Supplementary Material to "Hypothesis Testing of the Q-matrix"

The supplementary material is organized as follows. The proofs of the main results are included in Section A. Analysis of the Q-matrix specified in de la Torre and Chiu (2016) is given in Section B, and additional simulation results are given in Section C.

A Theoretical derivations

A.1 Proof of Theorem 1

Write the true attribute profile probabilities as

$$\mathbf{p}_0 = (p_{0,\boldsymbol{\alpha}}, \boldsymbol{\alpha} \in \{0,1\}^K)^\top$$

with the first element defined as $p_{0,\mathbf{0}}$ and the other part as $\mathbf{p}_0^*$ (see the definition of $\mathbf{p}$ in Section 2). Under Q_0, the likelihood function taking the form of

$$L_N(\boldsymbol{\theta}, \mathbf{p}^*) = \prod_{i=1}^{N} \left\{ \sum_{\boldsymbol{\alpha}\in\{0,1\}^K} \left(p_{\boldsymbol{\alpha}} \prod_{j=1}^{J} \theta_{j,\boldsymbol{\alpha}}^{R_{i,j}} (1-\theta_{j,\boldsymbol{\alpha}})^{1-R_{i,j}} \right) \right\}. \tag{A.1}$$

Here recall that $\theta_{j,\boldsymbol{\alpha}} = P(R_{i,j} = 1 \,|\, Q_0, \boldsymbol{\alpha}, \boldsymbol{\theta})$ is a function of $\boldsymbol{\theta}$ and Q_0 as described in Section 2. Note that L_N is written as a function of $\mathbf{p}^*$ instead of $\mathbf{p}$ due to the constraint that $\sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} = 1$. If the model parameters are identifiable, as $N \to \infty$,

$$\sqrt{N}\begin{pmatrix} \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 \\ \hat{\mathbf{p}}^* - \mathbf{p}_0^* \end{pmatrix} \xrightarrow{d} \mathcal{N}(0, \mathbf{I}_0^{-1}),$$

where $\mathbf{I}_0$ is the Fisher information of the likelihood function (A.1) evaluated at $(\boldsymbol{\theta}_0, \mathbf{p}_0^*)$.

When the true parameters $\boldsymbol{\theta}_0$ are unknown, we use the plug-in method and replace $\boldsymbol{\theta}_0$ with the MLE $\hat{\boldsymbol{\theta}}$ and the test statistic $S_{\hat{\boldsymbol{\theta}},\hat{\mathbf{p}},\mathcal{W}}(Q_0)$ takes the form

$$S_{\hat{\boldsymbol{\theta}},\hat{\mathbf{p}},\mathcal{W}}(Q_0) = \left|\mathcal{W}^{1/2}(T(Q_0, \hat{\boldsymbol{\theta}})\hat{\mathbf{p}} - \boldsymbol{\gamma})\right|^2.$$

Then, since $(\hat{\boldsymbol{\theta}}, \hat{\mathbf{p}})$ are consistent and $\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0, \hat{\mathbf{p}}^* - \mathbf{p}_0^*)$ is normally distributed, the following approximation holds

$$\begin{aligned} S_{\hat{\boldsymbol{\theta}},\hat{\mathbf{p}},\mathcal{W}}(Q_0) &= \left|\mathcal{W}^{1/2}(\boldsymbol{\gamma} - T(Q_0, \hat{\boldsymbol{\theta}})\hat{\mathbf{p}})\right|^2 \\ &= \Big|\mathcal{W}^{1/2}\Big\{\boldsymbol{\gamma} - T(Q_0, \boldsymbol{\theta}_0)\mathbf{p}_0 + \Big\{T(Q_0, \boldsymbol{\theta}_0) - T(Q_0, \hat{\boldsymbol{\theta}})\Big\}\,\mathbf{p}_0 - T(Q_0, \boldsymbol{\theta}_0)(\hat{\mathbf{p}} - \mathbf{p}_0) \\ &\quad - \Big\{T(Q_0, \hat{\boldsymbol{\theta}}) - T(Q_0, \boldsymbol{\theta}_0)\Big\}\,(\hat{\mathbf{p}} - \mathbf{p}_0)\Big\}\Big|^2 \\ &= \{1 + o(1)\} \\ &\quad \times\Big|\mathcal{W}^{1/2}\Big\{\{\boldsymbol{\gamma} - T(Q_0, \boldsymbol{\theta}_0)\mathbf{p}_0\} + \Big\{T(Q_0, \boldsymbol{\theta}_0) - T(Q_0, \hat{\boldsymbol{\theta}})\Big\}\,\mathbf{p}_0 - T(Q_0, \boldsymbol{\theta}_0)(\hat{\mathbf{p}} - \mathbf{p}_0)\Big\}\Big|^2. \end{aligned} \tag{A.2}$$

It can be seen that the distribution of $S_{\hat{\boldsymbol{\theta}},\hat{\mathbf{p}},\mathcal{W}}(Q_0)$ depends on the joint distribution of $(\boldsymbol{\gamma}, T(Q_0, \hat{\boldsymbol{\theta}}), \hat{\mathbf{p}})$, which further depends on the joint distribution of $(\boldsymbol{\gamma}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{p}})$.

In the following, we derive the joint distribution of $(\boldsymbol{\gamma}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{p}})$ for general CDMs. From (5), the vector $\boldsymbol{\gamma}$ converges weakly to a multivariate normal distribution. Therefore, to derive the joint distribution of $(\boldsymbol{\gamma}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{p}})$, we only need to specify the asymptotic correlation between $\boldsymbol{\gamma}$ and $(\hat{\boldsymbol{\theta}}, \hat{\mathbf{p}})$. This is given in the following lemma, which states that the MLE derived from (A.1) can be expressed as a linear function of the saturated response vector $\boldsymbol{\gamma}_{all}$.

Before stating the lemma, we introduce some more notation. Let $\boldsymbol{\beta}$ be a $(2^J - 1) \times 1$ vector of true proportions of all the possible response patterns other than zero, defined as

$$\boldsymbol{\beta} = \left(P(\mathbf{R} = \mathbf{r}), \mathbf{r} \in \{0,1\}^J \backslash \mathbf{0} \right)^\top,$$

and let $\hat{\boldsymbol{\beta}}$ be the corresponding observed proportions, defined as

$$\hat{\boldsymbol{\beta}} = \frac{1}{N} \left(\sum_{i=1}^{N} I(\mathbf{R}_i = \mathbf{r}); \mathbf{r} \in \{0,1\}^J \backslash \mathbf{0} \right)^\top.$$

By the definition, there exists a one-to-one mapping between vectors $\boldsymbol{\gamma}_{use}$ and the observed proportions $\hat{\boldsymbol{\beta}}$. We write the $n \times (2^J - 1)$ mapping matrix as U, which satisfies

$$\boldsymbol{\gamma}_{use} = U \hat{\boldsymbol{\beta}}. \tag{A.3}$$

Then we have the following result whose proof is given in Section A.2.

Lemma 1 *Under the null hypothesis $H_0 : Q = Q_0$, suppose that MLE $(\hat{\boldsymbol{\theta}}, \hat{\mathbf{p}}^*)$ is consistent. Then as $N \to \infty$,*

$$\sqrt{N} \begin{pmatrix} \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 \\ \hat{\mathbf{p}}^* - \mathbf{p}_0^* \end{pmatrix} = \{1 + o(1)\} \, \mathbf{I}_0^{-1} \boldsymbol{\eta}^\top \times \sqrt{N} \left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} \right),$$

where $\boldsymbol{\eta}$ *is a* $2^J \times (dim(\boldsymbol{\theta}) + 2^K - 1)$ *matrix defined as*

$$\boldsymbol{\eta} = \left(\eta_{\theta_1}, \cdots, \eta_{\theta_{dim(\boldsymbol{\theta})}}, \eta_{p_{\boldsymbol{\alpha}_1}}, \cdots, \eta_{p_{\boldsymbol{\alpha}_{2^K-1}}}\right).$$

The column vectors of $\boldsymbol{\eta}$ *are defined as:*

$$\eta_{\theta_i} = \left(\frac{\partial P(\mathbf{R}|Q_0, \boldsymbol{\theta}, \mathbf{p})/\partial\theta_i|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0, \mathbf{p}=\mathbf{p}_0}}{P(\mathbf{R}|Q_0, \boldsymbol{\theta}_0, \mathbf{p}_0)} \;:\; \mathbf{R} \in \{0,1\}^J\right)^\top,$$

$$\eta_{p_{\boldsymbol{\alpha}_h}} = \left(\frac{P(\mathbf{R}|\boldsymbol{\alpha} = \boldsymbol{\alpha}_h, Q_0, \boldsymbol{\theta}_0, \mathbf{p}_0) - P(\mathbf{R}|\boldsymbol{\alpha} = \mathbf{0}, Q_0, \boldsymbol{\theta}_0, \mathbf{p}_0)}{P(\mathbf{R}|Q_0, \boldsymbol{\theta}_0, \mathbf{p}_0)} \;:\; \mathbf{R} \in \{0,1\}^J\right)^\top.$$

