Online Supplement to
"Asymptotics for Time–Varying Vector MA(∞) Processes"

Yayi Yan[*,†], Jiti Gao[†] and Bin Peng[†]

[*]School of Statistics and Management, Shanghai University of Finance and Economics

[†]Department of Econometrics and Business Statistics, Monash University

In this appendix, we first discuss some impulse response analyses in Appendix B.1, and then present several preliminary lemmas in Appendix B.2, which are helpful to the development of the main results. The omitted proofs of the main results and the proofs of the preliminary lemmas are given in Appendices B.3 and B.4, respectively.

B.1 Impulse Responses for Time–Varying VARX Models

In this subsection, we focus on the marginal impact of changes in the exogenous variables, which is measured by the impulse response functions, and is of great interest in VAR analysis. We first note that after some trivial but tedious development, successive substitution for lagged $\boldsymbol{y}_t$'s of (4.1) gives

$$\boldsymbol{y}_t = \boldsymbol{B}_0(\tau_t)\boldsymbol{x}_t + \sum_{j=1}^{\infty} \bar{\boldsymbol{J}} \prod_{k=0}^{j-1} \boldsymbol{\Gamma}(\tau_{t-k})\boldsymbol{\Upsilon}(\tau_{t-j})\boldsymbol{x}_{t-j} + \boldsymbol{\eta}_t + \sum_{j=1}^{\infty} \bar{\boldsymbol{J}} \prod_{k=0}^{j-1} \boldsymbol{\Gamma}(\tau_{t-k})\bar{\boldsymbol{J}}^\top \boldsymbol{\eta}_{t-j},$$

where $\bar{\boldsymbol{J}} = [\boldsymbol{I}_d, \boldsymbol{0}_{d\times(d(p-1)+mq)}]$, $\boldsymbol{\Upsilon}(\tau) = (\boldsymbol{B}_0(\tau)^\top, \underbrace{\boldsymbol{0}_{d\times m}^\top, \ldots, \boldsymbol{0}_{d\times m}^\top}_{p-1 \text{ matrices}}, \boldsymbol{I}_m, \underbrace{\boldsymbol{0}_m, \ldots, \boldsymbol{0}_m}_{q-1 \text{ matrices}})^\top$

$$\boldsymbol{\Gamma}(\tau) = \begin{pmatrix} \boldsymbol{A}_1(\tau) & \cdots & \boldsymbol{A}_{p-1}(\tau) & \boldsymbol{A}_p(\tau) & \boldsymbol{B}_1(\tau) & \cdots & \boldsymbol{B}_{q-1}(\tau) & \boldsymbol{B}_q(\tau) \\ \boldsymbol{I}_d & \cdots & \boldsymbol{0}_d & \boldsymbol{0}_d & \boldsymbol{0}_{d\times m} & \cdots & \boldsymbol{0}_{d\times m} & \boldsymbol{0}_{d\times m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0}_d & \cdots & \boldsymbol{I}_d & \boldsymbol{0}_d & \boldsymbol{0}_{d\times m} & \cdots & \boldsymbol{0}_{d\times m} & \boldsymbol{0}_{d\times m} \\ & & & & \boldsymbol{0}_m & \cdots & \boldsymbol{0}_m & \boldsymbol{0}_m \\ & & \boldsymbol{0} & & \boldsymbol{I}_m & \cdots & \boldsymbol{0}_m & \boldsymbol{0}_m \\ & & & & \vdots & \ddots & \vdots & \vdots \\ & & & & \boldsymbol{0}_m & \cdots & \boldsymbol{I}_m & \boldsymbol{0}_m \end{pmatrix}.$$

Then it is easy to see that

$$\boldsymbol{D}_{j,t}^{\boldsymbol{x}} \equiv \bar{\boldsymbol{J}} \prod_{k=0}^{j-1} \boldsymbol{\Gamma}(\tau_{t-k})\boldsymbol{\Upsilon}(\tau_{t-j}) \quad \text{with} \quad j = 1, 2, \ldots \tag{B.1}$$

are the impulse response functions associated with the exogenous shocks. We can prove that $\boldsymbol{D}_{j,t}^{\boldsymbol{x}}$ can be approximated by $\boldsymbol{D}_j^{\boldsymbol{x}}(\tau)$ with negligible errors, where

$$\boldsymbol{D}_j^{\boldsymbol{x}}(\tau) = \bar{\boldsymbol{J}}\boldsymbol{\Gamma}^j(\tau)\boldsymbol{\Upsilon}(\tau). \tag{B.2}$$

In addition, we define the nonparametric estimator of (B.2) as

$$\widehat{\boldsymbol{D}}_j^{\boldsymbol{x}}(\tau) = \bar{\boldsymbol{J}}\widehat{\boldsymbol{\Gamma}}^j(\tau)\widehat{\boldsymbol{\Upsilon}}(\tau), \tag{B.3}$$

in which we replace $\boldsymbol{A}_j(\tau)$ and $\boldsymbol{B}_j(\tau)$ of $\boldsymbol{\Gamma}(\tau)$ and $\boldsymbol{\Upsilon}(\tau)$ of (B.2) with the corresponding nonparametric estimators obtained from (4.5).

The next theorem summarizes the asymptotic properties associated with (B.2) and (B.3).

Theorem B.1. *Suppose that Assumptions 4 and 5 hold. As* $T \to \infty$, $\sup_{t\geq 1}\left\|\boldsymbol{D}_{j,t}^{\boldsymbol{x}} - \boldsymbol{D}_j^{\boldsymbol{x}}(\tau_t)\right\| = O(T^{-1})$. *Moreover, for* $\forall \tau \in (0,1)$,

$$\sqrt{Th}\left(\mathrm{vec}[\widehat{\boldsymbol{D}}_j^{\boldsymbol{x}}(\tau) - \boldsymbol{D}_j^{\boldsymbol{x}}(\tau)] - \frac{1}{2}h^2\widetilde{c}_2\boldsymbol{D}_j^{\boldsymbol{x},(2)}(\tau)\right) \to_D N(\boldsymbol{0}, \boldsymbol{\Delta}_{\boldsymbol{D}_j^{\boldsymbol{x}}}(\tau)),$$

where $\boldsymbol{D}_j^{\boldsymbol{x},(2)}(\tau) = \boldsymbol{G}_j(\tau)\boldsymbol{\beta}^{(2)}(\tau)$, $\boldsymbol{\Delta}_{\boldsymbol{D}_j^{\boldsymbol{x}}}(\tau) = \boldsymbol{G}_j(\tau)\boldsymbol{V}(\tau)\boldsymbol{G}_j^\top(\tau)$, $\boldsymbol{G}_0(\tau) = (\boldsymbol{0}_{dm\times(d^2p+dmq)}, \boldsymbol{I}_{dm})$, *and for* $j \geq 1$

$$\boldsymbol{G}_j(\tau) = \left(\sum_{k=0}^{j-1}\boldsymbol{\Upsilon}(\tau)^\top(\boldsymbol{\Gamma}(\tau)^\top)^{j-1-k} \otimes \bar{\boldsymbol{J}}\boldsymbol{\Gamma}^k(\tau)\bar{\boldsymbol{J}}^\top,\ \boldsymbol{I}_m \otimes \bar{\boldsymbol{J}}\boldsymbol{\Gamma}^j(\tau)\bar{\boldsymbol{J}}^\top\right).$$

We now comment on how to construct the confidence interval. By Theorem 4.1, it is straightforward to have

$$\widehat{\boldsymbol{\Delta}}_{\boldsymbol{D}_j^{\boldsymbol{x}}}(\tau) \to_P \boldsymbol{\Delta}_{\boldsymbol{D}_j^{\boldsymbol{x}}}(\tau), \tag{B.4}$$

where $\widehat{\boldsymbol{\Delta}}_{\boldsymbol{D}_j^{\boldsymbol{x}}}(\tau)$ has a form identical to $\boldsymbol{\Delta}_{\boldsymbol{D}_j^{\boldsymbol{x}}}(\tau)$ but replacing $\boldsymbol{\Gamma}(\tau)$, $\boldsymbol{\Upsilon}(\tau)$ and $\boldsymbol{V}(\tau)$ with their estimators, respectively.

B.2 Preliminary Lemmas

Lemma B.1. *Suppose* $\{Z_t, \mathcal{F}_t\}$ *is a martingale difference sequence,* $S_T = \sum_{t=1}^T Z_t$, $U_T = \sum_{t=1}^T Z_t^2$ *and* $s_T^2 = E(U_T^2) = E(S_T^2)$. *If* $s_T^{-2}U_T^2 \to_P 1$ *and* $\sum_{t=1}^T E[Z_{T,t}^2 I(|Z_{T,t}| > \nu)] \to 0$ *for any* $\nu > 0$ *with* $Z_{T,t} = s_T^{-1}Z_t$, *then as* $T \to \infty$, $s_T^{-1}S_T \to_D N(0,1)$.

Lemma B.1 is Corollary 3.1 of Hall and Heyde (1980).

Lemma B.2. *Let* $\{Z_t, \mathcal{F}_t\}$ *be a martingale difference sequence. Suppose that* $|Z_t| \leq M$ *for a constant*

M, $t = 1, \dots, T$. *Let* $V_T = \sum_{t=1}^{T} \operatorname{Var}(Z_t|\mathcal{F}_{t-1}) \le V$ *for some* $V > 0$. *Then for any given* $\nu > 0$,

$$\Pr\left(\left|\sum_{t=1}^{T} Z_t\right| > \nu\right) \le \exp\left\{-\frac{\nu^2}{2(V + M\nu)}\right\}.$$

Lemma B.2 is Proposition 2.1 of Freedman (1975).

Lemma B.3. *The following algebraic decompositions hold true.*

1. $\mathbb{B}_t(L) = \sum_{j=0}^{\infty} \boldsymbol{B}_{j,t} L^j$ *can be decomposed as* $\mathbb{B}_t(L) = \mathbb{B}_t(1) - (1 - L)\widetilde{\mathbb{B}}_t(L)$, *where* $\widetilde{\mathbb{B}}_t(L) = \sum_{j=0}^{\infty} \widetilde{\boldsymbol{B}}_{j,t} L^j$ *and* $\widetilde{\boldsymbol{B}}_{j,t} = \sum_{k=j+1}^{\infty} \boldsymbol{B}_{k,t}$.

2. $\mathbb{B}_t^r(L) = \sum_{j=0}^{\infty} (\boldsymbol{B}_{j+r,t} \otimes \boldsymbol{B}_{j,t}) L^j$ *can be decomposed as* $\mathbb{B}_t^r(L) = \mathbb{B}_t^r(1) - (1 - L)\widetilde{\mathbb{B}}_t^r(L)$, *where* $\widetilde{\mathbb{B}}_t^r(L) = \sum_{j=0}^{\infty} \widetilde{\boldsymbol{B}}_{j,t}^r L^j$ *and* $\widetilde{\boldsymbol{B}}_{j,t}^r = \sum_{k=j+1}^{\infty} (\boldsymbol{B}_{k+r,t} \otimes \boldsymbol{B}_{k,t})$.

In addition, let Assumption 1 hold, then

3. $\sup_{t\ge1} \sum_{j=0}^{\infty} \|\widetilde{\boldsymbol{B}}_{j,t}\| < \infty$;

4. $\lim_{T\to\infty} \sum_{t=1}^{T-1} \|\widetilde{\mathbb{B}}_{t+1}(1) - \widetilde{\mathbb{B}}_t(1)\| < \infty$;

5. $\sup_{t\ge1} \sum_{j=0}^{\infty} \|\widetilde{\boldsymbol{B}}_{j,t}^r\| < \infty$;

6. $\sup_{t\ge1} \sum_{r=1}^{\infty} \|\widetilde{\mathbb{B}}_t^r(1)\| < \infty$;

7. $\lim_{T\to\infty} \sum_{t=1}^{T-1} \sum_{r=0}^{\infty} \|\widetilde{\mathbb{B}}_{t+1}^r(1) - \widetilde{\mathbb{B}}_t^r(1)\| < \infty$.

Lemma B.4. *Let Assumptions 1 and 2 hold, and let* $\{\boldsymbol{W}_{T,t}(\cdot)\}_{t=1}^{T}$ *be a sequence of* $m \times d$ *matrices of functions, where* $m \ge 1$ *is fixed, and each functional component is Lipschitz continuous and defined on a compact set* $[a, b]$. *Moreover, suppose that (1)* $\sup_{\tau\in[a,b]} \sum_{t=1}^{T} \|\boldsymbol{W}_{T,t}(\tau)\| = O(1)$, *and (2)* $T^{\frac{2}{\delta}} d_T \log T \to 0$, *where* $d_T = \sup_{\tau\in[a,b],t\ge1} \|\boldsymbol{W}_{T,t}(\tau)\|$. *As* $T \to \infty$, $\sup_{\tau\in[a,b]} \left\|\sum_{t=1}^{T} \boldsymbol{W}_{T,t}(\tau)\mathbb{B}_t(1)\boldsymbol{\epsilon}_t\right\| = O_P\left(\sqrt{d_T \log T}\right)$.

Lemma B.5. *Let the conditions of Lemma B.4 hold. Suppose* $T^{\frac{4}{\delta}} d_T \log T \to 0$, $\sup_{t\ge1} E[\|\boldsymbol{\epsilon}_t\|^4|\mathcal{F}_{t-1}] < \infty$ *a.s. and* $\sup_{\tau\in[a,b]} \sum_{t=1}^{T-1} \|\boldsymbol{W}_{T,t+1}(\tau) - \boldsymbol{W}_{T,t}(\tau)\| = O(d_T)$. *As* $T \to \infty$

1. $\sup_{\tau\in[a,b]} \left\|\sum_{t=1}^{T} (\boldsymbol{I}_d \otimes \boldsymbol{W}_{T,t}(\tau))\, \mathbb{B}_t^0(1) \left(\operatorname{vec}[\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t^\top] - \operatorname{vec}[\boldsymbol{I}_d]\right)\right\| = O_P\left(\sqrt{d_T \log T}\right)$;

2. $\sup_{\tau\in[a,b]} \left\|\sum_{t=1}^{T} (\boldsymbol{I}_d \otimes \boldsymbol{W}_{T,t}(\tau))\, \boldsymbol{\zeta}_t\boldsymbol{\epsilon}_t\right\| = O_P\left(\sqrt{d_T \log T}\right)$;

where $\boldsymbol{\zeta}_t = \sum_{r=1}^{\infty} \sum_{s=0}^{\infty} \{\boldsymbol{B}_{s+r,t}\boldsymbol{\epsilon}_{t-r}\} \otimes \boldsymbol{B}_{s,t}$.

Lemma B.6. *Under Assumption 5, there exists a time–varying VMA(∞) process*

$$\widetilde{\boldsymbol{y}}_t = \boldsymbol{\mu}^*(\tau_t) + \sum_{j=0}^{\infty} \boldsymbol{D}_j^{\epsilon}(\tau_t)\boldsymbol{\epsilon}_{t-j} + \sum_{j=0}^{\infty} \boldsymbol{D}_j^{v}(\tau_t)\boldsymbol{v}_{t-j}$$

such that $\max_{t\geq 1}\{E\left\|\boldsymbol{y}_t-\widetilde{\boldsymbol{y}}_t\right\|^{\delta}\}^{1/\delta}=O(T^{-1})$, *where*

$$\boldsymbol{\mu}^*(\tau)=\sum_{j=0}^{\infty}\boldsymbol{\Psi}_j(\tau)\left(\boldsymbol{\mu}(\tau)+\sum_{l=0}^{q}\boldsymbol{B}_l(\tau)\boldsymbol{g}(\tau)\right),\quad \boldsymbol{\Psi}_j(\tau)=\boldsymbol{J}\boldsymbol{\Phi}^j(\tau)\boldsymbol{J}^\top,$$

$$\boldsymbol{\Phi}(\tau)=\begin{pmatrix}\boldsymbol{A}_1(\tau) & \cdots & \boldsymbol{A}_{p-1}(\tau) & \boldsymbol{A}_p(\tau)\\ \boldsymbol{I}_d & \cdots & \boldsymbol{0}_d & \boldsymbol{0}_d\\ \vdots & \ddots & \vdots & \vdots\\ \boldsymbol{0}_d & \cdots & \boldsymbol{I}_d & \boldsymbol{0}_d\end{pmatrix},\quad \boldsymbol{J}=\left[\boldsymbol{I}_d,\boldsymbol{0}_{d\times d(p-1)}\right],$$

$$\boldsymbol{D}_j^{\boldsymbol{\epsilon}}(\tau)=\boldsymbol{\Psi}_j(\tau)\boldsymbol{\omega}(\tau),\quad \boldsymbol{D}_j^{\boldsymbol{v}}(\tau)=\sum_{b=\max(0,j-q)}^{j}\boldsymbol{D}_{b,j-b}^{\boldsymbol{v}}(\tau),$$

$$\boldsymbol{D}_{j,l}^{\boldsymbol{v}}(\tau)=\sum_{k=0}^{j}\boldsymbol{\Psi}_k(\tau)\boldsymbol{B}_l(\tau)\boldsymbol{C}_{j-k}(\tau).$$

Moreover, $\widetilde{\boldsymbol{y}}_t$ *and* $\boldsymbol{x}_t$ *admit the following expression*

$$\begin{pmatrix}\widetilde{\boldsymbol{y}}_t\\ \boldsymbol{x}_t\end{pmatrix}=\begin{pmatrix}\boldsymbol{\mu}^*(\tau_t)\\ \boldsymbol{g}(\tau_t)\end{pmatrix}+\sum_{j=0}^{\infty}\boldsymbol{D}_j(\tau_t)\begin{pmatrix}\boldsymbol{\epsilon}_{t-j}\\ \boldsymbol{v}_{t+1-j}\end{pmatrix},$$

where $\boldsymbol{D}_j(\tau)=\begin{pmatrix}\boldsymbol{D}_j^{\boldsymbol{\epsilon}}(\tau) & \boldsymbol{D}_{j-1}^{\boldsymbol{v}}(\tau)\\ \boldsymbol{0} & \boldsymbol{C}_{j-1}(\tau)\end{pmatrix}$, *and* $\boldsymbol{D}_j^{\boldsymbol{v}}(\tau)=0$ *and* $\boldsymbol{C}_j(\tau)=0$ *for* $j<0$. *Here, all* $\boldsymbol{D}_j(\cdot)$ *satisfy the same conditions as those in Assumption 3.*

