Electronic Supplementary Material to

Quantifying Adventitious Error in a Covariance Structure as a Random Effect

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This file contains the descriptions of the three MATLAB functions in the supplementary material.

1. File `MBLE.m`

This file involves a function `MBLE` that computes the maximum beta likelihood estimates of parameters from a given sample covariance matrix. It invokes another function `FgH` inside the same file that computes the value, gradient and Hessian matrix of the per-sample-size descrepancy function F. To be consistent with the symbol in the MATLAB file, we use s as the number of structural parameters instead of q.

1.1. Input and output

```
function [theta,v,err,Ohat,H]=...
MBLE(S,df,nn,theta0,v0,funn,vfree,pr,bd,thrs,btol,maxiter,varargin)
```

Input arguments to the function `MBLE`

- `df` is the degrees of freedom. For unadjusted discrepancy function, use $p(p+1)/2$.
- `S` is the sample covariance matrix $\mathbf{S}$.
- `nn` is the df of the Wishart distribution $(\mathbf{S}|\mathbf{\Sigma}) \sim \mathrm{W}_p(\mathbf{\Sigma}/n, n)$.

- `theta0` is the initial value of structure parameters $\boldsymbol{\xi}$ in the model.
- `v0` is the initial value of $v = 1/m$.
- `funn` is a function handle of the covariance structure of the form

  `[O,err,R,dOdx,iRdOdx]=fun(theta,...)`

  whose input and output arguments must be in the following format:

  - `theta`: a row vector of parameters, $\boldsymbol{\xi}'$.
  - `O`: the model implied covariance matrix $\boldsymbol{\Omega}$.
  - `[R,err]=chol(O);` i.e. `R` is the upper triangular Cholesky factor $\mathbf{R}_{\boldsymbol{\Omega}}$ of $\boldsymbol{\Omega}$ and `err` is 0 if $\boldsymbol{\Omega}$ is positive definite but greater than 0 if not.
  - `dOdx`: the derivative of $\boldsymbol{\Omega}$, $\boldsymbol{\Delta}$, a p^2 by s matrix
  - `iRdOdx`: $(\mathbf{R}'_{\boldsymbol{\Omega}} \otimes \mathbf{R}'_{\boldsymbol{\Omega}})^{-1}\boldsymbol{\Delta}$, an expression to be used frequently.
- `vfree` denotes whether $v = 1/m$ is treated as fixed or not:
  - If `vfree=0`, v is fixed at `v0`;
  - If it takes other values, v is estimated.
- `pr` prints iteration details:
  - 0: none;
  - 1: only `F`, `Res.Cos`, `cond.#.H`, `NPB` and `NEC`;
  - 2: only parameter values;
  - 3: both details in 1 and 2.
- `bd` is a matrix of size $s \times 2$, giving boundary conditions of the structural parameters. Default value is $[-\infty, \infty]$ for all structural parameters. Boundaries of v $[0, 1/(p-1)]$ are automatically added.
- `thrs` is the threshold to stop the iterations. Default value is 10^{-7}.

- `btol` is matrix of boundary thresholds. It can be a scalar, a column vector or a 2-columned matrix.
  - a scalar: to be used for both the lower and upper boundaries of all structural parameters.
  - a column vector of length s: to be used for both the lower and upper boundaries.
  - a 2-columned matrix: specifies the threshold of each boundary of each structural parameter.

  Default value is 10^{-8}. The boundary thresholds for v is 0 and 10^{-8} for the lower and upper bounds.
- `maxiter` is the maximum iterations allowed. Default value is 200.
- `varargin` passes arguments to `funn`, the function handle.

Output arguments from the function `MBLE`

- `theta` is the parameter estimate.
- `v` is the estimate $\hat{v}_0$. It allows negative values.
- `err>0` means nonconvergence.
- `Ohat` is the estimated covariance matrix.
- `H` is the approximate Hessian for the covariance structure parameters.

Screen Outputs from the function `MBLE`

- `#`: iteration number
- `F`: discrepancy function value (per sample size)
- `Res.Cos`: residual cosine
- `cond.#.H`: (approximate) condition number of `H` (block for unconstrained parameters)

- `NPB`: number of parameters on boundary
- `NEC`: number of effective constraints
- `parameters`: parameter values, the first one is v, followed by structural parameters.