We now prove Theorem 1. Thanks to Lemma 1, we can replace the last two terms in equation (A.2) with a linear transformation of $\boldsymbol{\beta}$. Note that $N \cdot \boldsymbol{\beta}_{all}$ follows multinomial distribution with mean vector $N \cdot (P(\mathbf{R}_i = \mathbf{R}); \mathbf{R} \in \{0,1\}^J)^\top$. We consider the three terms in (A.2) one by one. Note that generally, for any parameters $(\beta_1, \cdots, \beta_h)$ and their consistent estimators $(\hat\beta_1, \cdots, \hat\beta_h)$ such that $\sqrt{N}\{(\hat\beta_1, \cdots, \hat\beta_h) - (\beta_1, \cdots, \beta_h)\}$ follows a multivariate normal distribution, we have the approximation

$$\hat\beta_1 \cdots \hat\beta_h - \beta_1 \cdots \beta_h = \{1 + o(1)\} \sum_{l=1}^{h} \beta_1 \cdots \beta_{l-1}\beta_{l+1} \cdots \beta_h(\hat\beta_l - \beta_l).$$

This implies that there exists an $n \times 2J$ matrix $W_{\boldsymbol{\theta}_0, \mathbf{p}_0}$ such that

$$\left\{T(Q_0, \hat{\boldsymbol{\theta}}) - T(Q_0, \boldsymbol{\theta}_0)\right\} \mathbf{p}_0 \;=\; \{1 + o(1)\} W_{\boldsymbol{\theta}_0, \mathbf{p}_0}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0). \tag{A.4}$$

In addition, we have that

$$T(Q_0, \boldsymbol{\theta}_0)(\hat{\mathbf{p}} - \mathbf{p}_0) \;=\; T(Q_0, \boldsymbol{\theta}_0) \begin{pmatrix} -\mathbf{1}^\top \\ \mathcal{I}_{2^K-1} \end{pmatrix} (\hat{\mathbf{p}}^* - \mathbf{p}_0^*).$$

The above results imply that Equation (A.2) equals

$$
\begin{aligned}
& \sqrt{N}\left\{\boldsymbol{\gamma} - T(Q_0,\boldsymbol{\theta}_0)\mathbf{p}_0 + \left(T(Q_0,\boldsymbol{\theta}_0) - T(Q_0,\hat{\boldsymbol{\theta}})\right)\mathbf{p}_0 - T(Q_0,\boldsymbol{\theta}_0)(\hat{\mathbf{p}} - \mathbf{p}_0)\right\} \\
= \; & \sqrt{N}\left\{\boldsymbol{\gamma} - T(Q_0,\boldsymbol{\theta}_0)\mathbf{p}_0 - \left(W_{\boldsymbol{\theta}_0,\mathbf{p}_0}, T(Q_0,\boldsymbol{\theta}_0)\begin{pmatrix} -\mathbf{1}^\top \\ \mathcal{I}_{2^K-1}\end{pmatrix}\right)\begin{pmatrix}\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 \\ \hat{\mathbf{p}}^* - \mathbf{p}_0^*\end{pmatrix}\right\} \\
= \; & (1+o(1))\sqrt{N}\left\{A\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right)\right\},
\end{aligned}
$$

where A is an $n \times (2^J - 1)$ matrix defined by

$$
A = U - \left(W_{\boldsymbol{\theta}_0,\mathbf{p}_0}\ ,\ T(Q_0,\boldsymbol{\theta}_0)\begin{pmatrix} -\mathbf{1}^\top \\ \mathcal{I}_{2^K-1}\end{pmatrix}\right)\mathbf{I}_0^{-1}\boldsymbol{\eta}^\top. \tag{A.5}
$$

Then by the central limit theorem, we have

$$
\sqrt{N}\left\{A\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right)\right\} \ \xrightarrow{d}\ \mathcal{N}(\mathbf{0},\Xi),\ \text{as } N \to \infty.
$$

where

$$
\Xi = A\ Cov(\boldsymbol{\beta})\ A^\top \tag{A.6}
$$

Therefore, we have that as $N \to \infty$,

$$
NS_{\hat{\boldsymbol{\theta}},\hat{\mathbf{p}},\mathcal{W}}(Q_0) \xrightarrow{d} \sum_{l=1}^{n}\lambda_l Z_l^2,
$$

where Z_1, ..., Z_n are i.i.d. standard normal random variables, and $\lambda_1 \geq \cdots \geq \lambda_n$ are the eigenvalues of $\Xi_{\mathcal{W}} = \mathcal{W}^{1/2}\Xi\mathcal{W}^{1/2}$. This concludes the proof.

A.2 Proof of Lemma 1

Under a general diagnostic model, the likelihood takes the form of

$$L_N(\boldsymbol{\theta}, \mathbf{p}^*) = \prod_{i=1}^{N} \left\{ \sum_{\boldsymbol{\alpha}} \left(p_{\boldsymbol{\alpha}} \prod_{j=1}^{J} P(R_{i,j} \mid Q_0, \boldsymbol{\alpha}, \boldsymbol{\theta}) \right) \right\}.$$

The log-likelihood function is

$$l_N(\boldsymbol{\theta}, \mathbf{p}^*) = \sum_{i=1}^{N} \log \left\{ \sum_{\boldsymbol{\alpha}} \left(p_{\boldsymbol{\alpha}} \prod_{j=1}^{J} P(R_{i,j} \mid Q_0, \boldsymbol{\alpha}, \boldsymbol{\theta}) \right) \right\}.$$

We start with the derivative of l_N with respect to θ_i, $i = 1, \ldots, dim(\boldsymbol{\theta})$, which takes the following form

$$\begin{aligned} \left.\frac{\partial l_N(\boldsymbol{\theta}, \mathbf{p}^*)}{\partial \theta_i}\right|_{\hat{\boldsymbol{\theta}}, \hat{\mathbf{p}}^*} &= \left.\sum_{i=1}^{N} \frac{\sum_{\boldsymbol{\alpha}} \left(\partial \left(p_{\boldsymbol{\alpha}} \prod_{j=1}^{J} P(R_{i,j}|Q_0, \boldsymbol{\alpha}, \boldsymbol{\theta}) \right) / \partial \theta_i \right)}{\sum_{\boldsymbol{\alpha}} \left(p_{\boldsymbol{\alpha}} \prod_{j=1}^{J} P(R_{i,j}|Q_0, \boldsymbol{\alpha}, \boldsymbol{\theta}) \right)} \right|_{\hat{\boldsymbol{\theta}}, \hat{\mathbf{p}}^*} \\ &= \sum_{\mathbf{R} \in \{0,1\}^J} \left(\sum_{i=1}^{N} I(\mathbf{R}_i = \mathbf{R}) \right) \left\{ \frac{\partial P(\mathbf{R}|Q_0, \boldsymbol{\theta}, \mathbf{p}) / \partial \theta_i |_{\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}, \mathbf{p} = \hat{\mathbf{p}}}}{P(\mathbf{R}|Q_0, \boldsymbol{\theta}_0, \mathbf{p}_0)} \right\}. \end{aligned}$$

Let

$$\boldsymbol{\beta}_{all} = \left(P(\mathbf{R} = \mathbf{0}) \ , \ \boldsymbol{\beta}^\top \right)^\top.$$

$$\hat{\boldsymbol{\beta}}_{all} = \left(\frac{1}{N} \sum_{i=1}^{N} I(\mathbf{R}_i = \mathbf{0}) \ , \ \hat{\boldsymbol{\beta}}^\top \right)^\top.$$

That is, the first the element of $\hat{\boldsymbol{\beta}}_{all}$ is the proportion of subjects whose response vectors are zero. Then $N \cdot \hat{\boldsymbol{\beta}}_{all}$ follows multinomial distribution with parameters $\{N, \ (P(\mathbf{R}_i = \mathbf{r}), \mathbf{r} \in \{0,1\}^J)^\top\}$.