Lemma B.7. *Let Assumptions 4–5 hold. Suppose further* $\sup_{t\geq 1}E\left[\|\boldsymbol{e}_t\|^4|\mathcal{F}_{t-1}\right]<\infty$ *a.s. and* $\frac{T^{1-4/\delta}h}{\log T}\to\infty$ *as* $T\to\infty$,

1. $\sup_{\tau\in[h,1-h]}\left\|\frac{1}{T}\boldsymbol{Z}_{2,\tau}^\top\boldsymbol{K}_\tau\boldsymbol{Z}_{2,\tau}-\boldsymbol{\Sigma}_{\boldsymbol{Z}_2}(\tau)\otimes\boldsymbol{\Lambda}_1\right\|=O_P\left(h^2+\sqrt{\frac{\log T}{Th}}\right)$, *where* $\boldsymbol{\Lambda}_1=\mathrm{diag}(\tilde{c}_0,\tilde{c}_2)$ *and* $\boldsymbol{\Sigma}_{\boldsymbol{Z}_2}(\tau)=\boldsymbol{\Sigma}_{\boldsymbol{z}_2}(\tau)\otimes\boldsymbol{I}_d$;
2. $\sup_{\tau\in[h,1-h]}\left\|\frac{1}{T}\boldsymbol{Z}_{2,\tau}^\top\boldsymbol{K}_\tau\boldsymbol{Z}_1-\boldsymbol{\Sigma}_{\boldsymbol{Z}_{1,2}}^\top(\tau)\otimes\boldsymbol{\Lambda}_2\right\|=O_P\left(h^2+\sqrt{\frac{\log T}{Th}}\right)$, *where* $\boldsymbol{\Lambda}_2=[\tilde{c}_0,0]^\top$ *and* $\boldsymbol{\Sigma}_{\boldsymbol{Z}_{1,2}}(\tau)=\boldsymbol{\Sigma}_{\boldsymbol{z}_{1,2}}(\tau)\otimes\boldsymbol{I}_d$;
3. $\sup_{\tau\in[0,1]}\left\|\frac{1}{T}\boldsymbol{Z}_\tau\boldsymbol{K}_\tau\boldsymbol{\eta}\right\|=O_P\left(\sqrt{\frac{\log T}{Th}}\right)$, *where* $\boldsymbol{\eta}=(\boldsymbol{\eta}_1,\ldots,\boldsymbol{\eta}_T)^\top$;
4. $\sup_{\tau\in[0,1]}\|\widehat{\boldsymbol{\beta}}(\tau)-\boldsymbol{\beta}(\tau)\|=O_P(h^2+\sqrt{\log T/(Th)})$; *and*
5. $\sup_{\tau\in[0,1]}\left\|\widehat{\boldsymbol{\Omega}}(\tau)-\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{\Omega}(\tau_t)K_h(\tau_t-\tau)\right\|=O_P\left(h^2+\sqrt{\frac{\log T}{Th}}\right)$.

Lemma B.8. *Suppose Assumptions 4–6 hold. As* $T\to\infty$,

1. $T^{-1}\sum_{t=1}^{T}\widetilde{\boldsymbol{Z}}_{1t}\widehat{\boldsymbol{\Omega}}^{-1}(\tau_t)\widetilde{\boldsymbol{Z}}_{1t}^\top\to_P\int_0^1(\boldsymbol{\Sigma}_{\boldsymbol{z}_1}(\tau)-\boldsymbol{\Sigma}_{\boldsymbol{z}_{1,2}}(\tau)\boldsymbol{\Sigma}_{\boldsymbol{z}_2}^{-1}(\tau)\boldsymbol{\Sigma}_{\boldsymbol{z}_{1,2}}^\top(\tau))\otimes\boldsymbol{\Omega}^{-1}(\tau)\mathrm{d}\tau$;
2. $\sum_{t=1}^{T}\widetilde{\boldsymbol{Z}}_{1t}\widehat{\boldsymbol{\Omega}}^{-1}(\tau_t)\left(\boldsymbol{Z}_{2t}^\top\boldsymbol{\theta}(\tau_t)-\boldsymbol{Z}_{2t}^\top\boldsymbol{s}(\tau_t)\left[\boldsymbol{\theta}(\tau_1)^\top\boldsymbol{Z}_{21},...,\boldsymbol{\theta}(\tau_T)^\top\boldsymbol{Z}_{2T}\right]^\top\right)=o_P(\sqrt{T})$;

3. $\sum_{t=1}^T \widetilde{\boldsymbol{Z}}_{1t}\widehat{\boldsymbol{\Omega}}^{-1}(\tau_t)\boldsymbol{Z}_{2t}^\top \boldsymbol{s}(\tau_t)\boldsymbol{\eta} = o_P(\sqrt{T})$; and

4. $\sum_{t=1}^T \widetilde{\boldsymbol{Z}}_{1t}\widehat{\boldsymbol{\Omega}}^{-1}(\tau_t)\boldsymbol{\eta}_t = \sum_{t=1}^T \left(\boldsymbol{Z}_{1t} - \boldsymbol{\Sigma}_{\boldsymbol{Z}_{1,2}}(\tau_t)\boldsymbol{\Sigma}_{\boldsymbol{Z}_2}^{-1}(\tau_t)\boldsymbol{Z}_{2t}\right)\boldsymbol{\Omega}^{-1}(\tau_t)\boldsymbol{\eta}_t + o_P(\sqrt{T})$.

B.3 Omitted Proofs of the Main Results

Proof of Proposition 2.1.

(1). Let ρ denote the largest eigenvalue of $\boldsymbol{\Phi}_t$ uniformly over t. Then, $\rho < 1$ by the condition in Proposition 2.1. Similar to the proof of Proposition 2.4 in Dahlhaus and Polonik (2009), we have $\max_{t\geq 1}\|\prod_{i=0}^{j-1}\boldsymbol{\Phi}_{t-i}\| \leq M\rho^j$, which yields that

$$\max_{t\geq 1}\sum_{j=1}^\infty j\,\|\boldsymbol{B}_{j,t}\| = \max_{t\geq 1}\sum_{j=1}^\infty j\left\|\boldsymbol{J}\prod_{i=0}^{j-1}\boldsymbol{\Phi}_{t-i}\boldsymbol{J}^\top\right\| \leq M\sum_{j=1}^\infty j\rho^j = O(1).$$

In addition, for any conformable matrices $\{\boldsymbol{A}_i\}$ and $\{\boldsymbol{B}_i\}$, since

$$\prod_{i=1}^r \boldsymbol{A}_i - \prod_{i=1}^r \boldsymbol{B}_i = \sum_{j=1}^r\left(\prod_{k=1}^{j-1}\boldsymbol{A}_k\right)(\boldsymbol{A}_j - \boldsymbol{B}_j)\left(\prod_{k=j+1}^r \boldsymbol{B}_k\right),$$

we then obtain that

$$\begin{aligned}
&\lim_{T\to\infty}\sum_{t=1}^{T-1}\sum_{j=1}^\infty j\left\|\boldsymbol{J}\left(\prod_{i=0}^{j-1}\boldsymbol{\Phi}_{t+1-i} - \prod_{i=0}^{j-1}\boldsymbol{\Phi}_{t-i}\right)\boldsymbol{J}^\top\right\| \\
&= \lim_{T\to\infty}\sum_{t=1}^{T-1}\sum_{j=1}^\infty j\left\|\boldsymbol{J}\sum_{m=1}^j\left(\prod_{k=1}^{m-1}\boldsymbol{\Phi}_{t+2-k}\right)(\boldsymbol{\Phi}_{t+2-m} - \boldsymbol{\Phi}_{t+1-m})\left(\prod_{k=m}^j \boldsymbol{\Phi}_{t+1-k}\right)\boldsymbol{J}^\top\right\| \\
&\leq M\sum_{j=1}^\infty j^2\rho^{j-1}\lim_{T\to\infty}\sum_{t=1}^{T-1}\|\boldsymbol{\Phi}_{t+1} - \boldsymbol{\Phi}_t\| = O(1)
\end{aligned}$$

given the condition in Proposition 2.1. Based on the above two results, we have

$$\max_{t\geq 1}\sum_{b=1}^\infty b\,\|\boldsymbol{D}_{b,t}\| \leq M\max_{t\geq 1}\sum_{b=1}^\infty b\sum_{j=b-q}^b \|\boldsymbol{B}_{j,t}\| \leq M\sum_{b=1}^\infty b\rho^b = O(1).$$

We also have

$$\begin{aligned}
&\lim_{T\to\infty}\sum_{t=1}^{T-1}\sum_{b=1}^\infty b\,\|\boldsymbol{D}_{b,t+1} - \boldsymbol{D}_{b,t}\| \leq \lim_{T\to\infty}\sum_{t=1}^{T-1}\sum_{b=1}^\infty b\sum_{j=\max(0,b-q)}^b \|\boldsymbol{B}_{j,t+1} - \boldsymbol{B}_{j,t}\|\,\|\boldsymbol{\Theta}_{b-j,t+1-j}\| \\
&+ \lim_{T\to\infty}\sum_{t=1}^{T-1}\sum_{b=1}^\infty b\sum_{j=\max(0,b-q)}^b \|\boldsymbol{B}_{j,t}\|\,\|\boldsymbol{\Theta}_{b-j,t+1-j} - \boldsymbol{\Theta}_{b-j,t-j}\| \\
&\leq \max_{m,t}\|\boldsymbol{\Theta}_{m,t}\|\cdot q\cdot\lim_{T\to\infty}\sum_{t=1}^{T-1}\sum_{b=1}^\infty b\,\|\boldsymbol{B}_{b,t+1} - \boldsymbol{B}_{b,t}\| \\
&+ \left(\max_{t\geq 1}\sum_{b=1}^\infty b\sum_{j=\max(0,b-q)}^b \|\boldsymbol{B}_{j,t}\|\right)\cdot\left(\max_m\lim_{T\to\infty}\sum_{t=1}^{T-1}\|\boldsymbol{\Theta}_{m,t+1} - \boldsymbol{\Theta}_{m,t}\|\right) = O(1).
\end{aligned}$$

(2). By part (1) and the condition of Proposition 2.1, it suffices to show that $\max_{t\geq 1}\sum_{j=1}^{\infty} j\left\|\boldsymbol{D}_{j,t}\right\| < \infty$ and $\lim_{T\to\infty}\sum_{t=1}^{T-1}\sum_{j=1}^{\infty} j\left\|\boldsymbol{D}_{j,t+1}-\boldsymbol{D}_{j,t}\right\| < \infty$. Write

$$\begin{aligned}
&\max_{t\geq 1}\sum_{j=1}^{\infty} j\left\|\boldsymbol{D}_{j,t}\right\| \leq M\max_{t\geq 1}\sum_{j=1}^{\infty} j\sum_{k=0}^{j}\left\|\boldsymbol{B}_{k,t}\right\|\left\|\boldsymbol{C}_{j-k,t-k}\right\| \\
&= M\max_{t\geq 1}\sum_{k=0}^{\infty}\sum_{j=k}^{\infty} j\left\|\boldsymbol{B}_{k,t}\right\|\left\|\boldsymbol{C}_{j-k,t-k}\right\| = M\max_{t\geq 1}\sum_{k=0}^{\infty}\left\|\boldsymbol{B}_{k,t}\right\|\sum_{j=0}^{\infty}(k+j)\left\|\boldsymbol{C}_{j,t-k}\right\| \\
&= M\max_{t\geq 1}\sum_{j=0}^{\infty} j\left\|\boldsymbol{B}_{j,t}\right\|\sum_{k=1}^{\infty}\left\|\boldsymbol{C}_{k,t-j}\right\| + M\max_{t\geq 1}\sum_{k=0}^{\infty}\left\|\boldsymbol{B}_{k,t}\right\|\sum_{j=0}^{\infty} j\left\|\boldsymbol{C}_{j,t-k}\right\| = O(1).
\end{aligned}$$

In addition,

$$\begin{aligned}
&\lim_{T\to\infty}\sum_{t=1}^{T-1}\sum_{j=1}^{\infty} j\left\|\boldsymbol{D}_{j,t+1}-\boldsymbol{D}_{j,t}\right\| \leq \lim_{T\to\infty}\sum_{t=1}^{T-1}\sum_{j=1}^{\infty} j\sum_{k=0}^{j}\left\|\boldsymbol{B}_{k,t}\right\|\cdot\left\|\boldsymbol{\Theta}_{t-k}\right\|\cdot\left\|\boldsymbol{C}_{j-k,t+1-k}-\boldsymbol{C}_{j-k,t-k}\right\| \\
&+\lim_{T\to\infty}\sum_{t=1}^{T-1}\sum_{j=1}^{\infty} j\sum_{k=0}^{j}\left\|\boldsymbol{B}_{k,t}\right\|\cdot\left\|\boldsymbol{\Theta}_{t+1-k}-\boldsymbol{\Theta}_{t-k}\right\|\cdot\left\|\boldsymbol{C}_{j-k,t+1-k}\right\| \\
&+\lim_{T\to\infty}\sum_{t=1}^{T-1}\sum_{j=1}^{\infty} j\sum_{k=0}^{j}\left\|\boldsymbol{B}_{k,t+1}-\boldsymbol{B}_{k,t}\right\|\cdot\left\|\boldsymbol{\Theta}_{t+1-k}\right\|\cdot\left\|\boldsymbol{C}_{j-k,t+1-k}\right\| := M_{T,1}+M_{T,2}+M_{T,3}.
\end{aligned}$$

We need only to show that $M_{T,1}$ is bounded below, as the proofs of $M_{T,2}$ and $M_{T,3}$ can be established similarly. Observe that

$$\begin{aligned}
M_{T,1} &\leq \max_{t\geq 1}\left\|\boldsymbol{\Theta}_t\right\|\cdot\lim_{T\to\infty}\sum_{t=1}^{T-1}\sum_{j=1}^{\infty} j\sum_{k=0}^{j}\left\|\boldsymbol{B}_{k,t}\right\|\cdot\left\|\boldsymbol{C}_{j-k,t+1-k}-\boldsymbol{C}_{j-k,t-k}\right\| \\
&= \max_{t\geq 1}\left\|\boldsymbol{\Theta}_t\right\|\cdot\lim_{T\to\infty}\sum_{t=1}^{T-1}\sum_{k=0}^{\infty}\sum_{j=k}^{\infty} j\left\|\boldsymbol{B}_{k,t}\right\|\cdot\left\|\boldsymbol{C}_{j-k,t+1-k}-\boldsymbol{C}_{j-k,t-k}\right\| \\
&= \max_{t\geq 1}\left\|\boldsymbol{\Theta}_t\right\|\cdot\lim_{T\to\infty}\sum_{t=1}^{T-1}\sum_{k=0}^{\infty}\left\|\boldsymbol{B}_{k,t}\right\|\sum_{j=0}^{\infty}(j+k)\left\|\boldsymbol{C}_{j,t+1-k}-\boldsymbol{C}_{j,t-k}\right\| \\
&\leq \max_{t\geq 1}\left\|\boldsymbol{\Theta}_t\right\|\cdot\max_{t\geq 1}\sum_{k=0}^{\infty}\left\|\boldsymbol{B}_{k,t}\right\|\cdot\lim_{T\to\infty}\sum_{t=1}^{T-1}\sum_{j=0}^{\infty} j\left\|\boldsymbol{C}_{j,t+1-k}-\boldsymbol{C}_{j,t-k}\right\| \\
&\quad +\max_{t\geq 1}\left\|\boldsymbol{\Theta}_t\right\|\cdot\max_{t\geq 1}\sum_{k=0}^{\infty} k\left\|\boldsymbol{B}_{k,t}\right\|\cdot\lim_{T\to\infty}\sum_{t=1}^{T-1}\sum_{j=0}^{\infty}\left\|\boldsymbol{C}_{j,t+1-k}-\boldsymbol{C}_{j,t-k}\right\| = O(1).
\end{aligned}$$

The proof is now completed. □

Proof of Theorem 3.1.

Since $\frac{1}{T}\sum_{t=1}^{T} K_h(\tau_t-\tau) = 1+O\left(\frac{1}{Th}\right)$, we have

$$\widehat{\boldsymbol{\mu}}(\tau)-E\left(\widehat{\boldsymbol{\mu}}(\tau)\right) = \frac{1}{T}\sum_{t=1}^{T}(\boldsymbol{x}_t-\boldsymbol{\mu}(\tau_t))K_h\left(\tau_t-\tau\right)+O_P\left(\frac{1}{Th}\right),$$

which follows that $\sqrt{Th}\left(\widehat{\boldsymbol{\mu}}(\tau)-E\left(\widehat{\boldsymbol{\mu}}(\tau)\right)\right)=\frac{1}{\sqrt{Th}}\sum_{t=1}^{T}(\boldsymbol{x}_t-\boldsymbol{\mu}(\tau_t))K\left(\frac{\tau_t-\tau}{h}\right)+o_P(1)$.