1.2. Details of file `MBLE.m`

Global parameters

- `n`=`nn`.
- `vfr`=`vfree`
- `fun`=`funn`.
- `other`=`varargin`.
- `p`: The number of variables.
- `s`: The number of structural parameters.
- `CS`: The upper triangular Cholesky factor $\mathbf{R_S}$ of $\mathbf{S}$.
- `detS`: The determinant of $\mathbf{S}$.
- `alpha`: The correction factor $\alpha = 2df/p(p+1)$.
- `coef`: A row vector of coefficients $(c_0, c_1, c_2, 2c_3)'$ in Lemma 5 of the paper.

Algorithm: Newton-Raphson with approximate Hessian.

- Target function `f` is in fact the per-sample-size discrepancy function F/n, where F is given by Equation 18 in the paper.
- The gradient of F w.r.t v and $\boldsymbol{\xi}$ are given in Section 5.2 of the paper. The actual gradient used is $(\frac{\partial(F/n)}{\partial(nv)}, \frac{\partial(F/n)}{\partial\boldsymbol{\xi}'})'$.

- The approximate Hessian matrix is $\operatorname{diag}\left\{\frac{\partial^2(F/n)}{\partial(nv)^2}, \mathrm{E}\frac{\partial^2(F/n)}{\partial\boldsymbol{\xi}\partial\boldsymbol{\xi}'}\right\}$, where the elements are given in the Appendix B of the paper. Note the first element takes its (approximated) expected value if it is negative.

- Convergence criterion: $\min\left\{f, \max_i \frac{g_i^2}{h_{ii}f}\right\}$, where the maximum only concerns the parameters not subject to an active boundary.

- If one move results in a higher, infinite or `NA` target function value, the step is halved until a value smaller than the current target function value is obtained.

- If one move results in one or more parameters outside their boundary, the move is shortened so that the resultant parameter value is on the boundary.

- When boundary parameters are present at one iteration, an unconstrained search is first attempted. If this search moves the iteration away from the boundary, it is executed; if not, the search is contrained on the boundary.

1.3. Sub-function ***FgH***

```
function [f,err,g,H]=FgH(theta,f0)
```

Input arguments to the sub-function `FgH`

- `theta`: proposed next position of the full parameter vector, the first element being v.

- `f0`: the target function value at the current iteration.

Output arguments from the sub-function `FgH`

- `f`: target function value.

- `err`: whether the proposed move is admissible

  - > 0: The new target function value is infinite, `NA` or higher than `f0`.

  - 0: The new target function value is admissible.

- `g`: the derivative of `f`, or F/n, w.r.t. $(nv, \boldsymbol{\xi})$.

- `H`: the approximate Hessian matrix.

Calculation of `f`, `g` and `H` in the sub-function `FgH`:

- Variables defined in the intermediate steps:

  - `fac`$= 1 + nv$.
  - `R`: The upper triangular Cholesky factor $\mathbf{R}_{\mathbf{\Omega}}$ of $\mathbf{\Omega}$.
  - `A`$= \mathbf{R}_{\mathbf{S}}\mathbf{R}_{\mathbf{\Omega}}^{-1}$.
  - `iOS`$= \mathbf{A}'\mathbf{A} = \mathbf{R}_{\mathbf{\Omega}}^{-1\prime}\mathbf{S}\mathbf{R}_{\mathbf{\Omega}}^{-1}$.
  - `iOaveOS`$= (\mathbf{I} + nv\mathbf{R}_{\mathbf{\Omega}}^{-1\prime}\mathbf{S}\mathbf{R}_{\mathbf{\Omega}}^{-1})/(1+nv) = \mathbf{R}_{\mathbf{\Omega}}^{-1\prime}\overline{\mathbf{\Sigma}}\mathbf{R}_{\mathbf{\Omega}}^{-1}$.
  - `dif`$= \mathbf{R}_{\mathbf{\Omega}}^{-1\prime}\mathbf{S}\mathbf{R}_{\mathbf{\Omega}}^{-1} - \mathbf{I}$.
  - `WLS`$= \mathrm{tr}(\mathbf{\Omega}^{-1}\mathbf{S} - \mathbf{I})^2$.
  - `B`$= \dfrac{1}{1+nv}\mathbf{R}_{\mathbf{\Omega}}^{-1\prime}(\mathbf{\Omega} - \mathbf{S})\overline{\mathbf{\Sigma}}^{-1}\mathbf{R}_{\mathbf{\Omega}} = \dfrac{m}{n}\left\{\mathbf{R}_{\mathbf{\Omega}}\overline{\mathbf{\Sigma}}^{-1}\mathbf{R}_{\mathbf{\Omega}} - \mathbf{I}\right\}$

- Gradient and Hessian w.r.t $\boldsymbol{\xi}$.

