < 1$;*

(2) $|w_{ij,n}| \le c d_{ij}^{-\alpha}$ for some constants $c > 0$ and $\alpha > d$;

(3) for some $p \ge 1$, $\left\{\left(X'_{i,n}, \epsilon_{i,n}\right)' : i \in D_n, n \ge 1\right\}$ is L^p-FD on an independent random field $u \equiv \{u_{i,n} : i \in D_n, n \ge 1\}$ with the spatial FDM $\delta_{X\epsilon,p}(i, I, n)$ and the L^p-FD coefficient $\Delta_{X\epsilon,p}(s)$ satisfying $\Delta_{X\epsilon,p}(s) = O\left(s^{-(\alpha-d)}(\log s)^{\alpha-d}\right)$ as $s \to \infty$ and $\Delta_{X\epsilon,p}(0) < \infty$; $Z_{i,n}$'s are independent over i;

(4) $\|\epsilon\|_{L^p} = \sup_{n,i} \|\epsilon_{i,n}\|_{L^p} < \infty$, $\|X\|_{L^p} = \sup_{n,i} \left\|X'_{i,n}\beta\right\|_{L^p} < \infty$ and $\|Z\|_{L^p} = \sup_{n,i} \|m(Z_{i,n})\|_{L^p} < \infty$.

In Su and Jin (2010, Assumption 1), they assume that $X_{i,n}$'s and $Z_{i,n}$'s are nonstochastic, which is a special case of our Assumption S.2. Thus, our conclusion directly applies to their settings.

Proposition S.1. *Under Assumptions 1 and S.2, the $\{Y_{i,n}\}$ generated by the model (S.44) is L^p-FD on $\left\{\left(u'_{i,n}, Z'_{i,n}\right)'\right\}$ with the L^p-FD coefficient $\Delta_p(s) = O\left(s^{-(\alpha-d)}(\log s)^{\alpha-d}\right)$ as $s \to \infty$.*

Proof. By Assumption S.2(3), $\left\{\left(X'_{i,n}, m(Z_{i,n}) + \epsilon_{i,n}\right)' : i \in D_n, n \geq 1\right\}$ is L^p-FD on $\left\{\left(u'_{i,n}, Z'_{i,n}\right)'\right\}$ with the L^p-FD coefficient $\Delta_{X\epsilon Z,p}(s)$ satisfying $\Delta_{X\epsilon Z,p}(s) = \Delta_{X\epsilon,p}(s)$ for $s > 0$ and $\Delta_{X\epsilon Z,p}(0) \leq \Delta_{X\epsilon,p}(0) + 2\|Z\|_{L^p} < \infty$. We regard $m(Z_{i,n}) + \epsilon_{i,n}$ as a whole. Then, the conditions of Proposition 4.3(6) are satisfied. Then, the result follows from Proposition 4.3(6). ■

S.8.2. The Functional-coefficient SAR model in Sun (2016)

In this section, we consider the functional-coefficient SAR model with nonparametric spatial weights in Sun (2016). The model can be written as

$$Y_n = W_n Y_n + \text{mtx}\{X_n, \theta(D_n)\} + \epsilon_n, \tag{S.45}$$

where $X_n = (X_{1,n}, X_{2,n}, \ldots, X_{n,n})' \in \mathbb{R}^{n \times K_1}$, $Z_n = (Z_{1,n}, Z_{2,n}, \ldots, Z_{n,n})' \in \mathbb{R}^{n \times K_2}$ and $D_n = (D_{1,n}, D_{2,n}, \ldots, D_{n,n})' \in \mathbb{R}^{n \times 1}$ are covariates, $\text{mtx}\{X_n, \theta(D_n)\} \equiv (X'_{1,n}\theta(D_{1,n}), \cdots, X'_{n,n}\theta(D_{n,n}))'$, $W_n = (w_{ij,n})_{n \times n}$, $w_{ij,n} = g(Z_{i,n}, Z_{j,n})$, $\epsilon_n = (\epsilon_{1,n}, \epsilon_{2,n}, \ldots, \epsilon_{n,n})' \in \mathbb{R}^n$ is the disturbance term, and both $\theta(\cdot) : \mathbb{R} \to \mathbb{R}^{K_1}$ and $g(\cdot, \cdot) : \mathbb{R}^{K_2} \times \mathbb{R}^{K_2} \to \mathbb{R}$ are unknown functions. It is interesting to notice that this setup allows the spatial weights matrix W_n to be endogenous under the following conditions.

Assumption S.3. *(1) $(X'_{i,n}, Z'_{i,n}, D_{i,n}, \epsilon_{i,n})$'s are independent over i;*

(2) $\|W_n\|_\infty \leq \zeta < 1$ a.s. for some constant ζ;

(3) $|g(Z_{i,n}, Z_{j,n})| \leq c d_{ij}^{-\alpha}$ a.s. for some constants $c > 0$ and $\alpha > d$;

(4) $\|\epsilon\|_{L^p} = \sup_{n,i} \|\epsilon_{i,n}\|_{L^p} < \infty$ *and* $\|X\theta\|_{L^p} = \sup_{n,i} \left\| X'_{i,n}\theta(D_{i,n}) \right\|_{L^p} < \infty$.

Proposition S.2. *Under Assumptions 1 and S.3, the* $\{Y_{i,n}\}$ *generated by (S.45) is* L^p*-FD on* $\left\{(X'_{i,n}, Z'_{i,n}, D_{i,n}, \epsilon_{i,n})\right\}$ *with the* L^p*-FD coefficient* $\Delta_p(s) = O\left(s^{-(\alpha-d)} (\log s)^{\alpha-d}\right)$ *as* $s \to \infty$.