As $\frac{1}{T}\sum_{t=1}^{T}K_h(\tau_t-\tau)=O(1)$, $\max_t\frac{1}{T}K_h(\tau_t-\tau)=O(1/(Th))$ and $\frac{1}{T}\sum_{t=1}^{T-1}(K_h(\tau_{t+1}-\tau)-K_h(\tau_t-\tau))=O(1/(Th))$, by Lemma 2.4, we have

$$
\begin{aligned}
&\frac{1}{\sqrt{Th}}\sum_{t=1}^{T}(\boldsymbol{x}_t-\boldsymbol{\mu}(\tau_t))K\left(\frac{\tau_t-\tau}{h}\right)\\
&=\frac{1}{\sqrt{Th}}\sum_{t=1}^{T}\mathbb{B}_t(1)\boldsymbol{\epsilon}_tK\left(\frac{\tau_t-\tau}{h}\right)+\frac{1}{\sqrt{Th}}\widetilde{\mathbb{B}}_1(L)\boldsymbol{\epsilon}_0K\left(\frac{\tau_1-\tau}{h}\right)-\frac{1}{\sqrt{Th}}\widetilde{\mathbb{B}}_T(L)\boldsymbol{\epsilon}_TK\left(\frac{\tau_T-\tau}{h}\right)\\
&\quad+\frac{1}{\sqrt{Th}}\sum_{t=1}^{T-1}\left(\widetilde{\mathbb{B}}_{t+1}(L)K\left(\frac{\tau_{t+1}-\tau}{h}\right)-\widetilde{\mathbb{B}}_t(L)K\left(\frac{\tau_t-\tau}{h}\right)\right)\boldsymbol{\epsilon}_t\\
&\to_D N\left(\mathbf{0},\widetilde{v}_0\left\{\sum_{j=0}^{\infty}\boldsymbol{B}_j(\tau)\right\}\left\{\sum_{j=0}^{\infty}\boldsymbol{B}_j^{\top}(\tau)\right\}\right).
\end{aligned}
$$

For the bias term, we have for any $\tau\in(0,1)$

$$
\frac{1}{Th}\sum_{t=1}^{T}\boldsymbol{\mu}(\tau_t)K\left(\frac{\tau_t-\tau}{h}\right)=\boldsymbol{\mu}(\tau)+\frac{1}{2}h^2\widetilde{c}_2\boldsymbol{\mu}^{(2)}(\tau)+o(h^2)+O\left(\frac{1}{Th}\right).
$$

The proof is now completed. $\square$

Proof of Theorem 3.2.

Note that $\boldsymbol{x}_t^*=\widetilde{\boldsymbol{\mu}}(\tau_t)+\boldsymbol{e}_t^*$, so we can write

$$
\widehat{\boldsymbol{\mu}}^*(\tau)-\widetilde{\boldsymbol{\mu}}(\tau)=\left(\sum_{t=1}^{T}W_{T,t}(\tau)\widetilde{\boldsymbol{\mu}}(\tau_t)-\widetilde{\boldsymbol{\mu}}(\tau)\right)+\sum_{t=1}^{T}W_{T,t}(\tau)\boldsymbol{e}_t^*:=K_{T,1}+K_{T,2},
$$

where $W_{T,t}(\tau)=K(\frac{\tau_t-\tau}{h})/\sum_{t=1}^{T}K(\frac{\tau_t-\tau}{h})$.

We start our investigation from $K_{T,1}$, and write

$$
\begin{aligned}
K_{T,1}&=\left(\frac{1}{Th}\sum_{t=1}^{T}K\left(\frac{\tau_t-\tau}{h}\right)\frac{1}{T\widetilde{h}}\sum_{s=1}^{T}\boldsymbol{\mu}(\tau_s)K\left(\frac{\tau_s-\tau_t}{\widetilde{h}}\right)-\frac{1}{T\widetilde{h}}\sum_{s=1}^{T}\boldsymbol{\mu}(\tau_s)K\left(\frac{\tau_s-\tau}{\widetilde{h}}\right)\right)\\
&\quad+\frac{1}{\sqrt{T\widetilde{h}}}\left(\sum_{t=1}^{T}K\left(\frac{\tau_t-\tau}{h}\right)\boldsymbol{Z}_T(\tau_t)-\boldsymbol{Z}_T(\tau)\right)+O_P\left(\frac{1}{Th}\right)\\
&:=K_{T,11}+K_{T,12}+O_P\left(\frac{1}{Th}\right),
\end{aligned}
$$

where the definitions of $K_{T,11}$ and $K_{T,12}$ should be obvious, $\boldsymbol{Z}_T(\tau)=\frac{1}{\sqrt{T\widetilde{h}}}\sum_{t=1}^{T}\boldsymbol{e}_tK\left(\frac{\tau_t-\tau}{\widetilde{h}}\right)$ and $\boldsymbol{e}_t=\sum_{j=0}^{\infty}\boldsymbol{B}_j(\tau_t)\boldsymbol{\epsilon}_{t-j}$. Similar to the development of Lemma 2.2, we can show that $\|K_{T,12}\|=O_P((T\widetilde{h})^{-1/2})$, which, along with the conditions of Theorem 3.2, yields

$$
\sqrt{Th}\|K_{T,12}\|=O_P((h/\widetilde{h})^{1/2})=o_P(1).
$$

For $K_{T,11}$, by the definition of Riemann integral, we have

$$
\begin{aligned}
K_{T,11} &= \int_{-1}^{1} K(u) \int_{-1}^{1} K(v) \left(\boldsymbol{\mu}(\tau + v\widetilde{h} + uh) - \boldsymbol{\mu}(\tau + v\widetilde{h})\right) dvdu + O\left(\frac{1}{Th}\right) \\
&= \frac{1}{2} h^2 \widetilde{c}_2 \boldsymbol{\mu}^{(2)}(\tau) + O(h^2(h + \widetilde{h})) + O\left(\frac{1}{Th}\right).
\end{aligned}
$$

Thus, we need only to focus on $K_{T,2}$ and then show that

$$
\frac{1}{\sqrt{Th}} \sum_{t=1}^{T} K\left(\frac{\tau_t - \tau}{h}\right) \boldsymbol{e}_t^* \to_{D^*} N\left(\boldsymbol{0}, \tilde{v}_0 \left\{\sum_{j=0}^{\infty} \boldsymbol{B}_j(\tau)\right\} \left\{\sum_{j=0}^{\infty} \boldsymbol{B}_j^\top(\tau)\right\}\right).
$$

Using the Cramér-Wold device, this is enough to show for any conformable unit vector $\boldsymbol{d}$,

$$
\frac{1}{\sqrt{Th}} \sum_{t=1}^{T} K\left(\frac{\tau_t - \tau}{h}\right) \boldsymbol{d}^\top \boldsymbol{e}_t^* \to_{D^*} N\left(\boldsymbol{0}, \tilde{v}_0 \boldsymbol{d}^\top \left\{\sum_{j=0}^{\infty} \boldsymbol{B}_j(\tau)\right\} \left\{\sum_{j=0}^{\infty} \boldsymbol{B}_j^\top(\tau)\right\} \boldsymbol{d}\right).
$$

For $\forall \tau \in [h + \widetilde{h}, 1 - h - \widetilde{h}]$, we write

$$
\begin{aligned}
\frac{1}{\sqrt{Th}} \sum_{t=1}^{T} K\left(\frac{\tau_t - \tau}{h}\right) \boldsymbol{d}^\top \widetilde{\boldsymbol{e}}_t \xi_t^* &= Z_T^*(\tau) + \frac{1}{\sqrt{Th}} \sum_{t=1}^{T} K\left(\frac{\tau_t - \tau}{h}\right) \boldsymbol{d}^\top (\widetilde{\boldsymbol{e}}_t - \boldsymbol{e}_t) \xi_t^* \\
&= Z_T^*(\tau) + o_{P^*}(1),
\end{aligned} \tag{B.1}
$$

where $Z_T^*(\tau) = \frac{1}{\sqrt{Th}} \sum_{t=1}^{T} K\left(\frac{\tau_t - \tau}{h}\right) \boldsymbol{d}^\top \boldsymbol{e}_t \xi_t^*$, and the second equality follows from

$$
\begin{aligned}
& EE^* \left\| \frac{1}{\sqrt{Th}} \sum_{t=1}^{T} K\left(\frac{\tau_t - \tau}{h}\right) \boldsymbol{d}^\top (\widetilde{\boldsymbol{e}}_t - \boldsymbol{e}_t) \xi_t^* \right\|^2 \\
\leq & \max_{\lfloor T(\tau - h) \rfloor \leq t \leq \lceil T(\tau + h) \rceil} E \left\| \widetilde{\boldsymbol{e}}_t - \boldsymbol{e}_t \right\|^2 \left(\frac{1}{Th} \sum_{t=1}^{T} \sum_{s=1}^{T} K\left(\frac{\tau_t - \tau}{h}\right) K\left(\frac{\tau_s - \tau}{h}\right) E^*(\xi_t^* \xi_s^*) \right) \\
= & O\left(\widetilde{h}^4 + 1/(T\widetilde{h})\right) O(l) = o(1),
\end{aligned}
$$

where $EE^*[\cdot]$ stands for taking the expectation of the variables with respect to the bootstrap draws, and then taking the exception with respect to the original sample.

By Lemma 2.5, we already have

$$
Z_T^*(\tau) \to_{D^*} N\left(0, \tilde{v}_0 \boldsymbol{d}^\top \left\{\sum_{j=0}^{\infty} \boldsymbol{B}_j(\tau)\right\} \left\{\sum_{j=0}^{\infty} \boldsymbol{B}_j^\top(\tau)\right\} \boldsymbol{d}\right).
$$

Combining the above results, we have

$$
\sqrt{Th} \left(\widehat{\boldsymbol{\mu}}^*(\tau) - \widetilde{\boldsymbol{\mu}}(\tau)\right) \to_{D^*} N\left(\boldsymbol{\mu}_b(\tau), \widetilde{v}_0 \boldsymbol{\Sigma}_{\boldsymbol{\mu}}(\tau)\right). \tag{B.2}
$$

The proof is now completed. □

Proof of Theorem 4.1.

(1). For notational simplicity, let $\boldsymbol{Z}_{\tau,t}$ be the transpose of the t^{th} row of $\boldsymbol{Z}_{\tau}$. Also, we define

$$\boldsymbol{S}_{T,k}(\tau) = \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{Z}_t\boldsymbol{Z}_t^{\top}\left(\frac{\tau_t-\tau}{h}\right)^k K_h(\tau_t-\tau) \text{ for } 0\leq k\leq 3,$$
$$\boldsymbol{M}(\tau_t) = \boldsymbol{\beta}(\tau_t)-\boldsymbol{\beta}(\tau)-\boldsymbol{\beta}^{(1)}(\tau)(\tau_t-\tau)-\frac{1}{2}\boldsymbol{\beta}^{(2)}(\tau)(\tau_t-\tau)^2,$$
$$\boldsymbol{S}_T(\tau) = \begin{pmatrix}\boldsymbol{S}_{T,0}(\tau) & \boldsymbol{S}_{T,1}(\tau)\\ \boldsymbol{S}_{T,1}(\tau) & \boldsymbol{S}_{T,2}(\tau)\end{pmatrix}.$$

Since $\boldsymbol{y}_t = \boldsymbol{Z}_t^{\top}\left(\boldsymbol{\beta}(\tau)+\boldsymbol{\beta}^{(1)}(\tau)(\tau_t-\tau)+\frac{1}{2}\boldsymbol{\beta}^{(2)}(\tau)(\tau_t-\tau)^2+\boldsymbol{M}(\tau_t)\right)+\boldsymbol{\eta}_t$, we can write

$$\widehat{\boldsymbol{\beta}}(\tau)-\boldsymbol{\beta}(\tau) = [\boldsymbol{I}_s,\boldsymbol{0}_s]\left(\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{Z}_{\tau,t}\boldsymbol{Z}_{,\tau,t}^{\top}K_h(\tau_t-\tau)\right)^{-1}\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{Z}_{\tau,t}\boldsymbol{y}_tK_h(\tau_t-\tau)-\boldsymbol{\beta}(\tau)$$
$$= [\boldsymbol{I}_s,\boldsymbol{0}_s]\boldsymbol{S}_T^{-1}(\tau)\begin{bmatrix}\boldsymbol{S}_{T,2}(\tau)\\ \boldsymbol{S}_{T,3}(\tau)\end{bmatrix}\frac{1}{2}h^2\boldsymbol{\beta}^{(2)}(\tau)+[\boldsymbol{I}_s,\boldsymbol{0}_s]\boldsymbol{S}_T^{-1}(\tau)\cdot\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{Z}_{\tau,t}\boldsymbol{Z}_t^{\top}\boldsymbol{M}(\tau_t)K_h(\tau_t-\tau)$$
$$+[\boldsymbol{I}_s,\boldsymbol{0}_s]\boldsymbol{S}_T^{-1}(\tau)\cdot\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{Z}_{\tau,t}\boldsymbol{\eta}_tK_h(\tau_t-\tau) = J_{T,1}+J_{T,2}+J_{T,3},$$

where the definitions of $J_{T,1}$ to $J_{T,3}$ should be obvious.

By Lemma B.6 and Lemma 2.2, we have

$$J_{T,1} = \frac{1}{2}h^2\widetilde{c}_2\boldsymbol{\beta}^{(2)}(\tau)+O_P\left(h^2(h^2+(Th)^{-1/2})\right),$$
$$J_{T,2} = o_P(h^2).$$

Thus, we focus on $J_{T,3}$ below. For any $\tau\in(0,1)$, as $\{\boldsymbol{Z}_t\boldsymbol{\eta}_t\}$ is a sequence of martingale differences, by Lemma 2.2 and the martingale central limit theory, we have

$$\sqrt{Th}J_{T,3} = \left(\boldsymbol{\Sigma}_{\boldsymbol{z}}^{-1}(\tau)\otimes\boldsymbol{I}_d\right)\left(\frac{\sqrt{Th}}{T}\sum_{t=1}^{T}\boldsymbol{Z}_t\boldsymbol{\eta}_tK_h(\tau_t-\tau)\right)+o_P(1)\to_D N\left(\boldsymbol{0},\widetilde{v}_0\boldsymbol{V}(\tau)\right).$$

The proof of the first result of this theorem is now completed.

(2). By Lemma 2.2, we have

$$\left\|\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{z}}(\tau)-\boldsymbol{\Sigma}_{\boldsymbol{z}}(\tau)\right\| = o_P(1).$$

Then we need only to focus on the rate associated with $\widehat{\boldsymbol{\Omega}}(\tau)$. For notational simplicity, we ignore the $\frac{1}{T}\sum_{t=1}^{T}K_h(\tau_t-\tau)$, because of $\frac{1}{T}\sum_{t=1}^{T}K_h(\tau_t-\tau)=1+O((Th)^{-1})$.

Write

$$
\begin{aligned}
\widehat{\boldsymbol{\Omega}}(\tau) &= \frac{1}{Th}\sum_{t=1}^{T}\widehat{\boldsymbol{\eta}}_t\widehat{\boldsymbol{\eta}}_t^{\top}K\left(\frac{\tau_t-\tau}{h}\right) \\
&= \frac{1}{Th}\sum_{t=1}^{T}\left(\boldsymbol{\eta}_t+\widehat{\boldsymbol{\eta}}_t-\boldsymbol{\eta}_t\right)\left(\boldsymbol{\eta}_t+\widehat{\boldsymbol{\eta}}_t-\boldsymbol{\eta}_t\right)^{\top}K\left(\frac{\tau_t-\tau}{h}\right) \\
&= \frac{1}{Th}\sum_{t=1}^{T}\boldsymbol{\eta}_t\boldsymbol{\eta}_t^{\top}K\left(\frac{\tau_t-\tau}{h}\right)+\frac{1}{Th}\sum_{t=1}^{T}(\widehat{\boldsymbol{\eta}}_t-\boldsymbol{\eta}_t)(\widehat{\boldsymbol{\eta}}_t-\boldsymbol{\eta}_t)^{\top}K\left(\frac{\tau_t-\tau}{h}\right) \\
&\quad +\frac{1}{Th}\sum_{t=1}^{T}\boldsymbol{\eta}_t(\widehat{\boldsymbol{\eta}}_t-\boldsymbol{\eta}_t)^{\top}K\left(\frac{\tau_t-\tau}{h}\right)+\frac{1}{Th}\sum_{t=1}^{T}(\widehat{\boldsymbol{\eta}}_t-\boldsymbol{\eta}_t)\boldsymbol{\eta}_t^{\top}K\left(\frac{\tau_t-\tau}{h}\right) \\
&:= J_{T,21}+J_{T,22}+J_{T,23}+J_{T,24}.
\end{aligned}
$$

Consider $J_{T,21}$. Since $\{\boldsymbol{\eta}_t\boldsymbol{\eta}_t^{\top}-E\left(\boldsymbol{\eta}_t\boldsymbol{\eta}_t^{\top}\right)\}$ is a sequence of martingale differences, we have

$$
\left\|\frac{1}{T}\sum_{t=1}^{T}\left[\boldsymbol{\eta}_t\boldsymbol{\eta}_t^{\top}-E\left(\boldsymbol{\eta}_t\boldsymbol{\eta}_t^{\top}\right)\right]K_h(\tau_t-\tau)\right\| = o_P(1).
$$

Next, consider $J_{T,22}$. By Lemma B.7 (4)

$$
\|J_{T,22}\| \leq \sup_{\tau\in[0,1]}\|\widehat{\boldsymbol{\beta}}(\tau)-\boldsymbol{\beta}(\tau)\|^2\cdot\frac{1}{T}\sum_{t=1}^{T}\|\boldsymbol{Z}_t\|^2K_h(\tau_t-\tau) = o_P(1).
$$

Similarly, for $J_{T,23}$ and $J_{T,24}$, we have

$$
\|J_{T,23}\| \leq \sup_{\tau\in[0,1]}\left\|\widehat{\boldsymbol{\beta}}(\tau)-\boldsymbol{\beta}(\tau)\right\|\cdot\frac{1}{T}\sum_{t=1}^{T}\|\boldsymbol{Z}_t\boldsymbol{\eta}_t\|K_h(\tau_t-\tau) = o_P(1).
$$

The proof is now completed. □

Proof of Theorem 4.2.