  - `dFdx`: $\partial f/\partial\boldsymbol{\xi} = m\mathbf{\Delta}'(\overline{\mathbf{\Sigma}} \otimes \mathbf{\Omega})^{-1}(\boldsymbol{\omega} - \boldsymbol{s})/(m+n)$
  - `d2Fdxdx`: $\partial^2 f/\partial\boldsymbol{\xi}\partial\boldsymbol{\xi}' = m\mathbf{\Delta}'(\mathbf{\Omega} \otimes \mathbf{\Omega})^{-1}\mathbf{\Delta}/(m+n)$.

- Target function value `f`$= F/n$.

$$
\begin{aligned}
F_1 =& 2\sum_i \left(\ln\Gamma\left[\frac{m-i+1}{2}\right] - \ln\Gamma\left[\frac{m-i+n+1}{2}\right]\right) - mp\ln\frac{m}{2} + (m+n)p\ln\frac{m+n}{2} - np \\
F_2/n =& \ln(|\mathbf{\Omega}|/|\mathbf{S}|) + (1+m/n)\ln|\overline{\mathbf{\Sigma}}\mathbf{\Omega}^{-1}| \\
F_{(\alpha)}/n =& \alpha F_1/n + F_2/n
\end{aligned}
$$

  - If $v = 0$: $F/n = \ln(|\mathbf{\Omega}|/|\mathbf{S}|) + \mathrm{tr}(\mathbf{\Omega}^{-1}\mathbf{S} - \mathbf{I})$
  - If ($p = 1$ and $v \leq 0.02$) or ($p > 1$ and $(p-1)v < .02$): F_1 is approximated by

$$
F_1/n = \frac{c_0\ln(1+nv)}{n} + \frac{c_1}{m(m+n)} + \frac{c_2(2m+n)}{m^2(m+n)^2} + \frac{2c_3(3m(m+n)+n^2)}{m^3(m+n)^3}
$$

  where c_0 - c_3 are defined in Lemma 5 of the paper.

- Gradient w.r.t. v: `dFdv`$= \partial(F_{(\alpha)}/n)/\partial v$.

$$\frac{\partial F_1}{\partial m} = \sum_i \left(\psi \left[\frac{m-i+1}{2} \right] - \psi \left[\frac{m-i+n+1}{2} \right] + \ln(1+nv) \right)$$
$$\frac{\partial F_2}{\partial m} = \ln |\mathbf{R}_{\mathbf{\Omega}}^{-1\prime} \overline{\mathbf{\Sigma}} \mathbf{R}_{\mathbf{\Omega}}^{-1}| + \mathrm{tr} \left\{ \mathbf{R}_{\mathbf{\Omega}} \overline{\mathbf{\Sigma}}^{-1} \mathbf{R}_{\mathbf{\Omega}} - \mathbf{I} \right\}$$
$$\frac{\partial (F_{(\alpha)}/n)}{\partial v} = -\frac{m^2}{n} \left(\alpha \frac{\partial F_1}{\partial m} + \frac{\partial F_2}{\partial m} \right)$$

  - If v is fixed in the optimization, `dFdv` is set to 0.
  - If $v = 0$: $\partial(F/n)/\partial v = \alpha p(p+1)/2 - n\mathrm{tr}(\mathbf{\Omega}^{-1}\mathbf{S} - \mathbf{I})^2/2$.
  - If ($p = 1$ and $v \le 0.02$) or ($p > 1$ and $(p-1)v < .02$) we use the following approximation:

$$-\frac{m^2}{n}\frac{\partial F_1}{\partial m} = c_0 \frac{m}{m+n} + c_1 \frac{2m+n}{(m+n)^2} + c_2 \frac{2(3m(m+n)+n^2)}{m(m+n)^3} + 2c_3 \frac{3(m^2(4m+6n)+n^2(4m+n))}{m^2(m+n)^4}$$

    where c_0 - c_3 are defined in Lemma 5 of the paper.

