Supplementary Material to "Second-Order Probability Matching Priors for the Person Parameter in Unidimensional IRT Models"

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A Proofs

A.1 Assumptions

Posterior expansion. Regularity conditions C1–C5 of Crowder (1988) ensure a valid asymptotic expansion of the posterior cumulative distribution function (cdf) when the data are independent but not necessarily identically distributed (i.n.i.d.), which extends the corresponding conditions of Johnson (1970) for independent and identically distributed (i.i.d.) data. Note that all the convergence in P_{θ_0}-probability in the aformentioned conditions should be strengthened to convergence with P_θ probability $1 + o(n^{-1})$ uniformly for θ on compact sets, which parallels a comment of Datta and Mukerjee (2004, p. 11). To further accommodate data-dependant priors in the form of Equation 16, Condition C3 needs to be modified: Substituting $\pi_n(\cdot, \cdot)$ for $p(\cdot)$ in Crowder's notation, we assume that, as $n \to \infty$,

$$P_\theta\{\pi_n^{(0,a)}(\theta_1, \theta_2) \leq N_1,\ \forall \theta_1, \theta_2 \in C_n\} = 1 + o(n^{-1}) \tag{C3$'$}$$

uniformly for θ on compact sets, in which $a = 0, 1, 2, 3,$ $N_1 < \infty$, and C_n is a neighborhood of the maximum likelihood (ML) estimator $\hat{\theta}_n$ defined on p. 299 of Crowder (1988). Note that the reference to $p(\theta)$ in Condition C5(b) should be changed to $\pi_n(\tilde{\theta}_n, \theta)$ as well.

Edgeworth expansion. If the square-root Wald statistic U_n can be expressed as a sufficiently smooth function of some other statistics that possess a valid Edgeworth expansion, then the desired expansion of U_n follows provided Assumption 3.1 of Skovgaard (1981b) is satisfied. For instance, the ML estimator, and consequently the square-root Wald statistic, can often be expressed as a smooth function of the log-likelihood derivatives due to the Implicit Function Theorem (Rudin, 1976, pp. 223–228). Bhattacharya and Ghosh (1978, Assumptions A1–A6) and, more generally, Skovgaard (1981a, Assumption 3.1) supplied regularity conditions to formalize the argument; unfortunately, their assumptions are not directly applicable to discrete data problems.

Initial estimator and modifier function. The initial estimator is not limited to but should behave sufficiently similar to the ML estimator. In particular, it is assumed that, uniformly for θ on compact sets, (1) $P_\theta\{\tilde{\theta}_n \in C_n\} = 1 + o(n^{-1})$, in which C_n is defined in Crowder (1988, p. 299) as before, and (2) $E_\theta \tilde{\theta}_n = \theta + o(n^{-1/2})$. In addition, it is assumed that $D_n(\theta) = H_n^{(0,1)}(\theta, \theta)$ is continuously differentiable for all n and θ.

Example: The two-parameter logistic model. For the two-parameter logistic (2PL) model, sufficient conditions for the posterior expansion can be formulated in a fashion similar to the Poisson loglinear model example in Crowder (1988, Section 3.1). To obtain an Edgeworth expansion for the square-root Wald statistic U_n, we note that U_n can be expressed as an implicit smooth function of $T_n = \sum_{i_1}^n a_i Y_i$, which is denoted $U_n = u_n(T_n)$ (e.g., Biehler, Holling, & Doebler, 2015, Equation A.11). Lemma 2.2 of Albers, Bickel, and van Zwet (1976) warrants a desired expansion for weighted sums of i.n.i.d. dichotomous data, which directly applies to T_n. Assumption 3.1 of Skovgaard (1981b) can be verified by calculating the asymptotic orders for the cumulants of T_n (Albers et al., 1976, Equations 2.6 and 2.7) and the derivatives of $u_n(\cdot)$. For example, suppose that a_i's and b_i's are bounded sequences, and that a_i's satisfy Lemma 2.2 of Albers et al. (1976). In this case, the third and fourth cumulants of T_n are of orders $n^{-1/2}$ and n^{-1}; $u''(\cdot)$ and $u'''(\cdot)$ are of orders $n^{-1/2}$ and n^{-1} as well. Hence, Assumption 3.1 of Skovgaard (1981b) holds, and there exists an Edgeworth expansion of U_n with an $o(n^{-1})$ error margin. Finally, those $H_n(\cdot,\cdot)$'s considered in the present work are sufficiently smooth in both arguments.