(1). Since

$$
\begin{aligned}
\widetilde{\boldsymbol{y}}_t &= \boldsymbol{y}_t-\boldsymbol{Z}_{2t}^{\top}\boldsymbol{s}(\tau_t)\boldsymbol{y} \\
&= (\boldsymbol{Z}_{1t}-\boldsymbol{Z}_1\boldsymbol{s}^{\top}(\tau_t)\boldsymbol{Z}_{2t})^{\top}\boldsymbol{c}+\boldsymbol{\eta}_t-\boldsymbol{Z}_{2t}^{\top}\boldsymbol{s}(\tau_t)\boldsymbol{\eta}+\boldsymbol{Z}_{2t}^{\top}\boldsymbol{\theta}(\tau_t)-\boldsymbol{Z}_{2t}^{\top}\boldsymbol{s}(\tau_t)\left[\boldsymbol{\theta}(\tau_1)^{\top}\boldsymbol{Z}_{21},...,\boldsymbol{\theta}(\tau_T)^{\top}\boldsymbol{Z}_{2T}\right]^{\top}
\end{aligned}
$$

with $\boldsymbol{\eta}=[\boldsymbol{\eta}_1^{\top},...,\boldsymbol{\eta}_T^{\top}]^{\top}$, we can decompose $\widehat{\boldsymbol{c}}_{WLS}-\boldsymbol{c}$ into

$$
\begin{aligned}
&\sqrt{T}\left(\widehat{\boldsymbol{c}}_{WLS}-\boldsymbol{c}\right) \\
&= \left(T^{-1}\sum_{t=1}^{T}\widetilde{\boldsymbol{Z}}_{1t}\widehat{\boldsymbol{\Omega}}^{-1}(\tau_t)\widetilde{\boldsymbol{Z}}_{1t}^{\top}\right)^{-1}T^{-1/2}\sum_{t=1}^{T}\widetilde{\boldsymbol{Z}}_{1t}\widehat{\boldsymbol{\Omega}}^{-1}(\tau_t)(\boldsymbol{\eta}_t-\boldsymbol{Z}_{2t}^{\top}\boldsymbol{s}(\tau_t)\boldsymbol{\eta})
\end{aligned}
$$

$$
+\left(T^{-1}\sum_{t=1}^{T}\widetilde{\boldsymbol{Z}}_{1t}\widehat{\boldsymbol{\Omega}}^{-1}(\tau_t)\widetilde{\boldsymbol{Z}}_{1t}^{\top}\right)^{-1} T^{-1/2}\sum_{t=1}^{T}\widetilde{\boldsymbol{Z}}_{1t}\widehat{\boldsymbol{\Omega}}^{-1}(\tau_t)
$$

$$
\times\left(\boldsymbol{Z}_{2t}^{\top}\boldsymbol{\theta}(\tau_t)-\boldsymbol{Z}_{2t}^{\top}\boldsymbol{s}(\tau_t)\left[\boldsymbol{\theta}(\tau_1)^{\top}\boldsymbol{Z}_{21},...,\boldsymbol{\theta}(\tau_T)^{\top}\boldsymbol{Z}_{2T}\right]^{\top}\right).
$$

By Lemma B.8, we have

$$
\begin{aligned}
&(1).\ T^{-1}\sum_{t=1}^{T}\widetilde{\boldsymbol{Z}}_{1t}\widehat{\boldsymbol{\Omega}}^{-1}(\tau_t)\widetilde{\boldsymbol{Z}}_{1t}^{\top}\to_P \boldsymbol{\Delta}_{\boldsymbol{c}},\\
&(2).\ T^{-1/2}\sum_{t=1}^{T}\widetilde{\boldsymbol{Z}}_{1t}\widehat{\boldsymbol{\Omega}}^{-1}(\tau_t)\boldsymbol{\eta}_t = T^{-1/2}\sum_{t=1}^{T}\left(\boldsymbol{Z}_{1t}-\boldsymbol{\Sigma}_{\boldsymbol{Z}_{1,2}}(\tau_t)\boldsymbol{\Sigma}_{\boldsymbol{Z}_2}^{-1}(\tau_t)\boldsymbol{Z}_{2t}\right)\boldsymbol{\Omega}^{-1}(\tau_t)\boldsymbol{\eta}_t+o_P(1)\\
&(3).\ T^{-1/2}\sum_{t=1}^{T}\widetilde{\boldsymbol{Z}}_{1t}\widehat{\boldsymbol{\Omega}}^{-1}(\tau_t)\boldsymbol{Z}_{2t}^{\top}\boldsymbol{s}(\tau_t)\boldsymbol{\eta}\to_P 0,\\
&(4).\ T^{-1/2}\sum_{t=1}^{T}\widetilde{\boldsymbol{Z}}_{1t}\widehat{\boldsymbol{\Omega}}^{-1}(\tau_t)\left(\boldsymbol{Z}_{2t}^{\top}\boldsymbol{\theta}(\tau_t)-\boldsymbol{Z}_{2t}^{\top}\boldsymbol{s}(\tau_t)\left[\boldsymbol{\theta}(\tau_1)^{\top}\boldsymbol{Z}_{21},...,\boldsymbol{\theta}(\tau_T)^{\top}\boldsymbol{Z}_{2T}\right]^{\top}\right)\to_P 0,
\end{aligned}
$$

which follows that

$$
\sqrt{T}\left(\widehat{\boldsymbol{c}}_{WLS}-\boldsymbol{c}\right)=\boldsymbol{\Delta}_{\boldsymbol{c}}^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\boldsymbol{Z}_{1t}-\boldsymbol{\Sigma}_{\boldsymbol{Z}_{1,2}}(\tau_t)\boldsymbol{\Sigma}_{\boldsymbol{Z}_2}^{-1}(\tau_t)\boldsymbol{Z}_{2t}\right)\boldsymbol{\Omega}^{-1}(\tau_t)\boldsymbol{\eta}_t+o_P(1).
$$

Since $\left\{\left(\boldsymbol{Z}_{1t}-\boldsymbol{\Sigma}_{\boldsymbol{Z}_{1,2}}(\tau_t)\boldsymbol{\Sigma}_{\boldsymbol{Z}_2}^{-1}(\tau_t)\boldsymbol{Z}_{2t}\right)\boldsymbol{\Omega}^{-1}(\tau_t)\boldsymbol{\eta}_t\right\}$ is a sequence of martingale differences, the result follows by the central limit theorem for martingale differences. Note that the convergence of conditional variance can be proved by Lemma 2.2.

(2). Let $\boldsymbol{\Theta}(\tau)=[\boldsymbol{\theta}(\tau)^{\top},h\boldsymbol{\theta}^{(1)}(\tau)^{\top}]^{\top}$. Note that

$$
\begin{aligned}
\widehat{\boldsymbol{\theta}}(\tau)-\boldsymbol{\theta}(\tau) &= [\boldsymbol{I}_{s_2},\boldsymbol{0}_{s_2}](\boldsymbol{Z}_{2,\tau}^{\top}\boldsymbol{K}_{\tau}\boldsymbol{Z}_{2,\tau})^{-1}\boldsymbol{Z}_{2,\tau}^{\top}\boldsymbol{K}_{\tau}\left(\boldsymbol{y}-\boldsymbol{Z}_1\widehat{\boldsymbol{c}}_{WLS}-\boldsymbol{Z}_{2,\tau}\boldsymbol{\Theta}(\tau)\right)\\
&= [\boldsymbol{I}_{s_2},\boldsymbol{0}_{s_2}](\boldsymbol{Z}_{2,\tau}^{\top}\boldsymbol{K}_{\tau}\boldsymbol{Z}_{2,\tau})^{-1}\boldsymbol{Z}_{2,\tau}^{\top}\boldsymbol{K}_{\tau}\boldsymbol{Z}_1(\boldsymbol{c}-\widehat{\boldsymbol{c}}_{WLS})\\
&\quad +[\boldsymbol{I}_{s_2},\boldsymbol{0}_{s_2}](\boldsymbol{Z}_{2,\tau}^{\top}\boldsymbol{K}_{\tau}\boldsymbol{Z}_{2,\tau})^{-1}\boldsymbol{Z}_{2,\tau}^{\top}\boldsymbol{K}_{\tau}\left(\begin{bmatrix}\boldsymbol{Z}_{21}^{\top}\boldsymbol{\theta}(\tau_1)\\ \vdots\\ \boldsymbol{Z}_{2T}^{\top}\boldsymbol{\theta}(\tau_T)\end{bmatrix}-\boldsymbol{Z}_{2,\tau}\boldsymbol{\Theta}(\tau)\right)\\
&\quad +[\boldsymbol{I}_{s_2},\boldsymbol{0}_{s_2}](\boldsymbol{Z}_{2,\tau}^{\top}\boldsymbol{K}_{\tau}\boldsymbol{Z}_{2,\tau})^{-1}\boldsymbol{Z}_{2,\tau}^{\top}\boldsymbol{K}_{\tau}\boldsymbol{\eta}\\
&:= J_{T,31}+J_{T,32}+J_{T,33}.
\end{aligned}
$$

By part (1), we have $J_{T,31}=O_P(T^{-1/2})$. By using standard arguments of the local linear kernel method, we have $J_{T,32}=\frac{1}{2}h^2\widetilde{c}_2\boldsymbol{\theta}^{(2)}(\tau)+o_P(h^2)$. Then, it suffices to show that

$$
[\boldsymbol{I}_{s_2},\boldsymbol{0}_{s_2}]\frac{\sqrt{h}}{\sqrt{T}}\boldsymbol{Z}_{2,\tau}^{\top}\boldsymbol{K}_{\tau}\boldsymbol{\eta}\to_D N\left(\boldsymbol{0},\tilde{v}_0\boldsymbol{\Sigma}_{\boldsymbol{z}_2}(\tau)\otimes\boldsymbol{\Omega}(\tau)\right),
$$

which follows directly from the martingale central limit theorem. Also, the convergence of conditional variance can be proved by Lemma 2.2.

□

Proof of Theorem B.1.

(1). To proceed, define

$$\mathbf{\Gamma}(\tau) = \begin{bmatrix} \mathbf{\Phi}(\tau) & * \\ \mathbf{0} & \mathbf{I}^* \end{bmatrix} \quad \text{and} \quad \mathbf{I}^* = \begin{bmatrix} \mathbf{0}_m & \cdots & \mathbf{0}_m & \mathbf{0}_m \\ \mathbf{I}_m & \cdots & \mathbf{0}_m & \mathbf{0}_m \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_m & \cdots & \mathbf{I}_m & \mathbf{0}_m \end{bmatrix}.$$

Let ρ denote the largest eigenvalue of $\mathbf{\Phi}(\tau)$ uniformly over $\tau \in [0,1]$. Then we have $\rho < 1$ by Assumption 5.1. By construction, it is easy to show that the largest eigenvalue of $\mathbf{\Gamma}(\tau)$ is ρ uniformly over $\tau \in [0,1]$, since all eigenvalues of $\mathbf{I}^*$ are zeros. Similar to the proof of Proposition 2.4 in Dahlhaus and Polonik (2009), we have $\max_{t\geq 1} \left\| \prod_{m=0}^{j-1} \mathbf{\Gamma}(\tau_{t-m}) \right\| \leq M\rho^j$.

Note that for any conformable matrices $\{\mathbf{A}_i\}$ and $\{\mathbf{B}_i\}$, since

$$\prod_{i=1}^{r} \mathbf{A}_i - \prod_{i=1}^{r} \mathbf{B}_i = \sum_{j=1}^{r} \left(\prod_{k=1}^{j-1} \mathbf{A}_k \right) (\mathbf{A}_j - \mathbf{B}_j) \left(\prod_{k=j+1}^{r} \mathbf{B}_k \right),$$

we have

$$\begin{aligned}
\left\| \mathbf{D}_{j,t}^{\boldsymbol{x}} - \mathbf{D}_j^{\boldsymbol{x}}(\tau_t) \right\| &= \left\| \bar{\mathbf{J}} \prod_{k=0}^{j-1} \mathbf{\Gamma}(\tau_{t-k}) \mathbf{\Upsilon}(\tau_{t-j}) - \bar{\mathbf{J}} \mathbf{\Gamma}^j(\tau_t) \mathbf{\Upsilon}(\tau_t) \right\| \\
= \left\| \bar{\mathbf{J}} \left(\prod_{k=0}^{j-1} \mathbf{\Gamma}(\tau_{t-k}) - \mathbf{\Gamma}^j(\tau_t) \right) \mathbf{\Upsilon}(\tau_t) \right. &+ \left. \bar{\mathbf{J}} \prod_{k=0}^{j-1} \mathbf{\Gamma}(\tau_{t-k}) \left(\mathbf{\Upsilon}(\tau_{t-j}) - \mathbf{\Upsilon}(\tau_t) \right) \right\| \\
\leq O(1) \sum_{i=1}^{j-1} \left\| \mathbf{\Gamma}^i(\tau_t) \left(\mathbf{\Gamma}(\tau_{t-i}) - \mathbf{\Gamma}(\tau_t) \right) \prod_{k=i+1}^{j-1} \mathbf{\Gamma}(\tau_{t-k}) \right\| &+ M\rho^j \frac{j}{T} \\
\leq O(1) \sum_{i=1}^{j-1} \frac{i}{T} \rho^{j-1} + O(1) \rho^j \frac{j}{T} &= O(T^{-1}).
\end{aligned}$$

The proof of the first result is then complete.

(2). Part (2) follows directly from the Delta method (see Lütkepohl, 2005, p.408) and

$$\mathbf{G}_j(\tau) = \frac{\partial \text{vec}\left[\mathbf{D}_j^{\boldsymbol{x}}(\tau) \right]}{\partial \boldsymbol{\beta}(\tau)^\top} = \left[\sum_{k=0}^{j-1} \mathbf{\Upsilon}(\tau)^\top (\mathbf{\Gamma}(\tau)^\top)^{j-1-k} \otimes \mathbf{J} \mathbf{\Gamma}^k(\tau) \mathbf{J}^\top, \mathbf{I}_m \otimes \mathbf{J} \mathbf{\Gamma}^k(\tau) \mathbf{J}^\top \right].$$

□

B.4 Proofs of the Preliminary Lemmas

Proof of Lemma B.3.

(1). The first result follows from the standard BN decomposition (e.g., Phillips and Solo, 1992), so the details are omitted. (2). For the second decomposition, write

$$
\begin{aligned}
(1-L)\widetilde{\mathbb{B}}_t^r(L) &= \sum_{j=0}^{\infty}\left(L^j \sum_{k=j+1}^{\infty}(\boldsymbol{B}_{k+r,t}\otimes \boldsymbol{B}_{k,t}) - L^{j+1}\sum_{k=j+1}^{\infty}(\boldsymbol{B}_{k+r,t}\otimes \boldsymbol{B}_{k,t})\right)\\
&= \sum_{j=0}^{\infty}\left(L^{j+1} \sum_{k=j+2}^{\infty}(\boldsymbol{B}_{k+r,t}\otimes \boldsymbol{B}_{k,t}) - L^{j+1}\sum_{k=j+1}^{\infty}(\boldsymbol{B}_{k+r,t}\otimes \boldsymbol{B}_{k,t})\right) + \sum_{k=1}^{\infty}(\boldsymbol{B}_{k+r,t}\otimes \boldsymbol{B}_{k,t})\\
&= -\sum_{j=0}^{\infty} L^{j+1}(\boldsymbol{B}_{j+1+r,t}\otimes \boldsymbol{B}_{j+1,t}) + \sum_{k=1}^{\infty}(\boldsymbol{B}_{k+r,t}\otimes \boldsymbol{B}_{k,t})\\
&= -\sum_{j=0}^{\infty} L^{j}(\boldsymbol{B}_{j+r,t}\otimes \boldsymbol{B}_{j,t}) + \sum_{k=0}^{\infty}(\boldsymbol{B}_{k+r,t}\otimes \boldsymbol{B}_{k,t}) = \mathbb{B}_t^r(1) - \mathbb{B}_t^r(L).
\end{aligned}
$$

(3). By Assumption 1,

$$
\max_{t\geq 1}\sum_{j=0}^{\infty}\left\|\widetilde{\boldsymbol{B}}_{j,t}\right\| \leq \max_{t\geq 1}\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}\|\boldsymbol{B}_{k,t}\| = \max_{t\geq 1}\sum_{j=1}^{\infty} j\,\|\boldsymbol{B}_{j,t}\| < \infty.
$$

(4). By Assumption 1,

$$
\sum_{t=1}^{T-1}\|\widetilde{\mathbb{B}}_{t+1}(1) - \widetilde{\mathbb{B}}_t(1)\| \leq \sum_{t=1}^{T-1}\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}\|\boldsymbol{B}_{k,t+1} - \boldsymbol{B}_{k,t}\| = \sum_{t=1}^{T-1}\sum_{j=1}^{\infty} j\,\|\boldsymbol{B}_{j,t+1} - \boldsymbol{B}_{j,t}\| < \infty.
$$

(5). By Assumption 1,

$$
\begin{aligned}
\max_{t\geq 1}\sum_{j=0}^{\infty}\left\|\widetilde{\boldsymbol{B}}_{j,t}^r\right\| &\leq \max_{t\geq 1}\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}\|\boldsymbol{B}_{k+r,t}\otimes \boldsymbol{B}_{k,t}\| = \max_{t\geq 1}\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}\|\boldsymbol{B}_{k+r,t}\|\cdot\|\boldsymbol{B}_{k,t}\|\\
&= \max_{t\geq 1}\sum_{j=1}^{\infty} j\,\|\boldsymbol{B}_{j+r,t}\|\cdot\|\boldsymbol{B}_{j,t}\| \leq M\max_{t\geq 1}\sum_{j=1}^{\infty} j\,\|\boldsymbol{B}_{j,t}\| < \infty.
\end{aligned}
$$

(6). Write

$$
\begin{aligned}
&\max_{t\geq 1}\sum_{r=1}^{\infty}\left\|\widetilde{\mathbb{B}}_t^r(1)\right\| \leq \max_{t\geq 1}\sum_{r=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}\|\boldsymbol{B}_{k+r,t}\|\cdot\|\boldsymbol{B}_{k,t}\|\\
&= \max_{t\geq 1}\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}\|\boldsymbol{B}_{k,t}\|\cdot\left(\sum_{r=1}^{\infty}\|\boldsymbol{B}_{k+r,t}\|\right) \leq \max_{t\geq 1}\left(\sum_{r=1}^{\infty}\|\boldsymbol{B}_{r,t}\|\right)\cdot\left(\sum_{j=1}^{\infty} j\,\|\boldsymbol{B}_{j,t}\|\right) < \infty.
\end{aligned}
$$

(7). Write

$$
\begin{aligned}
&\sum_{t=1}^{T-1}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\left\|\widetilde{\mathbb{B}}_{t+1}^{r}(1)-\widetilde{\mathbb{B}}_{t}^{r}(1)\right\| \leq \sum_{t=1}^{T-1}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\left\|\sum_{k=j+1}^{\infty}\boldsymbol{B}_{k+r,t+1}\otimes\boldsymbol{B}_{k,t+1}-\boldsymbol{B}_{k+r,t}\otimes\boldsymbol{B}_{k,t}\right\| \\
&\leq \sum_{t=1}^{T-1}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}\left(\|\boldsymbol{B}_{k+r,t+1}-\boldsymbol{B}_{k+r,t}\|\cdot\|\boldsymbol{B}_{k,t+1}\|+\|\boldsymbol{B}_{k+r,t}\|\cdot\|\boldsymbol{B}_{k,t+1}-\boldsymbol{B}_{k,t}\|\right) \\
&= \sum_{t=1}^{T-1}\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}\left(\|\boldsymbol{B}_{k,t+1}\|\cdot\sum_{r=0}^{\infty}\|\boldsymbol{B}_{k+r,t+1}-\boldsymbol{B}_{k+r,t}\|+\|\boldsymbol{B}_{k,t+1}-\boldsymbol{B}_{k,t}\|\cdot\sum_{r=0}^{\infty}\|\boldsymbol{B}_{k+r,t}\|\right) \\
&\leq \left(\sum_{t=1}^{T-1}\sum_{r=1}^{\infty}\|\boldsymbol{B}_{r,t+1}-\boldsymbol{B}_{r,t}\|\right)\cdot\left(\max_{t\geq 1}\sum_{k=1}^{\infty}k\,\|\boldsymbol{B}_{k,t}\|\right) \\
&\quad+\left(\sum_{t=1}^{T-1}\sum_{k=1}^{\infty}k\,\|\boldsymbol{B}_{k,t+1}-\boldsymbol{B}_{k,t}\|\right)\cdot\left(\max_{t\geq 1}\sum_{r=1}^{\infty}\|\boldsymbol{B}_{r,t}\|\right)<\infty.
\end{aligned}
$$

The proof is now completed. □

Proof of Lemma B.4.