- Approximate Hessian w.r.t. v: `d2Fdv2`$= \partial^2(F/n)/\partial v^2$..

$$\frac{\partial^2 F_1}{\partial m^2} = \frac{1}{2} \sum \left((\psi_1 \left[\frac{m-i+1}{2} \right] - 2v) - (\psi_1 \left[\frac{m-i+n+1}{2} \right] - \frac{2}{m+n}) \right)$$
$$\frac{\partial^2 F_2}{\partial m^2} = -\frac{1}{m+n} \mathrm{tr} \left\{ \mathbf{R}_{\mathbf{\Omega}} \overline{\mathbf{\Sigma}}^{-1} \mathbf{R}_{\mathbf{\Omega}} - \mathbf{I} \right\}^2$$
$$\frac{\partial^2 (F_{(\alpha)}/n)}{\partial v^2} = \frac{\alpha}{n} \left(m^4 \frac{\partial^2 F_1}{\partial m^2} + 2m^3 \frac{\partial F_1}{\partial m} \right) + \frac{m^4}{n} \frac{\partial^2 F_2}{\partial m^2} + 2\frac{m^3}{n}\frac{\partial F_2}{\partial m}$$

  - If v is fixed in the optimization, `d2Fdv2` is set to 1.
  - If $v = 0$: $\partial^2(F/n)/\partial v^2 = n^2\{\mathrm{tr}(\mathbf{\Omega}^{-1}\mathbf{S} - \mathbf{I})^2 - \frac{2}{3}\mathrm{tr}(\mathbf{\Omega}^{-1}\mathbf{S} - \mathbf{I})^3\} + \alpha\{-c_0 n + 2c_1\}$.
  - If ($p = 1$ and $v \le 0.02$) or ($p > 1$ and $(p-1)v < .02$) we use the following approximation:

$$\frac{m^4}{n}\frac{\partial^2 F_1}{\partial m^2} + \frac{2m^3}{n}\frac{\partial F_1}{\partial m} = -\frac{c_0 n m^2}{(n+m)^2} + \frac{2c_1 m^3}{(n+m)^3} + \frac{2c_2(3m^3 + (m+n)(n^2 + 3m(m+n)))}{(m+n)^4} + \frac{(2c_3)6[2m^4 + (n+m)(n^3 - 2m^2 n + 4m(m+n)^2)]}{m(m+n)^5}$$

  - If the value obtained above is not positive, we use

$$\frac{\partial^2 (F_{(\alpha)}/n)}{\partial v^2} = \alpha \left(-\frac{c_0 m^2 n^2}{(m+n)^2} + \frac{2c_1 m^3 n}{(m+n)^3} \right) + \frac{2m^2 n(c_0 n + c_1)}{(m+n)^2} + \frac{2c_1 mn}{m+n} + \frac{m^3 np(p+3)}{(m+n)^3}$$

2. File MWLE.m

This file involves a function MWLE that computes the maximum Wishart likelihood estimates of parameters from a given sample covariance matrix. It is a simplified version of MBLE.m for the special case where v is set to 0. In this case, the algorithm becomes a Fisher scoring algorithm. We only list the input and output of this function. For details see notes to the file MBLE.m

```
function [theta,err,F,Ohat,H]=...
MWLE(S,theta0,fun,pr,bd,thrs,btol,maxiter,varargin)
```

Input arguments to the function MWLE

- S is the sample covariance matrix $\mathbf{S}$.
- theta0 is the initial value of structure parameters $\boldsymbol{\xi}$ in the model.
- fun is a function handle of the covariance structure of the form

  `[O,err,R,dOdx,iRdOdx]=fun(theta,...)`

  whose input and output arguments must be in the following format:

  - theta: a row vector of parameters, $\boldsymbol{\xi}'$.
  - O: the model implied covariance matrix $\boldsymbol{\Omega}$.
  - `[R,err]=chol(O);` i.e. R is the upper triangular Cholesky factor $\mathbf{R}_{\boldsymbol{\Omega}}$ of $\boldsymbol{\Omega}$ and err is 0 if $\boldsymbol{\Omega}$ is positive definite but greater than 0 if not.
  - dOdx: the derivative of $\boldsymbol{\Omega}$, $\boldsymbol{\Delta}$, a p^2 by s matrix
  - iRdOdx: $(\mathbf{R}'_{\boldsymbol{\Omega}} \otimes \mathbf{R}'_{\boldsymbol{\Omega}})^{-1}\boldsymbol{\Delta}$, an expression to be used frequently.
- pr prints iteration details:
  - 0: none;
  - 1: only F, Res.Cos, cond.#.H, NPB and NEC;
  - 2: only parameter values;

- 3: both details in 1 and 2.

