Supplementary Material to "Spurious factors in data with local-to-unit roots"

Alexei Onatski* and Chen Wang†
*Faculty of Economics, University of Cambridge.
†Department of Statistics and Actuarial Science,
University of Hong Kong.

March 7, 2024

Abstract

This note, first, derives explicit expressions for the eigenvalues and eigenfunctions of $K_{\mathcal{F}}$ for the special case of no heterogeneity in local-to-unit roots (see equation (3) of the main text). Next, it proves lemmas 2, 3, 4, 5, and equations (20) and (30). Finally, it provides the reader with a detailed analysis of the Markov chain example. All internal references have the prefix "A", whereas all references to the main text do not have any prefixes. For example, (A1) refers to the first numbered equation in this note, whereas (1) refers to the first numbered equation in the main text.

A1 Eigenstructure of $K_{\mathcal{F}}$ when $\phi_i = \phi > 0$, $i = 1, \ldots, N$.

When the heterogeneity in ϕ_i is absent, the kernel of $K_{\mathcal{F}}$ is proportional to

$$k_\phi(s,t) = \frac{e^{-\phi|t-s|}}{2\phi} + \frac{e^{-\phi t} + e^{-\phi(1-t)}}{2\phi^2} + \frac{e^{-\phi s} + e^{-\phi(1-s)}}{2\phi^2} + \frac{e^{-\phi} - 1 - \phi}{\phi^3},$$

which is the covariance kernel of the demeaned OU process with parameter $\phi > 0$. Let λ and $f(t)$ be an eigenvalue-eigenfunction pair for the integral operator with kernel $k_\phi(s,t)$. Then

$$\int_0^1 k_\phi(s,t) f(t) \mathrm{d}t = \lambda f(s).$$

We note that since the covariance kernel corresponds to the demeaned OU, we must have $\int f(t)\mathrm{d}t = 0$ for any $\lambda \neq 0$. Therefore, we can rewrite the above identity as

$$\frac{1}{2\phi^2} \int_0^1 \left(\phi e^{-\phi|t-s|} + e^{-\phi t} + e^{-\phi(1-t)} \right) f(t) \mathrm{d}t = \lambda f(s),$$

or equivalently,

$$\int_0^1 \left(\phi e^{-\phi(t-s)} + e^{-\phi t} + e^{-\phi(1-t)}\right) f(t)\mathrm{d}t + \int_0^s \phi \left(e^{\phi(t-s)} - e^{-\phi(t-s)}\right) f(t)\mathrm{d}t = 2\phi^2 \lambda f(s).$$

Differentiating both sides with respect to s and dividing by ϕ^2, we obtain

$$\int_0^1 e^{-\phi(t-s)} f(t)\mathrm{d}t - \int_0^s \left(e^{\phi(t-s)} + e^{-\phi(t-s)}\right) f(t)\mathrm{d}t = 2\lambda f'(s). \tag{A1}$$

Differentiating once more, we get

$$\int_0^1 \phi e^{-\phi(t-s)} f(t)\mathrm{d}t - 2f(s) + \int_0^s \phi \left(e^{\phi(t-s)} - e^{-\phi(t-s)}\right) f(t)\mathrm{d}t = 2\lambda f''(s). \tag{A2}$$

Differentiating one last time, we get

$$\int_0^1 \phi^2 e^{-\phi(t-s)} f(t)\mathrm{d}t - 2f'(s) - \int_0^s \phi^2 \left(e^{\phi(t-s)} + e^{-\phi(t-s)}\right) f(t)\mathrm{d}t = 2\lambda f'''(s).$$

Using (A1) in the latter display yields

$$\left(\phi^2 - 1/\lambda\right) f'(s) = f'''(s).$$

Reparameterizing the problem by letting $\lambda = \frac{1}{\phi^2+\omega^2}$ (we will consider a possibility $\lambda = \frac{1}{\phi^2-\omega^2}$ later), we get

$$f'''(s) = -\omega^2 f'(s),$$

so that

$$f'(s) = A\cos(\omega s) + B\sin(\omega s),$$

and therefore,

$$f(s) = a\cos(\omega s) + b\sin(\omega s) + c.$$

We need to find a, b, c, and ω. To this end, we will use the following boundary conditions (obtained from (A1) and (A2))

$$\lambda f''(0) = \lambda\phi f'(0) - f(0), \tag{A3}$$

$$\lambda f''(1) = -\lambda\phi f'(1) - f(1). \tag{A4}$$

These conditions imply the following relationships

$$(1-\lambda\omega^2)a - \lambda\phi\omega b + c = 0,$$
$$((1-\lambda\omega^2)\cos\omega - \lambda\phi\omega\sin\omega)a + ((1-\lambda\omega^2)\sin\omega + \lambda\phi\omega\cos\omega)b + c = 0.$$

Now, the fact that the integral of $f(s)$ over $[0,1]$ is zero yields

$$a\sin(\omega)/\omega - b(\cos(\omega)-1)/\omega + c = 0.$$

Using this in the previous display, we get

$$\left(1-\lambda\omega^2 - \frac{\sin\omega}{\omega}\right)a - \frac{1+\lambda\phi\omega^2-\cos\omega}{\omega}b = 0, \tag{A5}$$
$$\left((1-\lambda\omega^2)\cos\omega - \frac{1+\lambda\phi\omega^2}{\omega}\sin\omega\right)a + \left((1-\lambda\omega^2)\sin\omega + \frac{1+\lambda\phi\omega^2}{\omega}\cos\omega - \frac{1}{\omega}\right)b = 0.$$

For this system to have a nonzero solution, it must be degenerate, with the determinant of the matrix of the coefficients equal to zero. That is, we must have

$$-2(1-\lambda\omega^2) + \omega\left[(1-\lambda\omega^2)^2 - \lambda\phi(2+\lambda\phi\omega^2)\right]\sin(\omega) + 2(1+\lambda\phi\omega^2)(1-\lambda\omega^2)\cos(\omega) = 0.$$

Recalling that $\lambda = (\omega^2+\phi^2)^{-1}$, we note that

$$1-\lambda\omega^2 = \frac{\phi^2}{\omega^2+\phi^2}.$$

Hence, the above equation simplifies to the following form

$$-2(\omega^2+\phi^2) + \omega(\phi^2 - 2\phi - \omega^2 - 2\omega^2/\phi)\sin(\omega) + 2(\omega^2+\phi^2+\phi\omega^2)\cos(\omega) = 0. \tag{A6}$$

Figure A1 shows the graph of the left hand side (divided by $1+\omega^2$ for better scaling) for the special case where $\phi = 1$. The smallest positive root is approximately $\omega = 3.68$. It corresponds to the largest eigenvalue $\lambda \approx 1/(1+3.68^2) \approx 0.069$. This can be compared to the largest eigenvalue $1/\pi^2 \approx 0.10$ of the covariance operator of the Brownian motion (that corresponds to the case $\phi = 0$).

Once ω is chosen as a root of (A6), we obtain (from (A5) and the parameterization

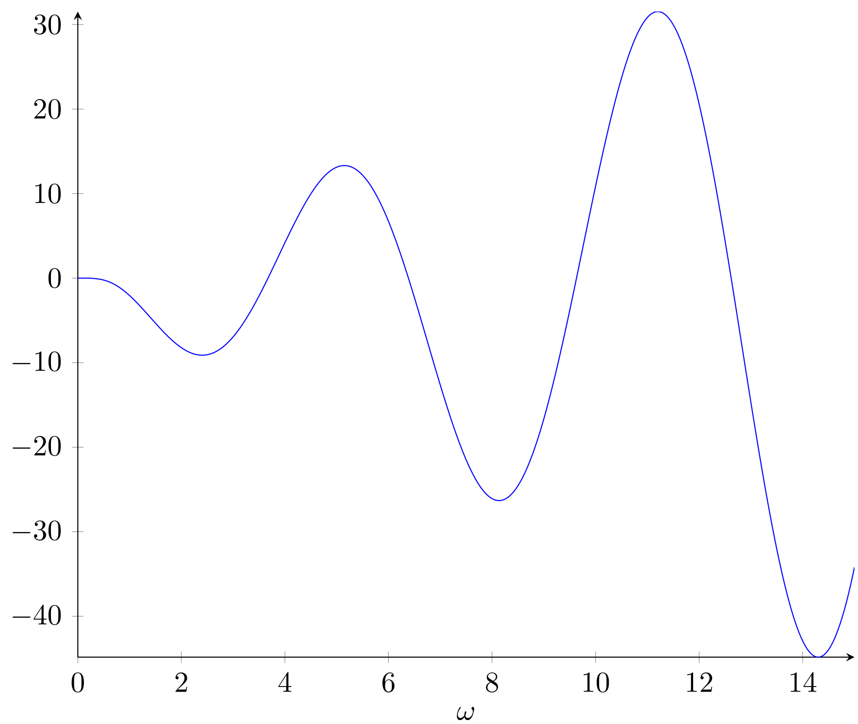

Figure 1: Plot of the (scaled) function of ω defined by the left hand side of (A6) when $\phi = 1$. The graph is scaled by dividing the function by $1 + \omega^2$ to help visualization. The first three positive roots are, approximately, $\omega = 3.68$, $\omega = 6.39$, and $\omega = 9.53$.