In the following proof, we cover the interval $[a, b]$ by a finite number of subintervals $\{S_l\}$, which are centred at s_l with the length denoted by δ_T. Denoting the number of these intervals by N_T, then $N_T = O(\delta_T^{-1})$. In addition, let $\delta_T = O(T^{-1}\gamma_T)$ with $\gamma_T = \sqrt{d_T \log T}$.

Write

$$
\begin{aligned}
\sup_{\tau\in[a,b]}\left\|\sum_{t=1}^{T}\boldsymbol{W}_{T,t}(\tau)\mathbb{B}_t(1)\boldsymbol{\epsilon}_t\right\| &\leq \max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}\boldsymbol{W}_{T,t}(s_l)\mathbb{B}_t(1)\boldsymbol{\epsilon}_t\right\| \\
&+\max_{1\leq l\leq N_T}\sup_{\tau\in S_l}\left\|\sum_{t=1}^{T}\left(\boldsymbol{W}_{T,t}(\tau)-\boldsymbol{W}_{T,t}(s_l)\right)\mathbb{B}_t(1)\boldsymbol{\epsilon}_t\right\| := J_{T,41}+J_{T,42}.
\end{aligned}
$$

For $J_{T,42}$, since $\boldsymbol{W}_{T,t}(\cdot)$ is Lipschitz continuous and $\max_{t\geq 1}\|\mathbb{B}_t(1)\| < \infty$ by Assumption 1, we have

$$
\begin{aligned}
E|J_{T,42}| &\leq \sum_{t=1}^{T}\max_{1\leq l\leq N_T}\sup_{\tau\in S_l}\|\boldsymbol{W}_{T,t}(\tau)-\boldsymbol{W}_{T,t}(s_l)\|\,E\,\|\mathbb{B}_t(1)\boldsymbol{\epsilon}_t\| \\
&\leq MT\delta_T\max_{t\geq 1}E\|\mathbb{B}_t(1)\boldsymbol{\epsilon}_t\| = O(\gamma_T).
\end{aligned}
$$

For $J_{T,41}$, we apply the truncation method. Define $\boldsymbol{\epsilon}_t' = \boldsymbol{\epsilon}_t I(\|\boldsymbol{\epsilon}_t\| \leq T^{\frac{1}{\delta}})$ and $\boldsymbol{\epsilon}_t'' = \boldsymbol{\epsilon}_t - \boldsymbol{\epsilon}_t'$, where δ is defined in Assumption 2, and $I(\cdot)$ is the indicator function. Write

$$
\begin{aligned}
J_{T,41} &= \max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}\boldsymbol{W}_{T,t}(s_l)\mathbb{B}_t(1)\left(\boldsymbol{\epsilon}_t'+\boldsymbol{\epsilon}_t''-E\left(\boldsymbol{\epsilon}_t'+\boldsymbol{\epsilon}_t''|\mathcal{F}_{t-1}\right)\right)\right\| \\
&\leq \max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}\boldsymbol{W}_{T,t}(s_l)\mathbb{B}_t(1)\left(\boldsymbol{\epsilon}_t'-E(\boldsymbol{\epsilon}_t'|\mathcal{F}_{t-1})\right)\right\| + \max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}\boldsymbol{W}_{T,t}(s_l)\mathbb{B}_t(1)\boldsymbol{\epsilon}_t''\right\|
\end{aligned}
$$

$$+\max_{1\le l\le N_T}\left\|\sum_{t=1}^{T}\boldsymbol{W}_{T,t}(s_l)\mathbb{B}_t(1)E(\boldsymbol{\epsilon}_t''|\mathcal{F}_{t-1})\right\|:=J_{T,51}+J_{T,52}+J_{T,53}.$$

Start from $J_{T,52}$. By Hölder's inequality and Markov's inequality,

$$\begin{aligned}
E|J_{T,52}| &\le O(1)d_T\sum_{t=1}^{T}E\left\|\boldsymbol{\epsilon}_t''\right\| = O(1)d_T\sum_{t=1}^{T}E\|\boldsymbol{\epsilon}_t I(\|\boldsymbol{\epsilon}_t\|\ge T^{\frac{1}{\delta}})\| \\
&\le O(1)d_T\sum_{t=1}^{T}\left\{E\|\boldsymbol{\epsilon}_t\|^{\delta}\right\}^{\frac{1}{\delta}}\left\{E\|I(\|\boldsymbol{\epsilon}_t\|\ge T^{\frac{1}{\delta}})\|\right\}^{\frac{\delta-1}{\delta}} \\
&= O(1)d_T\sum_{t=1}^{T}\left\{E\|\boldsymbol{\epsilon}_t\|^{\delta}\right\}^{\frac{1}{\delta}}\left\{\Pr(\|\boldsymbol{\epsilon}_t\|\ge T^{\frac{1}{\delta}})\right\}^{\frac{\delta-1}{\delta}} \\
&\le O(1)d_T\sum_{t=1}^{T}\left\{E\|\boldsymbol{\epsilon}_t\|^{\delta}\right\}^{\frac{1}{\delta}}\left\{\frac{E\|\boldsymbol{\epsilon}_t\|^{\delta}}{T}\right\}^{\frac{\delta-1}{\delta}} = O(T^{\frac{1}{\delta}}d_T)=o\left(\sqrt{d_T\log T}\right),
\end{aligned}$$

where the second inequality follows from Hölder's inequality, and the third inequality follows from Markov's inequality. Similarly, $J_{T,53}=O_P(T^{\frac{1}{\delta}}d_T)=o_P\left(\sqrt{d_T\log T}\right)$.

We now turn to $J_{T,51}$. For notational simplicity, let $\boldsymbol{Y}_t=\boldsymbol{W}_{T,t}(s_l)\mathbb{B}_t(1)\left(\boldsymbol{\epsilon}_t'-E(\boldsymbol{\epsilon}_t'|\mathcal{F}_{t-1})\right)$ for $1\le t\le T$ and $A_T=2T^{\frac{1}{\delta}}d_T\max_{t\ge1}\|\mathbb{B}_t(1)\|$. Simple algebra shows that $\|\boldsymbol{Y}_t\|\le A_T$ uniformly in t and s_l. By Assumption 2 and the first condition in the body of this lemma,

$$\max_{1\le l\le N_T}\sum_{t=1}^{T}E\left(\|\boldsymbol{Y}_t\|^2|\mathcal{F}_{t-1}\right)\le Md_T\max_{1\le l\le N_T}\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\,E\left(\|\boldsymbol{\epsilon}_t\|^2|\mathcal{F}_{t-1}\right)=O_{a.s.}(d_T).$$

By Lemma B.2 and $T^{\frac{2}{\delta}}d_T\log T\to0$, choose some $\beta>0$ (such as $\beta=4$), and write

$$\begin{aligned}
&\Pr\left(J_{T,51}>\sqrt{\beta M}\gamma_T\right)\\
&=\Pr\left(J_{T,51}>\sqrt{\beta M}\gamma_T,\ \max_{1\le l\le N_T}\left\|\sum_{t=1}^{T}E(\boldsymbol{Y}_t\boldsymbol{Y}_t^{\top}|\mathcal{F}_{t-1})\right\|\le Md_T\right)\\
&\quad+\Pr\left(J_{T,51}>\sqrt{\beta M}\gamma_T,\ \max_{1\le l\le N_T}\left\|\sum_{t=1}^{T}E(\boldsymbol{Y}_t\boldsymbol{Y}_t^{\top}|\mathcal{F}_{t-1})\right\|>Md_T\right)\\
&\le\Pr\left(J_{T,51}>\sqrt{\beta M}\gamma_T,\ \max_{1\le l\le N_T}\left\|\sum_{t=1}^{T}E(\boldsymbol{Y}_t\boldsymbol{Y}_t^{\top}|\mathcal{F}_{t-1})\right\|\le Md_T\right)\\
&\quad+\Pr\left(\max_{1\le l\le N_T}\left\|\sum_{t=1}^{T}E(\boldsymbol{Y}_t\boldsymbol{Y}_t^{\top}|\mathcal{F}_{t-1})\right\|>Md_T\right)\\
&\le N_T\exp\left(-\frac{\beta M\gamma_T^2}{2(Md_T+\gamma_T 2A_T)}\right)+o(1)\le N_T\exp\left(-\frac{\beta}{2}\log T\right)+o(1)\to0.
\end{aligned}$$

Based on the above development, the proof is now completed. □

Proof of Lemma B.5.

(1). Similar to the proof of Lemma B.4, we use a finite number of subintervals $\{S_l\}$ to cover the interval $[a,b]$, which are centered at s_l with the length δ_T. Denote the number of these intervals by N_T then $N_T = O(\delta_T^{-1})$. In addition, let $\delta_T = O(T^{-1}\gamma_T)$ with $\gamma_T = \sqrt{d_T \log T}$.

$$
\begin{aligned}
&\sup_{\tau\in[a,b]}\left\|\sum_{t=1}^{T}(\boldsymbol{I}_d\otimes \boldsymbol{W}_{T,t}(\tau))\mathbb{B}_t^0(1)\left(\mathrm{vec}\left(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t^\top\right)-\mathrm{vec}(\boldsymbol{I}_d)\right)\right\| \\
\leq\ & \max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}(\boldsymbol{I}_d\otimes \boldsymbol{W}_{T,t}(s_l))\mathbb{B}_t^0(1)\left(\mathrm{vec}\left(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t^\top\right)-\mathrm{vec}(\boldsymbol{I}_d)\right)\right\| \\
&+\max_{1\leq l\leq N_T}\sup_{\tau\in S_l}\left\|\sum_{t=1}^{T}\left(\boldsymbol{I}_d\otimes(\boldsymbol{W}_{T,t}(\tau)-\boldsymbol{W}_{T,t}(s_l))\right)\mathbb{B}_t^0(1)\left(\mathrm{vec}\left(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t^\top\right)-\mathrm{vec}(\boldsymbol{I}_d)\right)\right\| \\
:=\ & J_{T,61}+J_{T,62}.
\end{aligned}
$$

Start from $J_{T,62}$. Similar to the proof of Lemma B.4, since

$$
\left\|\mathbb{B}_t^0(1)\right\| \leq \sum_{j=0}^{\infty}\|\boldsymbol{B}_{j,t}\|^2 \leq \left(\sum_{j=0}^{\infty}\|\boldsymbol{B}_{j,t}\|\right)^2 < \infty
$$

by Assumption 1, we have

$$
E|J_{T,62}| \leq MT\delta_T \max_{t\geq 1} E\left\|\mathbb{B}_t^0(1)\left(\mathrm{vec}\left(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t^\top\right)-\mathrm{vec}(\boldsymbol{I}_d)\right)\right\| = O(\gamma_T).
$$

We then apply the truncation method. Define $\boldsymbol{u}_t = \mathbb{B}_t^0(1)\left(\mathrm{vec}\left(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t^\top\right)-\mathrm{vec}\left(\boldsymbol{I}_d\right)\right)$, $\boldsymbol{u}_t' = \boldsymbol{u}_t I(\|\boldsymbol{u}_t\| \leq T^{\frac{2}{\delta}})$ and $\boldsymbol{u}_t'' = \boldsymbol{u}_t - \boldsymbol{u}_t'$. For $J_{T,1}$, write

$$
\begin{aligned}
J_{T,61} =\ & \max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}(\boldsymbol{I}_d\otimes \boldsymbol{W}_{T,t}(s_l))\left(\boldsymbol{u}_t'+\boldsymbol{u}_t''-E(\boldsymbol{u}_t'+\boldsymbol{u}_t''|\mathcal{F}_{t-1})\right)\right\| \\
\leq\ & \max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}(\boldsymbol{I}_d\otimes \boldsymbol{W}_{T,t}(s_l))\left(\boldsymbol{u}_t'-E(\boldsymbol{u}_t'|\mathcal{F}_{t-1})\right)\right\| + \max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}(\boldsymbol{I}_d\otimes \boldsymbol{W}_{T,t}(s_l))\boldsymbol{u}_t''\right\| \\
&+\max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}(\boldsymbol{I}_d\otimes \boldsymbol{W}_{T,t}(s_l))E(\boldsymbol{u}_t''|\mathcal{F}_{t-1})\right\| := J_{T,71}+J_{T,72}+J_{T,73}.
\end{aligned}
$$

As in the proof of Lemma B.4, we can show that $J_{T,72} = o_P(\sqrt{d_T\log T})$ and $J_{T,73} = o_P(\sqrt{d_T\log T})$ respectively. We focus on $J_{T,71}$ below.

For any $1\leq l\leq N_T$, let $\boldsymbol{Y}_t = (\boldsymbol{I}_d\otimes \boldsymbol{W}_{T,t}(s_l))(\boldsymbol{u}_t' - E(\boldsymbol{u}_t'|\mathcal{F}_{t-1}))$. We then have $E\left(\boldsymbol{Y}_t|\mathcal{F}_{t-1}\right)=0$ and $\|\boldsymbol{Y}_t\| \leq 2T^{2/\delta}d_T\max_t\left\|\mathbb{B}_t^0(1)\right\|$. Since $\max_{t\geq 1}E\left(\|\boldsymbol{\epsilon}_t\|^4|\mathcal{F}_{t-1}\right) < \infty$ a.s., we can write

$$
\max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}E(\boldsymbol{Y}_t\boldsymbol{Y}_t^\top|\mathcal{F}_{t-1})\right\| \leq Md_T\max_{t\geq 1}E\left(\|\boldsymbol{u}_t\|^2|\mathcal{F}_{t-1}\right)\max_{1\leq l\leq N_T}\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\| = O_{a.s.}\left(d_T\right).
$$

Similar to Lemma B.2, choose $\beta > 0$ (such as $\beta = 4$). In view of the fact that $T^{\frac{4}{\delta}}d_T\log T \to 0$, we write

$$\begin{aligned}
\Pr\left(J_{T,71} > \sqrt{\beta M}\gamma_T\right) &= \Pr\left(J_{T,71} > \sqrt{\beta M}\gamma_T, \max_{1\le l\le N_T}\left\|\sum_{t=1}^{T} E(\boldsymbol{Y}_t\boldsymbol{Y}_t^\top|\mathcal{F}_{t-1})\right\| \le Md_T\right) \\
&\quad + \Pr\left(J_{T,71} > \sqrt{\beta M}\gamma_T, \max_{1\le l\le N_T}\left\|\sum_{t=1}^{T} E(\boldsymbol{Y}_t\boldsymbol{Y}_t^\top|\mathcal{F}_{t-1})\right\| > Md_T\right) \\
&\le \Pr\left(J_{T,71} > \sqrt{\beta M}\gamma_T, \max_{1\le l\le N_T}\left\|\sum_{t=1}^{T} E(\boldsymbol{Y}_t\boldsymbol{Y}_t^\top|\mathcal{F}_{t-1})\right\| \le Md_T\right) \\
&\quad + \Pr\left(\max_{1\le l\le N_T}\left\|\sum_{t=1}^{T} E(\boldsymbol{Y}_t\boldsymbol{Y}_t^\top|\mathcal{F}_{t-1})\right\| > Md_T\right) \\
&\le N_T \exp\left(-\frac{\beta M\gamma_T^2}{2(Md_T + M\gamma_T T^{\frac{2}{\delta}} d_T)}\right) + o(1) \\
&\le N_T \exp\left(-\frac{\beta}{2}\log T\right) + o(1) = N_T T^{-\frac{\beta}{2}} + o(1) = o(1).
\end{aligned}$$

The first result then follows.