A.2 Expanding the posterior quantile

Let $c_n = -\ell_n''(\hat{\theta}_n, \mathbf{Y})$, $m_n = \ell_n'''(\hat{\theta}_n, \mathbf{Y})$ and $r_n = \ell_n''''(\hat{\theta}_n, \mathbf{Y})$ be the derivatives of the observed log-likelihood. Also let $u = (nc_n)^{1/2}(\theta - \hat{\theta}_n)$ and $\underline{\pi}_n(u|\mathbf{Y})$ be the posterior probability density function (pdf) of u corresponding to the prior $\pi_n(\hat{\theta}_n, \cdot)$ (Equation 16). Under the assumptions sketched out in Section A.1, we have the following expansion for $\underline{\pi}_n(u|\mathbf{Y})$:

$$\begin{aligned}\underline{\pi}_n(u|\mathbf{Y}) = \phi(u)\Big\{1 &+ n^{-1/2}[G_1 J_1(u) + G_3 J_3(u)] \\ &+ n^{-1}\left[G_2 J_2(u) + G_4 J_4(u) + G_6 J_6(u)\right]\Big\} + o_p(n^{-1}),\end{aligned} \tag{A.1}$$

in which $J_a(\cdot)$ is Hermite polynomial of degree a ($a \in \mathbb{N}$) defined by $d^a\phi(u)/du^a = (-1)^a J_a(u)\phi(u)$. Quantities G_1 to G_6 in Equation A.1 are given by

$$G_1 = A_1 + 3A_3,\ G_2 = A_2 + 6A_4 + 45A_6,\ G_4 = A_4 + 15A_6,\ G_3 = A_3,\ G_6 = A_6, \tag{A.2}$$

in which

$$A_1 = c_n^{-1/2}\frac{\pi_n^{(0,1)}(\tilde{\theta}_n,\hat{\theta}_n)}{\pi_n(\tilde{\theta}_n,\hat{\theta}_n)},\ A_2 = \frac{1}{2}c_n^{-1}\frac{\pi_n^{(0,2)}(\tilde{\theta}_n,\hat{\theta}_n)}{\pi_n(\tilde{\theta}_n,\hat{\theta}_n)}$$
$$A_3 = \frac{1}{6}c_n^{-3/2}m_n,\ A_4 = A_1A_3 + \frac{1}{24}c_n^{-2}r_n,\ A_6 = \frac{1}{2}A_3^2. \tag{A.3}$$

Integrating Equation A.1 with respect to u on the domain $(-\infty, z]$ leads to the following expansion of the posterior cdf

$$P^{\pi_n}\{u \le z|\mathbf{Y}\} = \Phi(z) - n^{-1/2}\phi(z)\left[G_1 + G_3J_2(z)\right] - n^{-1}\phi(z)\left[G_2J_1(z) + G_4J_3(z) + G_6J_5(z)\right] + o_p(n^{-1}), \tag{A.4}$$

in which P^{π} stands for the posterior probability measure with respect to the prior π. Define

$$\beta_1 = G_1 + G_3J_2(z_\alpha),$$
$$\beta_2 = 2z_\alpha\beta_1G_3 - \frac{1}{2}\beta_1^2z_\alpha + G_2J_1(z_\alpha) + G_4J_3(z_\alpha) + G_6J_5(z_\alpha), \tag{A.5}$$

and an approximation of the αth posterior quantile

$$\theta_\alpha(\pi_n,\mathbf{Y}) = \hat{\theta}_n + (nc_n)^{-1/2}\left[z_\alpha + n^{-1/2}\beta_1 + n^{-1}\beta_2\right]. \tag{A.6}$$

It can be deduced from Equation A.4 that $P^{\pi_n}\{\theta \le \theta_\alpha(\pi_n,\mathbf{Y})|\mathbf{Y}\} = \alpha + o_p(n^{-1})$.