$\lambda = (\omega^2 + \phi^2)^{-1})$

$$\begin{aligned} a &= h\left(\frac{1}{\omega} + \frac{\phi\omega}{\phi^2+\omega^2} - \frac{\cos\omega}{\omega}\right), \\ b &= h\left(\frac{\phi^2}{\phi^2+\omega^2} - \frac{\sin\omega}{\omega}\right), \\ c &= h\frac{\phi^2}{\phi^2+\omega^2}\left(\frac{\cos\omega - 1}{\omega} - \frac{\sin\omega}{\phi}\right), \end{aligned}$$

where the coefficient of proportionality h should be chosen so that the eigenfunction $f(s)$ has unit L_2 norm.

To finish our derivations, we need to consider the possibility that $\lambda = \frac{1}{\phi^2-\omega^2}$, as mentioned above. If this is the case, then we must have

$$f(s) = ae^{\omega s} + be^{-\omega s} + c.$$

Using initial conditions (A3)-(A4) and the requirement that $\int_0^1 f(s)\mathrm{d}s = 0$, we obtain the following system

$$\left(\lambda\omega^2 - \lambda\phi\omega + 1 - \frac{e^{\omega}-1}{\omega}\right)a + \left(\lambda\omega^2 + \lambda\phi\omega + 1 + \frac{e^{-\omega}-1}{\omega}\right)b = 0,$$
$$e^{\omega}\left(\lambda\omega^2 + \lambda\phi\omega + 1 + \frac{e^{-\omega}-1}{\omega}\right)a + e^{-\omega}\left(\lambda\omega^2 - \lambda\phi\omega + 1 - \frac{e^{\omega}-1}{\omega}\right)b = 0.$$

For this system to have a non-zero solution, we must have

$$e^{\omega}\left(\lambda\omega^2 + \lambda\phi\omega + 1 + \frac{e^{-\omega}-1}{\omega}\right)^2 = e^{-\omega}\left(\lambda\omega^2 - \lambda\phi\omega + 1 - \frac{e^{\omega}-1}{\omega}\right)^2.$$

Recalling that $\lambda = 1/(\phi^2 - \omega^2)$, and that it must be positive (since the covariance operator is non-negative definite), we obtain

$$e^{\omega}\left(\frac{\phi}{\phi-\omega} + \frac{e^{-\omega}-1}{\omega}\right)^2 = e^{-\omega}\left(\frac{\phi}{\phi+\omega} - \frac{e^{\omega}-1}{\omega}\right)^2.$$

This implies that

$$e^{\omega}\left(\frac{\phi}{\phi-\omega} + \frac{e^{-\omega}-1}{\omega}\right) = \pm\left(\frac{\phi}{\phi+\omega} - \frac{e^{\omega}-1}{\omega}\right).$$

If the sign on the right hand side is '+', then we have

$$\frac{e^{\omega}}{\phi-\omega} = \frac{1}{\phi+\omega},$$

which is clearly impossible for $\phi > \omega > 0$. If the sign is '$-$', then we have, after some algebra,

$$e^{\omega} = \frac{\phi-\omega}{\phi+\omega}\frac{2\phi/(2+\phi)+\omega}{2\phi/(2+\phi)-\omega},$$

which is only possible if $\omega < 2\phi/(2+\phi)$. Suppose that the latter inequality holds. Then, taking the logarithm of both sides of the above display and rearranging, we obtain

$$\ln(\phi-\omega) - \ln(\phi+\omega) + \ln\left(\frac{2\phi}{2+\phi} + \omega\right) - \ln\left(\frac{2\phi}{2+\phi} - \omega\right) - \omega = 0.$$

This equation does hold for $\omega = 0$, but not for $\omega > 0$. Indeed, the derivative of the left hand side with respect ω is

$$-\frac{2\phi}{\phi^2-\omega^2} + \frac{\frac{4\phi}{2+\phi}}{\left(\frac{2\phi}{2+\phi}\right)^2 - \omega^2} - 1 = \omega^2\frac{\left(\frac{2\phi}{2+\phi}\right)^2\frac{12+6\phi+\phi^2}{4} - \omega^2}{(\phi^2-\omega^2)\left(\left(\frac{2\phi}{2+\phi}\right)^2 - \omega^2\right)} > 0,$$

where the last inequality holds because $\omega < 2\phi/(2+\phi)$. To summarise, the parameterization $\lambda = \frac{1}{\phi^2-\omega^2}$ is incompatible with the initial conditions, and the solutions that we have found for the case $\lambda = \frac{1}{\phi^2+\omega^2}$ are the only possible ones.

A2 Proof of lemma 2

Let l be the T-dimensional vector of ones. A direct derivation yields, for $\phi > 0$,

$$\begin{aligned}
\frac{1}{T}\left(MU_\phi' U_\phi M\right)_{ij} &= \frac{1}{T}\left(U_\phi' U_\phi\right)_{ij} - \frac{1}{T^2}\left(U_\phi' U_\phi l\right)_i - \frac{1}{T^2}\left(U_\phi' U_\phi l\right)_j + \frac{1}{T^3} l' U_\phi' U_\phi l \\
&= \frac{e^{-\phi|j-i|/T} - e^{-2\phi T^2} e^{-\phi(j+i)/T}}{T\left(1-e^{-2\phi/T}\right)} \\
&\quad - \frac{e^{\phi/T}\left(1-e^{-\phi i/T}\right) + 1 - e^{-\phi} e^{\phi i/T} - e^{-2\phi T^2} e^{-\phi i/T}\left(1-e^{-\phi}\right)}{T^2\left(1-e^{-2\phi/T}\right)\left(e^{\phi/T}-1\right)} \\
&\quad - \frac{e^{\phi/T}\left(1-e^{-\phi j/T}\right) + 1 - e^{-\phi} e^{\phi j/T} - e^{-2\phi T^2} e^{-\phi j/T}\left(1-e^{-\phi}\right)}{T^2\left(1-e^{-2\phi/T}\right)\left(e^{\phi/T}-1\right)} \\
&\quad + \frac{2e^{\phi/T}\left(e^{-\phi}-1\right) + T\left(e^{2\phi/T}-1\right) - e^{-2\phi T^2}\left(1-e^{-\phi}\right)^2}{T^3\left(1-e^{-2\phi/T}\right)\left(e^{\phi/T}-1\right)^2}.
\end{aligned}$$

Rearranging the terms on the right hand side of the latter equation yields the lemma for $\phi > 0$. For $\phi = 0$, we have

$$\begin{aligned}
\frac{1}{T}\left(MU_0' U_0 M\right)_{ij} &= \min\{i/T, j/T\} + \frac{(i/T)^2}{2} - i/T - (i/T)/(2T) \\
&\quad + \frac{(j/T)^2}{2} - j/T - (j/T)/(2T) + T(T+1)(2T+1)/\left(6T^3\right) \\
&= -|i/T - j/T|/2 + \frac{(i/T)^2}{2} - \frac{i/T}{2} + \frac{1}{4} + \frac{(j/T)^2}{2} - \frac{j/T}{2} + \frac{1}{4} \\
&\quad - (i/T)/(2T) - (j/T)/(2T) + T(T+1)(2T+1)/\left(6T^3\right) - 1/2,
\end{aligned}$$

which yields the lemma for $\phi = 0$. □

A3 Proof of lemma 3

The uniform boundedness of $|k_\phi(s,t)|$ follows from that of $|a_\phi(s,t)| = -a_\phi(s,t)$ and the definitions $b_\phi(s) = \int_0^1 a_\phi(s,t)\,\mathrm{d}t$ and $c_\phi = \int_0^1\int_0^1 a_\phi(s,t)\,\mathrm{d}t\mathrm{d}s$. The uniform boundedness of $|a_\phi(s,t)|$ follows from the inequality $e^{-\phi x} \geq 1-\phi$. This inequality implies that the maximum of $|a_\phi(s,t)|$ over $s,t \in [0,1]^2$ is no larger than $1/2$. The uniform bound on $|k_\phi(s,t)|$ equals 1 because $k_\phi(s,t) \leq -b_\phi(s) - b_\phi(t) \leq 1$ and $-k_\phi(s,t) \leq -a_\phi(s,t) - c_\phi \leq 1$.

