

**Supplemental Material for
“Semiparametric Factor Analysis for Item-Level Response Time Data”**

A Estimating the Proportional Hazard Factor Model

In the proportional hazard (PH) factor model (Ranger & Ortner, 2012), the conditional hazard function of the manifest variable (MV) $Y_{ij}^* = y^*$, $y^* \geq 0$, given the latent variable (LV) $X_i^* = x^*$ is subject to the following factorization:

$$h_j(y^*|x^*) = h_{j0}(y^*) \cdot \exp(\beta_j x^*), \quad (\text{A.1})$$

in which $h_{j0}(y^*)$ is a non-parametric baseline hazard function, and the incremental hazard $\exp(\beta_j x^*)$ is a log-linear regression on the LV. The corresponding conditional density function can then be expressed as

$$f_j^*(y^*|x^*) = h_{j0}(y^*) \exp(\beta_j x^* - \exp(\beta_j x^*) H_{j0}(y^*)), \quad (\text{A.2})$$

in which $H_{j0}(y^*) = \int_0^{y^*} h_{j0}(t) dt$ is termed the cumulative baseline hazard function. Assuming that $X_i^* \sim \mathcal{N}(0, 1)$ with density $\phi(\cdot)$, the marginal likelihood for an observed MV vector $\mathbf{y}_i^* = (y_{i1}^*, \dots, y_{im}^*)^\top$ can be expressed as

$$f^*(\mathbf{y}_i^*) = \int_{-\infty}^{\infty} \prod_{j=1}^m f_j^*(y_{ij}^*|x^*) \phi(x^*) dx^*. \quad (\text{A.3})$$

To find β_j and $h_{j0}(\cdot)$, $j = 1, \dots, m$, that maximize Equation A.2, Ranger and Ortner (2012) proposed an Expectation-Maximization (EM) algorithm. Similar to our algorithm detailed in Section 4.1 of the main text, Ranger and Ortner’s EM algorithm iterates between an E-step, where posterior weights are computed at each quadrature node for each individual, and an M-step, where a weighted sum of log conditional densities (Equation A.2) is maximized for each j . Due to the presence of a non-parametric component $h_{j0}(\cdot)$, the M-step optimization (for each j) is done in two stages. Let r denote the current EM iteration. After computing $e_{iq}^{(r)}$, i.e., the posterior weight at quadrature node $q = 1, \dots, Q$ for person i (computed in a fashion similar to Equation 24 in the main document) in the E-step, we first compute the (weighted) Breslow estimator of the cumulative baseline hazard:

$$H_{0j}^{(r+1)}(y^*) = \sum_{i=1}^n I\{y_{ij}^* \leq y^*\} h_{0j}^{(r+1)}(y_{ij}^*), \quad (\text{A.4})$$

in which

$$h_{0j}^{(r+1)}(y_{ij}^*) = \sum_{i=1}^n \frac{I\{y_{ij}^* \leq y^*\}}{\sum_{k=1}^n I\{y_{kj}^* \geq y_{ij}^*\} \left[\sum_{q=1}^Q e_{kq}^{(r)} \exp(\beta_j^{(r)} x_q^*) \right]} \quad (\text{A.5})$$

is an estimate of the baseline hazard at the observed data points y_{ij}^* , $i = 1, \dots, n$. In Equations A.4 and A.5, $I\{\cdot\}$ stands for the indicator function, x_q^* is the q th quadrature

node for the LV, and $\beta_j^{(r)}$ denotes the current regression coefficient. Then the updated coefficient $\beta_j^{(r+1)}$ is obtained by numerically maximizing

$$\sum_{i=1}^n \sum_{q=1}^Q e_{iq}^{(r)} \left\{ \log h_{j0}^{(r+1)}(y_{ij}^*) + \beta_j x_q^* - \exp(\beta_j x_q^*) H_{j0}^{(r+1)}(y_{ij}^*) \right\} \quad (\text{A.6})$$

with respect to β_j .

We implemented the aforementioned EM algorithm in R (R Core Team, 2020). We verified on simulated data sets that our code and Dr. Jochen Ranger’s code produced identical estimates for β_j and the baseline hazard functions.