- `bd` is a matrix of size $s \times 2$, giving boundary conditions of the structural parameters. Default value is $[-\infty, \infty]$ for all structural parameters.
- `thrs` is the threshold to stop the iterations. Default value is 10^{-7}.
- `btol` is matrix of boundary thresholds. It can be a scalar, a column vector or a 2-columned matrix.
  - a scalar: to be used for both the lower and upper boundaries of all structural parameters.
  - a column vector of length s: to be used for both the lower and upper boundaries.
  - a 2-columned matrix: specifies the threshold of each boundary of each structural parameter.

  Default value is 10^{-8}. The boundary thresholds for v is 0 and 10^{-8} for the lower and upper bounds.
- `maxiter` is the maximum iterations allowed. Default value is 200.
- `varargin` passes arguments to `funn`, the function handle.

Output arguments from the function `MWLE`

- `theta` is the parameter estimate.
- `err>0` means nonconvergence.
- `F` is the MWL discrepancy function value.
- `Ohat` is the estimated covariance matrix.
- `H` is the approximate Hessian for the covariance structure parameters.

Screen outputs (first stage) from the function `MWLE`

- `#`: iteration number
- `F`: inverse Wishart discrepancy function value
- `Res.Cos`: residual cosine
- `cond.#.H`: (approximate) condition number of `H` (block for unconstrained parameters)
- `NPB`: number of parameters on boundary
- `NEC`: number of effective constraints
- `parameters`: values of the structural parameters.

Screen outputs (second stage) from the function `MWLE`

- `#`: iteration number
- `F`: inverse Wishart -2 log-likelihood value (less a constant)
- `Res.Cos`: residual cosine
- `H`: value of h_{vv}, the second derivative.
- `v`: value of v

3. File `MIWLE.m`

This file involves a function `MIWL` that computes the maximum inverse Wishart likelihood estimates of parameters from a given population covariance matrix. It involves two subfunctions: `FgHtheta` computes the value, gradient and Hessian of the MIWL discrepancy function; `FgHv` computes the gradient and Hessian used in the second stage of optimization.

3.1. Input and Output

```
function [theta,v,err,F0,Ohat,H0]=...
MIWLE(SS,df,theta0,v0,funn,vfr,pr,bd,thrs,btol,maxiter,varargin)
```

Input arguments to the function `MIWL.m`

- `df` is the degrees of freedom. For unadjusted discrepancy function, use $p(p+1)/2$.
- `SS` is the input covariance matrix $\mathbf{\Sigma}$.
- `theta0` is the initial value of structure parameters $\boldsymbol{\xi}$ in the model.
- `v0` is the initial value of $v = 1/m$.
- `funn` is a function handle of the covariance structure of the form

  `[O,err,R,dOdx,iRdOdx]=fun(theta,...)`

  whose input and output arguments must be in the following format:

  - `theta`: a row vector of parameters, $\boldsymbol{\xi}'$.
  - `O`: the model implied covariance matrix $\mathbf{\Omega}$.
  - `[R,err]=chol(O);` i.e. `R` is the upper triangular Cholesky factor $\mathbf{R}_{\mathbf{\Omega}}$ of $\mathbf{\Omega}$ and `err` is 0 if $\mathbf{\Omega}$ is positive definite but greater than 0 if not.
  - `dOdx`: the derivative of $\mathbf{\Omega}$, $\mathbf{\Delta}$, a p^2 by s matrix
  - `iRdOdx`: $(\mathbf{R}'_{\mathbf{\Omega}} \otimes \mathbf{R}'_{\mathbf{\Omega}})^{-1}\mathbf{\Delta}$, an expression to be used frequently.
- `vfr` denotes whether $v = 1/m$ is treated as fixed or not:
  - If `vfr=0`, v is fixed at `v0`;
  - If it takes other values, v is estimated.
- `pr` prints iteration details:
  - 0: none;
  - 1: only `F`, `Res.Cos`, `cond.#.H`, `NPB` and `NEC`;
  - 2: only parameter values;
  - 3: both details in 1 and 2.
- `bd` is a matrix of size $s \times 2$, giving boundary conditions of the structural parameters. Default value is $[-\infty, \infty]$ for all structural parameters.

- `thrs` is the threshold to stop the iterations. Default value is 10^{-7}.
- `btol` is matrix of boundary thresholds. It can be a scalar, a column vector or a 2-columned matrix.
  - a scalar: to be used for both the lower and upper boundaries of all structural parameters.
  - a column vector of length s: to be used for both the lower and upper boundaries.
  - a 2-columned matrix: specifies the threshold of each boundary of each structural parameter.