To establish the uniform boundedness of $|k_{\phi,T}(s,t)|$, we will prove that $|\omega_{\phi 1,T} a_\phi(s,t)|$, $|\omega_{\phi 2,T}(b_\phi(t)+b_\phi(s))|$, $|d_{\phi,T}|$, and $|e_{\phi,T}(s,t)|$ are uniformly bounded. For $\phi>0$, we have

$$|\omega_{\phi 1,T} a_\phi(s,t)| = \frac{1-e^{-\phi|t-s|}}{T\left(1-e^{-2\phi/T}\right)} = \frac{1-\rho^{T|t-s|}}{T\left(1-\rho^2\right)},$$

where $\rho = e^{-\phi/T}$. This yields

$$|\omega_{\phi 1,T} a_\phi(s,t)| \leq \frac{1-\rho^T}{T\left(1-\rho^2\right)} \leq 1.$$

Clearly, $|\omega_{01,T} a_0(s,t)| = |t-s|/2 < 1$. Hence, $|\omega_{\phi 1,T} a_\phi(s,t)| \leq 1$ for all $\phi \geq 0$ and all positive integers T.

Note that

$$\omega_{\phi 2,T} b_\phi(t) = \frac{\phi/T}{e^{\phi/T}-1}\omega_{\phi 1,T} b_\phi(t)$$

for $\phi>0$ and $\omega_{02,T} b_0(t) = \omega_{01,T} b_0(t)$. Since $\omega_{\phi 1,T} b_\phi(t) = \int_0^1 \omega_{\phi 1,T} a_\phi(s,t)\,\mathrm{d}s$ and $|\omega_{\phi 1,T} a_\phi(s,t)| \leq 1$ for all $\phi \geq 0$ and T, we have $|\omega_{\phi 1,T} b_\phi(t)| \leq 1$. But $\left|\frac{\phi/T}{e^{\phi/T}-1}\right| \leq 1$. Therefore, $|\omega_{\phi 2,T} b_\phi(t)| \leq 1$ for all $\phi \geq 0$ and T. Hence, $|\omega_{\phi 2,T}(b_\phi(s)+b_\phi(t))| \leq 2$ for all $\phi \geq 0$ and T.

Next, by definition, for $\phi>0$,

$$d_{\phi,T} = \frac{2\rho\left(\rho^T-1\right)+T\left(1-\rho^2\right)-T^2\left(1-\rho\right)^2}{T^3\left(1-\rho^2\right)\left(1-\rho\right)^2},$$

where $\rho = e^{-\phi/T}$. This yields, after some algebra (see section A9),

$$d_{\phi,T} = -\frac{\sum_{j=0}^{T-2}(T-j)(T-j-1)\rho^j}{T^3(1+\rho)}. \tag{A7}$$

Therefore, $|d_{\phi,T}| \leq 1$ for all $\phi>0$ and T. For $\phi=0$, $d_{0,T} = (T+1)(2T+1)/\left(6T^2\right) - 1/2$, and hence, $|d_{0,T}| \leq 1/2$ for all T. To summarize, $|d_{\phi,T}| \leq 1$ for all $\phi \geq 0$ and T.

Finally, for $\phi>0$, we have

$$e_{\phi,T}(s,t) = e_{\phi 1,T}(s,t) + e^{-2\phi T^2} e_{\phi 2,T}(s,t) \tag{A8}$$

with

$$e_{\phi 1,T}\left(s,t\right) = \frac{2-e^{-\phi s}-e^{-\phi t}}{T^{2}\left(1-e^{-2\phi/T}\right)}-\frac{2(e^{\phi/T}-1-\phi/T)}{T(1-e^{-2\phi/T})(e^{\phi/T}-1)} \text{ and} \tag{A9}$$

$$e_{\phi 2,T}\left(s,t\right) = \frac{e^{-\phi(t+s)}-1}{T\left(1-e^{-2\phi/T}\right)}+\frac{\left(2-e^{-\phi s}-e^{-\phi t}\right)e^{-\phi/T}\left(1-e^{-\phi}\right)}{T^{2}\left(1-e^{-2\phi/T}\right)\left(1-e^{-\phi/T}\right)} + \frac{\left(T\left(1-e^{-\phi/T}\right)-\left(1-e^{-\phi}\right)e^{-\phi/T}\right)^{2}}{T^{3}\left(1-e^{-2\phi/T}\right)\left(1-e^{-\phi/T}\right)^{2}}. \tag{A10}$$

For term $e_{\phi 1,T}\left(s,t\right)$, we have

$$e_{\phi 1,T}\left(s,t\right)\leq\frac{2-\rho^{Ts}-\rho^{Tt}}{T^{2}\left(1-\rho^{2}\right)}\leq\frac{2\left(1-\rho^{T}\right)}{T^{2}\left(1-\rho^{2}\right)}\leq\frac{2}{T},$$

and

$$-e_{\phi 1,T}\left(s,t\right)\leq\frac{2(e^{\phi/T}-1-\phi/T)}{T(1-e^{-2\phi/T})(e^{\phi/T}-1)}\leq\frac{2(1-e^{-\phi/T})}{T(1-e^{-2\phi/T})}\leq\frac{2}{T},$$

so that

$$|e_{\phi 1,T}\left(s,t\right)|\leq\frac{2}{T}. \tag{A11}$$

For the components of the term $e_{\phi 2,T}\left(s,t\right)$, we have

$$\frac{1-\rho^{T(t+s)}}{T\left(1-\rho^{2}\right)}\leq\frac{1-\rho^{2T}}{T\left(1-\rho^{2}\right)}=\frac{1+\rho^{2}+\cdots+\rho^{2(T-1)}}{T}\leq 1, \tag{A12}$$

$$\frac{\left(2-\rho^{Ts}-\rho^{Tt}\right)\rho\left(1-\rho^{T}\right)}{T^{2}\left(1-\rho^{2}\right)\left(1-\rho\right)}\leq\frac{2\rho\left(1+\cdots+\rho^{T-1}\right)^{2}}{T^{2}\left(1+\rho\right)}\leq 1, \tag{A13}$$

and (see a derivation in the next section)

$$\frac{\left(T(1-\rho)-(1-\rho^{T})\rho\right)^{2}}{T^{3}(1-\rho^{2})(1-\rho)^{2}}\leq 1. \tag{A14}$$

These bounds yield

$$|e_{\phi 2,T}\left(s,t\right)|\leq 2. \tag{A15}$$

Combining this with the above bound for $e_{\phi 1,T}\left(s,t\right)$ yields

$$\left|e_{\phi,T}\left(s,t\right)\right|\leq 4$$

for all $\phi>0$ and all T. For $\phi=0$, we obviously have $\left|e_{0,T}\left(s,t\right)\right|=\left|s+t\right|/\left(2T\right)\leq 1.$

Summing up the above results, we obtain

$$\sup_{\phi\geq 0}\sup_{T\geq 1}\max_{s,t\in[0,1]^2}|k_{\phi,T}(s,t)|\leq 1+2+1+4=8. \qquad \square$$

A4 Proof of inequality (A14)