A.3 A shrinkage argument

We now proceed with a standard shrinkage argument, which consists of the following three steps:

1. Define an auxiliary data-free prior $\bar{\pi}$ which is sufficiently smooth and has a compact support that contains the true θ_0 in the interior; both $\bar{\pi}$ and its derivatives vanish at the boundary of the support. Find an expansion of $P^{\bar{\pi}}\{\theta \le \theta_\alpha(\pi_n,\mathbf{Y})|\mathbf{Y}\}$ with an error term of order $o_p(n^{-1})$.
2. Take the P_θ-expectation of the expansion of $P^{\bar{\pi}}\{\theta \le \theta_\alpha(\pi_n,\mathbf{Y})|\mathbf{Y}\}$ obtained in Step 1, resulting in an expansion of $E_\theta\left[P^{\bar{\pi}}\{\theta \le \theta_\alpha(\pi_n,\mathbf{Y})|\mathbf{Y}\}\right]$ with an $o(n^{-1})$ error term.
3. Integrate the expansion of $E_\theta\left[P^{\bar{\pi}}\{\theta \le \theta_\alpha(\pi_n,\mathbf{Y})|\mathbf{Y}\}\right]$ obtained in Step 2 with respect to the auxiliary prior $\bar{\pi}$. Sending $\bar{\pi}$ to a point mass concentrated at θ_0 eventually yields the desired expansion of $P_{\theta_0}\{\theta_0 \le \theta_\alpha(\pi_n,\mathbf{Y})\}$.

Step 1 entails expanding the posterior cdf associated with the auxiliary prior $\bar{\pi}$ (similar to the derivation of Equation A.4) and evaluating the expansion at $\theta_\alpha(\pi_n,\mathbf{Y})$ (Equation A.6). Define $\bar{G}_1$ and $\bar{G}_2$ in a similar manner as G_1 and G_2

(Equations A.2 and A.3) by substituting the auxiliary prior $\bar{\pi}$ for the prior of interest π_n. After simplification, we have

$$\begin{aligned}
&P^{\bar{\pi}}\{\theta \le \theta_\alpha(\pi_n, \mathbf{Y})|\mathbf{Y}\} \\
&= \alpha + n^{-1/2}\phi(z_\alpha)(G_1 - \bar{G}_1) + n^{-1}z_\alpha\phi(z_\alpha)\Big[G_2 - \bar{G}_2 - (G_1 - \bar{G}_1)(G_1 + 2G_3)\Big] + o_p(n^{-1}) \\
&= \alpha + n^{-1/2}\phi(z_\alpha)c_n^{-1/2}\left\{\frac{\pi_n^{(0,1)}(\tilde{\theta}_n, \hat{\theta}_n)}{\pi_n(\tilde{\theta}_n, \hat{\theta}_n)} - \frac{\bar{\pi}'(\hat{\theta}_n)}{\bar{\pi}(\hat{\theta}_n)}\right\} \\
&\quad + n^{-1}z_\alpha\phi(z_\alpha)\left\{\left[\frac{1}{6}c_n^{-2}m_n - c_n^{-1}\frac{\pi_n^{(0,1)}(\tilde{\theta}_n, \hat{\theta}_n)}{\pi_n(\tilde{\theta}_n, \hat{\theta}_n)}\right]\left[\frac{\pi_n^{(0,1)}(\tilde{\theta}_n, \hat{\theta}_n)}{\pi_n(\tilde{\theta}_n, \hat{\theta}_n)} - \frac{\bar{\pi}'(\hat{\theta}_n)}{\bar{\pi}(\hat{\theta}_n)}\right]\right. \\
&\quad \left. + \frac{1}{2}c_n^{-1}\left[\frac{\pi_n^{(0,2)}(\tilde{\theta}_n, \hat{\theta}_n)}{\pi_n(\tilde{\theta}_n, \hat{\theta}_n)} - \frac{\bar{\pi}''(\hat{\theta}_n)}{\bar{\pi}(\hat{\theta}_n)}\right]\right\} + o_p(n^{-1}). \qquad \text{(A.7)}
\end{aligned}$$

Next, we take a P_θ-expectation on the right-hand side of Equation A.7 in Step 2. Our assumptions allow to replace $\hat{\theta}_n$ by θ and the observed log-likelihood derivatives by their expected counterpart after taking the expectation (see Datta & Mukerjee, 2004, Chapter 2):

$$\begin{aligned}
&E_\theta\left[P^{\bar{\pi}}\{\theta \le \theta_\alpha(\pi_n, \mathbf{y})|\mathbf{y}\}\right] \\
&= \alpha + n^{-1/2}\phi(z_\alpha)I_n(\theta)^{-1/2}\left\{\frac{\pi_n^{(0,1)}(\theta, \theta)}{\pi_n(\theta, \theta)} - \frac{\bar{\pi}'(\theta)}{\bar{\pi}(\theta)}\right\} \\
&\quad + n^{-1}z_\alpha\phi(z_\alpha)\left\{\left[\frac{1}{6}I_n(\theta)^{-2}M_n(\theta) - I_n(\theta)^{-1}\frac{\pi_n^{(0,1)}(\theta, \theta)}{\pi_n(\theta, \theta)}\right]\left[\frac{\pi_n^{(0,1)}(\theta, \theta)}{\pi_n(\theta, \theta)} - \frac{\bar{\pi}'(\theta)}{\bar{\pi}(\theta)}\right]\right. \\
&\quad \left. + \frac{1}{2}I_n(\theta)^{-1}\left[\frac{\pi_n^{(0,2)}(\theta, \theta)}{\pi_n(\theta, \theta)} - \frac{\bar{\pi}''(\theta)}{\bar{\pi}(\theta)}\right]\right\} + o(n^{-1}). \qquad \text{(A.8)}
\end{aligned}$$