(2). Let $\{S_l\}$ be a finite number of subintervals covering the interval $[a,b]$, which are centered at s_l with the length δ_T. Denote the number of these intervals by N_T then $N_T = O(\delta_T^{-1})$. In addition, let $\delta_T = O(T^{-1}\gamma_T)$ with $\gamma_T = \sqrt{d_T \log T}$. Then

$$\begin{aligned}
\sup_{\tau\in[a,b]}\left\|\sum_{t=1}^{T}\left(\boldsymbol{I}_d\otimes \boldsymbol{W}_{T,t}(\tau)\right)\boldsymbol{\zeta}_t\boldsymbol{\epsilon}_t\right\| &\le \max_{1\le l\le N_T}\left\|\sum_{t=1}^{T}\left(\boldsymbol{I}_d\otimes \boldsymbol{W}_{T,t}(s_l)\right)\boldsymbol{\zeta}_t\boldsymbol{\epsilon}_t\right\| \\
+ \max_{1\le l\le N_T}\sup_{\tau\in S_l}\left\|\sum_{t=1}^{T}\left(\boldsymbol{I}_d\otimes (\boldsymbol{W}_{T,t}(\tau) - \boldsymbol{W}_{T,t}(s_l))\right)\boldsymbol{\zeta}_t\boldsymbol{\epsilon}_t\right\| &:= J_{T,81} + J_{T,82}.
\end{aligned}$$

Consider $J_{T,82}$. By the fact that $|\mathrm{tr}(\boldsymbol{A})| \le d\,\|\boldsymbol{A}\|$ for any $d\times d$ matrix $\boldsymbol{A}$ and Assumption 1,

$$\begin{aligned}
E\left\|\boldsymbol{\zeta}_t\boldsymbol{\epsilon}_t\right\| &= E\left\|\sum_{r=1}^{\infty}\left(\sum_{s=0}^{\infty}\boldsymbol{B}_{s+r,t}\otimes\boldsymbol{B}_{s,t}\right)\mathrm{vec}\left(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_{t-r}^\top\right)\right\| \\
&\le \left(E\left\|\sum_{r=1}^{\infty}\left(\sum_{s=0}^{\infty}\boldsymbol{B}_{s+r,t}\otimes\boldsymbol{B}_{s,t}\right)\mathrm{vec}\left(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_{t-r}^\top\right)\right\|^2\right)^{1/2} \\
&\le \left\{\mathrm{tr}\left[\sum_{r=1}^{\infty}\left(\sum_{s=0}^{\infty}\boldsymbol{B}_{s+r,t}\otimes\boldsymbol{B}_{s,t}\right)\cdot(\boldsymbol{I}_d\otimes\boldsymbol{I}_d)\cdot\left(\sum_{s=0}^{\infty}\boldsymbol{B}_{s+r,t}\otimes\boldsymbol{B}_{s,t}\right)^\top\right]\right\}^{1/2} \\
&\le M\left(\sum_{r=1}^{\infty}\left\|\sum_{s=0}^{\infty}\boldsymbol{B}_{s+r,t}\otimes\boldsymbol{B}_{s,t}\right\|^2\right)^{1/2} \le M\left(\sum_{r=1}^{\infty}\left(\sum_{s=0}^{\infty}\|\boldsymbol{B}_{s+r,t}\|^2\right)\cdot\left(\sum_{s=0}^{\infty}\|\boldsymbol{B}_{s,t}\|^2\right)\right)^{1/2} \\
&= M\left(\left(\sum_{r=1}^{\infty} r\,\|\boldsymbol{B}_{r,t}\|^2\right)\cdot\left(\sum_{s=0}^{\infty}\|\boldsymbol{B}_{s,t}\|^2\right)\right)^{1/2} \\
&\le M\left(\left(\sum_{r=1}^{\infty} r\,\|\boldsymbol{B}_{r,t}\|\right)^2\cdot\left(\sum_{s=0}^{\infty}\|\boldsymbol{B}_{s,t}\|^2\right)\right)^{1/2} < \infty.
\end{aligned}$$

Similarly, we have $E|J_{T,82}| \leq MT\delta_T \max_{t\geq 1} E\left\|\boldsymbol{\zeta}_t\boldsymbol{\epsilon}_t\right\| = O(\gamma_T)$.

Before investigating $J_{T,81}$, we first show that

$$\max_{1\leq l\leq N_T}\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\,E\left(\|\boldsymbol{\zeta}_t\boldsymbol{\epsilon}_t\|^2\,|\mathcal{F}_{t-1}\right) = O_P(1). \tag{B.1}$$

Note that

$$\begin{aligned}
&\max_{1\leq l\leq N_T}\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\,E\left(\|\boldsymbol{\zeta}_t\boldsymbol{\epsilon}_t\|^2\,|\mathcal{F}_{t-1}\right)\\
\leq\ &\max_{1\leq l\leq N_T}\left|\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\left(E\left(\|\boldsymbol{\zeta}_t\boldsymbol{\epsilon}_t\|^2\,|\mathcal{F}_{t-1}\right) - E\left\|\boldsymbol{\zeta}_t\boldsymbol{\epsilon}_t\right\|^2\right)\right|\\
&+\max_{1\leq l\leq N_T}\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\,E\left\|\boldsymbol{\zeta}_t\boldsymbol{\epsilon}_t\right\|^2
\end{aligned}$$

and $\max_{1\leq l\leq N_T}\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\,E\left\|\boldsymbol{\zeta}_t\boldsymbol{\epsilon}_t\right\|^2 = O(1)$. Thus, to prove (B.1), it is sufficient to show

$$\max_{1\leq l\leq N_T}\left|\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\left(E\left(\|\boldsymbol{\zeta}_t\boldsymbol{\epsilon}_t\|^2\,|\mathcal{F}_{t-1}\right) - E\left\|\boldsymbol{\zeta}_t\boldsymbol{\epsilon}_t\right\|^2\right)\right| = o_P(1).$$

In order to do so, we write

$$\begin{aligned}
&\max_{1\leq l\leq N_T}\left|\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\left(E\left(\|\boldsymbol{\zeta}_t\boldsymbol{\epsilon}_t\|^2\,|\mathcal{F}_{t-1}\right) - E\left\|\boldsymbol{\zeta}_t\boldsymbol{\epsilon}_t\right\|^2\right)\right|\\
=\ &\max_{1\leq l\leq N_T}\left|\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\,\mathrm{tr}\left(\sum_{r,r^*=1}^{\infty}\mathbb{B}_t^r(1)(\boldsymbol{\epsilon}_{t-r}\boldsymbol{\epsilon}_{t-r^*}^\top\otimes\boldsymbol{I}_d)\mathbb{B}_t^{r^*,\top}(1) - \sum_{r=1}^{\infty}\mathbb{B}_t^r(1)\mathbb{B}_t^{r,\top}(1)\right)\right|\\
\leq\ & d^2\cdot\max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\sum_{r=1}^{\infty}(\mathbb{B}_t^r(1)\otimes\mathbb{B}_t^r(1))\left(\mathrm{vec}\left(\boldsymbol{\epsilon}_{t-r}\boldsymbol{\epsilon}_{t-r}^\top\otimes\boldsymbol{I}_d\right) - \mathrm{vec}(\boldsymbol{I}_{d^2})\right)\right\|\\
&+2d^2\cdot\max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\sum_{r=1}^{\infty}\sum_{j=1}^{\infty}\left(\mathbb{B}_t^{r+j}(1)\otimes\mathbb{B}_t^r(1)\right)\mathrm{vec}\left(\boldsymbol{\epsilon}_{t-r}\boldsymbol{\epsilon}_{t-r-j}^\top\otimes\boldsymbol{I}_d\right)\right\|\\
&:= J_{T,91}+J_{T,92}.
\end{aligned}$$

Let $\mathbb{F}_{r,t}(L) = \sum_{j=1}^{\infty}\mathbb{B}_t^{r+j}(1)\otimes\mathbb{B}_t^j(1)L^j$. Similar to the second result of Lemma B.3, we have

$$\mathbb{F}_{r,t}(L) = \mathbb{F}_{r,t}(1) - (1-L)\widetilde{\mathbb{F}}_{r,t}(L) \tag{B.2}$$

where $\widetilde{\mathbb{F}}_{r,t}(L) = \sum_{j=1}^{\infty} \widetilde{\mathbb{F}}_{rj,t} L^j$ and $\widetilde{\mathbb{F}}_{rj,t} = \sum_{k=j+1}^{\infty} \mathbb{B}_t^{r+k}(1) \otimes \mathbb{B}_t^k(1)$. For notational simplicity, denote

$$\widetilde{X}_{at} = \sum_{j=1}^{\infty} \left(\mathbb{B}_t^j(1) \otimes \mathbb{B}_t^j(1)\right) \text{vec}\left(\boldsymbol{\epsilon}_{t-j}\boldsymbol{\epsilon}_{t-j}^\top \otimes \boldsymbol{I}_d\right),$$

$$\widetilde{X}_{bt} = \sum_{r=1}^{\infty}\sum_{j=1}^{\infty} \left(\mathbb{B}_t^{r+j}(1) \otimes \mathbb{B}_t^j(1)\right) \text{vec}\left(\boldsymbol{\epsilon}_{t-j}\boldsymbol{\epsilon}_{t-r-j}^\top \otimes \boldsymbol{I}_d\right).$$

Applying (B.2) to $\widetilde{X}_{at}$ and $\widetilde{X}_{bt}$ yields that

$$\widetilde{X}_{at} = \mathbb{F}_{0,t}(1)\text{vec}\left(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t^\top \otimes \boldsymbol{I}_d\right) - (1-L)\widetilde{\mathbb{F}}_{0,t}(L)\text{vec}\left(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t^\top \otimes \boldsymbol{I}_d\right),$$

$$\widetilde{X}_{bt} = \sum_{r=1}^{\infty} \mathbb{F}_{r,t}(1)\text{vec}\left(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_{t-r}^\top \otimes \boldsymbol{I}_d\right) - (1-L)\sum_{r=1}^{\infty} \widetilde{\mathbb{F}}_{r,t}(L)\text{vec}\left(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_{t-r}^\top \otimes \boldsymbol{I}_d\right).$$

For $J_{T,91}$, summing up $\widetilde{X}_{at}$ over t yields

$$\begin{aligned}
&\max_{1\le l\le N_T}\left\|\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\sum_{r=1}^{\infty}\left(\mathbb{B}_t^r(1)\otimes\mathbb{B}_t^r(1)\right)\left(\text{vec}\left(\boldsymbol{\epsilon}_{t-r}\boldsymbol{\epsilon}_{t-r}^\top\otimes\boldsymbol{I}_d\right)-\text{vec}(\boldsymbol{I}_{d^2})\right)\right\| \\
\le &\max_{1\le l\le N_T}\left\|\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\,\mathbb{F}_{0,t}(1)\left(\text{vec}\left(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t^\top\otimes\boldsymbol{I}_d\right)-\text{vec}\left(\boldsymbol{I}_{d^2}\right)\right)\right\| \\
&+\max_{1\le l\le N_T}\left\|\|\boldsymbol{W}_{T,1}(s_l)\|\,\widetilde{\mathbb{F}}_{0,1}(L)\text{vec}\left(\boldsymbol{\epsilon}_0\boldsymbol{\epsilon}_0^\top\otimes\boldsymbol{I}_d\right)\right\|+\sup_{0\le\tau\le1}\left\|\|\boldsymbol{W}_{T,T}(s_l)\|\,\widetilde{\mathbb{F}}_{0,T}(L)\text{vec}\left(\boldsymbol{\epsilon}_T\boldsymbol{\epsilon}_T^\top\otimes\boldsymbol{I}_d\right)\right\| \\
&+\max_{1\le l\le N_T}\left\|\sum_{t=1}^{T-1}\left(\|\boldsymbol{W}_{T,t+1}(s_l)\|\,\widetilde{\mathbb{F}}_{0,t+1}(L)-\|\boldsymbol{W}_{T,t}(s_l)\|\,\widetilde{\mathbb{F}}_{0,t}(L)\right)\text{vec}\left(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t^\top\otimes\boldsymbol{I}_d\right)\right\| \\
:= &J_{T,101}+J_{T,102}+J_{T,103}+J_{T,104}.
\end{aligned}$$

Similar to the proof of Lemma B.5.1, we can show that $J_{T,101} = O_P\left(\sqrt{d_T \log T}\right)$, since

$$\begin{aligned}
\max_{t\ge1}\|\mathbb{F}_{0,t}(1)\| &\le \max_{t\ge1}\sum_{j=1}^{\infty}\left\|\sum_{k=0}^{\infty}\boldsymbol{B}_{k+j,t}\otimes\boldsymbol{B}_{k,t}\right\|^2 \le \max_{t\ge1}\sum_{j=1}^{\infty}\left(\sum_{k=0}^{\infty}\|\boldsymbol{B}_{k+j,t}\|^2\right)\left(\sum_{k=0}^{\infty}\|\boldsymbol{B}_{k,t}\|^2\right) \\
&\le \max_{t\ge1}\left(\sum_{k=0}^{\infty}\|\boldsymbol{B}_{k,t}\|^2\right)\left(\sum_{j=1}^{\infty}j\,\|\boldsymbol{B}_{j,t}\|^2\right) < \infty.
\end{aligned}$$

Also, we can show that $J_{T,102} = O_P(d_T)$ and $J_{T,103} = O_P(d_T)$, since

$$\begin{aligned}
\max_{t\ge1}\left\|\widetilde{\mathbb{F}}_{0,t}(1)\right\| &\le \max_{t\ge1}\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}\left\|\mathbb{B}_t^k(1)\right\|^2 \le \max_{t\ge1}\sum_{r=1}^{\infty}\sum_{k=r+1}^{\infty}\left(\sum_{j=0}^{\infty}\|\boldsymbol{B}_{j+k,t}\|^2\right)\left(\sum_{j=0}^{\infty}\|\boldsymbol{B}_{j,t}\|^2\right) \\
&\le \max_{t\ge1}\left(\sum_{j=0}^{\infty}\|\boldsymbol{B}_{j,t}\|^2\right)\left(\sum_{r=1}^{\infty}\sum_{k=r+1}^{\infty}(k-r)\,\|\boldsymbol{B}_{k,t}\|^2\right) \\
&\le \max_{t\ge1}\left(\sum_{j=0}^{\infty}\|\boldsymbol{B}_{j,t}\|^2\right)\left(\sum_{r=1}^{\infty}\frac{r(r+1)}{2}\,\|\boldsymbol{B}_{r+1,t}\|^2\right)
\end{aligned}$$

$$\leq \max_{t\geq 1}\left(\sum_{j=0}^{\infty}\|\boldsymbol{B}_{j,t}\|^2\right)\left(\sum_{j=1}^{\infty}j^2\|\boldsymbol{B}_{j,t}\|^2\right)<\infty.$$

We can easily show $J_{T,104}=o_P(1)$, since

$$\sup_{\tau\in[a,b]}\left(\sum_{t=1}^{T-1}\|\boldsymbol{W}_{T,t+1}(\tau)\|-\|\boldsymbol{W}_{T,t}(\tau)\|\right)\leq \sup_{\tau\in[a,b]}\sum_{t=1}^{T-1}\|\boldsymbol{W}_{T,t+1}(\tau)-\boldsymbol{W}_{T,t}(\tau)\|=o(1)$$

and

$$\begin{aligned}
&\sum_{t=1}^{T-1}\left\|\widetilde{\mathbb{F}}_{0,t+1}(1)-\widetilde{\mathbb{F}}_{0,t}(1)\right\|\leq\sum_{t=1}^{T-1}\sum_{r=1}^{\infty}\sum_{k=r+1}^{\infty}\left\|\mathbb{B}_{t+1}^k(1)\otimes\mathbb{B}_{t+1}^k(1)-\mathbb{B}_t^k(1)\otimes\mathbb{B}_t^k(1)\right\|\\
&\leq\sum_{t=1}^{T-1}\sum_{r=1}^{\infty}\sum_{k=r+1}^{\infty}\left\|\mathbb{B}_{t+1}^k(1)-\mathbb{B}_t^k(1)\right\|\cdot\left(\left\|\mathbb{B}_{t+1}^k(1)\right\|+\left\|\mathbb{B}_t^k(1)\right\|\right)\\
&\leq M\sum_{t=1}^{T-1}\sum_{r=1}^{\infty}\sum_{k=r+1}^{\infty}\sum_{j=0}^{\infty}\|\boldsymbol{B}_{j+k,t+1}\otimes\boldsymbol{B}_{j,t+1}-\boldsymbol{B}_{j+k,t}\otimes\boldsymbol{B}_{j,t}\|\\
&\leq M\sum_{t=1}^{T-1}\sum_{r=1}^{\infty}\sum_{k=r+1}^{\infty}\sum_{j=0}^{\infty}(\|\boldsymbol{B}_{j+k,t+1}-\boldsymbol{B}_{j+k,t}\|\cdot\|\boldsymbol{B}_{j,t+1}\|+\|\boldsymbol{B}_{j,t+1}-\boldsymbol{B}_{j,t}\|\cdot\|\boldsymbol{B}_{j+k,t}\|)\\
&\leq M\sum_{t=1}^{T-1}\sum_{r=1}^{\infty}\sum_{k=r+1}^{\infty}\left(\|\boldsymbol{B}_{k,t+1}-\boldsymbol{B}_{k,t}\|\cdot\sum_{j=0}^{\infty}\|\boldsymbol{B}_{j,t+1}\|+\|\boldsymbol{B}_{k,t}\|\cdot\sum_{j=0}^{\infty}\|\boldsymbol{B}_{j,t+1}-\boldsymbol{B}_{j,t}\|\right)\\
&\leq M\left(\sum_{t=1}^{T-1}\sum_{k=1}^{\infty}k\|\boldsymbol{B}_{k,t+1}-\boldsymbol{B}_{k,t}\|\right)\cdot\left(\max_{t\geq 1}\sum_{j=0}^{\infty}\|\boldsymbol{B}_{j,t+1}\|\right)\\
&\quad+M\left(\max_{t\geq 1}\sum_{k=1}^{\infty}k\|\boldsymbol{B}_{k,t}\|\right)\cdot\left(\sum_{t=1}^{T-1}\sum_{j=0}^{\infty}\|\boldsymbol{B}_{j,t+1}-\boldsymbol{B}_{j,t}\|\right)=O(1).
\end{aligned}$$