Then by the definition of η_{θ_i}, we have

$$\frac{1}{\sqrt{N}}\frac{\partial l_N(\boldsymbol{\theta},\mathbf{p}^*)}{\partial\theta_i}\bigg|_{\hat{\boldsymbol{\theta}},\hat{\mathbf{p}}^*} = \sqrt{N}\eta_{\theta_i}^\top\hat{\boldsymbol{\beta}}_{all}.$$

Some basic calculation implies that $\sqrt{N}\eta_{\theta_i}^\top\boldsymbol{\beta}_{all}=0$. Therefore, we have

$$\frac{1}{\sqrt{N}}\frac{\partial l_N(\boldsymbol{\theta},\mathbf{p}^*)}{\partial\theta_i}\bigg|_{\hat{\boldsymbol{\theta}},\hat{\mathbf{p}}^*} = \sqrt{N}\eta_{\theta_i}^\top(\hat{\boldsymbol{\beta}}_{all}-\boldsymbol{\beta}_{all}).$$

Similarly, for derivatives of l_N with respect to parameters $\{p_{\boldsymbol{\alpha}_h},\,\boldsymbol{\alpha}_h\in\{0,1\}^K\backslash\mathbf{0}\}$,

$$\frac{1}{\sqrt{N}}\frac{\partial l_N(\boldsymbol{\theta},\mathbf{p}^*)}{\partial p_{\boldsymbol{\alpha}_h}}\bigg|_{\hat{\boldsymbol{\theta}},\hat{\mathbf{p}}^*} = \sqrt{N}\eta_{p_{\boldsymbol{\alpha}_h}}^\top(\hat{\boldsymbol{\beta}}_{all}-\boldsymbol{\beta}_{all}),$$

where $\eta_{p_{\boldsymbol{\alpha}_h}}$ is defined as in the statement of Lemma 1.

By Taylor's expansion, we have

$$\begin{aligned}
&\sqrt{N}\begin{pmatrix}\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0\\ \hat{\mathbf{p}}^*-\mathbf{p}_0^*\end{pmatrix}\\
=&\ \{1+o(1)\}\ \mathbf{I}_0^{-1}\ \frac{1}{\sqrt{N}}\frac{\partial l_N(\boldsymbol{\theta},\mathbf{p})}{\partial(\boldsymbol{\theta}^\top,\mathbf{p}^\top)^\top}\bigg|_{\hat{\boldsymbol{\theta}},\hat{\mathbf{p}}^*}\\
=&\ \{1+o(1)\}\ \mathbf{I}_0^{-1}\ \boldsymbol{\eta}^\top\ \cdot\sqrt{N}\left(\hat{\boldsymbol{\beta}}_{all}-\boldsymbol{\beta}_{all}\right),
\end{aligned}$$

where Fisher information matrix $\mathbf{I}_0=-\boldsymbol{\eta}^\top\Sigma_{\boldsymbol{\beta}}\boldsymbol{\eta}$ is the negative Hessian matrix of l_N with resect to $(\boldsymbol{\theta},\mathbf{p}^*)$ evaluated at $(\boldsymbol{\theta}_0,\mathbf{p}_0^*)$, where $\Sigma_{\boldsymbol{\beta}}$ is the covariance matrix of $\boldsymbol{\beta}$. This completes the proof.

A.3 Proof of Proposition 1

We first introduce some useful notations. For two vectors $\mathbf{a} = (a_1, \ldots, a_K)$ and $\mathbf{b} = (b_1, \ldots, b_K)$ of same length K, denote $\mathbf{a} \succeq \mathbf{b}$ if $a_k \geq b_k$ for all $k = 1, \ldots, K$. For any attribute profile $\boldsymbol{\alpha} \in \{0,1\}^K$, denote its J-dimensional ideal response vector under the DINA model by $\xi_{\bullet,\boldsymbol{\alpha}}(Q) = (\mathbf{1}(\boldsymbol{\alpha} \succeq \mathbf{q}_1), \ldots, \mathbf{1}(\boldsymbol{\alpha} \succeq \mathbf{q}_J))^\top$. Note that for any item j, $\mathbf{1}(\boldsymbol{\alpha} \succeq \mathbf{q}_j) = \xi^{DINA}_{j,\boldsymbol{\alpha}}(Q)$ by definition.

Recall the definition of Q-equivalent, then it actually says Q induces an equivalence relation in the sense that $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ are Q-equivalent, if $\xi_{\bullet,\boldsymbol{\alpha}}(Q) = \xi_{\bullet,\boldsymbol{\alpha}'}(Q)$. We define a set of attribute profiles $\mathcal{R}^Q$ following Gu and Xu (2018),

$$\mathcal{R}^Q = \{\mathbf{0}\} \cup \{\boldsymbol{\alpha} = \vee_{h \in S}\, \mathbf{q}_h : S \subseteq \{1, \ldots, J\}\}. \tag{A.7}$$

Then $\mathcal{R}^Q$ is a subset of the attribute profile space $\{0,1\}^K$ and it has the following two important properties. First, for any $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in \mathcal{R}^Q$ and $\boldsymbol{\alpha}_1 \neq \boldsymbol{\alpha}_2$, we must have $\xi_{\bullet,\boldsymbol{\alpha}_1}(Q) \neq \xi_{\bullet,\boldsymbol{\alpha}_2}(Q)$. Second, for any $\boldsymbol{\alpha} \in \{0,1\}^K$, there exists $\boldsymbol{\alpha}' \in \mathcal{R}^Q$ such that $\xi_{\bullet,\boldsymbol{\alpha}'}(Q) = \xi_{\bullet,\boldsymbol{\alpha}}(Q)$. These two properties indicate that $\mathcal{R}$ gives a complete set of representatives under the equivalence relation induced by Q. Given an incomplete Q-matrix under DINA, the cardinality of $\mathcal{R}^Q$ is less than $\#\{0,1\}^K = 2^K$, because lacking any single attribute item with $\mathbf{q}$-vector $\mathbf{e}_k$ would result in attribute profile $\boldsymbol{\alpha} = \mathbf{e}_k$ not included in $\mathcal{R}^Q$. Without loss of generality, denote $\#\mathcal{R}^Q = C$ and denote the elements in $\mathcal{R}^Q$ by $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_C$ with $\boldsymbol{\alpha}_i \in \mathcal{A}_i$ for all i.

$\mathcal{R}^Q$ provides a basis for constructing the submatrix $T^{eq}(Q, \boldsymbol{\theta})$ of size $2^J \times C$. First note that under DINA, any two Q-equivalent attribute profiles $\boldsymbol{\alpha} \stackrel{Q}{\sim} \boldsymbol{\alpha}'$ have identical item parameter vectors, i.e. $\boldsymbol{\theta}_{j,\boldsymbol{\alpha}} = \boldsymbol{\theta}_{j,\boldsymbol{\alpha}'} (= 1 - s_j \text{ or } g_j)$ for all item j. This further implies $T_{\bullet,\boldsymbol{\alpha}}(Q, \boldsymbol{\theta}) = T_{\bullet,\boldsymbol{\alpha}'}(Q, \boldsymbol{\theta})$ by the definition of the T-matrix. Then $T^{eq}(Q, \boldsymbol{\theta})$ is a submatrix of $T(Q, \boldsymbol{\theta})$ by just extracting C columns, out of 2^K columns of $T(Q, \boldsymbol{\theta})$, indexed by $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_C \in \mathcal{R}^Q$.

With the introduced notations and relations, we next prove $T(Q,\boldsymbol{\theta})\mathbf{p} = T^{eq}(Q,\boldsymbol{\theta})\boldsymbol{\nu}^Q$ as follows.

$$\begin{aligned} T(Q,\boldsymbol{\theta})\mathbf{p} &= \sum_{\boldsymbol{\alpha}\in\{0,1\}^K} T_{\bullet,\boldsymbol{\alpha}}(Q,\boldsymbol{\theta})p_{\boldsymbol{\alpha}} \\ &= \sum_{i=1}^{C}\sum_{\boldsymbol{\alpha}\in\mathcal{A}_i} T_{\bullet,\boldsymbol{\alpha}}(Q,\boldsymbol{\theta})p_{\boldsymbol{\alpha}} \\ &= \sum_{i=1}^{C} T_{\bullet,\boldsymbol{\alpha}_i}(Q,\boldsymbol{\theta})\sum_{\boldsymbol{\alpha}\in\mathcal{A}_i} p_{\boldsymbol{\alpha}} \\ &= \sum_{i=1}^{C} T_{\bullet,\boldsymbol{\alpha}_i}(Q,\boldsymbol{\theta})\nu_{\mathcal{A}_i} = T^{eq}(Q,\boldsymbol{\theta})\boldsymbol{\nu}^Q. \end{aligned}$$

This also proves the claim that $T(Q_0,\boldsymbol{\theta})\mathbf{p}$ and $S_{\boldsymbol{\theta},\mathbf{p}}(Q_0)$ depend on $\mathbf{p}$ only through $\boldsymbol{\nu}^Q$.