For easy reference, let

$$\beta(\theta) = \frac{1}{6}I_n(\theta)^{-2}M_n(\theta) - I_n(\theta)^{-1}\frac{\pi_n^{(0,1)}(\theta, \theta)}{\pi_n(\theta, \theta)}.$$

It follows upon integration by parts that

$$\int f(\theta)\frac{\bar{\pi}^{(a)}(\theta)}{\bar{\pi}(\theta)}d\theta \to (-1)^a f^{(a)}(\theta_0) \qquad \text{(A.9)}$$

for any a-time continuously differentiable function f $(a \in \mathbb{N})$ as $\bar{\pi}$ approaches a point mass at θ_0. In the final step, we apply Equation A.9 to Equation A.8 and conclude after some algebra that

$$P_{\theta_0}\{\theta_0 \le \theta_\alpha(\pi_n, \mathbf{Y})\} = \alpha + n^{-1/2}\phi(z_\alpha)\left[I_n(\theta_0)^{-1/2}\frac{\pi_n^{(0,1)}(\theta_0,\theta_0)}{\pi_n(\theta_0,\theta_0)} + (I_n^{-1/2})'(\theta_0)\right]$$
$$+ n^{-1}z_\alpha\phi(z_\alpha)\left\{\beta(\theta_0)\frac{\pi_n^{(0,1)}(\theta_0,\theta_0)}{\pi_n(\theta_0,\theta_0)} + \beta'(\theta_0)\right.$$
$$\left. + \frac{1}{2}I_n(\theta_0)^{-1}\frac{\pi_n^{(0,2)}(\theta_0,\theta_0)}{\pi_n(\theta_0,\theta_0)} - \frac{1}{2}(I_n^{-1})''(\theta_0)\right\} + o(n^{-1}). \quad \text{(A.10)}$$

Equation A.10 simplifies to Equation 17 after some algebra. Corollaries 1 and 2 can be established by setting the first- and second-order terms in Equation 17 to zero.

B Expected log-likelihood derivatives

The expected log-likelihood derivatives appeared in Equation 20 can be expressed as functions of the first to fourth derivatives of the pmfs. The pmf derivatives for the 2PL, 3PL, graded, and nominal models can be found in the example R code in the supplementary material. We also comment on the propriety of the Jeffreys prior for the four types of models.

B.1 Derivatives

For conciseness, we write $f_{iy} = f_i(y;\theta)$ and drop the dependencies on θ from the notation. After some straightforward algebra, we have

$$I_n = \frac{1}{n}\sum_{i=1}^{n}\sum_{y=0}^{K_i-1}\frac{(f'_{iy})^2}{f_{iy}}, \quad \text{(B.11)}$$

$$I'_n = \frac{1}{n}\sum_{i=1}^{n}\sum_{y=0}^{K_i-1}\left[-\frac{(f'_{iy})^3}{f_{iy}^2} + \frac{2f'_{iy}f''_{iy}}{f_{iy}}\right], \quad \text{(B.12)}$$

$$I''_n = \frac{1}{n}\sum_{i=1}^{n}\sum_{y=0}^{K_i-1}\left[\frac{2(f'_{iy})^4}{f_{iy}^3} - \frac{5(f'_{iy})^2f''_{iy}}{f_{iy}^2} + \frac{2(f''_{iy})^2 + 2f'_{iy}f'''_{iy}}{f_{iy}}\right], \quad \text{(B.13)}$$

$$M_n = \frac{1}{n}\sum_{i=1}^{n}\sum_{y=0}^{K_i-1}\left[\frac{2(f'_{iy})^3}{f_{iy}^2} - \frac{3f'_{iy}f''_{iy}}{f_{iy}} + f'''_{iy}\right], \quad \text{(B.14)}$$

$$M'_n = \frac{1}{n}\sum_{i=1}^{n}\sum_{y=0}^{K_i-1}\left[-\frac{4(f'_{iy})^4}{f_{iy}^3} + \frac{9(f'_{iy})^2f''_{iy}}{f_{iy}^2} - \frac{3(f''_{iy})^2 + 3f'_{iy}f'''_{iy}}{f_{iy}} + f''''_{iy}\right]. \quad \text{(B.15)}$$

B.2 Jeffreys prior

The Jeffreys prior is proper for the four types of IRT models under study, which further leads to proper posterior distributions due to the boundedness of the likelihood function.

