Supplementary materials

Here, we describe the steps of the Gibbs Sampler for estimating the models presented in the paper.

Step 1. For each combination of person $p \in [1:N]$ and item $i \in [1:K]$ sample the augmented continuous response y_{pi}:

$$y_{pi} \sim \begin{cases} \mathcal{N}(y_{pi}; \alpha_i\theta_p + \beta_p, 1)\mathcal{I}(y_{pi} \geq 0), \text{ if } x_{pi} = 1; \\ \mathcal{N}(y_{pi}; \alpha_i\theta_p + \beta_i, 1)\mathcal{I}(y_{pi} < 0), \text{ if } x_{pi} = 0. \end{cases} \tag{1}$$

Step 2. For each person $p \in [1:N]$ sample the person parameters: 1) in the case of the two-dimensional models $[\theta_p, \eta_p]^T$ is sampled from a bivariate normal distribution with the covariance matrix equal to

$$\mathbf{\Omega}_p = \left(\mathbf{\Sigma}^{-1} + \mathbf{z}^T\mathbf{z} + \mathbf{v}_p^T\mathbf{v}_p\right)^{-1}, \tag{2}$$

where $\mathbf{z} = [\boldsymbol{\alpha}\ \mathbf{0}]$ and $\mathbf{v}_p = \left[\mathbf{0}\ \left(\boldsymbol{\lambda}_{\cdot 0} \circ \boldsymbol{\sigma}_{\cdot 0}^{-1} \circ (1 - \mathbf{x}_{p\cdot}) + \boldsymbol{\lambda}_{\cdot 1} \circ \boldsymbol{\sigma}_{\cdot 1}^{-1} \circ \mathbf{x}_{p\cdot}\right)\right]$, and the mean vector equal to

$$\mathbf{\Omega}_p \left(\mathbf{z}^T(\mathbf{y}_{p\cdot} - \boldsymbol{\beta}) + \mathbf{v}_p^T \left(\left(\frac{1 - \mathbf{x}_{p\cdot}}{\boldsymbol{\sigma}_{\cdot 0}} + \frac{\mathbf{x}_{p\cdot}}{\boldsymbol{\sigma}_{\cdot 1}}\right) \circ \left(\mathbf{t}_{p\cdot}^* - (1 - \mathbf{x}_{p\cdot}) \circ \boldsymbol{\xi}_{\cdot 0} - \mathbf{x}_{p\cdot} \circ \boldsymbol{\xi}_{\cdot 1}\right)\right)\right); \tag{3}$$

2) in the three-dimensional model $[\theta_p\ \eta_{p0}\ \eta_{p1}]^T$ is sampled from a multivariate normal distribution with the covariance matrix:

$$\mathbf{\Omega}_p = \left(\mathbf{\Sigma}^{-1} + \mathbf{z}^T\mathbf{z} + \mathbf{v}_{p0}^T\mathbf{v}_{p0} + \mathbf{v}_{p1}^T\mathbf{v}_{p1}\right)^{-1}, \tag{4}$$

where $\mathbf{z} = [\boldsymbol{\alpha}\ \mathbf{0}\ \mathbf{0}]$, $\mathbf{v}_{p0} = \left[\mathbf{0}\ \left(\boldsymbol{\lambda}_{\cdot 0} \circ \boldsymbol{\sigma}_{\cdot 0}^{-1} \circ (1 - \mathbf{x}_{p\cdot})\right)\ \mathbf{0}\right]$, and $\mathbf{v}_{p1} = \left[\mathbf{0}\ \mathbf{0}\ \left(\boldsymbol{\lambda}_{\cdot 1} \circ \boldsymbol{\sigma}_{\cdot 1}^{-1} \circ \mathbf{x}_{p\cdot}\right)\right]$;

and the mean vector:

$$\boldsymbol{\Omega}_p\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}+\mathbf{z}^T(\mathbf{y}_{p\cdot}-\boldsymbol{\beta})+\mathbf{v}_{p0}^T\left(\left(\frac{1-\mathbf{x}_{p\cdot}}{\boldsymbol{\sigma}_{\cdot 0}}\right)\circ\left(\mathbf{t}_{p\cdot}^*-(1-\mathbf{x}_{p\cdot})\circ\boldsymbol{\xi}_{\cdot 0}\right)\right)+\mathbf{v}_{p1}^T\left(\frac{\mathbf{x}_{p\cdot}}{\boldsymbol{\sigma}_{\cdot 1}}\circ(\mathbf{t}_{p\cdot}^*-\mathbf{x}_{p\cdot}\boldsymbol{\xi}_{\cdot 1})\right)\right) \tag{5}$$