A.4 Theory of p-Partial Identifiability of Two-Parameter CDMs

Before stating the theoretical result in Gu and Xu (2018), we introduce two new definitions. First, given a Q-matrix, define the non-basis item set S_{non} and the basis item set S_{basis} as follows.

$$S_{non} = \{j : \exists h \in \{1,\dots,J\}\setminus\{j\} \text{ s.t. } \mathbf{q}_h \preceq \mathbf{q}_j\} \quad \text{and} \quad S_{basis} = \{1,\dots,J\}\setminus S_{non}. \tag{A.8}$$

In other words, an item j is a non-basis item if there is some other item h whose required attributes are all required by j. Second, for an item j and a set of items $S \subseteq \{1,\dots,J\}$, j is said to be *S-differentiable* if there exist S^+, $S^- \subseteq S$ such that

$$\mathbf{0} \precneqq \vee_{h\in S^+}\mathbf{q}_h - \vee_{h\in S^-}\mathbf{q}_h \preceq \mathbf{q}_j. \tag{A.9}$$

Theorem 1 (Gu and Xu (2018)) *If a Q-matrix satisfies the following conditions (C1) and (C2), then the Two-Parameter CDM is* $\mathbf{p}$*-partially identifiable.*

(C1) For each item j, there exist two disjoint sets of items S_j^1, $S_j^2 \subseteq \{1, \ldots, J\} \setminus \{j\}$ such that

$$\mathbf{q}_j \preceq \vee_{h \in S_j^1} \mathbf{q}_h \quad \text{and} \quad \mathbf{q}_j \preceq \vee_{h \in S_j^2} \mathbf{q}_h.$$

(C2) Each basis item $j \in S_{basis}$ is S_{non}-differentiable.

A.5 Computation of the G-DINA model

For notational convenience, let the first K_j^* attributes be the required attributes for item j, and $\boldsymbol{\alpha}_j^*$ be the reduced attribute vector for item j. The formulation of the G-DINA model (de la Torre, 2011) can be written into the sum of the effects due to the presence of specific attributes and their interactions. Specifically,

$$P(R_j = 1 \mid \boldsymbol{\alpha}, Q) = \delta_{j0} + \sum_{k=1}^{K_j^*} \delta_{jk}\alpha_k + \sum_{k'=k+1}^{K_j^*} \sum_{k=1}^{K_j^*-1} \delta_{jkk'}\alpha_k\alpha_{k'} + \cdots + \delta_{j12\cdots K_j^*} \prod_{k=1}^{K_j^*} \alpha_k.$$

Under the G-DINA model, the item parameters are

$$\boldsymbol{\theta}_{GDINA} = (\delta_{j,0}, \delta_{j,1}, \cdots;\ j = 1, \cdots, J)^\top.$$

Under the null hypothesis $H_0 : Q = Q_0$, let $(\boldsymbol{\theta}_{GDINA,0}, \mathbf{p}_0)$ be the true model parameters. For a response vector $\mathbf{R}$, denote the corresponding probability mass function by $P_0(\mathbf{R}) = P(\mathbf{R}|Q_0, \mathbf{p}_0, \boldsymbol{\theta}_{GDINA,0})$. Moreover, let $\mathbf{R}_{-j} = (R_1, \cdots, R_{j-1}, R_{j+1}, \cdots, R_J)^\top$ and write $P_0(\mathbf{R}_{-j}) = P(\mathbf{R}_{-j}|Q_0, \mathbf{p}_0, \boldsymbol{\theta}_{GDINA,0})$.

The following result specifies the form of the $\boldsymbol{\eta}$ matrix under the GDINA model.

Corollary 1 *Under the GDINA model and the conditions of Lemma 1, the $\boldsymbol{\eta}$ matrix is defined as*

$$\boldsymbol{\eta}_{GDINA} = \left(\{\eta_{\delta_{j,0}}, \cdots, j = 1, \cdots, J\}; \eta_{p_{\boldsymbol{\alpha}_1}}, \cdots, \eta_{p_{\boldsymbol{\alpha}_{2^K-1}}}\right). \tag{A.10}$$

Assume $\mathbf{R}$ *arranged in the same order as in the response vector* $\boldsymbol{\beta}$*. For notational convenience, let the first* K_j^* *attributes be the required attributes for item* j*, then for any* $1 \leq l \leq K_j^*$ *and any* $1 \leq k_1 < \cdots < k_l \leq K_j^*$

$$\eta_{\theta_{j,0}} = \left(\{I(R_j = 1) - I(R_j = 0)\} \cdot \frac{\sum_{\{\boldsymbol{\alpha}:\alpha_k=0,\forall k=1,\ldots,K_j^*\}} p_{0,\boldsymbol{\alpha}} P_0(\mathbf{R}_{-j}|\boldsymbol{\alpha})}{P_0(\mathbf{R})} \ ; \ \mathbf{R} \in \{0,1\}^J \right)^\top, \tag{A.11}$$

$$\eta_{\theta_{j,k_1,\cdots,k_l}} = \left(\{I(R_j = 1) - I(R_j = 0)\} \cdot \frac{\sum_{\{\boldsymbol{\alpha}:\{k\leq K_j^*:\alpha_k=1\}=\{k_1,\ldots,k_l\}\}} p_{0,\boldsymbol{\alpha}} P_0(\mathbf{R}_{-j}|\boldsymbol{\alpha})}{P_0(\mathbf{R})} \ ; \ \mathbf{R} \in \{0,1\}^J \right)^\top. \tag{A.12}$$

and

$$\eta_{p_{\boldsymbol{\alpha}_h}} = \left(\frac{P_0(\mathbf{R}|\boldsymbol{\alpha}_h) - P_0(\mathbf{R}|\boldsymbol{\alpha} = \mathbf{0})}{P_0(\mathbf{R})} \ ; \ \mathbf{R} \in \{0,1\}^J \right)^\top. \tag{A.13}$$

The proof follows directly from Lemma 1. With the $\boldsymbol{\eta}$ matrix specified in (A.16)–(A.17), we can further calculate the matrices A and Ξ_{all} using (A.5) and (A.6), and get the asymptotic distribution of test statistic $S_{\hat{\boldsymbol{\theta}},\hat{\mathbf{p}}}(Q_0)$ from Theorem 1.

Proof of Corollary 1. Following the form of η_{θ_i} in Lemma 1, for $\eta_{\theta_{j,k_1,\cdots,k_l}}$, we have the numerator term equal to

$$\begin{aligned}
& \frac{\partial P(\mathbf{R}|Q_0, \boldsymbol{\theta}_{GDINA}, \mathbf{p})}{\partial \eta_{\theta_{j,k_1,\cdots,k_l}}}\Big|_{\boldsymbol{\theta}_{GDINA}=\boldsymbol{\theta}_{GDINA,0},\mathbf{p}=\mathbf{p}_0} \\
= & \sum_{\boldsymbol{\alpha}} \left(p_{\boldsymbol{\alpha}} \frac{\partial P(R_j|Q_0, \boldsymbol{\alpha}, \boldsymbol{\theta}_{GDINA})}{\partial \eta_{\theta_{j,k_1,\cdots,k_l}}} \prod_{h\neq j;\ h=1,\cdots,J} P(R_h|Q_0, \boldsymbol{\alpha}, \boldsymbol{\theta}_{GDINA}) \right)\Big|_{\boldsymbol{\theta}_{GDINA}=\boldsymbol{\theta}_{DINA,0},\mathbf{p}=\mathbf{p}_0} \\
= & \{I(R_j = 0) - I(R_j = 1)\} \sum_{\{\boldsymbol{\alpha}:\{k\leq K_j^*:\alpha_k=1\}=\{k_1,\ldots,k_l\}\}} \left(p_{0,\boldsymbol{\alpha}} \cdot \prod_{h\neq j;\ h=1,\cdots,J} P_0(R_h|\boldsymbol{\alpha}) \right) \\
= & \{I(R_j = 0) - I(R_j = 1)\} \sum_{\{\boldsymbol{\alpha}:\{k\leq K_j^*:\alpha_k=1\}=\{k_1,\ldots,k_l\}\}} (p_{0,\boldsymbol{\alpha}} P_0(\mathbf{R}_{-j}|\boldsymbol{\alpha})),
\end{aligned}$$

where $\mathbf{R}_{-j} := (R_1, \cdots, R_{j-1}, R_{j+1}, \cdots, R_J)^\top$. A similar argument gives the form of $\eta_{\theta_j,0}$. ■

A.6 Computation of the DINA model

Under the DINA model as introduced in Example 1, the item parameters are

$$\boldsymbol{\theta}_{DINA} = (s_1, \cdots, s_J, g_1, \cdots, g_J)^\top,$$

where s_j and g_j are the slipping and guessing parameters.

Under the null hypothesis $H_0 : Q = Q_0$, let $(\boldsymbol{\theta}_{DINA,0}, \mathbf{p}_0)$ be the true model parameters. For a response vector $\mathbf{R}$, denote the corresponding probability mass function by

$$P_0(\mathbf{R}) = P(\mathbf{R}|Q_0, \mathbf{p}_0, \boldsymbol{\theta}_{DINA,0}).$$

Moreover, let $\mathbf{R}_{-j} = (R_1, \cdots, R_{j-1}, R_{j+1}, \cdots, R_J)^\top$ and write

$$P_0(\mathbf{R}_{-j}) = P(\mathbf{R}_{-j}|Q_0, \mathbf{p}_0, \boldsymbol{\theta}_{DINA,0}).$$

Following Lemma 1 we have the following result, which specifies the form of the $\boldsymbol{\eta}$ matrix under the DINA model. With the $\boldsymbol{\eta}$ matrix specified in (A.15)–(A.17), we can easily calculate the matrices A and further Ξ_{all} using (A.5) and (A.6). From Theorem 1, we can get the asymptotic distribution of test statistic $S_{\hat{\boldsymbol{\theta}},\hat{\mathbf{p}}}(Q_0)$.