Proof. In this proof, we regard $\left\{(X'_{i,n}, Z'_{i,n}, D_{i,n}, \epsilon_{i,n})\right\}$ as the underlying random field and assume the weights $w_{ij,n}$'s are always nonnegative w.l.o.g. Denote $e_{i,n} \equiv X'_{i,n}\theta(D_{i,n}) + \epsilon_{i,n}$ and $e_n \equiv (e_{1,n}, \ldots, e_{n,n})'$.

For any fixed unit $k \in D_n$ and any $s \geq 0$, denote $I_s \equiv \{j \in D_n : d_{kj} \geq s\}$. Let $w_{ij,I_s} \equiv g(Z_{i,n,I_s}, Z_{j,n,I_s})$, $W_{n,I_s} \equiv (w_{ij,I_s})_{n\times n}$, $M_n \equiv (I - W_n)^{-1} \equiv (m_{ij})_{n\times n}$ and $M_{n,I_s} \equiv (I - W_{n,I_s})^{-1} \equiv (m_{ij,I_s})_{n\times n}$. By (S.45)

$$Y_n = M_n e_n.$$

Similarly, $Y_{n,I_s} = M_{n,I_s} e_{n,I_s}$. Let $M_{k\cdot,n}$ and $M_{k\cdot,n,I_s}$ be the kth row of M_n and M_{n,I_s}, respectively. Then,

$$Y_{k,n,I_s} - Y_{k,n} = M_{k\cdot,n,I_s} e_{n,I_s} - M_{k\cdot,n} e_n = \underbrace{M_{k\cdot,n,I_s}\left(e_{n,I_s} - e_n\right)}_{Q_1} + \underbrace{\left(M_{k\cdot,n,I_s} - M_{k\cdot,n}\right) e_n}_{Q_2}.$$

We handle Q_1 and Q_2 respectively. For Q_1, note that $e_{i,n,I_s} = e_{i,n}$ if $d_{ki} < s$, and $e_{i,n,I_s} = e^*_{i,n}$ otherwise. Thus,

$$\|Q_1\|_{L^p} \leq \left\| \sum_{j:d_{kj}\geq s} m_{kj,I_s}\left(e^*_{j,n} - e_{j,n}\right) \right\|_{L^p} \leq 2\left(\|\epsilon\|_{L^p} + \|X\theta\|_{L^p}\right) \sum_{j:d_{kj}\geq s} m_{kj,I_s}.$$

By Assumption S.3(2)-(3) and (S.33), $\|Q_1\|_{L^p} = O\left(s^{-(\alpha-d)} (\log s)^{\alpha-d}\right)$. For Q_2, by $M_n = \sum_{l=0}^{\infty} W_n^l$ and the formula

$$A^l - B^l = \sum_{h=0}^{l-1} B^h (A - B) A^{l-1-h} \tag{S.46}$$

for all square matrices A and B, we have

$$M_{n,I_s} - M_n = \sum_{l=0}^{\infty} W_{n,I_s}^l - \sum_{l=0}^{\infty} W_n^l = \sum_{l=1}^{\infty} \sum_{h=0}^{l-1} W_n^h \left(W_{n,I_s} - W_n\right) W_{n,I_s}^{l-1-h},$$

and

$$\sum_{k_3=1}^{n} |m_{kk_3,I_s} - m_{kk_3,n}| = \sum_{l=1}^{\infty} \sum_{h=0}^{l-1} \sum_{k_1,k_2,k_3=1}^{n} \left(W_n^h\right)_{kk_1} |w_{k_1k_2,I_s} - w_{k_1k_2,n}| \left(W_{n,I_s}^{l-1-h}\right)_{k_2k_3}. \tag{S.47}$$

When $0 < h \le l-1$, by Lemma S.8, (S.31) and Assumption S.3(2)-(3), $\phi_{W_n^h}(s) \le O\left(h\zeta^{h-1}\left(\frac{s}{h}\right)^{d-\alpha}\right)$, where $\phi(s)$ is defined in Lemma S.8. Besides, $\phi_{|W_{n,I_s}-W_n|}(s) = O\left(s^{d-\alpha}\right)$. Because $Z_{i,n,I_s} = Z_{i,n}$ if $d_{ki} < s$ and $Z_{i,n,I_s} = Z_{i,n}^*$ otherwise, when $d_{ki} < s$ and $d_{kj} < s$, we have $w_{ij,I_s} - w_{ij,n} = 0$. Thus, $\left(W_n^h\right)_{kk_1} |w_{k_1k_2,I_s} - w_{k_1k_2,n}| \left(W_{n,I_s}^{l-1-h}\right)_{k_2k_3} = 0$ if $d_{kk_1} < \frac{s}{2}$ and $d_{k_1k_2} < \frac{s}{2}$. As a result,

$$\begin{aligned}
&\sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n} \left(W_n^h\right)_{kk_1} |w_{k_1k_2,I_s} - w_{k_1k_2,n}| \left(W_{n,I_s}^{l-1-h}\right)_{k_2k_3} \\
\le &\sum_{k_1:d_{kk_1}\ge s/2}^{n} \left(W_n^h\right)_{kk_1} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n} |w_{k_1k_2,I_s} - w_{k_1k_2,n}| \left(W_{n,I_s}^{l-1-h}\right)_{k_2k_3} + \\
&\sum_{k_1=1}^{n} \left(W_n^h\right)_{kk_1} \sum_{k_2:d_{k_1k_2}\ge s/2}^{n} |w_{k_1k_2,I_s} - w_{k_1k_2,n}| \sum_{k_3=1}^{n} \left(W_{n,I_s}^{l-1-h}\right)_{k_2k_3} \\
\le &\phi_{W_n^h}\left(\frac{s}{2}\right) \left\|\left(W_{n,I_s} - W_n\right) W_{n,I_s}^{l-1-h}\right\|_\infty + \left\|W_n^h\right\|_\infty \phi_{|W_{n,I_s}-W_n|}\left(\frac{s}{2}\right) \left\|W_{n,I_s}^{l-1-h}\right\|_\infty \\
\le &O\left(h\zeta^{h-1}\left(\frac{s}{h}\right)^{d-\alpha}\right) \cdot 2\zeta \cdot \zeta^{l-1-h} + \zeta^{l-1} O\left(s^{d-\alpha}\right) = \zeta^{l-1} O\left(h\left(\frac{s}{h}\right)^{d-\alpha}\right).
\end{aligned}$$