Step 3. For each item $i \in [1:K]$ sample the item parameters in the RA model, and time intensity(ies) and factor loading(s) in the RT model: 1) for the $\mathcal{M}_1$ and $\mathcal{M}_{3b}$ (i.e., $\xi_{i0} = \xi_{i1} = \xi_i, \lambda_{i0} = \lambda_{i1} = \lambda_i$) sample $[\alpha_i\ \beta_i\ \xi_i\ \lambda_i]^T$, 2) for the $\mathcal{M}_2$ and $\mathcal{M}_{4b}$ (i.e., $\xi_{i0} \neq \xi_{i1}, \lambda_{i0} = \lambda_{i1} = \lambda_i$) sample $[\alpha_i\ \beta_i\ \xi_{i0}\ \xi_{i1}\ \lambda_i]^T$, 3) for the $\mathcal{M}_{3a}$ and $\mathcal{M}_{4a}$ (i.e., $\xi_{i0} \neq \xi_{i1}, \lambda_{i0} \neq \lambda_{i1}$) sample $[\alpha_i\ \beta_i\ \xi_{i0}\ \xi_{i1}\ \lambda_{i0}\ \lambda_{i1}]^T$. In each case the parameters are sampled from a multivariate normal distribution with the covariance matrix $\boldsymbol{\Omega}_i = \left(\frac{\mathbf{I}_M}{100^2} + \mathbf{u}^T\mathbf{u} + \mathbf{w}^T\mathbf{w}\right)^{-1}$, where for $\mathcal{M}_1$ and $\mathcal{M}_{3b}$: $M = 4$, $\mathbf{u} = [\boldsymbol{\theta}\,\mathbf{1}\,\mathbf{0}\,\mathbf{0}]$, and $\mathbf{w} = \left[\mathbf{0}\,\mathbf{0}\ \left(\frac{1-\mathbf{x}_{\cdot i}}{\sigma_{i0}}+\frac{\mathbf{x}_{\cdot i}}{\sigma_{i1}}\right)\ -\left(\frac{\boldsymbol{\eta}_{\cdot 0}\circ(1-\mathbf{x}_{\cdot i})}{\sigma_{i0}}+\frac{\boldsymbol{\eta}_{\cdot 1}\circ\mathbf{x}_{\cdot i}}{\sigma_{i1}}\right)\right]$, for $\mathcal{M}_2$ and $\mathcal{M}_{4b}$: $M = 5$, $\mathbf{u} = [\boldsymbol{\theta}\,\mathbf{1}\,\mathbf{0}\,\mathbf{0}\,\mathbf{0}]$, and $\mathbf{w} = \left[\mathbf{0}\,\mathbf{0}\ \frac{1-\mathbf{x}_{\cdot i}}{\sigma_{i0}}\ \frac{\mathbf{x}_{\cdot i}}{\sigma_{i1}}\ -\left(\frac{\boldsymbol{\eta}_{\cdot 0}\circ(1-\mathbf{x}_{\cdot i})}{\sigma_{i0}}+\frac{\boldsymbol{\eta}_{\cdot 1}\circ\mathbf{x}_{\cdot i}}{\sigma_{i1}}\right)\right]$, for $\mathcal{M}_{3a}$ and $\mathcal{M}_{4a}$: $M = 6$, $\mathbf{u} = [\boldsymbol{\theta}\,\mathbf{1}\,\mathbf{0}\,\mathbf{0}\,\mathbf{0}\,\mathbf{0}]$, and $\mathbf{w} = \left[\mathbf{0}\,\mathbf{0}\ \frac{1-\mathbf{x}_{\cdot i}}{\sigma_{i0}}\ \frac{\mathbf{x}_{\cdot i}}{\sigma_{i1}}\ -\frac{\boldsymbol{\eta}_{\cdot 0}\circ(1-\mathbf{x}_{\cdot i})}{\sigma_{i0}}\ -\frac{\boldsymbol{\eta}_{\cdot 1}\circ\mathbf{x}_{\cdot i}}{\sigma_{i1}}\right]$. The mean vector for the conditional posterior distribution is equal to

$$\boldsymbol{\Omega}_i\left(\mathbf{u}^T\mathbf{y}_{\cdot i} + \mathbf{w}^T\left(\mathbf{t}_{\cdot i}^*\circ\left(\frac{1-\mathbf{x}_{\cdot i}}{\sigma_{i0}}+\frac{\mathbf{x}_{\cdot i}}{\sigma_{i1}}\right)\right)\right). \tag{6}$$