Corollary 2 *Under the DINA model and the conditions of Lemma 1, the* $\boldsymbol{\eta}$ *matrix is a* $2^J \times (2J + 2^K - 1)$ *matrix defined as*

$$\boldsymbol{\eta}_{DINA} = \left(\eta_{s_1}, \cdots, \eta_{s_J}, \eta_{g_1}, \cdots, \eta_{g_J}, \eta_{p_{\boldsymbol{\alpha}_1}}, \cdots, \eta_{p_{\boldsymbol{\alpha}_{2^K-1}}}\right). \tag{A.14}$$

Here with $\mathbf{R}$ *arranged in the same order as in the response vector* $\boldsymbol{\beta}$ *and* $\xi^{DINA}_{j,\boldsymbol{\alpha}}(Q_0)$ *as* *defined in* (2)*, we have*

$$\eta_{s_j} = \left(\{I(R_j = 0) - I(R_j = 1)\} \cdot \frac{\sum_{\xi^{DINA}_{j,\boldsymbol{\alpha}}(Q_0)=1} p_{0,\boldsymbol{\alpha}} P_0(\mathbf{R}_{-j}|\boldsymbol{\alpha})}{P_0(\mathbf{R})} \ ; \ \mathbf{R} \in \{0,1\}^J \right)^\top, \quad \text{(A.15)}$$

$$\eta_{g_j} = \left(\{I(R_j = 1) - I(R_j = 0)\} \cdot \frac{\sum_{\xi^{DINA}_{j,\boldsymbol{\alpha}}(Q_0)=0} p_{0,\boldsymbol{\alpha}} P_0(\mathbf{R}_{-j}|\boldsymbol{\alpha})}{P_0(\mathbf{R})} \ ; \ \mathbf{R} \in \{0,1\}^J \right)^\top, \text{(A.16)}$$

and

$$\eta_{p_{\boldsymbol{\alpha}_h}} = \left(\frac{P_0(\mathbf{R}|\boldsymbol{\alpha}_h) - P_0(\mathbf{R}|\boldsymbol{\alpha} = \mathbf{0})}{P_0(\mathbf{R})} \ ; \ \mathbf{R} \in \{0,1\}^J \right)^\top. \quad \text{(A.17)}$$

Proof of Corollary 2. Following the form of η_{θ_i} in Lemma 1, for η_{s_j}, we have the numerator term equals

$$\begin{aligned}
& \frac{\partial P(\mathbf{R}|Q_0, \boldsymbol{\theta}_{DINA}, \mathbf{p})}{\partial s_j}\Big|_{\boldsymbol{\theta}_{DINA}=\boldsymbol{\theta}_{DINA,0}, \mathbf{p}=\mathbf{p}_0} \\
= & \sum_{\boldsymbol{\alpha}} \left(p_{\boldsymbol{\alpha}} \frac{\partial P(R_i^j|Q_0, \boldsymbol{\alpha}, \boldsymbol{\theta}_{DINA})}{\partial s_j} \prod_{h \neq j;\ h=1,\cdots,J} P(R_i^h|Q_0, \boldsymbol{\alpha}, \boldsymbol{\theta}_{DINA}) \right)\Big|_{\boldsymbol{\theta}_{DINA}=\boldsymbol{\theta}_{DINA,0}, \mathbf{p}=\mathbf{p}_0} \\
= & \left\{I(R_i^j = 0) - I(R_i^j = 1)\right\} \sum_{\xi^{DINA}_{j,\boldsymbol{\alpha}}(Q_0)=1} \left(p_{0,\boldsymbol{\alpha}} \cdot \prod_{h \neq j;\ h=1,\cdots,J} P_0(R_i^h|\boldsymbol{\alpha}) \right) \\
= & \{I(R_1 = 0) - I(R_1 = 1)\} \sum_{\xi^{DINA}_{j,\boldsymbol{\alpha}}(Q_0)=1} (p_{0,\boldsymbol{\alpha}} P_0(\mathbf{R}_{-1}|\boldsymbol{\alpha})),
\end{aligned}$$

where $\mathbf{R}_{-j} := (R_1, \cdots, R_{j-1}, R_{j+1}, \cdots, R_J)^\top$. A similar argument gives the form of η_{g_j}. ■

B Analysis of the Q-matrix specified in de la Torre and Chiu (2016)

In addition to the original 20×8 Q-matrix, we also test the Q-matrix specified in de la Torre and Chiu (2016). The authors used responses to a subset of 11 items and specified 4 attributes: (1) performing basic fraction subtraction operation, (2) simplifying/reducing, (3) separating whole number from fraction, and (4) borrowing one from whole number to fraction. The Q-matrix they used is shown in Table 1. The p-value corresponding to this Q-matrix is 0.15 under the DINA model and 0.89 under the G-DINA model. This suggests the Q-matrix fits the data well under both the DINA and the G-DINA models. To validate that type I error is well controlled under this Q-matrix, we further conduct simulations with this 11×4 Q-matrix under the G-DINA model to evaluate the performance of the testing procedure in "Uniform", $|\rho| \leq 0.25$, $|\rho| \leq 0.5$, and $|\rho| \leq 0.75$ settings. The results of the Type I errors are presented in Table 2. The Type I error is well controlled, which means the false rejection of a true Q-matrix is unlikely to happen and the testing procedure is safe to use.

C Additional Simulation Results

We also present the Q-Q plots of p-values in the correlated attribute case and incomplete Q-matrix case, for all the settings considered in the section of simulation studies with sample size $N = 500$. Figures 1, 3 and 5 correspond to Table 3 in the main text with correlation $\rho = 0.25, 0.5, 0.75$ and sample size $N = 500$, showing p-value distributions when testing the true Q-matrices. And Figures 2, 4 and 6 correspond to Table 4 in the main text with correlation $\rho = 0.25, 0.5, 0.75$ and sample size $N = 500$, showing p-value distributions when testing the misspecified Q-matrices. Figure 7 and Figure 8 correspond to the first row of

Item ID	Content	α_1	α_2	α_3	α_4
4	$3\frac{1}{2} - 2\frac{3}{2}$	1	1	1	1
6	$\frac{6}{7} - \frac{4}{7}$	1	0	0	0
9	$3\frac{7}{8} - 2$	1	0	1	0
10	$4\frac{4}{12} - 2\frac{7}{12}$	1	1	1	1
11	$4\frac{1}{3} - 2\frac{4}{3}$	1	1	1	1
12	$\frac{11}{8} - \frac{1}{8}$	1	1	0	0
14	$3\frac{4}{5} - 3\frac{2}{5}$	1	0	1	0
16	$4\frac{5}{7} - 1\frac{4}{7}$	1	0	1	0
17	$7\frac{3}{5} - \frac{4}{5}$	1	0	1	1
18	$4\frac{1}{10} - 2\frac{8}{10}$	1	1	1	1
20	$4\frac{1}{3} - 1\frac{5}{3}$	1	1	1	1