Based on the above development, we conclude that $J_{T,91}=o_P(1)$. Next, we focus on $J_{T,92}$, and write

$$\begin{aligned}
&\max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\sum_{r=1}^{\infty}\sum_{j=1}^{\infty}\mathbb{B}_t^{r+j}(1)\otimes\mathbb{B}_t^r(1)\mathrm{vec}\left(\boldsymbol{\epsilon}_{t-r}\boldsymbol{\epsilon}_{t-r-j}^\top\otimes\boldsymbol{I}_d\right)\right\|\\
&\leq\max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\sum_{r=1}^{\infty}\mathbb{F}_{r,t}(1)\mathrm{vec}\left(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_{t-r}^\top\otimes\boldsymbol{I}_d\right)\right\|\\
&\quad+\max_{1\leq l\leq N_T}\left\|\|\boldsymbol{W}_{T,1}(s_l)\|\sum_{r=1}^{\infty}\widetilde{\mathbb{F}}_{r,1}(L)\mathrm{vec}\left(\boldsymbol{\epsilon}_0\boldsymbol{\epsilon}_{-r}^\top\otimes\boldsymbol{I}_d\right)\right\|\\
&\quad+\max_{1\leq l\leq N_T}\left\|\|\boldsymbol{W}_{T,T}(s_l)\|\sum_{r=1}^{\infty}\widetilde{\mathbb{F}}_{r,T}(L)\mathrm{vec}\left(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_{T-r}^\top\otimes\boldsymbol{I}_d\right)\right\|\\
&\quad+\max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T-1}\sum_{r=1}^{\infty}\left(\|\boldsymbol{W}_{T,t+1}(s_l)\|\widetilde{\mathbb{F}}_{r,t+1}(L)-\|\boldsymbol{W}_{T,t}(s_l)\|\widetilde{\mathbb{F}}_{r,t}(L)\right)\mathrm{vec}\left(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_{t-r}^\top\otimes\boldsymbol{I}_d\right)\right\|\\
&:=J_{T,111}+J_{T,112}+J_{T,113}+J_{T,114}.
\end{aligned}$$

We can show that $J_{T,112}$ and $J_{T,113}$ are $O_P(d_T)$, since

$$\max_{t\geq 1}\left\|\sum_{r=1}^{\infty}\widetilde{\mathbb{F}}_{r,t}(1)\right\| \leq \max_{t\geq 1}\sum_{r=1}^{\infty}\sum_{j=1}^{\infty}\left\|\widetilde{\mathbb{F}}_{rj,t}\right\| \leq \max_{t\geq 1}\sum_{r=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}\left\|\mathbb{B}_t^{r+k}(1)\right\|\left\|\mathbb{B}_t^{k}(1)\right\|$$
$$\leq \max_{t\geq 1}\left(\sum_{r=1}^{\infty}\|\mathbb{B}_t^r(1)\|\right)\left(\sum_{j=1}^{\infty}j\left\|\mathbb{B}_t^j(1)\right\|\right) \leq M\max_{t\geq 1}\sum_{j=1}^{\infty}j\sum_{k=0}^{\infty}\|\boldsymbol{B}_{k+j,t}\|\,\|\boldsymbol{B}_{k,t}\|$$
$$\leq M\max_{t\geq 1}\left(\sum_{j=1}^{\infty}j\,\|\boldsymbol{B}_{j,t}\|\right)\left(\sum_{k=0}^{\infty}\|\boldsymbol{B}_{k,t}\|\right) < \infty.$$

Similar to $J_{T,104}$, we have $J_{T,114} = o_P(1)$, since

$$\sum_{t=1}^{T-1}\sum_{r=1}^{\infty}\left\|\widetilde{\mathbb{F}}_{r,t+1}(1)-\widetilde{\mathbb{F}}_{r,t}(1)\right\| \leq \sum_{t=1}^{T-1}\sum_{r=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}\left\|\mathbb{B}_{t+1}^{r+k}(1)\otimes\mathbb{B}_{t+1}^{k}(1)-\mathbb{B}_t^{r+k}(1)\otimes\mathbb{B}_t^k(1)\right\|$$
$$\leq \sum_{t=1}^{T-1}\sum_{r=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}\left(\left\|\mathbb{B}_{t+1}^{r+k}(1)-\mathbb{B}_t^{r+k}(1)\right\|\left\|\mathbb{B}_{t+1}^k(1)\right\|+\left\|\mathbb{B}_{t+1}^k(1)-\mathbb{B}_t^k(1)\right\|\left\|\mathbb{B}_t^{r+k}(1)\right\|\right)$$
$$\leq \left(\sum_{t=1}^{T-1}\sum_{r=1}^{\infty}\|\mathbb{B}_{t+1}^r(1)-\mathbb{B}_t^r(1)\|\right)\left(\max_{t\geq 1}\sum_{j=1}^{\infty}j\left\|\mathbb{B}_t^j\right\|\right)$$
$$+\left(\sum_{t=1}^{T-1}\sum_{j=1}^{\infty}j\left\|\mathbb{B}_{t+1}^j(1)-\mathbb{B}_t^j(1)\right\|\right)\left(\max_{t\geq 1}\sum_{r=1}^{\infty}\|\mathbb{B}_t^r\|\right) = O(1).$$

Now consider term $J_{T,111}$. Define $\boldsymbol{u}_t = \sum_{r=1}^{\infty}\mathbb{F}_{r,t}(1)\text{vec}\left(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_{t-r}^\top\otimes\boldsymbol{I}_d\right)$, $\boldsymbol{u}_t' = \boldsymbol{u}_tI(\|\boldsymbol{u}_t\|\leq T^{\frac{2}{\delta}})$ and $\boldsymbol{u}_t'' = \boldsymbol{u}_t-\boldsymbol{u}_t'$. Then we have

$$J_{T,111} = \max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\left(\boldsymbol{u}_t'+\boldsymbol{u}_t''-E(\boldsymbol{u}_t'+\boldsymbol{u}_t''|\mathcal{F}_{t-1})\right)\right\|$$
$$\leq \max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\left(\boldsymbol{u}_t'-E(\boldsymbol{u}_t'|\mathcal{F}_{t-1})\right)\right\| + \max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\,\boldsymbol{u}_t''\right\|$$
$$+\max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\,E(\boldsymbol{u}_t''|\mathcal{F}_{t-1})\right\| := J_{T,121}+J_{T,122}+J_{T,123}.$$

Using the same argument as that used in the proof of $J_{T,72}$ in Lemma B.4, we can show that $J_{T,122}$ and $J_{T,123}$ are $O_P\left(T^{\frac{2}{\delta}}d_T\right)$. Next, consider $J_{T,121}$. For any $1\leq l\leq N_T$, let $\boldsymbol{Y}_t = \|\boldsymbol{W}_{T,t}(s_l)\|\,(\boldsymbol{u}_t' - E(\boldsymbol{u}_t'|\mathcal{F}_{t-1}))$. We then have $E\left(\boldsymbol{Y}_t|\mathcal{F}_{t-1}\right)=0$ and $\|\boldsymbol{Y}_t\|\leq 2T^{2/\delta}d_T$. In addition, we have

$$\max_{1\leq l\leq N_T}\left\|\sum_{t=1}^{T}E(\boldsymbol{Y}_t\boldsymbol{Y}_t^\top|\mathcal{F}_{t-1})\right\| \leq 4\max_{1\leq l\leq N_T}\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|^2E\left(\|\boldsymbol{u}_t\|^2|\mathcal{F}_{t-1}\right)$$
$$\leq M\cdot d_T\max_{1\leq l\leq N_T}\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\sum_{r=1}^{\infty}\|\mathbb{F}_{r,t}(1)\|^2\|\boldsymbol{\epsilon}_{t-r}\|^2 \leq M\cdot d_T\max_{t\geq 1}\sum_{r=1}^{\infty}\|\mathbb{F}_{r,t}(1)\|^2\|\boldsymbol{\epsilon}_{t-r}\|^2$$

$$\leq M \cdot d_T \sum_{r=1}^{\infty} \max_{t\geq 1} \|\mathbb{F}_{r,t}(1)\|^2 \left(\sum_{t=1}^{T} \|\boldsymbol{\epsilon}_{t-r}\|^{\delta} \right)^{\frac{2}{\delta}} = O_P\left(d_T T^{\frac{2}{\delta}}\right).$$

Therefore, we have $\max_{1\leq l\leq N_T} \left\| \sum_{t=1}^T E(\boldsymbol{Y}_t \boldsymbol{Y}_t^\top | \mathcal{F}_{t-1}) \right\| = O_P(d_T T^{\frac{2}{\delta}})$. By Lemma B.2, and choosing $\beta = 4$, we have

$$\begin{aligned}
&\Pr\left(J_{T,121} > \sqrt{\beta M} \sqrt{d_T T^{\frac{2}{\delta}} \log T} \right) \\
&= \Pr\left(J_{T,121} > \sqrt{\beta M} \sqrt{d_T T^{\frac{2}{\delta}} \log T}, \max_{1\leq l\leq N_T} \left\| \sum_{t=1}^T E(\boldsymbol{Y}_t \boldsymbol{Y}_t^\top | \mathcal{F}_{t-1}) \right\| \leq M d_T T^{\frac{2}{\delta}} \right) \\
&\quad + \Pr\left(J_{T,121} > \sqrt{\beta M} \sqrt{d_T T^{\frac{2}{\delta}} \log T}, \max_{1\leq l\leq N_T} \left\| \sum_{t=1}^T E(\boldsymbol{Y}_t \boldsymbol{Y}_t^\top | \mathcal{F}_{t-1}) \right\| > M d_T T^{\frac{2}{\delta}} \right) \\
&\leq \Pr\left(J_{T,121} > \sqrt{\beta M} \sqrt{d_T T^{\frac{2}{\delta}} \log T}, \max_{1\leq l\leq N_T} \left\| \sum_{t=1}^T E(\boldsymbol{Y}_t \boldsymbol{Y}_t^\top | \mathcal{F}_{t-1}) \right\| \leq M d_T T^{\frac{2}{\delta}} \right) \\
&\quad + \Pr\left(\max_{1\leq l\leq N_T} \left\| \sum_{t=1}^T E(\boldsymbol{Y}_t \boldsymbol{Y}_t^\top | \mathcal{F}_{t-1}) \right\| > M d_T T^{\frac{2}{\delta}} \right) \\
&\leq N_T \exp\left(-\frac{\beta M d_T T^{\frac{2}{\delta}} \log T}{2(M d_T T^{\frac{2}{\delta}} + M\sqrt{d_T T^{\frac{2}{\delta}} \log T} T^{\frac{2}{\delta}} d_T)} \right) + o(1) \\
&\leq N_T \exp\left(-\frac{\beta}{2} \log T \right) = N_T T^{-\frac{\beta}{2}} = o(1)
\end{aligned}$$

given $d_T T^{\frac{4}{\delta}} \log T \to 0$. Hence, we have $J_{T,611} = O_P(\{d_T T^{\frac{2}{\delta}} \log T\}^{1/2})$. Combining the above results, we have proved that $\sup_{\tau\in[a,b]} \left| \sum_{t=1}^T \|\boldsymbol{W}_{T,t}(\tau)\| E\left(\|\boldsymbol{\zeta}_t \boldsymbol{\epsilon}_t\|^2 | \mathcal{F}_{t-1} \right) \right| = O_P(1)$.

Finally, we turn to $J_{T,81}$, and apply the truncation method. Let $\boldsymbol{u}_t = \boldsymbol{\zeta}_t \boldsymbol{\epsilon}_t$, $\boldsymbol{u}_t' = \boldsymbol{u}_t I\left(\|\boldsymbol{u}_t\| \leq T^{\frac{2}{\delta}} \right)$ and $\boldsymbol{u}_t'' = \boldsymbol{u}_t - \boldsymbol{u}_t'$. Then we have

$$\begin{aligned}
J_{T,81} &= \max_{1\leq l\leq N_T} \left\| \sum_{t=1}^T (\boldsymbol{I}_d \otimes \boldsymbol{W}_{T,t}(s_l)) \left(\boldsymbol{u}_t' + \boldsymbol{u}_t'' - E(\boldsymbol{u}_t' + \boldsymbol{u}_t'' | \mathcal{F}_{t-1}) \right) \right\| \\
&\leq \max_{1\leq l\leq N_T} \left\| \sum_{t=1}^T (\boldsymbol{I}_d \otimes \boldsymbol{W}_{T,t}(s_l)) \left(\boldsymbol{u}_t' - E(\boldsymbol{u}_t' | \mathcal{F}_{t-1}) \right) \right\| + \max_{1\leq l\leq N_T} \left\| \sum_{t=1}^T (\boldsymbol{I}_d \otimes \boldsymbol{W}_{T,t}(s_l)) \boldsymbol{u}_t'' \right\| \\
&\quad + \max_{1\leq l\leq N_T} \left\| \sum_{t=1}^T (\boldsymbol{I}_d \otimes \boldsymbol{W}_{T,t}(s_l)) E(\boldsymbol{u}_t'' | \mathcal{F}_{t-1}) \right\| = J_{T,131} + J_{T,132} + J_{T,133}.
\end{aligned}$$

It's easy to show that $J_{T,132} = O_P(T^{\frac{2}{\delta}} d_T)$ and $J_{T,133} = O_P(T^{\frac{2}{\delta}} d_T)$. Thus, we focus on $J_{T,131}$.

For any $1 \leq l \leq N_T$, let $\boldsymbol{Y}_t = (\boldsymbol{I}_d \otimes \boldsymbol{W}_{T,t}(s_l))(\boldsymbol{u}_t' - E(\boldsymbol{u}_t' | \mathcal{F}_{t-1}))$, then we have $E(\boldsymbol{Y}_t | \mathcal{F}_{t-1}) = 0$ and $\|\boldsymbol{Y}_t\| \leq 2T^{2/\delta} d_T$. Also,

$$\max_{1\leq l\leq N_T} \sum_{t=1}^T \|\boldsymbol{W}_{T,t}(s_l)\| E\left(\|\boldsymbol{\zeta}_t \boldsymbol{\epsilon}_t\|^2 | \mathcal{F}_{t-1} \right) = O_P(1),$$

which yields

$$\max_{1\le l\le N_T}\left\|\sum_{t=1}^{T} E(\boldsymbol{Y}_t\boldsymbol{Y}_t^\top|\mathcal{F}_{t-1})\right\| \le Md_T \max_{1\le l\le N_T}\sum_{t=1}^{T}\|\boldsymbol{W}_{T,t}(s_l)\|\, E\left(\|\boldsymbol{u}_t\|^2\,|\mathcal{F}_{t-1}\right) = O_P\left(d_T\right).$$

Therefore, we have $\max_{1\le l\le N_T}\left\|\sum_{t=1}^{T} E(\boldsymbol{Y}_t\boldsymbol{Y}_t^\top|\mathcal{F}_{t-1})\right\| = O_P(d_T)$. By Lemma B.2 and choosing $\beta = 4$, we have

$$\begin{aligned}
&\Pr\left(J_{T,131} > \sqrt{\beta M}\gamma_T\right) = \Pr\left(J_{T,31} > \sqrt{\beta M}\gamma_T, \max_{1\le l\le N_T}\left\|\sum_{t=1}^{T} E(\boldsymbol{Y}_t\boldsymbol{Y}_t^\top|\mathcal{F}_{t-1})\right\| \le Md_T\right)\\
&+\Pr\left(J_{T,131} > \sqrt{\beta M}\gamma_T, \max_{1\le l\le N_T}\left\|\sum_{t=1}^{T} E(\boldsymbol{Y}_t\boldsymbol{Y}_t^\top|\mathcal{F}_{t-1})\right\| > Md_T\right)\\
&\le \Pr\left(J_{T,131} > \sqrt{\beta M}\gamma_T, \max_{1\le l\le N_T}\left\|\sum_{t=1}^{T} E(\boldsymbol{Y}_t\boldsymbol{Y}_t^\top|\mathcal{F}_{t-1})\right\| \le Md_T\right)\\
&+\Pr\left(\max_{1\le l\le N_T}\left\|\sum_{t=1}^{T} E(\boldsymbol{Y}_t\boldsymbol{Y}_t^\top|\mathcal{F}_{t-1})\right\| > Md_T\right) \le N_T\exp\left(-\frac{\beta M\gamma_T^2}{2(Md_T + M\gamma_T T^{\frac{2}{\delta}}d_T)}\right) + o(1)\\
&\le N_T\exp\left(-\frac{\beta}{2}\log T\right) = N_T T^{-\frac{\beta}{2}} = o(1)
\end{aligned}$$

given $d_T T^{\frac{4}{\delta}}\log T \to 0$.

We now have completed the proof of the second result. □

Proof of Lemma B.6.

Let $\boldsymbol{\Psi}_j(\tau) = \boldsymbol{J}\boldsymbol{\Phi}^j(\tau)\boldsymbol{J}^\top$, where

$$\boldsymbol{\Phi}(\tau) = \begin{pmatrix} \boldsymbol{A}_1(\tau) & \cdots & \boldsymbol{A}_{p-1}(\tau) & \boldsymbol{A}_p(\tau)\\ \boldsymbol{I}_d & \cdots & \boldsymbol{0}_d & \boldsymbol{0}_d\\ \vdots & \ddots & \vdots & \vdots\\ \boldsymbol{0}_d & \cdots & \boldsymbol{I}_d & \boldsymbol{0}_d \end{pmatrix}$$

and $\boldsymbol{J} = \left[\boldsymbol{I}_d, \boldsymbol{0}_{d\times d(p-1)}\right]$.

