Supplementary Material to "The Sufficient and Necessary Condition for the Identifiability and Estimability of the DINA Model"

A1: Derivation of Equation (8) in Example 2

In Example 2, we claimed that, given the Q-matrix in the following form where there are J_0 items with $\boldsymbol{q}$-vectors being $(0,0)$ and $J-2-J_0$ items with $\boldsymbol{q}$-vectors being $(1,1)$,

$$Q = \begin{pmatrix} \multicolumn{2}{c}{\mathcal{I}_2} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{pmatrix}_{J\times 2},$$

to construct $(\bar{\boldsymbol{s}}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{p}}) \neq (\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$ satisfying Equation (2) where $\bar{\boldsymbol{s}} = \boldsymbol{s}$, $\bar{g}_j = g_j$ for all $j = 3, \ldots, J$, and $\bar{p}_{(1,1)} = p_{(1,1)}$, it suffices to ensure the Equations (8) hold. Now we prove this argument. Following the proof of the necessity of Conditions C1 and C2 in the Appendix, we can obtain the following equations in (S.1) from Equations (18) in the main text by replacing $(\alpha_1, \alpha_2, \boldsymbol{\alpha}^*)$ in (18) with (α_1, α_2) here, since in this case there are only two attributes. And similarly we have the conclusion that Equation (2) holds as long as Equations (S.1) hold,

$$
\begin{cases}
p_{(0,0)} + p_{(1,0)} + p_{(0,1)} = \bar{p}_{(0,0)} + \bar{p}_{(1,0)} + \bar{p}_{(0,1)}; \\
g_1[p_{(0,0)} + p_{(0,1)}] + (1-s_1)p_{(1,0)} = \bar{g}_1[\bar{p}_{(0,0)} + \bar{p}_{(0,1)}] + (1-s_1)\bar{p}_{(1,0)}; \\
g_2[p_{(0,0)} + p_{(1,0)}] + (1-s_2)p_{(0,1)} = \bar{g}_2[\bar{p}_{(0,0)} + \bar{p}_{(1,0)}] + (1-s_2)\bar{p}_{(0,1)}; \\
g_1 g_2 p_{(0,0)} + (1-s_1)g_2 p_{(1,0)} + g_1(1-s_2)p_{(0,1)} \\
\qquad = \bar{g}_1\bar{g}_2\bar{p}_{(0,0)} + (1-s_1)\bar{g}_2\bar{p}_{(1,0)} + \bar{g}_1(1-s_2)\bar{p}_{(0,1)}.
\end{cases} \tag{S.1}
$$

Adding $p_{(1,1)}$ to both hand sides of the first equation in (S.1), adding $(1-s_1)p_{(1,1)}$ to the second equation, adding $(1-s_2)p_{(1,1)}$ to the third equation and adding $(1-s_1)(1-s_2)p_{(1,1)}$ to the last equation, we exactly obtain (8) in Example 2.

A2: Proof of Corollary 1

When the identifiability conditions are satisfied, the maximum likelihood estimators of $\hat{\boldsymbol{s}}, \hat{\boldsymbol{g}}$, and $\hat{\boldsymbol{p}}$ are consistent as the sample size $N \to \infty$. Specifically, we introduce a 2^J-dimensional empirical response vector

$$
\begin{aligned}
\boldsymbol{\gamma} \;=\; & \Big\{1, N^{-1}\sum_{i=1}^{N} I(\boldsymbol{r}_i \succeq \boldsymbol{e}_1), \cdots, N^{-1}\sum_{i=1}^{N} I(\boldsymbol{r}_i \succeq \boldsymbol{e}_J), \\
& N^{-1}\sum_{i=1}^{N} I(\boldsymbol{r}_i \succeq \boldsymbol{e}_1 + \boldsymbol{e}_2), \cdots, N^{-1}\sum_{i=1}^{N} I(\boldsymbol{r}_i \succeq \boldsymbol{1})\Big\}^{\top},
\end{aligned}
$$

where elements of $\boldsymbol{\gamma}$ are indexed by response vectors arranged in the same order as the rows of the T-matrix. From the definition of the T-matrix and the law of large numbers, we know $\boldsymbol{\gamma} \to T(\boldsymbol{s}, \boldsymbol{g})\boldsymbol{p}$ almost surely as $N \to \infty$. On the other hand, the maximum likelihood estimators $\hat{\boldsymbol{s}}, \hat{\boldsymbol{g}}$, and $\hat{\boldsymbol{p}}$ satisfy $\|\boldsymbol{\gamma} - T(\hat{\boldsymbol{s}}, \hat{\boldsymbol{g}})\hat{\boldsymbol{p}}\| \to 0$, where $\|\cdot\|$ is the L_2 norm. Therefore,

$$
\|T(\boldsymbol{s}, \boldsymbol{g})\boldsymbol{p} - T(\hat{\boldsymbol{s}}, \hat{\boldsymbol{g}})\hat{\boldsymbol{p}}\| \to 0
$$

almost surely. Then from the proof of Theorem 1, we can obtain the consistency result that $(\hat{\boldsymbol{s}}, \hat{\boldsymbol{g}}, \hat{\boldsymbol{p}}) \to (\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$ almost surely as $N \to \infty$. $\square$