  Default value is 10^{-8}. The boundary threshold for v is 10^{-7} for both the lower and upper bounds.
- `maxiter` is the maximum iterations allowed. Default value is 200.
- `varargin` passes arguments to `funn`, the function handle.

Output arguments from the function `MIWLE`

- `theta` is the parameter estimate.
- `v` is the estimate of v_0. It allows negative values.
- `err>0` means nonconvergence.
- `F0` is the inverse Wishart discrepancy function value.
- `Ohat` is the estimated covariance matrix.
- `H0` is the approximate Hessian for the covariance structure parameters.

Screen Outputs from the function `MIWLE`

- `#`: iteration number
- `F`: discrepancy function value (per sample size)

- `Res.Cos`: residual cosine
- `cond.#.H`: (approximate) condition number of `H` (block for unconstrained parameters)
- `NPB`: number of parameters on boundary
- `NEC`: number of effective constraints
- `parameters`: structural parameter values

3.2. Details of file `MIWLE.m`

Global parameters

- `S`=`SS`.
- `fun`=`funn`.
- `other`=`varargin`.
- `p`: The number of variables.
- `CS`: The upper triangular Cholesky factor $\mathbf{R_S}$ of $\mathbf{S}$.
- `detS`: The determinant of $\mathbf{S}$.
- `alpha`: The correction factor $\alpha = 2df/p(p+1)$.
- `coef`: A row vector of coefficients $(c_0, c_1, c_2, 2c_3)'$ in Lemma 5 of the paper.

Algorithm Stage I: Estimating the structural parameters through Fisher-scoring

- Target function `f` is the MIWL discrepancy function given by Equation 12 in Section 4.3.1 of the paper.
- The gradient and Fisher information matrix of F w.r.t $\boldsymbol{\xi}$ are given in Section 4.3.1 of the paper.
- Convergence criterion: $\min\left\{f, \max_i \frac{g_i^2}{h_{ii}f}\right\}$, where the maximum only concerns the parameters not subject to an active boundary.

- If one move results in a higher, infinite or `NA` target function value, the step is halved until a value smaller than the current target function value is obtained.

- If one move results in one or more parameters outside their boundary, the move is shortened so that the resultant parameter value is on the boundary.

- When boundary parameters are present at one iteration, an unconstrained search is first attempted. If this search moves the iteration away from the boundary, it is executed; if not, the search is contrained on the boundary.

Algorithm Stage II: Estimating parameter v

- The target function $\alpha f(m) + m\hat{F}^{\text{IW}}$ minimized by Newton's method, where $\hat{F}^{\text{IW}}$ has been obtained through Stage I.

- The derivative is $g = -m^2(\alpha f'(m) + \hat{F}^{\text{IW}})$.

- The second derivative is $H = m^4\alpha f''(m) + 2m^3(\alpha f'(m) + \hat{F}^{\text{IW}})$. Note $H > 0$ because $f'(m)$ is monotonically increasing.

- Convergence criterion: g^2/H.

- Other details similar to Stage I.

3.3. Sub-function `FgHtheta`

```
function [f,err,g,H]=FgHtheta(theta,f0)
```

Input arguments to the sub-function `FgHtheta`

- `theta`: proposed next position of $\boldsymbol{\xi}$.

- `f0`: the MIWL discrepancy function value at the current iteration.

Output arguments from the sub-function `FgHtheta`

- `f`: MIWL discrepancy function value.

- `err`: whether the proposed move is admissible
  - > 0: The new target function value is infinite, `NA` or higher than `f0`.
  - 0: The new target function value is admissible.
- `g`: the derivative of F^{IW}.
- `H`: the Fisher information matrix.

3.4. Sub-function `FgHv`

```
function [f,errv,dfdv,d2fdv2]=FgHv(v,F0,f0)
```

Input arguments to the sub-function `FgHv`

- `v`: proposed next position of v.
- `f0`: the stage II target function value at the current iteration.
- `F0`: the inverted Wishart discrepancy function value obtained in stage I.

Output arguments from the sub-function `FgHv`

- `f`: the target function.
- `err`: whether the proposed move is admissible
  - > 0: The new target function value is infinite, `NA` or higher than `f0`.
  - 0: The new target function value is admissible.
- `g`: the derivative of the target function.
- `H`: the second derivative of the target function.