The 2PL model. Let $\eta_i = \exp[a_i(\theta - b_i)]$. The test information of the 2PL model can be written as

$$I_n = \frac{1}{n}\sum_{i=1}^{n}\frac{a_i^2\eta_i}{(1+\eta_i)^2}. \tag{B.16}$$

The propriety of the Jeffreys prior follows from the fact that $x^{1/2} + y^{1/2} \geq (x+y)^{1/2}$ for all $x, y \geq 0$, and that

$$\int_{-\infty}^{\infty}\frac{\exp(x/2)}{1+\exp(x)}dx = \pi. \tag{B.17}$$

The 3PL model. Let $x_i = \exp[a_i(\theta - b_i)]$. The test information of the 3PL model can be written as

$$I_n = \frac{1}{n}\sum_{i=1}^{n}\frac{a_i^2(1-c_i)\eta_i^2}{(1+\eta_i)^2(c_i+\eta_i)}. \tag{B.18}$$

Due to Equation B.17 and the fact that $x/(x+c) \in (0,1)$ for all $x, c > 0$, the Jeffreys prior is proper for the 3PL model.

The graded model. Let $\eta_{iy} = \exp[a_i(\theta - b_{iy})]$, $y = 1, \ldots, K_i - 1$, $K_i > 2$. The test information of the graded model is

$$I_n = \frac{1}{n}\sum_{i=1}^{n}a_i^2\left[\frac{\eta_{i1}^2}{(1+\eta_{i1})^3} + \sum_{y=1}^{K_i-2}\frac{(\eta_{iy}-\eta_{i,y+1})(\eta_{iy}\eta_{i,y+1}-1)^2}{(1+\eta_{iy})^3(1+\eta_{i,y+1})^3} + \frac{\eta_{i,K_i-1}}{(1+\eta_{i,K_i-1})^3}\right]. \tag{B.19}$$

Note that $(x_1x_2-1)^2/[(1+x_1)^2(1+x_2)^2] \in (0,1)$ for every $x_1, x_2 > 0$. By virtue of Equation B.17, the square root of each summand on the right-hand side of Equation B.19 is integrable.

The nominal model. Let $\eta_{iy} = \exp[\sum_{k=0}^{y}a_{ik}(\theta - b_{ik})]$ and $\alpha_{iy} = \sum_{k=0}^{y}a_{ik}$, $y = 0, \ldots, K_i - 1$. The test information of the nominal model is given by

$$\begin{aligned} I_n &= \frac{1}{n}\sum_{i=1}^{n}\frac{(\sum_{k=0}^{K_i-1}\eta_{ik})(\sum_{k=0}^{K_i-1}\eta_{ik}\alpha_{ik}^2) - (\sum_{k=0}^{K_i-1}\eta_{ik}\alpha_{ik})^2}{(\sum_{k=0}^{K_i-1}\eta_{ik})^2} \\ &= \frac{1}{n}\sum_{i=1}^{n}\sum_{k\neq l}\frac{\eta_{ik}\eta_{il}(\alpha_{ik}^2 - 2\alpha_{ik}\alpha_{il})}{(\sum_{k=0}^{K_i-1}\eta_{ik})^2}. \end{aligned} \tag{B.20}$$

Each summand on the right-hand side of Equation B.20, with the constant removed, satisfies

$$\frac{\eta_{ik}\eta_{il}}{(\sum_{k=0}^{K_i-1}\eta_{ik})^2} \leq \frac{\eta_{ik}\eta_{il}}{(\eta_{ik}+\eta_{il})^2} = \frac{\eta_{il}/\eta_{ik}}{(1+\eta_{il}/\eta_{ik})^2}.$$

The integrability of the Jeffreys prior follows again from Equation B.17.

C R code

C.1 Class `item` and methods

To facilitate applications to mixed response format tests, a parent S4 class `item` is defined and inherited by four derived classes: namely, `2pl`, `3pl`, `graded`, and `nom`, each of which implements the corresponding item type considered in the present paper. The source code can be found in `item.R`.