We have

$$\frac{(T(1-\rho)-(1-\rho^T)\rho)^2}{T^3(1-\rho^2)(1-\rho)^2}=\frac{\left(T-\frac{1-\rho^T}{1-\rho}\rho\right)^2}{T^3(1-\rho^2)}\leq\frac{T-\frac{1-\rho^T}{1-\rho}\rho}{T^2(1-\rho^2)}$$
$$=\frac{(1-\rho)+(1-\rho^2)+\cdots+(1-\rho^T)}{T^2(1-\rho^2)}. \qquad \text{(A16)}$$

If T is even, then (A16) is no larger than

$$\frac{2(1-\rho^2)+2(1-\rho^4)+\cdots+2(1-\rho^{2(T/2)})}{T^2(1-\rho^2)}.$$

On the other hand, $(1-\rho^{2k})/(1-\rho^2)=1+\rho^2+\cdots+\rho^{2(k-1)}\leq k$. Therefore, the expression in the above display is no larger than

$$\frac{2(1+2+\cdots+T/2)}{T^2}=\frac{T+2}{4T}\leq\frac{1}{2}$$

for $T\geq 2$. If T is odd and $T\geq 3$, then (A16) is no larger than

$$\frac{2(1-\rho^2)+2(1-\rho^4)+\cdots+2(1-\rho^{2((T+1)/2)})}{T^2(1-\rho^2)}$$
$$\leq\frac{2(1+\cdots+(T+1)/2)}{T^2}=\frac{(T+1)(T+3)}{4T^2}\leq\frac{2}{3}$$

for $T\geq 3$. Finally, for $T=1$, we have

$$\frac{(T(1-\rho)-(1-\rho^T)\rho)^2}{T^3(1-\rho^2)(1-\rho)^2}=\frac{1-\rho}{1+\rho}\leq 1.$$

To summarize, for all integer $T\geq 1$, we have

$$\frac{(T(1-\rho)-(1-\rho^T)\rho)^2}{T^3(1-\rho^2)(1-\rho)^2}\leq 1. \quad \square$$

A5 Proof of equation (20)

For $\phi = 0$,

$$|f_\epsilon(0)e_{0,T}(s,t)| = |s+t|/(2T) \leq 1/T,$$

so that the bound (20) obviously holds. Consider the case $\phi > 0$, or in terms of ρ, $\rho \in (0,1)$. Using representation (A8) of $e_{\phi,T}(s,t)$, and inequalities (A11) and (A15), we obtain

$$|f_\epsilon(\phi)\, e_{\phi,T}(s,t)| \leq |e_{\phi 1,T}(s,t)| + \rho^{2T^3} |e_{\phi 2,T}(s,t)| \leq \frac{2}{T} + 2\rho^{2T^3}.$$

For $\rho \in \left(0, 1 - 1/T^2\right]$, this yields

$$|f_\epsilon(\phi)\, e_{\phi,T}(s,t)| \leq \frac{2}{T} + 2e^{-2T} \leq \frac{4}{T},$$

so that (20) holds. It remains to consider the case $\rho \in (1 - 1/T^2, 1)$. For this case, we will show that

$$|e_{\phi 2,T}(s,t)| \leq \frac{2}{T}, \tag{A17}$$

so that

$$|f_\epsilon(\phi)\, e_{\phi,T}(s,t)| \leq |e_{\phi 1,T}(s,t)| + \rho^{2T^3} |e_{\phi 2,T}(s,t)| \leq \frac{2}{T} + \frac{2}{T} = \frac{4}{T},$$

which yields (20).

In the remaining part of the proof we establish (A17). Recall decomposition (A10):

$$e_{\phi 2,T}(s,t) = \frac{\rho^{T(t+s)} - 1}{T(1-\rho^2)} + \frac{(2 - \rho^{Ts} - \rho^{Tt})\rho(1-\rho^T)}{T^2(1-\rho^2)(1-\rho)} + \frac{(T(1-\rho) - (1-\rho^T)\rho)^2}{T^3(1-\rho^2)(1-\rho)^2}.$$

Rearranging the sum of the first two terms on the right hand side, and denoting $T(1-\rho) - (1-\rho^T)\rho$ as $S_{\rho T}$, we obtain

$$e_{\phi 2,T}(s,t) = \frac{(1-\rho^{Ts})(1-\rho^{Tt})}{T(1-\rho^2)} - \frac{(2 - \rho^{Ts} - \rho^{Tt})S_{\rho T}}{T^2(1-\rho^2)(1-\rho)} + \frac{S_{\rho T}^2}{T^3(1-\rho^2)(1-\rho)^2}. \tag{A18}$$

Now note that

$$S_{\rho T} = (1-\rho)^2 \sum_{j=0}^{T-1} (T-j)\rho^j \leq T^2(1-\rho)^2. \tag{A19}$$

To see that the first equality holds, consider the following arguments

$$\begin{aligned}\sum_{j=0}^{T-1}(T-j)\rho^j &= T\sum_{j=0}^{T-1}\rho^j - \sum_{j=0}^{T-1} j\rho^j = T\frac{1-\rho^T}{1-\rho} - \rho\frac{\mathrm{d}}{\mathrm{d}\rho}\sum_{j=0}^{T-1}\rho^j = T\frac{1-\rho^T}{1-\rho} - \rho\frac{\mathrm{d}}{\mathrm{d}\rho}\frac{1-\rho^T}{1-\rho} \\ &= T\frac{1-\rho^T}{1-\rho} - \rho\frac{-T\rho^{T-1}(1-\rho)+1-\rho^T}{(1-\rho)^2} = \frac{S_{\rho T}}{(1-\rho)^2}.\end{aligned}$$

For the last term on the right hand side of (A18), inequality (A19) yields

$$0 \le \frac{S_{\rho T}^2}{T^3(1-\rho^2)(1-\rho)^2} \le \frac{T(1-\rho)}{1+\rho}. \tag{A20}$$

For the negative of the second term, it yields

$$0 \le \frac{(2-\rho^{Ts}-\rho^{Tt})S_{\rho T}}{T^2(1-\rho^2)(1-\rho)} \le \frac{2(1-\rho^T)}{1+\rho} \le \frac{2T(1-\rho)}{1+\rho}. \tag{A21}$$

Finally, for the first term on the right hand side of (A18), we have

$$0 \le \frac{(1-\rho^{Ts})(1-\rho^{Tt})}{T(1-\rho^2)} \le \frac{(1-\rho^T)^2}{T(1-\rho^2)} \le \frac{T(1-\rho)}{1+\rho}. \tag{A22}$$

Using these bounds in (A18), and recalling that the first and the last terms enter the right hand side with positive sign while the second term enters the right hand side with negative sign, we obtain

$$|e_{\phi 2,T}(s,t)| \le \frac{2T(1-\rho)}{1+\rho}.$$

Since we are considering the case where $\rho \in (1-1/T^2, 1)$, this yields (A17). □

A6 Proof of lemma 4

First, let us prove the following lemma. Let $\mathbf{w}$ be the $T\times T$ orthogonal matrix with t-th column w_t, where for $t<T$, w_t is a vector with s-th coordinate $w_{ts} = -\sqrt{2/T}\cos\left(\left(s-1/2\right)\pi t/T\right)$, while $w_T = l/\sqrt{T}$. Here l is the T-dimensional vector of ones.