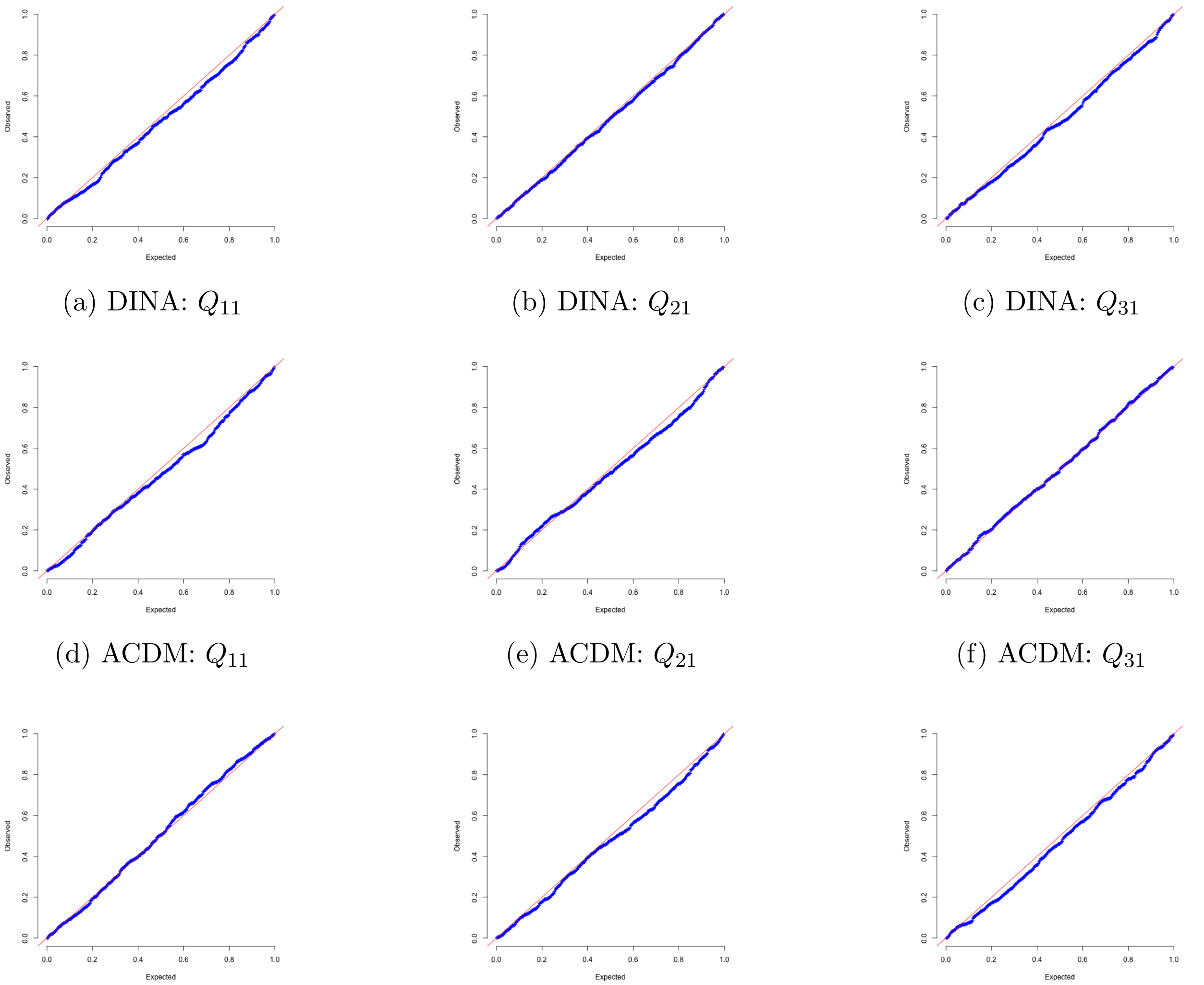
Table 1: The Q-matrix $Q_{11\times 4}$ specified in de la Torre and Chiu (2016)

	N	Uniform	$\lvert\rho\rvert \leq 0.25$	$\lvert\rho\rvert \leq 0.5$	$\lvert\rho\rvert \leq 0.75$
DINA	500	0.038	0.040	0.026	0.048
	1000	0.020	0.068	0.050	0.060
	2000	0.044	0.034	0.036	0.068
GDINA	500	0.044	0.042	0.036	0.042
	1000	0.022	0.044	0.046	0.044
	2000	0.040	0.042	0.036	0.042

Table 2: Type I Error Studies: Proportions of rejections for testing $Q_{11\times 4}$

Table 5 and Table 6 in the main text, respectively, showing p-value distributions when testing the true and misspecified incomplete Q-matrices.

The Q-Q plots further illustrate the good approximation of the asymptotic distribution in Theorem 1 to the "true" distribution with a relatively small sample size $N = 500$, when the attributes have low to high correlation levels ($\rho = 0.25, 0.5,$ and 0.75) and when the Q-matrices are incomplete (lacking single-attribute items).

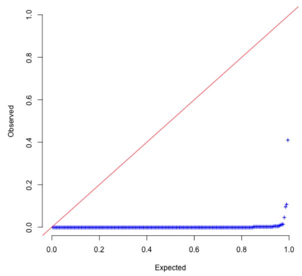


(a) DINA: Q_{11} (b) DINA: Q_{21} (c) DINA: Q_{31}

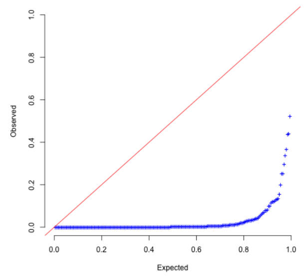
(d) ACDM: Q_{11} (e) ACDM: Q_{21} (f) ACDM: Q_{31}

(g) GDINA: Q_{11} (h) GDINA: Q_{21} (i) GDINA: Q_{31}

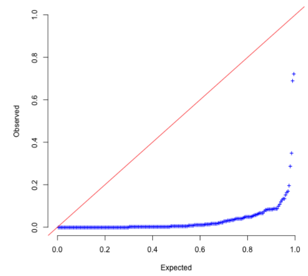
Figure 1: QQ-plots of p-values for testing True Q-matrices Q_{11}, Q_{21} and Q_{31} with $N = 500$, $\boldsymbol{\rho = 0.25}$.



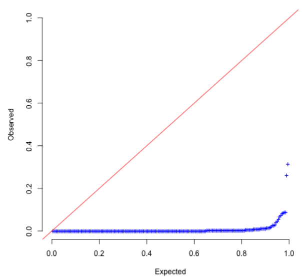
(a) DINA: Q_{12}



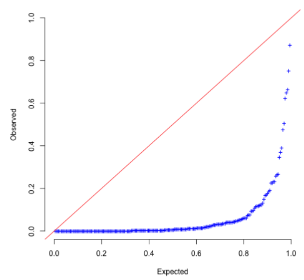
(b) DINA: Q_{22}



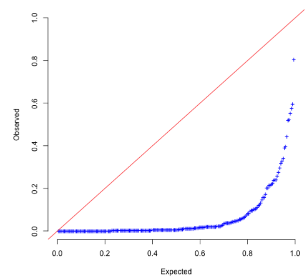
(c) DINA: Q_{32}



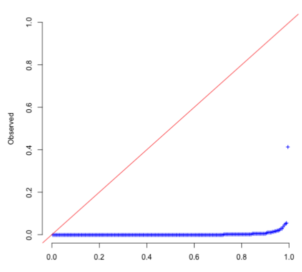
(d) ACDM: Q_{12}



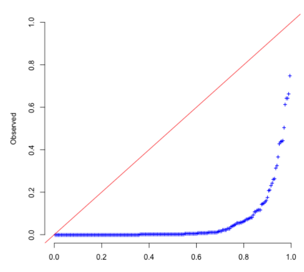
(e) ACDM: Q_{22}



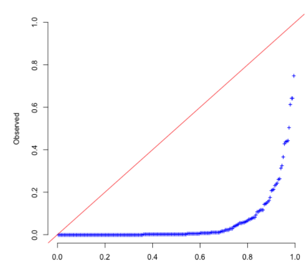
(f) ACDM: Q_{32}



(g) GDINA: Q_{12}

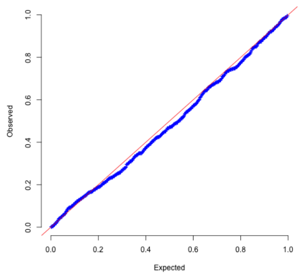


(h) GDINA: Q_{22}

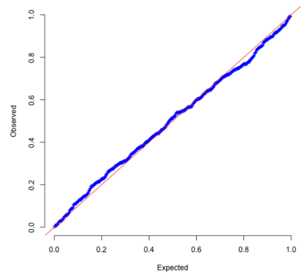


(i) GDINA: Q_{32}

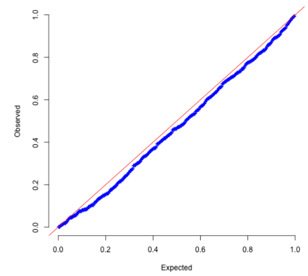
Figure 2: QQ-plots of p-values for testing Misspecified Q-matrices Q_{12}, Q_{22} and Q_{32} with $N = 500$, $\boldsymbol{\rho = 0.25}$.