An `item` object can be constructed by calling `new(_type_, par, theta)`: `_type_` refers to the name of a derived class, `par` is the vector of item parameters, and, optionally, `theta` is a grid of person parameter values at which we evaluate the pmf and its derivatives. The resulting object has the following slots.

- `par`: Item parameters.
- `ncat`: Number of categories.
- `P`: Pmf values (Equations 1–4).
- `dP`: First derivatives of `P`.
- `d2P`: Second derivatives of `P`.
- `d3P`: Third derivatives of `P`.
- `d4P`: Fourth derivatives of `P`.

Methods for the class `item` include

- `show`: Print information.
- `logLik`: Evaluate the log pmf.
- `info`: Evaluate the item information function (inner sum of Equation B.11).
- `dinfo`: Evaluate the first derivative of `info` (inner sum of Equation B.12).
- `d2info`: Evaluate the second derivative of `info` (inner sum of Equation B.13).
- `ed3`: Evaluate the expected third derivative of the log pmf (inner sum of Equation B.14).
- `ded3`: Evaluate the first derivative of `ed3` (inner sum of Equation B.15).
- `jterm`: Evaluate the item's contribution to $J(\cdot)$ in the weighted likelihood (WL) estimating equation (inner sum of Equation 6 in Magis, 2015).
- `djterm`: Evaluate the first derivative of `jterm`.

To create an item type inheriting from `item`, one should provide the following five methods. See the definition of derived classes `2pl`, `3pl`, `graded`, and `nom` for more details.

- `P`: Evaluate the pmf.
- `dP`: Evaluate the first derivative of `P`.

- `d2P`: Evaluate the second derivative of `P`.
- `d3P`: Evaluate the third derivative of `P`.
- `d4P`: Evaluate the fourth derivative of `P`.

C.2 Class `test` and methods

The source code file `test.R` contains the definition of the S4 class `test`, as well as the associated methods and functions that can be applied to the class.

The constructor of the class `test` is of form `new("test", thetalim, nint, itemtype, itempar)`, in which the arguments are

- `thetalim`: Integration limit for the person parameter.
- `nint`: Number of intervals used for the composite Simpson's rule (must be even).
- `itemtype`: Character vector of item types.
- `itempar`: List of item parameters.

The following slots are allocated for a `test` object upon construction.

- `nitem`: Number of items.
- `nint`: Number of intervals used for the composite Simpson's rule (must be even).
- `theta`: Vector of person parameter values (length = `nint` + 1).
- `items`: List of item objects.
- `info`: Test information function (Equation B.11) evaluated at `theta`.
- `kappa`: Function $\kappa_n(\cdot)$ (Equation 20) evaluated at `theta`.

The following methods are available for the class `test`.

- `show`: Print information.
- `logLik`: Evaluate the log-likelihood function.
- `info`: Evaluate the test information function (Equation B.11).
- `dinfo`: Evaluate the first derivative of `info` (Equation B.12).
- `d2info`: Evaluate the second derivative of `info` (Equation B.13).
- `ed3`: Evaluate the expected third derivative of the log-likelihood (Equation B.14).
- `ded3`: Evaluate the first derivative of `ed3` (Equation B.15).
- `jterm`: Evaluate the WL estimating equation (Equation 6 in Magis, 2015).
- `djterm`: Evaluate the first derivative of `jterm`.

Additional functions taking a `test` object as the first argument (`x`) include

- `kappa`: Evaluate function $\kappa_n(\cdot)$ (Equation 20)
- `djeff`: Evaluate the (log) Jeffreys prior (Equation 15)

- `ddet`: Evaluate the (log) DET prior (Equation 24)
- `dpost`: Evaluate the posterior pdf
- `ppost`: Evaluate the posterior cdf
- `qpost`: Evaluate the posterior quantile function
- `wlee`: Evaluate the WL estimating equation
- `wlse`: Evaluate the standard error function of the WL estimator
- `mle`: Compute the ML estimator and its standard error
- `wle`: Compute the WL estimator and its standard error

C.3 Examples

In `ex.R`, we create tests composed of the four basic types of items (`test1`—`test4`), upon loading the source code `test.R` and `item.R`. For selected response patterns, we compute the ML scores and standard errors (using function `mle`), the WL scores and standard errors (using function `wle`), the Jeffery CIs (using function `qpost` with prior `djeff`), and the DET CIs (using function `qpost` with prior `ddet`).

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