Lemma A1. *For any* $\phi \ge 0$,

$$\mathbf{w}' M U_\phi' U_\phi M \mathbf{w} = D_\phi - \Delta_\phi,$$

where D_ϕ *is a diagonal matrix with* t*-th diagonal element equal to* $\left|1-\exp\left\{\left(\mathrm{i}\pi t-\phi\right)/T\right\}\right|^{-2}$ *if* $t<T$ *and zero if* $t=T$*; and* Δ_ϕ *is a positive semi-definite matrix of rank two with* t,s*-th*

entry

$$\Delta_{\phi,ts} = \frac{2}{T}e^{-\phi/T}\left(1-e^{-\phi/T}\right)\frac{\cos\left(\pi t/2T\right)}{\left|e^{(-\phi+\mathrm{i}\pi t)/T}-1\right|^2}\frac{\cos\left(\pi s/2T\right)}{\left|e^{(-\phi+\mathrm{i}\pi s)/T}-1\right|^2}$$
$$\times\left(\frac{1+e^{-\phi/T}e^{-2\phi T^2}}{1+e^{-\phi/T}}\left(1-(-1)^t e^{-\phi}\right)\left(1-(-1)^s e^{-\phi}\right)+(-1)^{t+s}\frac{1-e^{-2\phi}}{1+e^{-\phi/T}}\right).$$

Proof: Let us partition U_ϕ into the upper $T^3\times T$ submatrix $U_\phi^{(1)}$ and the lower $T\times T$ matrix $U_\phi^{(2)}$. We have

$$U_\phi^{(1)} = \left(e^{-\phi T^2},\ldots,e^{-2\phi/T},e^{-\phi/T}\right)' v_1',$$

where v_1 is the T-dimensional vector with t-th coordinate $v_{1t}=e^{-\phi(t-1)/T}$. Obviously,

$$\mathbf{w}'MU_\phi^{(1)\prime}U_\phi^{(1)}M\mathbf{w} = 0 \text{ for } \phi=0. \tag{A23}$$

For $\phi>0$,

$$\mathbf{w}'MU_\phi^{(1)\prime}U_\phi^{(1)}M\mathbf{w} = \frac{1-e^{-2\phi T^2}}{e^{2\phi/T}-1}x_1x_1', \tag{A24}$$

where $x_1=\mathbf{w}'Mv_1$.

Next, note that

$$\left(U_\phi^{(2)}\right)^{-1} = \begin{pmatrix} 1 & -e^{-\phi/T} & & 0 \\ & 1 & \ddots & \\ & & \ddots & -e^{-\phi/T} \\ 0 & & & 1 \end{pmatrix}$$

and therefore,

$$\left(U_\phi^{(2)\prime}U_\phi^{(2)}\right)^{-1} = \begin{pmatrix} 1+e^{-2\phi/T} & -e^{-\phi/T} & 0 & \cdots & 0 \\ -e^{-\phi/T} & 1+e^{-2\phi/T} & -e^{-\phi/T} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \vdots & \ddots & 1+e^{-2\phi/T} & -e^{-\phi/T} \\ 0 & 0 & \cdots & -e^{-\phi/T} & 1 \end{pmatrix}.$$

It is straightforward to verify that w_t, $t=1,\ldots,T$ are eigenvectors of

$$\left(U_\phi^{(2)\prime}U_\phi^{(2)}\right)^{-1} - e^{-\phi/T}e_1e_1' - e^{-\phi/T}\left(1-e^{-\phi/T}\right)e_Te_T' - \mathbf{1}_{\phi=0}ll'/T$$

with corresponding eigenvalues equal to $d_{\phi t}^{-1} = \left|1-\exp\left\{\left(-\phi+\mathrm{i}\pi t\right)/T\right\}\right|^2$ for $t<T$ and

$d_{\phi T}^{-1} = |1 - \exp\{-\phi/T\}|^2 - \mathbf{1}_{\phi=0}$. Here e_t denotes the t-th column of the T-dimensional identity matrix, and $\mathbf{1}_{\phi=0}$ is the indicator of the event $\phi = 0$.

Let $\bar{D}_\phi = \operatorname{diag}\{d_{\phi 1}, \ldots, d_{\phi,T}\}$. Then

$$\left(\left(U_\phi^{(2)\prime} U_\phi^{(2)}\right)^{-1} - e^{-\phi/T} e_1 e_1' - e^{-\phi/T}\left(1 - e^{-\phi/T}\right) e_T e_T' - \mathbf{1}_{\phi=0} l l'/T\right)^{-1} = \mathbf{w}\bar{D}_\phi \mathbf{w}'.$$

Applying the Sherman-Morrison formula for the inverse of a low rank perturbation of an invertible matrix to the left hand side of the above equality yields, for $\phi > 0$,

$$\mathbf{w}\bar{D}_\phi \mathbf{w}' = U_\phi^{(2)\prime} U_\phi^{(2)} + \frac{e^{-\phi/T}}{1 - e^{-\phi/T}} v_1 v_1' + \frac{e^{-2\phi}}{\left(1 - e^{-2\phi/T}\right)\left(1 - e^{-2\phi}\right)} v_2 v_2', \tag{A25}$$

where v_1 is as defined above, and v_2 is the T-dimensional vector with t-th coordinate

$$v_{2t} = e^{\phi/2T} e^{\phi(t-1)/T} + e^{-\phi/2T} e^{-\phi(t-1)/T}.$$

Similarly, for $\phi = 0$, the Sherman-Morrison formula yields

$$\mathbf{w}\bar{D}_\phi \mathbf{w}' = U_\phi^{(2)\prime} U_\phi^{(2)} + lA + Bl', \tag{A26}$$

where A and B are some matrices, the exact form of which is of no consequence to what follows.

Mutiplying both sides of equation (A25) by $\mathbf{w}'M$ from left and by $M\mathbf{w}$ from right and rearranging, we obtain

$$\begin{aligned} \mathbf{w}'MU_\phi^{(2)\prime} U_\phi^{(2)} M\mathbf{w} &= \mathbf{w}'M\mathbf{w}\bar{D}_\phi \mathbf{w}'M\mathbf{w} - \frac{e^{-\phi/T}}{1 - e^{-\phi/T}} x_1 x_1' \\ &\quad - \frac{e^{-2\phi}}{\left(1 - e^{-2\phi/T}\right)\left(1 - e^{-2\phi}\right)} x_2 x_2', \end{aligned}$$

where $x_2 = \mathbf{w}'Mv_2$. Summing up with (A24) yields

$$\begin{aligned} \mathbf{w}'MU_\phi' U_\phi M\mathbf{w} &= \mathbf{w}'M\mathbf{w}\bar{D}_\phi \mathbf{w}'M\mathbf{w} - \frac{e^{-2\phi(T^2+1/T)} + e^{-\phi/T}}{1 - e^{-2\phi/T}} x_1 x_1' \\ &\quad - \frac{e^{-2\phi}}{\left(1 - e^{-2\phi/T}\right)\left(1 - e^{-2\phi}\right)} x_2 x_2' \end{aligned} \tag{A27}$$

for $\phi > 0$.

Note that $\mathbf{w}'M\mathbf{w} = I_T - e_T e_T'$, where e_T denotes the last column of the T-dimensional identity matrix, so that $\mathbf{w}'M\mathbf{w}\bar{D}_\phi \mathbf{w}'M\mathbf{w} = D_\phi$. Further, a direct calculation shows that the t-th coordinates of x_1 and x_2 equal

$$\begin{aligned} x_{1t} &= -\sqrt{\frac{2}{T}} \frac{\left(1-e^{-\phi/T}\right)\left(1-(-1)^t e^{-\phi}\right)\cos\left(\pi t/2T\right)}{\left|e^{(-\phi+\mathrm{i}\pi t)/T}-1\right|^2}, \\ x_{2t} &= -\sqrt{\frac{2}{T}} \frac{\left(1-e^{-\phi/T}\right)(-1)^t\left(e^{\phi}-e^{-\phi}\right)e^{-\phi/2T}\cos\left(\pi t/2T\right)}{\left|e^{(-\phi+\mathrm{i}\pi t)/T}-1\right|^2}. \end{aligned}$$

For $\phi > 0$, the lemma now follows from (A27) by verifying that

$$\frac{e^{-2\phi(T^2+1/T)}+e^{-\phi/T}}{1-e^{-2\phi/T}} x_1 x_1' + \frac{e^{-2\phi}}{\left(1-e^{-2\phi/T}\right)\left(1-e^{-2\phi}\right)} x_2 x_2' = \Delta_\phi.$$

For $\phi = 0$, multiplying both sides of equation (A26) by $\mathbf{w}'M$ from left and by $M\mathbf{w}$ from right and rearranging, we obtain $\mathbf{w}'MU_\phi^{(2)\prime}U_\phi^{(2)}M\mathbf{w} = \mathbf{w}'M\mathbf{w}\bar{D}_\phi\mathbf{w}'M\mathbf{w}$. Summing this up with (A23) yields

$$\mathbf{w}'MU_\phi'U_\phi M\mathbf{w} = \mathbf{w}'M\mathbf{w}\bar{D}_\phi\mathbf{w}'M\mathbf{w} = D_\phi.$$