Step 4. For each item $i \in [1:K]$ sample the residual variances in the RT model: for $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_{3a}$ (i.e., $\sigma_{i0}^2 = \sigma_{i1}^2 = \sigma_i^2$) sample σ_i^2:

$$\sigma_i^2 \sim \mathcal{IG}\left(\sigma_i^2; 0.001 + \frac{N}{2}, 0.001 + \frac{\sum_{p=1}^N(t_{pi}^* - \xi_{ix_{pi}} + \lambda_{ix_{pi}}\eta_{px_{pi}})^2}{2}\right); \tag{7}$$

and for the $\mathcal{M}_{3b}$, $\mathcal{M}_{4a}$ and $\mathcal{M}_{4b}$ sample σ_{i0}^2 from

$$\sigma_{i0}^2 \sim \mathcal{IG}\left(\sigma_{i0}^2; 0.001 + \frac{\sum_{p=1}^N(1-x_{pi})}{2}, 0.001 + \frac{\sum_{p=1}^N(1-x_{pi})(t_{pi}^* - \xi_{i0} + \lambda_{i0}\eta_{p0})^2}{2}\right); \tag{8}$$

and σ_{i1}^2 from:

$$\sigma_{i1}^2 \sim \mathcal{IG}\left(\sigma_{i1}^2; 0.001 + \frac{\sum_{p=1}^N x_{pi}}{2}, 0.001 + \frac{\sum_{p=1}^N x_{pi}(t_{pi}^* - \xi_{i1} + \lambda_{i1}\eta_{p1})^2}{2}\right). \tag{9}$$

Step 5. Sample the mean vector of the person parameters: in the two-dimensional model:

$$\boldsymbol{\mu} \sim \mathcal{N}_2\left(\boldsymbol{\mu}; \left[\frac{\sum_{p=1}^N \theta_p}{N} \; \frac{\sum_{p=1}^N \eta_p}{N}\right]^T, \frac{1}{N}\boldsymbol{\Sigma}\right), \tag{10}$$

and in the three-dimensional model:

$$\boldsymbol{\mu} \sim \mathcal{N}_3\left(\boldsymbol{\mu}; \left[\frac{\sum_{p=1}^N \theta_p}{N} \; \frac{\sum_{p=1}^N \eta_{p0}}{N} \; \frac{\sum_{p=1}^N \eta_{p1}}{N}\right]^T, \frac{1}{N}\boldsymbol{\Sigma}\right). \tag{11}$$

Step 6. Sample the covariance matrix of the person parameters: in the two-dimensional model:

$$\boldsymbol{\Sigma} \sim \mathcal{IW}\left(\boldsymbol{\Sigma}; 4 + N, \mathbf{I}_2 + \sum_{p=1}^N ([\theta_p \, \eta_p]^T - \boldsymbol{\mu})([\theta_p \, \eta_p] - \boldsymbol{\mu}^T)\right), \tag{12}$$

and in the three-dimensional model:

$$\boldsymbol{\Sigma} \sim \mathcal{IW}\left(\boldsymbol{\Sigma}; 5 + N, \mathbf{I}_3 + \sum_{p=1}^N ([\theta_p \, \eta_{p0} \, \eta_{p1}]^T - \boldsymbol{\mu})([\theta_p \, \eta_{p0} \, \eta_{p1}] - \boldsymbol{\mu}^T)\right). \tag{13}$$

Step 7. Re-scale the model parameter to satisfy the identification constraints:

$$\begin{array}{lcll} \theta_p & \rightarrow & \frac{\theta_p - \mu_1}{\sqrt{\Sigma_{11}}}, & p \in [1:N]; \\ \beta_i & \rightarrow & \beta_i + \alpha_i \mu_1, & i \in [1:K]; \\ \alpha_i & \rightarrow & \alpha_i \sqrt{\Sigma_{11}}, & i \in [1:K]; \end{array} \tag{14}$$

in the two-dimensional models:

$$
\begin{array}{lcll}
\eta_p & \rightarrow & \frac{\eta_p - \mu_2}{\sqrt{\Sigma_{22}}}, & p \in [1:N]; \\
\xi_{ik} & \rightarrow & \xi_{ik} + \lambda_{ik}\mu_2, & i \in [1:K], k \in \{0,1\}; \\
\lambda_{ik} & \rightarrow & \lambda_{ik}\sqrt{\Sigma_{22}}, & i \in [1:K], k \in \{0,1\}; \\
\boldsymbol{\mu} & \rightarrow & \mathbf{0}; & \\
\boldsymbol{\Sigma} & \rightarrow & \begin{bmatrix} 1 & \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \\ \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} & 1 \end{bmatrix}; &
\end{array}
\tag{15}
$$