When $h = 0$,

$$\begin{aligned}
&\sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n} \left(W_n^h\right)_{kk_1} |w_{k_1k_2,I_s} - w_{k_1k_2,n}| \left(W_{n,I_s}^{l-1-h}\right)_{k_2k_3} \\
= &\sum_{k_2=1}^{n} \sum_{k_3=1}^{n} |w_{kk_2,I_s} - w_{kk_2,n}| \left(W_{n,I_s}^{l-1}\right)_{k_2k_3} = \sum_{k_2:d_{kk_2\ge s}}^{n} |w_{kk_2,I_s} - w_{kk_2,n}| \sum_{k_3=1}^{n} \left(W_{n,I_s}^{l-1}\right)_{k_2k_3}
\end{aligned}$$

$$\leq \phi_{|W_{n,I_s}-W_n|}(s) \left\| W_{n,I_s}^{l-1} \right\|_{\infty} \leq \zeta^{l-1} O\left(s^{d-\alpha}\right).$$

The implicit constants in the $O(\cdot)$ in the above two inequalities depend neither on l nor n. Thus, by (S.47),

$$\sum_{k_3=1}^{n} |m_{kk_3,I_s} - m_{kk_3,n}| \leq \sum_{l=1}^{\infty} \left\{ \zeta^{l-1} O\left(s^{d-\alpha}\right) + \sum_{h=1}^{l-1} \zeta^{l-1} O\left(h\left(\frac{s}{h}\right)^{d-\alpha}\right) \right\}$$
$$\leq \sum_{l=1}^{\infty} \left\{ 1 + (l-1)^{\alpha-d+2} \right\} \zeta^{l-1} O(s^{d-\alpha}) = O(s^{d-\alpha}).$$

The implicit constants in the $O(\cdot)$ in the above inequalities do not depend on k or n. Hence,

$$\|Q_2\|_{L^p} = \left\| \sum_{k_3=1}^{n} (m_{kk_3,I_s} - m_{kk_3,n})\, e_{k_3,n} \right\|_{L^p} \leq (\|\epsilon\|_{L^p} + \|X\theta\|_{L^p}) \sum_{k_3=1}^{n} |m_{kk_3,I_s} - m_{kk_3,n}| = O(s^{d-\alpha}).$$

Therefore, $\|Y_{k,n,I_s} - Y_{k,n}\| \leq \|Q_1\|_{L^p} + \|Q_2\|_{L^p} \leq O\left(s^{-(\alpha-d)} (\log s)^{\alpha-d}\right)$ uniformly in k and n as $s \to \infty$, and the conclusion follows. ■

S.8.3. The Smooth-coefficient SAR model in Malikov and Sun (2017)

In this section, we consider the smooth-coefficient SAR model in Malikov and Sun (2017). The model can be written as

$$Y_n = \rho(Z_n) W_n Y_n + \mathrm{mtx}\left\{X_n, \beta(Z_n)\right\} + \epsilon_n, \tag{S.48}$$

where $W_n = (w_{ij,n})_{n\times n}$ is the nonstochastic spatial weights matrix, $X_n = (X_{1,n}, X_{2,n}, \ldots, X_{n,n})' \in \mathbb{R}^{n\times K_1}$ and $Z_n = (Z_{1,n}, Z_{2,n}, \ldots, Z_{n,n})' \in \mathbb{R}^{n\times K_2}$ are the covariates, $\epsilon_n = (\epsilon_{1,n}, \epsilon_{2,n}, \ldots, \epsilon_{n,n})' \in \mathbb{R}^n$ is the disturbance term, $\rho(\cdot) : \mathbb{R}^{K_2} \to \mathbb{R}$ and $\beta(\cdot) : \mathbb{R}^{K_2} \to \mathbb{R}^{K_1}$ are the unknown functions, and $\rho(Z_n) \equiv \mathrm{diag}\left\{\rho(Z_{1,n}), \ldots, \rho(Z_{n,n})\right\} \in \mathbb{R}^{n\times n}$ and $\beta(Z_n) \equiv (\beta(Z_{1,n}), \ldots, \beta(Z_{n,n}))' \in \mathbb{R}^{n\times K_1}$, and $\mathrm{mtx}\{X_n, \beta(Z_n)\} \equiv (X_{1,n}'\beta(Z_{1,n}), \ldots, X_{n,n}'\beta(Z_{n,n}))'$.

Assumption S.4. *(1)* $(X_{i,n}', Z_{i,n}', \epsilon_{i,n})$*'s are independent over* i*;*

(2) $\sup_u |\rho(u)| \sup_n \|W_n\|_\infty \leq \zeta < 1$ *for some constant* ζ*;*

(3) $|w_{ij,n}| \leq c d_{ij}^{-\alpha}$ *for some constants* $c > 0$ *and* $\alpha > d$*;*

(4) $\|\epsilon\|_{L^p} \equiv \sup_{n,i} \|\epsilon_{i,n}\|_{L^p} < \infty$ *and* $\|X\beta\|_{L^p} \equiv \sup_{n,i} \left\| X_{i,n}' \beta(Z_{i,n}) \right\|_{L^p} < \infty$.