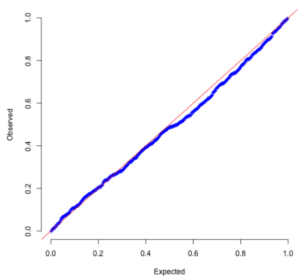
(a) DINA: Q_{11}



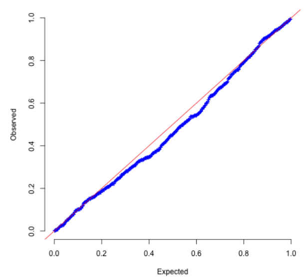
(b) DINA: Q_{21}



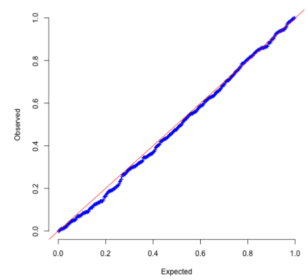
(c) DINA: Q_{31}



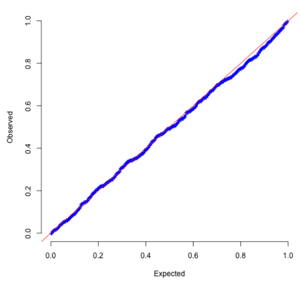
(d) ACDM: Q_{11}



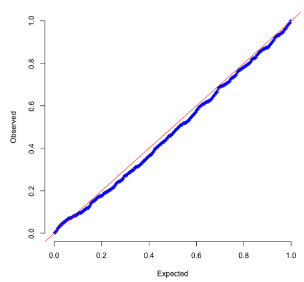
(e) ACDM: Q_{21}



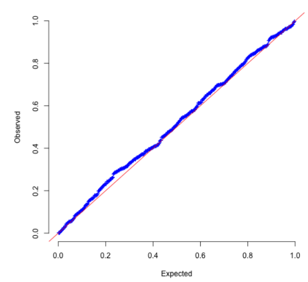
(f) ACDM: Q_{31}



(g) GDINA: Q_{11}

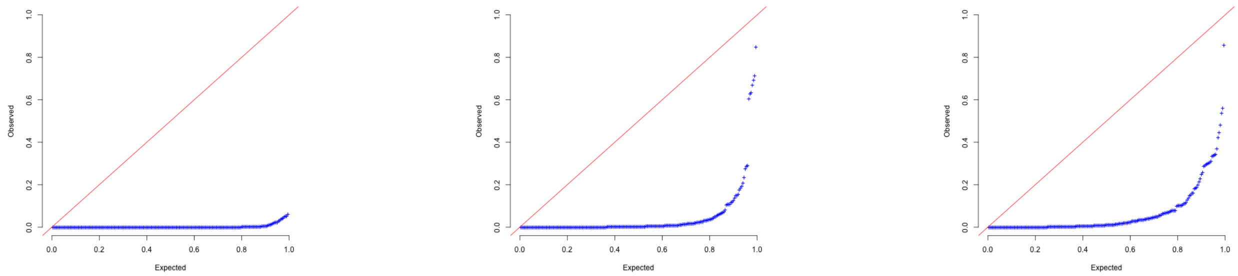


(h) GDINA: Q_{21}

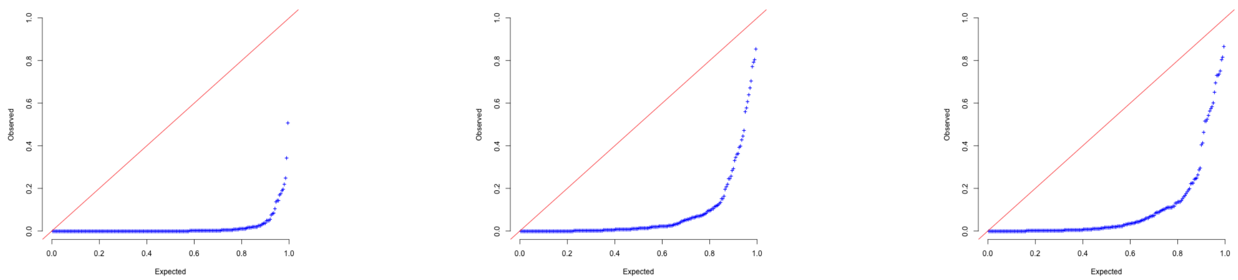


(i) GDINA: Q_{31}

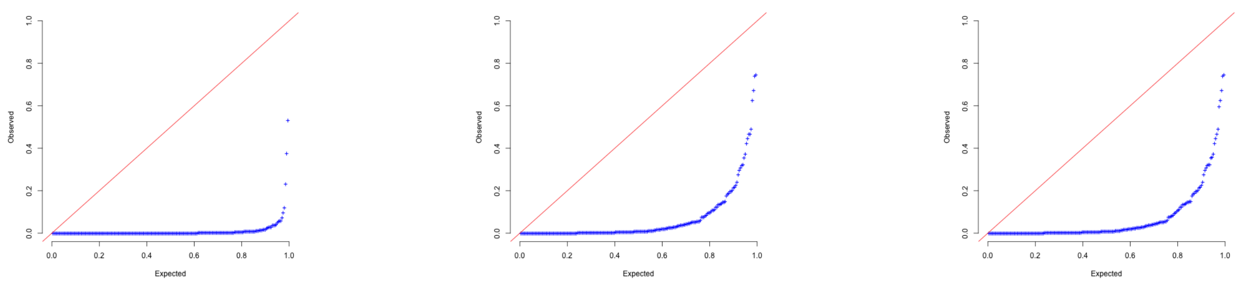
Figure 3: QQ-plots of p-values for testing True Q-matrices Q_{11}, Q_{21} and Q_{31} with $N = 500$, $\boldsymbol{\rho} = \mathbf{0.50}$.



(a) DINA: Q_{12} (b) DINA: Q_{22} (c) DINA: Q_{32}

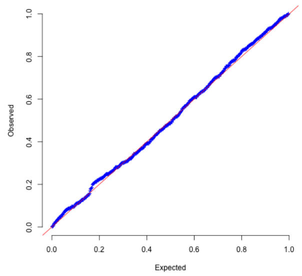


(d) ACDM: Q_{12} (e) ACDM: Q_{22} (f) ACDM: Q_{32}

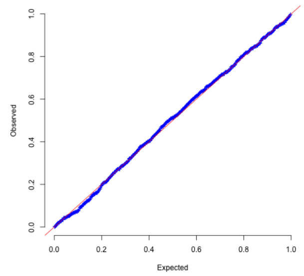


(g) GDINA: Q_{12} (h) GDINA: Q_{22} (i) GDINA: Q_{32}

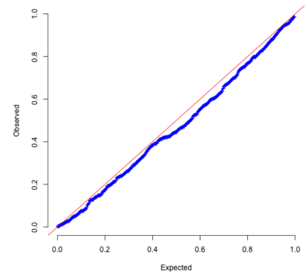
Figure 4: QQ-plots of p-values for testing Misspecified Q-matrices Q_{12}, Q_{22} and Q_{32} with $N = 500$, $\boldsymbol{\rho} = \mathbf{0.50}$.



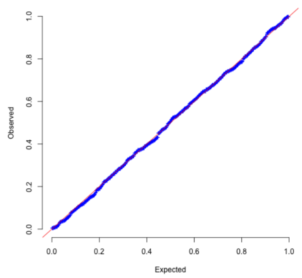
(a) DINA: Q_{11}



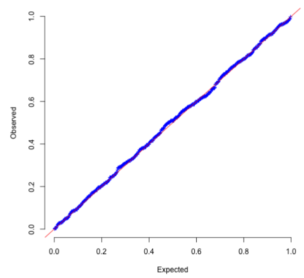
(b) DINA: Q_{21}



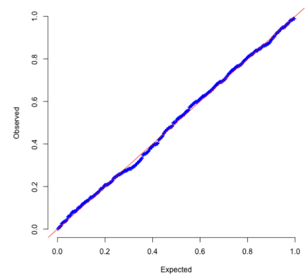
(c) DINA: Q_{31}



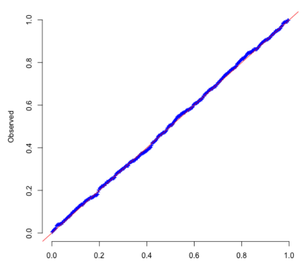
(d) ACDM: Q_{11}



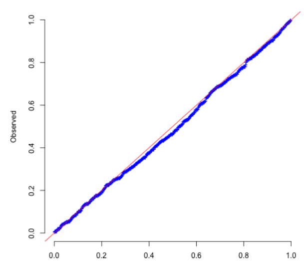
(e) ACDM: Q_{21}



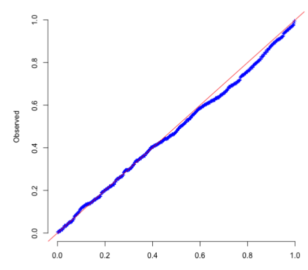
(f) ACDM: Q_{31}



(g) GDINA: Q_{11}

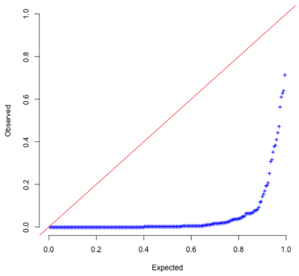


(h) GDINA: Q_{21}

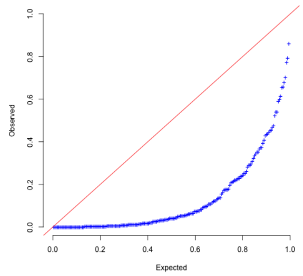


(i) GDINA: Q_{31}

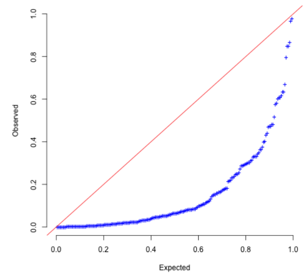
Figure 5: QQ-plots of p-values for testing True Q-matrices Q_{11}, Q_{21} and Q_{31} with $N = 500$, $\boldsymbol{\rho = 0.75}$.