This establishes the lemma for $\phi = 0$ because, as is easy to see, $\Delta_\phi = 0$ for $\phi = 0$. □

We now turn to the proof of lemma 4. By definition of $\mathbb{E}\tilde{\Sigma}$ and lemma A1,

$$\sum\nolimits_{j=J+1}^{T} \tilde{\mu}_j \leq \frac{1}{N}\sum\nolimits_{i=1}^{N}\sum\nolimits_{j=J+1}^{T} D_{\phi_i,jj}\Omega_{ii}.$$

On the other hand,

$$\begin{aligned} \sum\nolimits_{j=J+1}^{T} D_{\phi_i,jj} &= \sum\nolimits_{j=J+1}^{T-1} \left|1-\exp\left\{\left(-\phi_i+\mathrm{i}\pi j\right)/T\right\}\right|^{-2} \\ &\leq \sum\nolimits_{j=J+1}^{\infty} \frac{T^2}{\phi_i^2+\pi^2 j^2} + o\left(T^2\right) \\ &\leq \sum\nolimits_{j=J+1}^{\infty} \frac{T^2}{\pi^2 j^2} + o\left(T^2\right) \\ &\leq \frac{T^2}{\pi^2 J} + o\left(T^2\right) \leq \frac{T^2}{9J} \end{aligned}$$

for all sufficiently large T, uniformly over $\phi_i \geq 0$. Therefore,

$$\sum\nolimits_{j=J+1}^{T} \tilde{\mu}_j \leq \frac{T^2}{N}\sum\nolimits_{i=1}^{N}\frac{1}{9J}\Omega_{ii} = \frac{T^2}{9JN}\operatorname{tr}\Omega.$$

The lemma's second inequality is a straightforward consequence of the convergence $\tilde{\mu}_k/T^2 \to \mu_k > 0$ and the fact that, as implied by A4, $\operatorname{tr}\Omega/N$ is converging to a positive value as $N \to \infty$. □

A7 Proof of lemma 5

We have

$$\begin{aligned}\mathbb{E}\left(a'\varepsilon' A\varepsilon b\right) &= \sum\nolimits_{t,s=1}^{T}\sum\nolimits_{i,j=1}^{N}\mathbb{E}\left(a_t\varepsilon_{it}A_{ij}\varepsilon_{js}b_s\right)\\ &= \sum\nolimits_{t=1}^{T}\sum\nolimits_{i=1}^{N}a_tA_{ii}b_t = a'b\operatorname{tr}A.\end{aligned}$$

Further, denoting the i-th row of ε as $\varepsilon_{i\cdot}$, we have

$$\begin{aligned}&\mathbb{E}\left(a'\varepsilon' A\varepsilon bc'\varepsilon' B\varepsilon d\right)\\ =\;& \sum\nolimits_{i,j=1}^{N}\sum\nolimits_{p,l=1}^{N}\mathbb{E}\left(\varepsilon_{i\cdot}aA_{ij}\varepsilon_{j\cdot}b\varepsilon_{p\cdot}cB_{pl}\varepsilon_{l\cdot}d\right)\\ =\;& \sum\nolimits_{i=1}^{N}\sum\nolimits_{j\neq i}^{N}\mathbb{E}\left(\left(\varepsilon_{i\cdot}a\right)\left(\varepsilon_{i\cdot}b\right)\left(\varepsilon_{j\cdot}c\right)\left(\varepsilon_{j\cdot}d\right)A_{ii}B_{jj}\right)\\ &+\sum\nolimits_{i=1}^{N}\sum\nolimits_{j\neq i}^{N}\mathbb{E}\left(\left(\varepsilon_{i\cdot}a\right)\left(\varepsilon_{i\cdot}c\right)\left(\varepsilon_{j\cdot}b\right)\left(\varepsilon_{j\cdot}d\right)A_{ij}B_{ij}\right)\\ &+\sum\nolimits_{i=1}^{N}\sum\nolimits_{j\neq i}^{N}\mathbb{E}\left(\left(\varepsilon_{i\cdot}a\right)\left(\varepsilon_{i\cdot}d\right)\left(\varepsilon_{j\cdot}b\right)\left(\varepsilon_{j\cdot}c\right)A_{ij}B_{ji}\right)\\ &+\sum\nolimits_{i=1}^{N}\mathbb{E}\left(\left(\varepsilon_{i\cdot}a\right)\left(\varepsilon_{i\cdot}b\right)\left(\varepsilon_{i\cdot}c\right)\left(\varepsilon_{i\cdot}d\right)A_{ii}B_{ii}\right).\end{aligned}$$

We have, first,

$$\begin{aligned}&\sum\nolimits_{i=1}^{N}\sum\nolimits_{j\neq i}^{N}\mathbb{E}\left(\left(\varepsilon_{i\cdot}a\right)\left(\varepsilon_{i\cdot}b\right)\left(\varepsilon_{j\cdot}c\right)\left(\varepsilon_{j\cdot}d\right)A_{ii}B_{jj}\right)\\ =\;& \sum\nolimits_{i=1}^{N}\sum\nolimits_{j\neq i}^{N}\left(a'b\right)\left(c'd\right)A_{ii}B_{jj}\\ =\;& \left(a'b\right)\left(c'd\right)\left[\left(\operatorname{tr}A\right)\left(\operatorname{tr}B\right)-\sum\nolimits_{i=1}^{N}A_{ii}B_{ii}\right],\end{aligned}$$

second,

$$\begin{aligned}&\sum\nolimits_{i=1}^{N}\sum\nolimits_{j\neq i}^{N}\mathbb{E}\left(\left(\varepsilon_{i\cdot}a\right)\left(\varepsilon_{i\cdot}c\right)\left(\varepsilon_{j\cdot}b\right)\left(\varepsilon_{j\cdot}d\right)A_{ij}B_{ij}\right)\\ =\;& \sum\nolimits_{i=1}^{N}\sum\nolimits_{j\neq i}^{N}\left(a'c\right)\left(b'd\right)A_{ij}B_{ij}\\ =\;& \left(a'c\right)\left(b'd\right)\left[\operatorname{tr}\left(A'B\right)-\sum\nolimits_{i=1}^{N}A_{ii}B_{ii}\right],\end{aligned}$$

third,

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{j\neq i}^{N}\mathbb{E}\left(\left(\varepsilon_{i\cdot}a\right)\left(\varepsilon_{i\cdot}d\right)\left(\varepsilon_{j\cdot}b\right)\left(\varepsilon_{j\cdot}c\right)A_{ij}B_{ji}\right)\\
=\;&\sum_{i=1}^{N}\sum_{j\neq i}^{N}\left(a'd\right)\left(b'c\right)A_{ij}B_{ji}\\
=\;&\left(a'd\right)\left(b'c\right)\left[\operatorname{tr}\left(AB\right)-\sum_{i=1}^{N}A_{ii}B_{ii}\right],
\end{aligned}
$$

and finally,

$$
\begin{aligned}
&\sum_{i=1}^{N}\mathbb{E}\left(\left(\varepsilon_{i\cdot}a\right)\left(\varepsilon_{i\cdot}b\right)\left(\varepsilon_{i\cdot}c\right)\left(\varepsilon_{i\cdot}d\right)A_{ii}B_{ii}\right)\\
=\;&\sum_{i=1}^{N}\mathbb{E}\left(\sum_{t=1}^{T}\varepsilon_{it}a_t\sum_{t=1}^{T}\varepsilon_{it}b_t\sum_{t=1}^{T}\varepsilon_{it}c_t\sum_{t=1}^{T}\varepsilon_{it}d_tA_{ii}B_{ii}\right)\\
=\;&\sum_{i=1}^{N}A_{ii}B_{ii}\left(\sum_{t,s:t\neq s}^{T}a_tb_tc_sd_s+\sum_{t,s:t\neq s}^{T}a_tb_sc_td_s+\sum_{t,s:t\neq s}^{T}a_tb_sc_sd_t\right.\\
&\left.+\sum_{t=1}^{T}\mathbb{E}\varepsilon_{it}^4a_tb_tc_td_t\right)\\
=\;&\sum_{i=1}^{N}A_{ii}B_{ii}\left(\left(a'b\right)\left(c'd\right)+\left(a'c\right)\left(b'd\right)+\left(a'd\right)\left(b'c\right)+\sum_{t=1}^{T}\left(\mathbb{E}\varepsilon_{it}^4-3\right)a_tb_tc_td_t\right).
\end{aligned}
$$