Proposition S.3. *Under Assumptions 1 and S.4, the* $\{Y_{i,n}\}$ *generated by (S.48) is* L^p*-FD on* $\left\{(X_{i,n}', Z_{i,n}', \epsilon_{i,n})\right\}$ *with the* L^p*-FD coefficient* $\Delta_p(s) = O\left(s^{-(\alpha-d)} (\log s)^{\alpha-d}\right)$ *as* $s \to \infty$*.*

Proof. The proof is similar to that of Proposition S.2. We take $\left\{(X_{i,n}', Z_{i,n}', \epsilon_{i,n})\right\}$ as the underlying random field. Denote $e_{i,n} \equiv X_{i,n}'\beta(Z_{i,n}) + \epsilon_{i,n}$ and $e_n \equiv (e_{1,n}, \ldots, e_{n,n})'$. For any fixed unit $k \in D_n$ and any $s \geq 0$, denote $I_s \equiv \{j \in D_n : d_{kj} \geq s\}$. Let $M_n \equiv (I - \rho(Z_n)W_n)^{-1} \equiv (m_{ij,n})_{n\times n}$ and $M_{n,I_s} \equiv (I - \rho(Z_{n,I_s})W_n)^{-1} \equiv (m_{ij,n,I_s})_{n\times n}$. Denote $|A| \equiv (|a_{ij}|)$ for any matrix $A \equiv (a_{ij})$. By (S.48),

$$Y_n = M_n e_n \text{ and } Y_{n,I_s} = M_{n,I_s} e_{n,I_s}.$$

Let $M_{k\cdot,n}$ and $M_{k\cdot,n,I_s}$ be the kth row of M_n and M_{n,I_s}, respectively. Then,

$$Y_{k,n,I_s} - Y_{k,n} = M_{k\cdot,n,I_s} e_{n,I_s} - M_{k\cdot,n} e_n = M_{k\cdot,n,I_s}\left(e_{n,I_s} - e_n\right) + \left(M_{k\cdot,n,I_s} - M_{k\cdot,n}\right) e_n \equiv Q_1 + Q_2.$$

For Q_1, since $e_{i,n,I_s} = e_{i,n}$ if $d_{ki} < s$ and $e_{i,n,I_s} = e_{i,n}^*$ otherwise, we have

$$\|Q_1\|_{L^p} \leq \left\| \sum_{j:d_{kj}\geq s} m_{kj,I_s}\left(e_{j,n}^* - e_{j,n}\right) \right\|_{L^p} \leq 2\left(\|\epsilon\|_{L^p} + \|X\beta\|_{L^p}\right) \sum_{j:d_{kj}\geq s} m_{kj,I_s}. \tag{S.49}$$

By Assumption S.4(2)-(3) and (S.33), $\|Q_1\|_{L^p} = O\left(s^{-(\alpha-d)} (\log s)^{\alpha-d}\right)$ as $s \to \infty$.

For Q_2, by Neumann's expansion and (S.46),

$$M_{n,I_s} - M_n = \sum_{l=0}^{\infty} [\rho(Z_{n,I_s})W_n]^l - \sum_{l=0}^{\infty} [\rho(Z_n)W_n]^l$$

$$=\sum_{l=1}^{\infty}\sum_{h=0}^{l-1}\left[\rho(Z_n)W_n\right]^h\left\{\left[\rho(Z_{n,I_s})-\rho(Z_n)\right]W_n\right\}\left[\rho(Z_{n,I_s})W_n\right]^{l-1-h},$$

and

$$\sum_{k_3=1}^{n}\left|m_{kk_3,n,I_s}-m_{kk_3,n}\right|$$
$$=\sum_{l=1}^{\infty}\sum_{h=0}^{l-1}\underbrace{\sum_{k_1,k_2,k_3=1}^{n}\left(\left|\rho(Z_n)W_n\right|^h\right)_{kk_1}\left|\left[\rho(Z_{k_1,n,I_s})-\rho(Z_{k_1,n})\right]w_{k_1k_2,n}\right|\left(\left|\rho(Z_{n,I_s})W_n\right|^{l-1-h}\right)_{k_2k_3}}_{\equiv A_{lh,n}}. \tag{S.50}$$

When $h=0$, $A_{l0,n}=0$ because $Z_{k,n,I_s}=Z_{k,n}$. When $0<h\leq l-1$, by Lemma S.8, (S.31) and Assumption S.4(2), $\sup_{n,i}\sup_{Z_n}\sum_{j:d_{ij}\geq s}\left(\left|\rho(Z_n)W_n\right|^h\right)_{ij}\leq O\left(h\zeta^{h-1}\left(\frac{s}{h}\right)^{d-\alpha}\right)$ as $s\to\infty$. Since $Z_{k_1,n,I_s}=Z_{k_1,n}$ when $d_{kk_1}<s$,

$$A_{lh,n}=\sum_{k_1:d_{kk_1}\geq s}\left(\left|\rho(Z_n)W_n\right|^h\right)_{kk_1}\sum_{k_2=1}^{n}\sum_{k_3=1}^{n}\left|\left[\rho(Z_{k_1,n,I_s})-\rho(Z_{k_1,n})\right]w_{k_1k_2,n}\right|\left(\left|\rho(Z_{n,I_s})W_n\right|^{l-1-h}\right)_{k_2k_3}$$
$$\leq\sum_{k_1:d_{kk_1}\geq s}\left(\left|\rho(Z_n)W_n\right|^h\right)_{kk_1}\cdot\left\|\left[\rho(Z_{n,I_s})-\rho(Z_n)\right]W_n\left[\rho(Z_{n,I_s})W_n\right]^{l-1-h}\right\|_{\infty}$$
$$\leq\zeta^{h-1}h^{\alpha-d+1}\cdot O\left(s^{d-\alpha}\right)\cdot 2\zeta^{l-h}=\zeta^{l-1}h^{\alpha-d+1}\cdot O\left(s^{d-\alpha}\right),$$

where the implicit constant in the $O\left(s^{d-\alpha}\right)$ depends neither on l nor n. Combining the results for $h=0$ and $0<h\leq l-1$, by (S.50), we have

$$\sum_{k_3=1}^{n}\left|m_{kk_3,I_s}-m_{kk_3,n}\right|\leq O(s^{d-\alpha})\cdot\sum_{l=2}^{\infty}\sum_{h=1}^{l-1}\left(\zeta^{l-1}h^{\alpha-d+1}\right)\leq O(s^{d-\alpha})\cdot\sum_{l=2}^{\infty}\left[(l-1)^{\alpha-d+2}\zeta^{l-1}\right]=O(s^{d-\alpha}).$$