To proceed, we write $\boldsymbol{y_t}$ as a time–varying VMA(∞):

$$\boldsymbol{y_t} = \sum_{j=0}^{\infty}\boldsymbol{\Psi}_{j,t}\left(\sum_{l=0}^{q}\boldsymbol{B}_l(\tau_{t-j})\boldsymbol{x}_{t-l-j} + \boldsymbol{\eta}_{t-j}\right) = \boldsymbol{\mu}_t^* + \sum_{j=0}^{\infty}\boldsymbol{D}_{j,t}^{\boldsymbol{\epsilon}}\boldsymbol{\epsilon}_{t-j} + \sum_{l=0}^{q}\sum_{j=0}^{\infty}\boldsymbol{D}_{j,l,t}^{\boldsymbol{v}}\boldsymbol{v}_{t-l-j},$$

where $\boldsymbol{\mu}_t^* = \sum_{j=0}^{\infty}\boldsymbol{\Psi}_{j,t}(\boldsymbol{\mu}(\tau_{t-j}) + \sum_{l=0}^{q}\boldsymbol{B}_l(\tau_{t-j})\boldsymbol{g}(\tau_{t-l-j}))$, $\boldsymbol{\Psi}_{j,t} = \boldsymbol{J}\prod_{m=0}^{j-1}\boldsymbol{\Phi}(\tau_{t-m})\boldsymbol{J}^\top$, $\boldsymbol{D}_{j,t}^{\boldsymbol{\epsilon}} = \boldsymbol{\Psi}_{j,t}\boldsymbol{\omega}(\tau_{t-j})$, and $\boldsymbol{D}_{j,l,t}^{\boldsymbol{v}} = \sum_{k=0}^{j}\boldsymbol{\Psi}_{k,t}\boldsymbol{B}_l(\tau_{t-k})\boldsymbol{C}_{j-k}(\tau_{t-l-k})$.

Let ρ denote the largest eigenvalue of $\boldsymbol{\Phi}(\tau)$ uniformly over $\tau \in [0, 1]$. Then we have $\rho < 1$ by Assumption 1.1. In addition, similar to the proof of Proposition 2.4 in Dahlhaus and Polonik (2009), we have $\max_{t\geq 1} \| \prod_{m=0}^{j-1} \mathbf{\Phi}(\tau_{t-m}) \| \leq M\rho^j$.

Next, we will show that $\boldsymbol{y}_t$ can be approximated by a time-varying MA(∞) process $\widetilde{\boldsymbol{y}}_t$ satisfying $\{E\|\boldsymbol{y}_t - \widetilde{\boldsymbol{y}}_t\|^\delta\}^{1/\delta} = O(T^{-1})$, where $\widetilde{\boldsymbol{y}}_t$ has been defined in the body of this lemma. It follows that

$$\begin{aligned}\{E\|\boldsymbol{y}_t - \widetilde{\boldsymbol{y}}_t\|^\delta\}^{1/\delta} &\leq M \left(\|\boldsymbol{\mu}_t^* - \boldsymbol{\mu}^*(\tau_t)\| + \sum_{j=0}^{\infty} \left\|\boldsymbol{D}_{j,t}^{\boldsymbol{\epsilon}} - \boldsymbol{D}_j^{\boldsymbol{\epsilon}}(\tau_t)\right\| + \sum_{l=0}^{q}\sum_{j=0}^{\infty} \left\|\boldsymbol{D}_{j,l,t}^{\boldsymbol{v}} - \boldsymbol{D}_{j,l}^{\boldsymbol{v}}(\tau_t)\right\| \right) \\ &:= O(1)\cdot (J_{T,141} + J_{T,142} + J_{T,143}),\end{aligned}$$

where the definitions of $J_{T,141}$, $J_{T,142}$, and $J_{T,143}$ are obvious.

Consider $J_{T,141}$. Note that for any conformable matrices $\{\boldsymbol{A}_i\}$ and $\{\boldsymbol{B}_i\}$, since

$$\prod_{i=1}^{r} \boldsymbol{A}_i - \prod_{i=1}^{r} \boldsymbol{B}_i = \sum_{j=1}^{r} \left(\prod_{k=1}^{j-1} \boldsymbol{A}_k\right) (\boldsymbol{A}_j - \boldsymbol{B}_j) \left(\prod_{k=j+1}^{r} \boldsymbol{B}_k\right),$$

we obtain

$$\begin{aligned}\|\mathbf{\Psi}_{j,t} - \mathbf{\Psi}_j(\tau_t)\| &= \left\| \boldsymbol{J} \prod_{m=0}^{j-1} \mathbf{\Phi}(\tau_{t-m}) \boldsymbol{J}^\top - \boldsymbol{J}\mathbf{\Phi}^j(\tau_t)\boldsymbol{J}^\top \right\| \\ &\leq M \sum_{i=1}^{j-1} \left\| \mathbf{\Phi}^i(\tau_t)(\mathbf{\Phi}(\tau_{t-i}) - \mathbf{\Phi}(\tau_t)) \prod_{m=i+1}^{j-1} \mathbf{\Phi}(\tau_{t-m}) \right\| \leq M \sum_{i=1}^{j-1} \frac{i}{T}\rho^{j-1}.\end{aligned}$$

Hence, we have

$$\begin{aligned}J_{T,141} &\leq \sum_{j=0}^{\infty} \|\mathbf{\Psi}_{j,t} - \mathbf{\Psi}_j(\tau_t)\| \cdot \|\boldsymbol{\mu}(\tau_{t-j})\| + \sum_{j=0}^{\infty} \|\mathbf{\Psi}_j(\tau_t)\| \cdot \|\boldsymbol{\mu}(\tau_{t-j}) - \boldsymbol{\mu}(\tau_t)\| \\ &\quad + \sum_{j=0}^{\infty} \|\mathbf{\Psi}_{j,t} - \mathbf{\Psi}_j(\tau_t)\| \cdot \sum_{l=0}^{q} \|\boldsymbol{B}_l(\tau_{t-j})\boldsymbol{g}(\tau_{t-l-j})\| \\ &\quad + \sum_{j=0}^{\infty} \|\mathbf{\Psi}_j(\tau_t)\| \cdot \sum_{l=0}^{q} \|\boldsymbol{B}_l(\tau_{t-j})\boldsymbol{g}(\tau_{t-l-j}) - \boldsymbol{B}_l(\tau_t)\boldsymbol{g}(\tau_t)\| \\ &\leq M \sum_{j=0}^{\infty}\sum_{i=1}^{j-1} \frac{i}{T}\rho^{j-1} + M\sum_{j=0}^{\infty} \rho^j \frac{j}{T} = O(T^{-1}),\end{aligned}$$

where we have used the facts that $\|\mathbf{\Psi}_j(\tau)\| \leq M\rho^j$ and

$$\begin{aligned}&\|\boldsymbol{B}_l(\tau_{t-j})\boldsymbol{g}(\tau_{t-l-j}) - \boldsymbol{B}_l(\tau_t)\boldsymbol{g}(\tau_t)\| \\ &= \|\boldsymbol{B}_l(\tau_{t-j})\boldsymbol{g}(\tau_{t-l-j}) - \boldsymbol{B}_l(\tau_t)\boldsymbol{g}(\tau_{t-l-j}) + \boldsymbol{B}_l(\tau_t)\boldsymbol{g}(\tau_{t-l-j}) - \boldsymbol{B}_l(\tau_t)\boldsymbol{g}(\tau_t)\| \\ &\leq \|\boldsymbol{B}_l(\tau_{t-j}) - \boldsymbol{B}_l(\tau_t)\| \cdot \|\boldsymbol{g}(\tau_{t-l-j})\| + \|\boldsymbol{B}_l(\tau_t)\| \cdot \|\boldsymbol{g}(\tau_{t-l-j}) - \boldsymbol{g}(\tau_t)\| \leq M\frac{j}{T}.\end{aligned}$$

Similarly, we have $J_{T,142} = O(T^{-1})$.

For $J_{T,143}$, we have

$$\begin{aligned}
J_{T,143} &\leq \sum_{l=0}^{q}\sum_{j=0}^{\infty}\sum_{k=0}^{j} \|\mathbf{\Psi}_{k,t}\boldsymbol{B}_l(\tau_{t-k})\boldsymbol{C}_{j-k}(\tau_{t-l-k}) - \mathbf{\Psi}_k(\tau_t)\boldsymbol{B}_l(\tau_t)\boldsymbol{C}_{j-k}(\tau_t)\| \\
&= \sum_{l=0}^{q}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \|\mathbf{\Psi}_{j,t}\boldsymbol{B}_l(\tau_{t-j})\boldsymbol{C}_k(\tau_{t-l-j}) - \mathbf{\Psi}_j(\tau_t)\boldsymbol{B}_l(\tau_t)\boldsymbol{C}_k(\tau_t)\| \\
&\leq \sum_{l=0}^{q}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \|\mathbf{\Psi}_{j,t} - \mathbf{\Psi}_j(\tau_t)\| \cdot \|\boldsymbol{B}_l(\tau_{t-j})\| \cdot \|\boldsymbol{C}_k(\tau_{t-l-j})\| \\
&\quad + \sum_{l=0}^{q}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \|\mathbf{\Psi}_j(\tau_t)\| \cdot \|\boldsymbol{B}_l(\tau_{t-j}) - \boldsymbol{B}_l(\tau_t)\| \cdot \|\boldsymbol{C}_k(\tau_{t-l-j})\| \\
&\quad + \sum_{l=0}^{q}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \|\mathbf{\Psi}_j(\tau_t)\| \cdot \|\boldsymbol{B}_l(\tau_t)\| \cdot \|\boldsymbol{C}_k(\tau_{t-l-j}) - \boldsymbol{C}_k(\tau_t)\| \\
&\leq M\sum_{j=0}^{\infty}\sum_{i=1}^{j-1}\frac{i}{T}\rho^{j-1} \cdot \sup_{\tau\in[0,1]}\sum_{k=0}^{\infty}\|\boldsymbol{C}_k(\tau)\| \\
&\quad + \frac{M}{T}\left(\sum_{j=0}^{\infty} j\rho^j\right)\left(\sup_{\tau\in[0,1]}\sum_{k=0}^{\infty}\|\boldsymbol{C}_k(\tau)\| + \sup_{\tau\in[0,1]}\sum_{k=0}^{\infty}\left\|\boldsymbol{C}_k^{(1)}(\tau)\right\|\right) = O(T^{-1}).
\end{aligned}$$

In addition, it is straightforward to verify that

$$\sup_{\tau\in[0,1]}\sum_{j=0}^{\infty} j\|\boldsymbol{D}_j^{\boldsymbol{\epsilon},(k)}(\tau)\| < \infty \quad \text{and} \quad \sup_{\tau\in[0,1]}\sum_{j=0}^{\infty} j\|\boldsymbol{D}_{j,l}^{\boldsymbol{v},(k)}(\tau)\| < \infty$$

for $k = 0, 1$ (see, the proof of Propositions 2.1, for example.)

The proof is now completed. □

Proof of Lemma B.7.

(1)–(2). To prove parts (1) and (2), it suffices to show that

$$\sup_{\tau\in[0,1]}\left\|\frac{1}{T}\sum_{t=1}^{T}\left(\boldsymbol{Z}_t\boldsymbol{Z}_t^\top - E(\boldsymbol{Z}_t\boldsymbol{Z}_t^\top)\right)\left(\frac{\tau_t-\tau}{h}\right)^k K_h(\tau_t-\tau)\right\| = O_P\left(\sqrt{\frac{\log T}{Th}}\right)$$

for $k = 0, 1, 2$. Since $\boldsymbol{Z}_t$ can be approximated by a time-varying VMA(∞) process by Lemma B.6, then the uniform convergence results follow directly from Lemma 2.3.

(3). Part (3) follows directly from Lemmas B.6 and 2.3.

(4). The uniform convergence rate for $\widehat{\boldsymbol{\beta}}(\tau)$ follows directly from the proof of Theorem 4.1 (1), Lemma 2.3 and part (3).

(5). By the proof of Theorem 4.1 (2), we have

$$\sup_{\tau\in[0,1]}\left\|\widehat{\mathbf{\Omega}}(\tau) - T^{-1}\sum_{t=1}^{T}\boldsymbol{\eta}_t\boldsymbol{\eta}_t^\top K_h(\tau_t-\tau)\right\| = O_P(h^2 + \sqrt{\log T/(Th)}).$$

Similar to the proof of Lemma A.5 (1), we have

$$\sup_{\tau\in[0,1]}\left\|T^{-1}\sum_{t=1}^{T}\left[\boldsymbol{\eta}_t\boldsymbol{\eta}_t^\top - E(\boldsymbol{\eta}_t\boldsymbol{\eta}_t^\top)\right]K_h(\tau_t-\tau)\right\| = O_P(\sqrt{\log T/(Th)}).$$

The proof of part (5) is now complete. □

Proof of Lemma B.8.

(1). By Lemma B.7, we have

$$\sup_{\tau\in[0,1]}\left\|\boldsymbol{s}(\tau)\boldsymbol{Z}_1 - \boldsymbol{\Sigma}_{\boldsymbol{Z}_2}^{-1}(\tau)\boldsymbol{\Sigma}_{\boldsymbol{Z}_{1,2}}^\top(\tau)\right\| = o_P(1)$$

and $\sup_{\tau\in[0,1]}\left\|\widehat{\boldsymbol{\Omega}}(\tau)-\boldsymbol{\Omega}(\tau)\right\| = o_P(1)$. Hence, we have

$$\begin{aligned}&T^{-1}\sum_{t=1}^{T}\widetilde{\boldsymbol{Z}}_{1t}\widehat{\boldsymbol{\Omega}}^{-1}(\tau_t)\widetilde{\boldsymbol{Z}}_{1t}^\top\\ &= T^{-1}\sum_{t=1}^{T}\left(\boldsymbol{z}_{1t}-\boldsymbol{\Sigma}_{\boldsymbol{z}_{1,2}}(\tau_t)\boldsymbol{\Sigma}_{\boldsymbol{z}_2}^{-1}(\tau_t)\boldsymbol{z}_{2t}^\top\right)\left(\boldsymbol{z}_{1t}^\top-\boldsymbol{z}_{2t}^\top\boldsymbol{\Sigma}_{\boldsymbol{z}_2}^{-1}(\tau_t)\boldsymbol{\Sigma}_{\boldsymbol{z}_{1,2}}^\top(\tau_t)\right)\otimes\boldsymbol{\Omega}^{-1}(\tau_t)+o_P(1)\end{aligned}$$

Note that each element of $\boldsymbol{\Sigma}_{\boldsymbol{z}_2}(\tau)$ and $\boldsymbol{\Sigma}_{\boldsymbol{z}_{1,2}}(\tau)$ is Lipschitz continuous. Thus, by Lemmas 2.2 and B.6, the result holds.

(2). Let $\rho_T = h^2+\sqrt{\log T/(Th)}$. By Lemma B.7 (1), we have

$$\boldsymbol{Z}_{2t}^\top\boldsymbol{s}(\tau_t)\left[\boldsymbol{\theta}(\tau_1)^\top\boldsymbol{Z}_{21},...,\boldsymbol{\theta}(\tau_T)^\top\boldsymbol{Z}_{2T}\right]^\top = \boldsymbol{Z}_{2t}^\top\boldsymbol{\theta}(\tau_t)(1+O_P(\rho_T))$$

uniformly over $1\le t\le T$. In addition, by Lemma B.7 (4), we have

$$\begin{aligned}&\sum_{t=1}^{T}\widetilde{\boldsymbol{Z}}_{1t}\widehat{\boldsymbol{\Omega}}^{-1}(\tau_t)\left(\boldsymbol{Z}_{2t}^\top\boldsymbol{\theta}(\tau_t)-\boldsymbol{Z}_{2t}^\top\boldsymbol{s}(\tau_t)\left[\boldsymbol{\theta}(\tau_1)^\top\boldsymbol{Z}_{21},...,\boldsymbol{\theta}(\tau_T)^\top\boldsymbol{Z}_{2T}\right]^\top\right)\\ &=\sum_{t=1}^{T}\left[\left(\boldsymbol{z}_{1t}\boldsymbol{z}_{2t}^\top-\boldsymbol{\Sigma}_{\boldsymbol{z}_{1,2}}(\tau_t)\boldsymbol{\Sigma}_{\boldsymbol{z}_2}^{-1}(\tau_t)\boldsymbol{z}_{2t}\boldsymbol{z}_{2t}^\top(1+O_P(\rho_T))\right)\otimes(\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{\Omega}(\tau_s)K_h(\tau_s-\tau_t)+O_P(\rho_T))\right]\\ &\times\boldsymbol{\theta}(\tau_t)\cdot O_P(\rho_T)=\sum_{t=1}^{T}\left[\left(\boldsymbol{z}_{1t}\boldsymbol{z}_{2t}^\top-\boldsymbol{\Sigma}_{\boldsymbol{z}_{1,2}}(\tau_t)\boldsymbol{\Sigma}_{\boldsymbol{z}_2}^{-1}(\tau_t)\boldsymbol{z}_{2t}\boldsymbol{z}_{2t}^\top\right)\otimes(\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{\Omega}(\tau_s)K_h(\tau_s-\tau_t))\right]\\ &\times\boldsymbol{\theta}(\tau_t)\cdot O_P(\rho_T)+O_P(T\rho_T^2)=O_P(\sqrt{T}\rho_T)+O_P(T\rho_T^2)\end{aligned}$$

where the last equality follows from Lemma 2.2. Finally, the result holds since $O_P(T\rho_T^2)=o_P(\sqrt{T})$ by Assumption 6.

(3)–(4). By Lemma B.7, we have $\sup_{\tau\in[0,1]} \|\boldsymbol{s}(\tau)\boldsymbol{\eta}\| = O_P\left(\sqrt{\log T/(Th)}\right)$ and

$$\sup_{\tau\in[0,1]} \left\| \widehat{\boldsymbol{\Omega}}(\tau) - \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{\Omega}(\tau_t) K_h(\tau_t - \tau) \right\| = O_P\left(h^2 + \sqrt{\frac{\log T}{Th}}\right).$$

Then, parts (3) and (4) can be proved by using similar arguments of part (2). □

References

Dahlhaus, R. and Polonik, W. (2009), 'Empirical spectral processes for locally stationary time series', *Bernoulli* **15**(1), 1–39.

Freedman, D. A. (1975), 'On tail probabilities for martingales', *Annals of Probability* **3**(1), 100–118.

Hall, P. and Heyde, C. C. (1980), *Martingale Limit Theory and Its Application*, Academic Press.

Lütkepohl, H. (2005), *New Introduction to Multiple Time Series Analysis*, Springer Science & Business Media.

Phillips, P. C. B. and Solo, V. (1992), 'Asymptotics for linear processes', *Annals of Statistics* **20**(2), 971–1001.