Summing up,

$$
\begin{aligned}
&\mathbb{E}\left(a'\varepsilon'A\varepsilon bc'\varepsilon'B\varepsilon d\right)\\
=\;&\left(a'b\right)\left(c'd\right)\left(\operatorname{tr}A\right)\left(\operatorname{tr}B\right)+\left(a'c\right)\left(b'd\right)\operatorname{tr}\left(A'B\right)+\left(a'd\right)\left(b'c\right)\operatorname{tr}\left(AB\right)\\
&+\sum_{i=1}^{N}A_{ii}B_{ii}\sum_{t=1}^{T}\left(\mathbb{E}\varepsilon_{it}^4-3\right)a_tb_tc_td_t.
\end{aligned}
$$

Recall that $\mathbb{E}\left(a'\varepsilon'A\varepsilon b\right)=a'b\operatorname{tr}A$ and $\mathbb{E}\left(c'\varepsilon'B\varepsilon d\right)=c'd\operatorname{tr}B$. These equalities and the last display yield

$$
\begin{aligned}
Cov\left(a'\varepsilon'A\varepsilon b,c'\varepsilon'B\varepsilon d\right)\;=\;&\left(a'c\right)\left(b'd\right)\operatorname{tr}\left(A'B\right)+\left(a'd\right)\left(b'c\right)\operatorname{tr}\left(AB\right)\\
&+\sum_{i=1}^{N}A_{ii}B_{ii}\sum_{t=1}^{T}\left(\mathbb{E}\varepsilon_{it}^4-3\right)a_tb_tc_td_t.
\end{aligned}
$$

The inequality (29) follows because $\left|\mathbb{E}\varepsilon_{it}^4-3\right|$ is bounded by $2\varkappa_4$ uniformly over i and t. Indeed, by assumption A1, $\mathbb{E}\varepsilon_{it}^4\leq\varkappa_4$, and $\mathbb{E}\varepsilon_{it}^4-3\leq\varkappa_4$. On the other hand, $\mathbb{E}\varepsilon_{it}^4\geq\left(\mathbb{E}\varepsilon_{it}^2\right)^2=1$, and thus, $\varkappa_4\geq1$ and $\mathbb{E}\varepsilon_{it}^4-3\geq-2\geq-2\varkappa_4$. □

A8 Proof of equation (30)

Since $\|U_iM\|^2 \leq \mathrm{tr}\,(MU_i'U_iM)$ is obvious, it is sufficient to prove that

$$\sup_{\rho_i \in [0,1]} \mathrm{tr}\left(MU_i'U_iM\right) \leq 2T^2.$$

Let $U_i^{(1)}$ be the upper $T^3 \times T$ block of U_i and $U_i^{(2)}$ be the lower $T \times T$ block. Then,

$$\begin{aligned} \mathrm{tr}\left(MU_i'U_iM\right) &= \mathrm{tr}\left(MU_i^{(1)\prime}U_i^{(1)}M\right) + \mathrm{tr}\left(MU_i^{(2)\prime}U_i^{(2)}M\right) \\ &\leq \mathrm{tr}\left(MU_i^{(1)\prime}U_i^{(1)}M\right) + \mathrm{tr}\left(U_i^{(2)\prime}U_i^{(2)}\right) \\ &\leq \mathrm{tr}\left(MU_i^{(1)\prime}U_i^{(1)}M\right) + T^2, \end{aligned}$$

where the last inequality follows from the fact that $\mathrm{tr}\left(U_i^{(2)\prime}U_i^{(2)}\right)$ equals the sum of squared elements of the $T \times T$ matrix $U_i^{(2)}$ and all these elements are non-negative and no larger than 1. Hence, it is sufficient to prove that $\sup_{\rho_i \in [0,1]} \mathrm{tr}\left(MU_i^{(1)\prime}U_i^{(1)}M\right) \leq T^2$.

Note that

$$U_i^{(1)} = \begin{pmatrix} \rho_i^{T^3} & \rho_i^{T^3-1} & \cdots & \rho_i \end{pmatrix}' \begin{pmatrix} 1 & \rho_i & \cdots & \rho_i^{T-1} \end{pmatrix}.$$

Therefore, for $\rho_i = 1$, $U_i^{(1)}M = 0$ and $\mathrm{tr}\left(MU_i^{(1)\prime}U_i^{(1)}M\right) \leq T^2$ trivially holds. For $\rho_i < 1$, an elementary calculation yields

$$\begin{aligned} \mathrm{tr}\left(MU_i^{(1)\prime}U_i^{(1)}M\right) &= \rho_i^2 \frac{1-\rho_i^{2T^3}}{1-\rho_i^2}\left(\frac{1-\rho_i^{2T}}{1-\rho_i^2} - \frac{1}{T}\left(\frac{1-\rho_i^T}{1-\rho_i}\right)^2\right) \\ &\leq \frac{1}{1-\rho_i}\left(\frac{1-\rho_i^{2T}}{1-\rho_i^2} - \frac{1}{T}\left(\frac{1-\rho_i^T}{1-\rho_i}\right)^2\right) \\ &= \frac{1-\rho_i^T}{(1-\rho_i)^2}\left(\frac{1+\rho_i^T}{1+\rho_i} - \frac{1}{T}\frac{1-\rho_i^T}{1-\rho_i}\right) \\ &\leq \frac{T}{1-\rho_i}\left(\frac{1+\rho_i^T}{1+\rho_i} - \frac{1}{T}\frac{1-\rho_i^T}{1-\rho_i}\right). \end{aligned}$$

Since the term in the final bracket is no larger than unity, the obtained bound on $\mathrm{tr}\left(MU_i^{(1)\prime}U_i^{(1)}M\right)$ is no larger than T^2 for all non-negative $\rho_i \leq 1 - 1/T$. Hence, it is sufficient to show that

$$\sup_{\rho_i \in (1-1/T,1)} \frac{1}{1-\rho_i}\left(\frac{1+\rho_i^T}{1+\rho_i} - \frac{1}{T}\frac{1-\rho_i^T}{1-\rho_i}\right) \leq T.$$

Let us reparameterize the problem using $\rho_i = 1 - x/T$, where $x \in (0,1)$. It is sufficient to show that

$$\sup_{x\in(0,1)} \frac{1}{x}\left(\frac{1+(1-x/T)^T}{2-x/T} - \frac{1-(1-x/T)^T}{x}\right) \leq 1.$$

The Taylor expansion of $(1-x/T)^T$ at zero yields

$$(1-x/T)^T = 1 - x + \frac{T-1}{2T}\left(1-\frac{x^*}{T}\right)^{T-2} x^2,$$

where $x^* \in [0,x]$. Therefore, for all $T \geq 2$ and $x \in (0,1)$ we have

$$(1-x/T)^T = 1 - x + R_{x,T}x^2 \text{ with } |R_{x,T}| \leq 1/2.$$

This yields

$$\frac{1}{x}\left(\frac{1+(1-x/T)^T}{2-x/T} - \frac{1-(1-x/T)^T}{x}\right) = \frac{1}{x}\left(\frac{x/T - x + R_{x,T}x^2}{2-x/T} + R_{x,T}x\right).$$

But for $T \geq 2$ and $x \in (0,1)$, we have $x/T - x + R_{x,T}x^2 \leq 0$. Therefore, the right hand side of the displayed equality is no larger than $R_{x,T}$. Thus,

$$\sup_{x\in(0,1)} \frac{1}{x}\left(\frac{1+(1-x/T)^T}{2-x/T} - \frac{1-(1-x/T)^T}{x}\right) \leq 1/2 < 1.$$

This completes the proof of inequality (30) for all $T \geq 2$. A direct verification shows that the inequality also holds for $T = 1$.