By the above inequality,

$$\|Q_2\|_{L^p} \leq \sup_{n,k} \left\| \sum_{k_3=1}^{n} (m_{kk_3,I_s} - m_{kk_3,n})\, e_{k_3,n} \right\|_{L^p} \leq (\|\epsilon\|_{L^p} + \|X\beta\|_{L^p}) \sum_{k_3=1}^{n} |m_{kk_3,I_s} - m_{kk_3,n}| = O(s^{d-\alpha}). \tag{S.51}$$

Therefore, by (S.49) and (S.51), $\|Y_{k,n,I_s} - Y_{k,n}\|_{L^p} \leq \|Q_1\|_{L^p} + \|Q_2\|_{L^p} \leq O\left(s^{-(\alpha-d)} (\log s)^{\alpha-d}\right)$ uniformly in k and n as $s \to \infty$. And the conclusion follows. ■

S.8.4. The SDPD model in Shi and Lee (2017)

The SDPD model in Shi and Lee (2017) can be specified as

$$Y_{Nt} = \lambda W_N Y_{Nt} + \gamma Y_{N,t-1} + \rho W_N Y_{N,t-1} + X_{Nt}\beta + \Gamma_N f_t + U_{Nt}, \tag{S.52}$$

where $t = T, T-1, \ldots, i = 1, \ldots, N$, $Y_{Nt} = (y_{1t}, y_{2t}, \ldots, y_{Nt})'$, $W_N = (w_{ij,N})_{N\times N}$ is a nonstochastic spatial weights matrix and invariant as t changes, $X_{Nt} = (x_{1t}, \ldots, x_{Nt})' \in \mathbb{R}^{N\times p}$ is the regressor matrix, Γ_N is the $N \times r$ factor loading parameter matrix, f_t's are the time varying unknown common factors, $U_{Nt} = (u_{1t}, \ldots, u_{Nt})'$ is the disturbance term satisfying $U_{Nt} = \alpha \tilde{W}_N U_{Nt} + V_{Nt}$, $\tilde{W}_N \equiv (\tilde{w}_{ij,N})_{N\times N}$ is a nonstochastic spatial weights matrix, $V_{Nt} = (v_{1t}, \ldots, v_{Nt})'$, and v_{it}'s are i.i.d. random variables. Denote $S_N \equiv I_N - \lambda W_N$, $\tilde{S}_N \equiv I_N - \alpha \tilde{W}_N$, $A_N \equiv S_N^{-1}(\gamma I_N + \rho W_N)$, $B_N \equiv S_N^{-1}\tilde{S}_N^{-1}$, $\varepsilon_{Nt} \equiv (\epsilon_{1t}, \ldots, \epsilon_{Nt})' \equiv X_{Nt}\beta + \Gamma_N f_t$. Then (S.52) can be written as $Y_{Nt} = A_N Y_{N,t-1} + S_N^{-1}\varepsilon_{Nt} + B_N V_{Nt}$. Under some suitable conditions, by iterating the above equation, we have

$$Y_{Nt} = \sum_{h=0}^{\infty} A_N^h S_N^{-1} \varepsilon_{N,t-h} + \sum_{h=0}^{\infty} A_N^h B_N V_{N,t-h}. \tag{S.53}$$

To establish the FD properties for $\{y_{it}\}$, we need the following assumptions.

Assumption S.5. *Let $\mathcal{C} \equiv \left[\vee_{t=-\infty}^{\infty} \sigma(f_t)\right] \vee [\vee_{N=1}^{\infty} \sigma(\Gamma_N)]$ be the σ-field generated by all factors and factor loadings.*

(1) Conditional on $\mathcal{C}$, (x_{it}', v_{it})'s are independent over i and t;

(2) $\sup_N \|W_N\|_\infty \leq 1$ and $|\lambda| + |\gamma| + |\rho| < 1$;

(3) $\xi \equiv \sup_N \left\|\alpha \tilde{W}_N\right\|_\infty < 1$;

(4) $\|\epsilon\|_{L^p,\mathcal{C}} \equiv \sup_{N,T} \sup_{i,t} \|\epsilon_{it}\|_{L^p,\mathcal{C}} < \infty$ and $\|v\|_{L^p,\mathcal{C}} \equiv \sup_{N,T} \sup_{i,t} \|v_{it}\|_{L^p,\mathcal{C}} < \infty$ for some $p \geq 1$;

(5) $|w_{ij,N}| \leq c d_{ij}^{-\alpha}$ and $|\tilde{w}_{ij,N}| \leq c d_{ij}^{-\alpha}$ for some constants $c > 0$ and $\alpha > d$.

Remark. Under Assumption S.5, $\zeta \equiv \frac{|\gamma|+|\rho|}{1-|\lambda|} < 1$.