B Additional Simulation Results

We fit the PH model to the data sets generated in our simulation study (Section 5 of the main document). Similar to the configuration for fitting other candidate models, we used a 49-point (i.e., $Q = 49$) Gauss-Hermite quadrature to approximate the intractable marginal likelihood (Equation A.6) and terminate the algorithm when the log-likelihood change between consecutive iterations is less than 0.0001. Note that the raw data generated from the QMLV (i.e., quadratic mean function, log-linear variance function) model resemble log response time (RT) and thus can take negative values; therefore, the PH was fitted after the data was exponentiated.

Because the Breslow estimator (Equation A.4) is a step function, the corresponding estimator of the conditional density (Equation A.2) vanishes at y^* values that are not contained in $\{y_{ij}^* : i = 1, \dots, n\}$. We therefore cannot directly evaluate the empirical Kullback-Leibler (KL) risk (Equation 25 in the main document) at the maximum likelihood

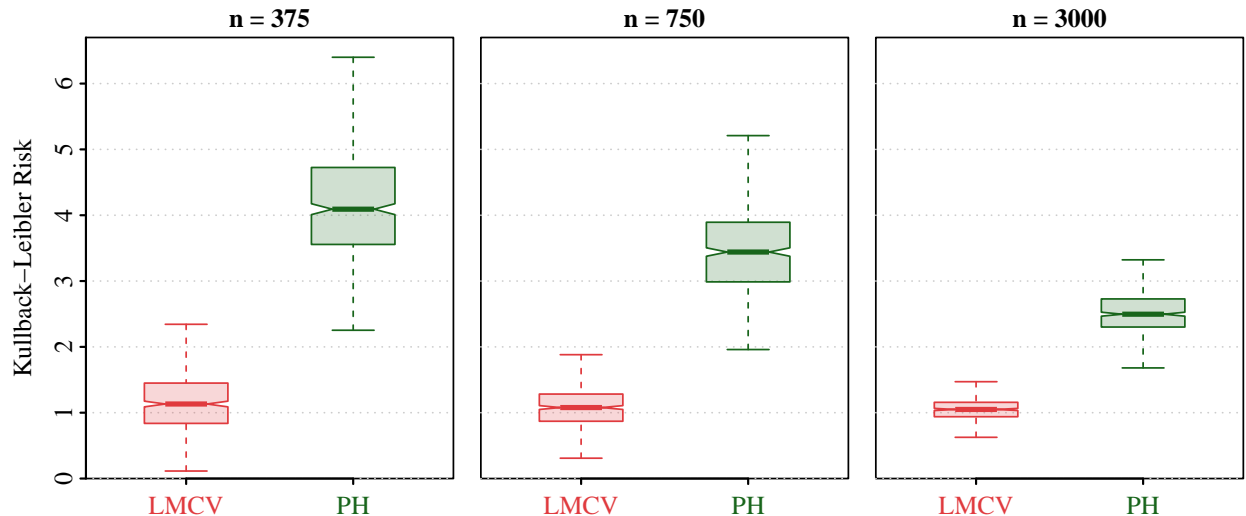


Figure B.1: Boxplots of empirical Kullback-Leibler (KL) risk across 500 replications. Results for LMCV and PH are shown in different colors, and the three sample sizes conditions ($n = 375, 750$ and 3000) are shown in separate panels. LMCV: Linear mean function, constant standard deviation (SD) function. PH: Proportional hazard.

estimator of the PH model, because the calibration and validation data typically contain different values. As a workaround, we fit a monotone polynomial of degree six (Murray, Müller, & Turlach, 2016) to the Breslow estimate of the cumulative baseline hazard in the final iteration; the conditional density was then evaluated using the smoothed version of the cumulative baseline hazard and its derivative. Finally, we back-transformed the estimated density to the log RT scale for comparison with other candidate models.

The KL risk estimates for PH and LMCV (linear mean, constant variance function) are summarized as boxplots in Figure B.1; the graphical display is similar to Figure 2 in the main text. It can be seen that PH exhibits a much higher empirical KL risk compared to LMCV. As we have seen in Section 5.2 that LMCV already fits substantially worse than QMLV (quadratic mean, log-linear variance function) and SP (semiparametric with the optimal λ), we then concluded that the overall fit of PH to the particular data-generating mechanism is not acceptable. Although the PH model features a nonparametric estimator for the baseline hazard, it does involve a rather restrictive log-linear parameterization for the incremental hazard (with respect to the LV), which is conjectured to be the cause of the undesirable fit.

References

- Murray, K., Müller, S., & Turlach, B. (2016). Fast and flexible methods for monotone polynomial fitting. *Journal of Statistical Computation and Simulation*, *86*(15), 2946–2966.
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