As a bi-product, we established the fact that function

$$h(\rho_i) = \begin{cases} \frac{1}{T}\frac{1}{1-\rho_i^2}\left(\frac{1+\rho_i^T}{1+\rho_i} - \frac{1}{T}\frac{1-\rho_i^T}{1-\rho_i}\right) & \text{for } \rho_i \in [0,1) \\ 0 & \text{for } \rho_i = 1 \end{cases} \tag{A28}$$

is non-negative, continuous, uniformly in T bounded, and such that, for all T, $h(\rho_i) \leq 1$ and $h(1-x/T) \leq x/4$ for $x \in [0,1)$. These facts are used in Section 4.4 of the main body of the paper.

A9 Proof of equation (A7)

We have

$$\sum\nolimits_{j=0}^{T-2}(T-j)(T-j-1)\rho^j = \rho^{T+1}\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}\sum_{j=0}^{T-2}\rho^{j-T+1} = \rho^{T+1}\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}\frac{\rho^{1-T}-1}{1-\rho}$$
$$= \rho^{T+1}\frac{\mathrm{d}}{\mathrm{d}\rho}\frac{(1-T)\rho^{-T}+T\rho^{1-T}-1}{(1-\rho)^2} = \frac{T(T-1)(1-\rho)^2+2((1-T)\rho+T\rho^2-\rho^{T+1})}{(1-\rho)^3}$$
$$= \frac{2\rho(1-\rho^T)-T(1-\rho^2)+T^2(1-\rho)^2}{(1-\rho)^3}.$$

A10 Detailed analysis of the Markov chain example

Recall that in the Markov chain example, we defined X_{it} as $Z_i(t/T)$, where $Z_i(s)$ were independent across $i=1,\ldots,N$ continuous time Markov chains with transition probabilities (57), and such that $Z_i(0)$ equals 0 or 1 with equal probabilities. Let X be an $N\times T$ matrix with entries X_{it}. Our goal is to establish an analogue of theorem 1 for the eigenvectors and eigenvalues of $\hat\Sigma = MX'XM/N$.

To this end, consider the integral operator $\check K_{\mathcal F}$ (acting on $C[0,1]$) with kernel

$$\check k_{\mathcal F}(s,t) = \int\int \frac{\phi}{2}k_\phi(s,t)\mathcal F(\mathrm{d}\omega,\mathrm{d}\phi).$$

Note that the integration with respect to ω is trivial. This is because there are no different variance weights on cross-sectional series in our Markov chain example. The following are analogues of assumption A4 and theorem 1.

Assumption A4'. *$\mathcal F_N$ weakly converges to $\mathcal F$ as $N,T\to\infty$. The supports of $\mathcal F_N$ and $\mathcal F$ belong to $[0,\bar\omega]\times[0,\bar\phi]$ for some $0<\bar\omega,\bar\phi<\infty$. The eigenvalues $\check\mu_1>\check\mu_2>\ldots$ of $\check K_{\mathcal F}$ are simple.*

Theorem A2. *Let $N,T\to\infty$ at arbitrary relative rates. Then under A4', for any fixed positive integer k,*
(i) $\left|\hat F_k' d_k\right| \xrightarrow{\mathrm{P}} 1$, where $d_k = (\check\varphi_k(1/T),\ldots,\check\varphi_k(T/T))/\sqrt{T}$ and $\check\varphi_k(s)$ is the k-th principal eigenfunction of $\check K_{\mathcal F}$.
(ii) $\hat\lambda_k/T \xrightarrow{\mathrm{P}} \check\mu_k$, where $\check\mu_k$ is the k-th principal eigenvalue of $\check K_{\mathcal F}$.
(iii) $\hat\lambda_k/\operatorname{tr}\hat\Sigma \xrightarrow{\mathrm{P}} \check\mu_k/\sum_{j=1}^\infty\check\mu_j$.

Remark A3. *In part (ii), we divide $\hat{\lambda}_k$ by T as opposed to T^2 in theorem 1. It is because the orders of the variances of a process with local-to-unit root and our Markov chain differ by T.*

Proof: The proof is similar to that of theorem 1 so we will be brief. Because of the special setup of the Markov chain example, we do not have analogues of Steps 1 and 4 of the proof of theorem 1. Thus, we start from Step 2 and finish by Step 3. Consider

$$\mathbb{E}\hat{\Sigma} = \frac{1}{N}\sum_{i=1}^{N} MW_iM,$$

where W_i is a $T \times T$ matrix with elements $W_{i,jk} = e^{-\phi_i|k-j|/T}/4$. It is useful to note that $4W_i/(1 - e^{-2\phi_i/T}) = \lim_{k\to\infty} U'_{k\phi_i} U_{k\phi_i}$, where

$$U_{k\phi} = \begin{pmatrix} e^{-T^k\phi} & e^{-\left(T^k+1/T\right)\phi} & \dots & e^{-\left(T^k+1-1/T\right)\phi} \\ \vdots & \vdots & & \vdots \\ e^{-\phi/T} & e^{-2\phi/T} & \dots & e^{-\phi} \\ 1 & e^{-\phi/T} & \dots & e^{-\phi(T-1)/T} \\ 0 & 1 & \dots & e^{-\phi(T-2)/T} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Matrix U_ϕ defined just before lemma 2 is identical to $U_{2\phi}$. The necessary changes to the proof of theorem 1 can often be obtained by just formally replacing U_ϕ with $U_{\infty\phi}$.

In particular, the quantity $\frac{4(MW_iM)_{jk}}{T(1-e^{-2\phi_i/T})}$ is almost equal to $k_{\phi_i,T}(s_j, t_k)$ with $s_j = j/T$ and $t_k = k/T$ (see (16) for the definition), except the term of $k_{\phi_i,T}$ which has multiplier $e^{-2\phi_i T^2}$ disappears because the multiplier becomes (formally) $e^{-2\phi_i T^\infty} = 0$. More precisely,

$$\frac{4(MW_iM)_{jk}}{T(1 - e^{-2\phi_i/T})} = k_{\phi_i,T}(s_j, t_k) + e^{-2\phi_i T^2} e_{\phi_i 2,T}(s_j, t_k),$$

where $e_{\phi_i 2,T}$ is as in (A10).

Step 2 of the proof then proceeds with only minor changes. One of them is that we compare the eigenvalues of the limiting operator to those of $\mathbb{E}\hat{\Sigma}/T$ instead of $\mathbb{E}\hat{\Sigma}/T^2$. Another one is that the upper and lower bounds in lemma 4 become, respectively, $\frac{\bar{\phi}}{2}\frac{T}{9J}$ and $C_k T$.

Turning to Step 3, the main change is that $A^{(i)}$ in (23) is now defined simply as $MX'_{i\cdot}X_{i\cdot}M$. Note that, for $q < T$,

$$|X_{i\cdot}M\tilde{\varphi}_q| = |X_{i\cdot}\tilde{\varphi}_q| \le \sqrt{\|X_{i\cdot}\|^2\|\varphi_q\|^2} \le \sqrt{T},$$

where the first equality holds because $M\tilde{\varphi}_q = \tilde{\varphi}_q$, and the last inequality holds because $\|\varphi_q\| = 1$ and X_{it} can take only values 0 and 1. Since we have assumed independence across $i = 1, \ldots, N$, the above display yields (cf. (31))

$$Var\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{\varphi}_r' A^{(i)}\tilde{\varphi}_q\right) = \frac{1}{N^2}\sum_{i=1}^{N} Var\left(\tilde{\varphi}_r' A^{(i)}\tilde{\varphi}_q\right) \leq \frac{T^2}{N}.$$

By Chebyshev's inequality, we have (cf. (32))

$$\tilde{\lambda}_{11} = \sum_{r=1}^{K}\lambda_r^2\tilde{\mu}_r + o_{\mathrm{P}}(T).$$

Further, instead of (34) we have

$$Var\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{r=K+1}^{T-1}(X_{i\cdot}M\tilde{\varphi}_r)^2\right) \leq \frac{1}{N}Var(\|X_{i\cdot}\|^2) \leq \frac{T^2}{N},$$

so that (cf. (35) and (36)),

$$\tilde{\lambda}_{12} \leq \sum_{r=K+1}^{T-1}\tilde{\mu}_r + o_{\mathrm{P}}(T) \leq (1 + o_{\mathrm{P}}(1))\frac{\bar{\phi}}{2}\frac{T}{9K}.$$

Step 3 is then completed with only relatively minor changes. □