Proposition S.4. *For model (S.52), under Assumptions 1 and S.5, $\{y_{it} : (i,t) \in D_{NT}\}$ is $\mathcal{C}$-conditionally L^p-FD on $\{(x_{it}, v_{it})\}$ with the $\mathcal{C}$-conditional L^p-FD coefficient $\Delta_p^{\mathcal{C}}(s) = O\left(s^{-(\alpha-d)} (\log s)^{\alpha-d}\right)$ almost surely as $s \to \infty$.*

Proof. To simplify the notation, we assume $\lambda, \gamma, \rho, \alpha \geq 0$ and all entries of W_n are nonnegative in the proof.[6] Note that all expectations and L^p-norms are taken conditional on $\mathcal{C}$, but we omit the subscript $\mathcal{C}$ to simplify the notation. It follows from $\sup_N \left\|S_N^{-1}\right\|_\infty = \sup_N \left\|\sum_{l=0}^\infty (\lambda W_N)^l\right\|_\infty \leq \sum_{l=0}^\infty \lambda^l = \frac{1}{1-\lambda}$ that $\sup_N \|A_N\|_\infty = \sup_N \left\|S_N^{-1}(\gamma I_N + \rho W_N)\right\|_\infty \leq \frac{\gamma+\rho}{1-\lambda} = \zeta < 1$. Similarly, $\sup_N \left\|\tilde{S}_N^{-1}\right\|_\infty \leq \frac{1}{1-\xi}$ and $\sup_N \|B_N\|_\infty \leq \frac{1}{(1-\xi)(1-\lambda)}$. Besides, S_N^{-1}, A_N, B_N and their products are all nonnegative[7]. Thus, Lemma S.8 is applicable. Since $y_{it} = \sum_{h=0}^\infty \sum_{j=1}^N \left(A_N^h S_N^{-1}\right)_{ij} \epsilon_{j,t-h} + \sum_{h=0}^\infty \sum_{j=1}^N \left(A_N^h B_N\right)_{ij} v_{j,t-h}$ and (ϵ_{jt}, v_{jt})'s are independent conditional on $\mathcal{C}$ by Assumption S.5(1), for any pair $((i_1,t_1),(i_2,t_2)) \in D_{NT}^2$,

$$
\begin{aligned}
\delta_p^{\mathcal{C}}(i_1 t_1, i_2 t_2) =& \left\|\left(A_N^{t_1-t_2} S_N^{-1}\right)_{i_1 i_2} \left(\epsilon_{i_2 t_2} - \epsilon_{i_2 t_2}^*\right) + \left(A_N^{t_1-t_2} B_N\right)_{i_1 i_2} \left(v_{i_2 t_2} - v_{i_2 t_2}^*\right)\right\|_{L^p} \\
\leq& 2\|\epsilon\|_{L^p} \left(A_N^{t_1-t_2} S_N^{-1}\right)_{i_1 i_2} + 2\|v\|_{L^p} \left(A_N^{t_1-t_2} B_N\right)_{i_1 i_2}.
\end{aligned}
$$

[6] This simplicity of the notation does not change the essence of the proof; without it, we would need to add many absolute value signs in the proof.

[7] A matrix is nonnegative if and only if all of its elements are nonnegative.

When $t_1 < t_2$, $\delta_p^{\mathcal{C}}(i_1 t_1, i_2 t_2) = 0$. So, in the following, it suffices to consider the case that $t_1 \geq t_2$. By Lemma S.4,

$$
\begin{aligned}
\Delta_p^{\mathcal{C}}(s) \leq & \sup_{(N,T)} \sup_{(i_1,t_1)\in D_{NT}} \sum_{(i_2,t_2)\in D_{NT}: d_{i_1t_1;i_2t_2}\geq s} \delta_p^{\mathcal{C}}(i_1t_1, i_2t_2) \\
\leq & 2 \sup_{(N,T)} \sup_{(i_1,t_1)\in D_{NT}} \sum_{(i_2,t_2)\in D_{NT}: d_{i_1t_1;i_2t_2}\geq s, t_2\leq t_1} \left[\|\epsilon\|_{L^p} \left(A_N^{t_1-t_2} S_N^{-1}\right)_{i_1i_2} + \|v\|_{L^p} \left(A_N^{t_1-t_2} B_N\right)_{i_1i_2} \right] \\
\leq & 2 \|\epsilon\|_{L^p} \sup_{(N,T)} \sup_{(i_1,t_1)\in D_{NT}} \left\{ \underbrace{\sum_{(i_2,t_2):t_1-t_2\geq s} \left(A_N^{t_1-t_2} S_N^{-1}\right)_{i_1i_2}}_{\text{Term 1}} + \underbrace{\sum_{(i_2,t_2):d_{i_1i_2}\geq s, 0\leq t_1-t_2<s} \left(A_N^{t_1-t_2} S_N^{-1}\right)_{i_1i_2}}_{\text{Term 3}} \right\} \\
& + 2 \|v\|_{L^p} \sup_{(N,T)} \sup_{(i_1,t_1)\in D_{NT}} \left\{ \underbrace{\sum_{(i_2,t_2):t_1-t_2\geq s} \left(A_N^{t_1-t_2} B_N\right)_{i_1i_2}}_{\text{Term 2}} + \underbrace{\sum_{(i_2,t_2):d_{i_1i_2}\geq s, 0\leq t_1-t_2<s} \left(A_N^{t_1-t_2} B_N\right)_{i_1i_2}}_{\text{Term 4}} \right\}
\end{aligned}
\tag{S.54}
$$

a.s. We bound the above four terms separately.

Term 1 For any $(i_1, t_1) \in D_{NT}$ and $s \in [0, \infty)$, because $\sup_N \left\|S_N^{-1}\right\|_\infty \leq \frac{1}{1-\lambda}$ and $\sup_N \|A_N\|_\infty \leq \zeta < 1$, we have

$$
\begin{aligned}
& \sum_{(i_2,t_2):t_1-t_2\geq s} \left(A_N^{t_1-t_2} S_N^{-1}\right)_{i_1i_2} \leq \sum_{h=\lfloor s\rfloor}^{\infty} \sum_{i_2=1}^{N} \left(A_N^h S_N^{-1}\right)_{i_1i_2} \leq \sum_{h=\lfloor s\rfloor}^{\infty} \sup_N \left\|A_N^h S_N^{-1}\right\|_\infty \\
\leq & \sum_{h=\lfloor s\rfloor}^{\infty} \|A_N\|_\infty^h \left\|S_N^{-1}\right\|_\infty \leq \sum_{h=\lfloor s\rfloor}^{\infty} \frac{\zeta^h}{1-\lambda} = \frac{\zeta^{\lfloor s\rfloor}}{(1-\lambda)(1-\zeta)} = \frac{\zeta^{\lfloor s\rfloor}}{1-\lambda-\gamma-\rho}.
\end{aligned}
\tag{S.55}
$$