in the three-dimensional $\mathcal{M}_1$ and $\mathcal{M}_{3b}$:

$$
\begin{array}{lcll}
\eta_{pk} & \rightarrow & \frac{\eta_{pk} - \mu_2}{\sqrt{\Sigma_{22}}}, & p \in [1:N], k \in \{0,1\}; \\
\xi_{ik} & \rightarrow & \xi_{ik} + \lambda_{ik}\mu_2, & i \in [1:K], k \in \{0,1\}; \\
\lambda_{ik} & \rightarrow & \lambda_{ik}\sqrt{\Sigma_{22}}, & i \in [1:K], k \in \{0,1\}; \\
\boldsymbol{\mu} & \rightarrow & [0\,0\,\frac{\mu_3 - \mu_2}{\sqrt{\Sigma_{22}}}]^T; & \\
\boldsymbol{\Sigma} & \rightarrow & \begin{bmatrix} 1 & \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} & \frac{\Sigma_{13}}{\sqrt{\Sigma_{11}}} \\ \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} & 1 & \frac{\Sigma_{23}}{\sqrt{\Sigma_{22}}} \\ \frac{\Sigma_{13}}{\sqrt{\Sigma_{11}}} & \frac{\Sigma_{23}}{\sqrt{\Sigma_{22}}} & \frac{\Sigma_{33}}{\Sigma_{22}} \end{bmatrix}; &
\end{array}
\tag{16}
$$

in the three-dimensional $\mathcal{M}_2$ and $\mathcal{M}_{4b}$:

$$
\begin{array}{lcll}
\eta_{pk} & \rightarrow & \frac{\eta_{pk} - \mu_{k+2}}{\sqrt{\Sigma_{22}}}, & p \in [1:N], k \in \{0,1\}; \\
\xi_{ik} & \rightarrow & \xi_{ik} + \lambda_{ik}\mu_{k+2}, & i \in [1:K], k \in \{0,1\}; \\
\lambda_{ik} & \rightarrow & \lambda_{ik}\sqrt{\Sigma_{22}}, & i \in [1:K], k \in \{0,1\}; \\
\boldsymbol{\mu} & \rightarrow & \mathbf{0}; & \\
\boldsymbol{\Sigma} & \rightarrow & \begin{bmatrix} 1 & \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} & \frac{\Sigma_{13}}{\sqrt{\Sigma_{11}}} \\ \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} & 1 & \frac{\Sigma_{23}}{\sqrt{\Sigma_{22}}} \\ \frac{\Sigma_{13}}{\sqrt{\Sigma_{11}}} & \frac{\Sigma_{23}}{\sqrt{\Sigma_{22}}} & \frac{\Sigma_{33}}{\Sigma_{22}} \end{bmatrix}; &
\end{array}
\tag{17}
$$

and in the three-dimensional $\mathcal{M}_{3a}$ and $\mathcal{M}_{4a}$:

$$
\begin{aligned}
\eta_{pk} &\rightarrow & \frac{\eta_{pk}-\mu_{k+2}}{\sqrt{\Sigma_{(k+2)(k+2)}}}, && p \in [1:N], k \in \{0,1\}; \\
\xi_{ik} &\rightarrow & \xi_{ik} + \lambda_{ik}\mu_{k+2}, && i \in [1:K], k \in \{0,1\}; \\
\lambda_{ik} &\rightarrow & \lambda_{ik}\sqrt{\Sigma_{(k+2)(k+2)}}, && i \in [1:K], k \in \{0,1\}; \\
\boldsymbol{\mu} &\rightarrow & \mathbf{0}; && \\
\boldsymbol{\Sigma} &\rightarrow & \begin{bmatrix} 1 & \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} & \frac{\Sigma_{13}}{\sqrt{\Sigma_{11}\Sigma_{33}}} \\ \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} & 1 & \frac{\Sigma_{23}}{\sqrt{\Sigma_{22}\Sigma_{33}}} \\ \frac{\Sigma_{13}}{\sqrt{\Sigma_{11}\Sigma_{33}}} & \frac{\Sigma_{23}}{\sqrt{\Sigma_{22}\Sigma_{33}}} & 1 \end{bmatrix}. &&
\end{aligned}
\tag{18}
$$