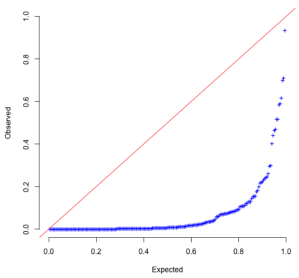
(a) DINA: Q_{12}



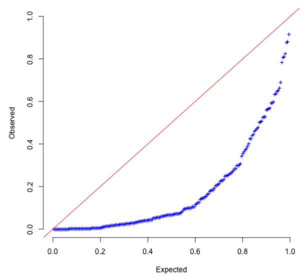
(b) DINA: Q_{22}



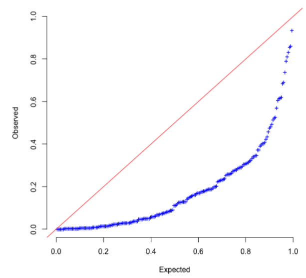
(c) DINA: Q_{32}



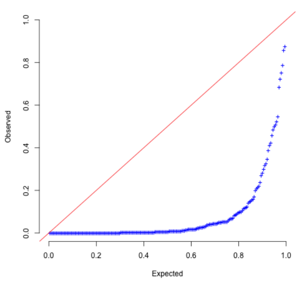
(d) ACDM: Q_{12}



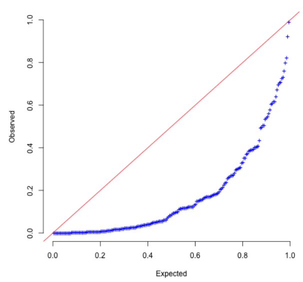
(e) ACDM: Q_{22}



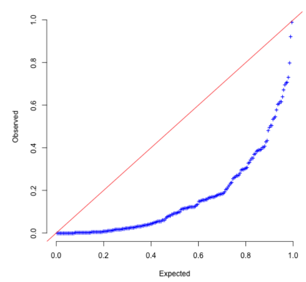
(f) ACDM: Q_{32}



(g) GDINA: Q_{12}

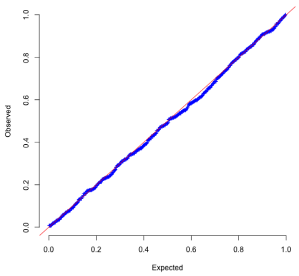


(h) GDINA: Q_{22}

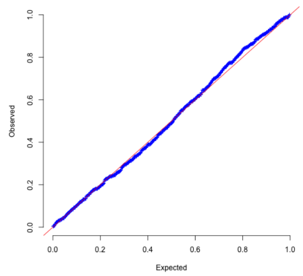


(i) GDINA: Q_{32}

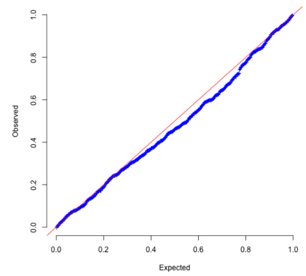
Figure 6: QQ-plots of p-values for testing Misspecified Q-matrices Q_{12}, Q_{22} and Q_{32} with $N = 500$, $\boldsymbol{\rho} = \mathbf{0.75}$.



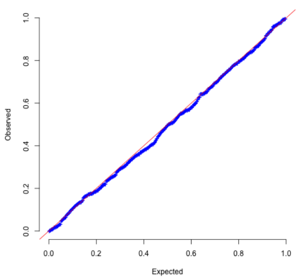
(a) DINA: $Q_{in,1}$



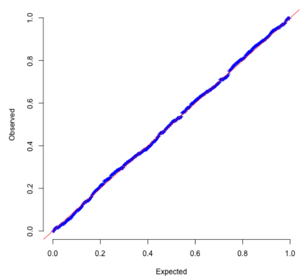
(b) DINA: $Q_{in,2}$



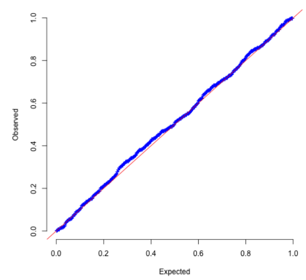
(c) DINA: $Q_{in,3}$



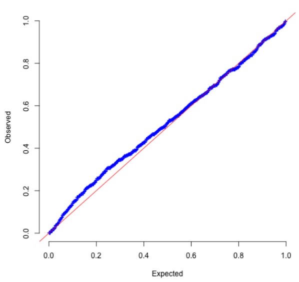
(d) ACDM: $Q_{in,1}$



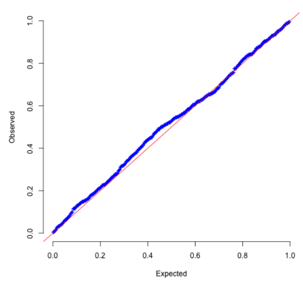
(e) ACDM: $Q_{in,2}$



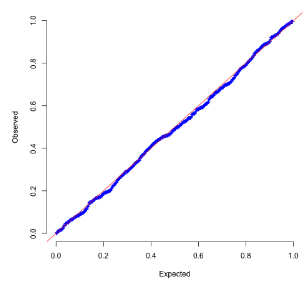
(f) ACDM: $Q_{in,3}$



(g) GDINA: $Q_{in,1}$

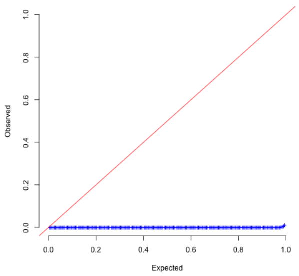


(h) GDINA: $Q_{in,2}$

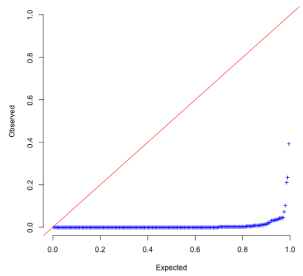


(i) GDINA: $Q_{in,3}$

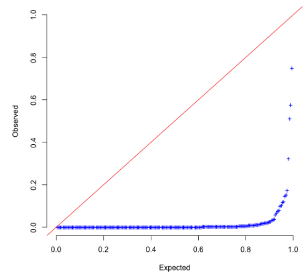
Figure 7: QQ-plots of p-values for testing True Incomplete Q-matrices $Q_{in,1}$, $Q_{in,2}$ and $Q_{in,3}$ with $N = 500$.



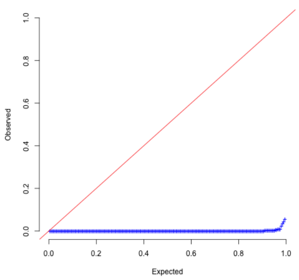
(a) DINA: $Q_{in,4}$



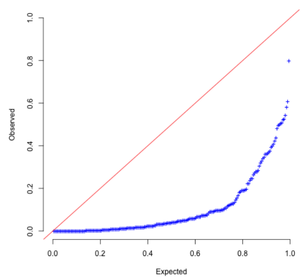
(b) DINA: $Q_{in,5}$



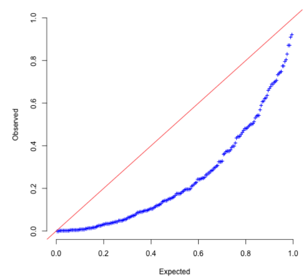
(c) DINA: $Q_{in,6}$



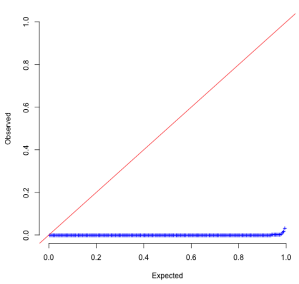
(d) ACDM: $Q_{in,4}$



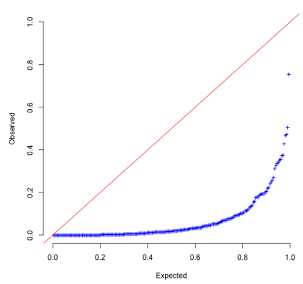
(e) ACDM: $Q_{in,5}$



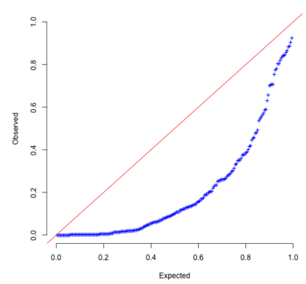
(f) ACDM: $Q_{in,6}$



(g) GDINA: $Q_{in,4}$



(h) GDINA: $Q_{in,5}$



(i) GDINA: $Q_{in,6}$

Figure 8: QQ-plots of p-values for testing Misspecified Incomplete Q-matrices $Q_{in,4}$, $Q_{in,5}$ and $Q_{in,6}$ with $N = 500$.

References

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Gu, Y. and Xu, G. (2018). Partial identifiability of restricted latent class models. *arXiv preprint arXiv:1803.04353*.