Term 2 Replacing S_N^{-1} in (S.55) by B_N, since $\sup_N \|B_N\|_\infty \leq \frac{1}{(1-\xi)(1-\lambda)}$, we have

$$
\sum_{(i_2,t_2):t_1-t_2\geq s} \left(A_N^{t_1-t_2} B_N\right)_{i_1i_2} \leq \frac{\zeta^{\lfloor s\rfloor}}{(1-\lambda)(1-\zeta)(1-\xi)}. \tag{S.56}
$$

Term 3 For any $(i_1, t_1) \in D_{NT}$,

$$\sum_{(i_2,t_2):d_{i_1 i_2}\geq s, 0\leq t_1-t_2<s} \left(A_N^{t_1-t_2} S_N^{-1}\right)_{i_1 i_2} \leq \sum_{h=0}^{\lfloor s \rfloor} \sum_{i_2:d_{i_1 i_2}\geq s} \left(A_N^h S_N^{-1}\right)_{i_1 i_2}. \tag{S.57}$$

Recall the definition of $\phi_M(s) = \sup_i \sum_{j:d_{ij}\geq s} m_{ij}$ for any square matrix $M = (m_{ij})$ in Lemma S.8. For any $s \in [0, \infty)$,

$$\phi_{\gamma I_N + \rho W_N}(s) = (\gamma I_N + \rho W_N)^h \gamma 1(s=0) + \sup_i \sum_{j:d_{ij}\geq s} \rho w_{ij,N} \leq C_1 \rho (s+1)^{-(\alpha-d)}, \tag{S.58}$$

for some constant C_1 that does not depend on s, where the inequality follows from (S.31). From (S.33), there exists a constant $C_2 > 0$ such that for any $s \in [0, \infty)$,

$$\phi_{S_N^{-1}}(s) \leq C_2 (s+1)^{-(\alpha-d)} (\log(s+2))^{\alpha-d}. \tag{S.59}$$

Because $A_N^h S_N^{-1} = S_N^{-h} (\gamma I_N + \rho W_N)^h S_N^{-1}$ for any $h \in \{0, 1, 2, \ldots\}$, $\zeta = \frac{\gamma+\rho}{1-\lambda}$, $\left\|S_N^{-1}\right\|_\infty \leq \frac{1}{1-\lambda}$, and $\|\gamma I_N + \rho W_N\|_\infty \leq \gamma + \rho$, by (S.58), (S.59) and Lemma S.8, for any $s \in [0, \infty)$, we have

$$\begin{aligned}
&\phi_{A_N^h S_N^{-1}}(s) \leq (h+1) \left(\frac{\gamma+\rho}{1-\lambda}\right)^h \phi_{S_N^{-1}}\left(\frac{s}{2h+1}\right) + \frac{h}{(1-\lambda)^2} \left(\frac{\gamma+\rho}{1-\lambda}\right)^{h-1} \phi_{\gamma I_N+\rho W_N}\left(\frac{s}{2h+1}\right) \\
\leq& C_3 \zeta^h (h+1) \left(\frac{s}{2h+1} + 1\right)^{-(\alpha-d)} \left[\log\left(\frac{s}{2h+1} + 2\right)\right]^{\alpha-d} \\
=& C_3 \zeta^h (h+1) (2h+1)^{\alpha-d} (s+2h+1)^{-(\alpha-d)} \left[\log\left(\frac{s}{2h+1} + 2\right)\right]^{\alpha-d} \\
\leq& C_4 \zeta^h (h+1)^{\alpha-d+1} (s+1)^{-(\alpha-d)} (\log(s+2))^{\alpha-d},
\end{aligned}$$

where $C_3, C_4 > 0$ are constants depending neither on s nor h, and the last inequality is because both $(s+2h+1)^{-(\alpha-d)}$ and $\left[\log\left(\frac{s}{2h+1}+2\right)\right]^{\alpha-d}$ are decreasing in $h\geq 0$. Thus, by (S.57),

$$\sup_{(i_1,t_1)\in D_{NT}}\sum_{(i_2,t_2):d_{i_1i_2}\geq s,0\leq t_1-t_2<s}\left(A_N^{t_1-t_2}S_N^{-1}\right)_{i_1i_2}\leq\sup_{(i_1,t_1)\in D_{NT}}\sum_{h=0}^{\lfloor s\rfloor}\phi_{A_N^hS_N^{-1}}(s)$$
$$\leq\sum_{h=0}^{\lfloor s\rfloor}C_4\zeta^h\left(h+1\right)^{\alpha-d+1}\left(s+1\right)^{-(\alpha-d)}\left(\log\left(s+2\right)\right)^{\alpha-d}\leq C_5\left(s+1\right)^{-(\alpha-d)}\left(\log\left(s+2\right)\right)^{\alpha-d}, \tag{S.60}$$

where $C_5>0$ is a constant not depending on s, and the last step follows from $\sum_{h=0}^{\infty}\zeta^h\left(h+1\right)^{\alpha-d+1}<\infty$.

Term 4 For any $(i_1,t_1)\in D_{NT}$,

$$\sum_{(i_2,t_2):d_{i_1i_2}\geq s,0\leq t_1-t_2<s}\left(A_N^{t_1-t_2}B_N\right)_{i_1i_2}\leq\sum_{h=0}^{\lfloor s\rfloor}\sum_{i_2:d_{i_1i_2}\geq s}\left(A_N^hB_N\right)_{i_1i_2}. \tag{S.61}$$

Under Assumption S.5(5), by the same argument as that for (S.59), for any $s\in[0,\infty)$, we have

$$\phi_{\tilde{S}_N^{-1}}(s)\leq C_2\left(s+1\right)^{-(\alpha-d)}\left(\log\left(s+2\right)\right)^{\alpha-d}. \tag{S.62}$$

Because $A_N^hB_N=S_N^{-h}\left(\gamma I_N+\rho W_N\right)^hS_N^{-1}\tilde{S}_N^{-1}$ for any $h\in\{0,1,2,\ldots\}$, $\left\|S_N^{-1}\right\|_\infty\leq\frac{1}{1-\lambda}$, $\|\gamma I_N+\rho W_N\|_\infty\leq\gamma+\rho$, and $\left\|\tilde{S}_N^{-1}\right\|_\infty\leq\frac{1}{1-\xi}$, by (S.62) and Lemma S.8, for any $s\in[0,\infty)$,

$$\phi_{A_N^hB_N}(s)\leq\frac{h+1}{1-\xi}\left(\frac{\gamma+\rho}{1-\lambda}\right)^h\phi_{S_N^{-1}}\left(\frac{s}{2h+2}\right)+\frac{h}{(1-\lambda)^2(1-\xi)}\left(\frac{\gamma+\rho}{1-\lambda}\right)^{h-1}\phi_{\gamma I_N+\rho W_N}\left(\frac{s}{2h+2}\right)$$
$$+\frac{1}{1-\lambda}\left(\frac{\gamma+\rho}{1-\lambda}\right)^h\phi_{\tilde{S}_N^{-1}}\left(\frac{s}{2h+2}\right)$$
$$\leq C_6\zeta^h\left(h+1\right)\left(\frac{s}{2h+2}+1\right)^{-(\alpha-d)}\left[\log\left(\frac{s}{2h+2}+2\right)\right]^{\alpha-d}$$
$$=C_6\zeta^h\left(h+1\right)\left(2h+2\right)^{\alpha-d}\left(s+2h+2\right)^{-(\alpha-d)}\left[\log\left(\frac{s}{2h+2}+2\right)\right]^{\alpha-d}$$
$$\leq C_7\zeta^h\left(h+1\right)^{\alpha-d+1}\left(s+2\right)^{-(\alpha-d)}\left(\log\left(\frac{s}{2}+2\right)\right)^{\alpha-d},$$

where $C_6, C_7 > 0$ are constants depending neither on s nor h, and the last inequality is because both $(s+2h+2)^{-(\alpha-d)}$ and $\left[\log\left(\frac{s}{2h+2}+2\right)\right]^{\alpha-d}$ are decreasing in $h \geq 0$. Thus, by (S.61),

$$\sup_{(i_1,t_1)\in D_{NT}} \sum_{(i_2,t_2):d_{i_1i_2}\geq s, 0\leq t_1-t_2<s} \left(A_N^{t_1-t_2}B_N\right)_{i_1i_2} \leq \sup_{(i_1,t_1)\in D_{NT}} \sum_{h=0}^{\lfloor s \rfloor} \phi_{A_N^h B_N}(s)$$

$$\leq \sum_{h=0}^{\lfloor s \rfloor} C_7\zeta^h (h+1)^{\alpha-d+1} (s+2)^{-(\alpha-d)} \left(\log\left(\frac{s}{2}+2\right)\right)^{\alpha-d} \leq C_8 (s+2)^{-(\alpha-d)} \left(\log\left(\frac{s}{2}+2\right)\right)^{\alpha-d}, \tag{S.63}$$

where $C_8 > 0$ is a constant not depending on s, and the last step follows from $\sum_{h=0}^{\infty} \zeta^h (h+1)^{\alpha-d+1} < \infty$.

Combining (S.55), (S.56), (S.60), (S.63), as $s \to \infty$, we have

$$\Delta_p^{\mathcal{C}}(s) \leq O\left(s^{-(\alpha-d)} (\log s)^{\alpha-d}\right).$$

■

References

Davidson, J. (1994) *Stochastic Limit Theory: An Introduction for Econometricians.* Oxford University Press, Oxford.

Jenish, N. and I. R. Prucha (2009) Central limit theorems and uniform laws of large numbers for arrays of random fields. *Journal of Econometrics* 150, 86–98.

Malikov, E. and Y. Sun (2017) Semiparametric estimation and testing of smooth coefficient spatial autoregressive models. *Journal of Econometrics* 199, 12–34.

Nagaev, S. V. (1979) Large deviations of sums of independent random variables. *The Annals of Probability* 7, 745–789.

Rio, E. (2009) Moment inequalities for sums of dependent random variables under projective conditions. *Journal of Theoretical Probability* 22, 146–163.

Shi, W. and L.-f. Lee (2017) Spatial dynamic panel data models with interactive fixed effects. *Journal of Econometrics* 197, 323–347.

Su, L. (2012) Semiparametric GMM estimation of spatial autoregressive models. *Journal of Econometrics* 167, 543–560.

Su, L. and S. Jin (2010) Profile quasi-maximum likelihood estimation of partially linear spatial autoregressive models. *Journal of Econometrics* 157, 18–33.

Sun, Y. (2016) Functional-coefficient spatial autoregressive models with nonparametric spatial weights. *Journal of Econometrics* 195, 134–153.

Wu, W. B. and Y. N. Wu (2016) Performance bounds for parameter estimates of high-dimensional linear models with correlated errors. *Electronic Journal of Statistics* 10, 352–379.

Xu, X. and L.-f. Lee (2015) Maximum likelihood estimation of a spatial autoregressive Tobit model. *Journal of Econometrics* 188, 264–280.

Xu, X. and L.-f. Lee (2018) Estimation of a binary choice game with network links. *Working Paper,* Xiamen University.

Xu, X., W. Wang, Y. Shin and C. Zheng (2022) Dynamic network quantile regression model. *Journal of Business & Economic Statistics* 0, 1–15.