A3: Proof of Proposition 2

Consider a Q-matrix of size $J \times K$ in the form

$$Q = \begin{pmatrix} Q' \\ \mathbf{0} \end{pmatrix}, \tag{S.2}$$

where Q' is of size $J' \times K$ and contains those nonzero $\boldsymbol{q}$-vectors of Q. Recall from the model setup in Section 2 of the main text, for any item $j \in \{J'+1, \ldots, J\}$ which has $\boldsymbol{q}_j = \mathbf{0}$, the guessing parameter is not needed by the DINA model and for notational convenience, we set $g_j \equiv \bar{g}_j \equiv 0$, so the slipping parameter s_j is the only unknown item parameter associated with such j. Taking the response pattern $\boldsymbol{r} = \boldsymbol{e}_j$ for any item $j \in \{J'+1, \ldots, J\}$ in Equation (12) gives

$$T_{\boldsymbol{e}_j, \boldsymbol{\cdot}}(\boldsymbol{s}, \boldsymbol{g})\boldsymbol{p} = (1 - s_j) \sum_{\boldsymbol{\alpha} \in \{0,1\}^K} p_{\boldsymbol{\alpha}} = (1 - \bar{s}_j) \sum_{\boldsymbol{\alpha} \in \{0,1\}^K} \bar{p}_{\boldsymbol{\alpha}} = T_{\boldsymbol{e}_j, \boldsymbol{\cdot}}(\bar{\boldsymbol{s}}, \bar{\boldsymbol{g}})\bar{\boldsymbol{p}},$$

then since $\sum_{\boldsymbol{\alpha} \in \{0,1\}^K} p_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha} \in \{0,1\}^K} \bar{p}_{\boldsymbol{\alpha}} = 1$, we have $s_j = \bar{s}_j$ for any $j \in \{J'+1, \ldots, J\}$.

Now denote $\boldsymbol{s}' = (s_1, \ldots, s_{J'})$, $\boldsymbol{g}' = (g_1, \ldots, g_{J'})$ and similarly denote $\bar{\boldsymbol{s}}'$, $\bar{\boldsymbol{g}}'$. Denote the $2^{J'} \times 2^K$ T-matrix associated with matrix Q' by $T'(\boldsymbol{s}', \boldsymbol{g}')$. For any response pattern $\boldsymbol{r} = (r_1, \ldots, r_{J'}, r_{J'+1}, \ldots, r_J) \in \{0,1\}^J$, denote $\boldsymbol{r}' = (r_1, \ldots, r_{J'})$ and $(\boldsymbol{r}', \mathbf{0}) =$

$(r_1, \ldots, r_{J'}, 0, \ldots, 0)$ of length J; then we have

$$T_{\boldsymbol{r},\boldsymbol{\cdot}}(\boldsymbol{s}, \boldsymbol{g})\boldsymbol{p} = \Big\{T_{(\boldsymbol{r}',\boldsymbol{0}),\boldsymbol{\cdot}}(\boldsymbol{s}, \boldsymbol{g})\boldsymbol{p}\Big\} \prod_{j>J'} (1 - s_j)^{r_j} = \Big\{T'_{\boldsymbol{r}',\boldsymbol{\cdot}}(\boldsymbol{s}', \boldsymbol{g}')\boldsymbol{p}\Big\} \prod_{j>J'} (1 - s_j)^{r_j},$$

$$T_{\boldsymbol{r},\boldsymbol{\cdot}}(\bar{\boldsymbol{s}}, \bar{\boldsymbol{g}})\bar{\boldsymbol{p}} = \Big\{T_{(\boldsymbol{r}',\boldsymbol{0}),\boldsymbol{\cdot}}(\bar{\boldsymbol{s}}, \bar{\boldsymbol{g}})\bar{\boldsymbol{p}}\Big\} \prod_{j>J'} (1 - s_j)^{r_j} = \Big\{T'_{\boldsymbol{r}',\boldsymbol{\cdot}}(\bar{\boldsymbol{s}}', \bar{\boldsymbol{g}}')\boldsymbol{p}\Big\} \prod_{j>J'} (1 - s_j)^{r_j}.$$

Using the above equalities, by Proposition 1, we have the following equivalent arguments,

$$\begin{aligned}
& (\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p}) \text{ associated with } Q \text{ are identifiable,} \\
\Longleftrightarrow\quad & \forall (\bar{\boldsymbol{s}}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{p}}) \neq (\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p}),\ \exists \boldsymbol{r} \in \{0,1\}^J \text{ such that } T_{\boldsymbol{r},\boldsymbol{\cdot}}(\boldsymbol{s}, \boldsymbol{g})\boldsymbol{p} \neq T_{\boldsymbol{r},\boldsymbol{\cdot}}(\bar{\boldsymbol{s}}, \bar{\boldsymbol{g}})\bar{\boldsymbol{p}}, \\
\Longleftrightarrow\quad & \forall (\bar{\boldsymbol{s}}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{p}}) \neq (\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p}),\ \exists \boldsymbol{r}' \in \{0,1\}^{J'} \text{ such that } T'_{\boldsymbol{r}',\boldsymbol{\cdot}}(\boldsymbol{s}', \boldsymbol{g}')\boldsymbol{p} \neq T'_{\boldsymbol{r}',\boldsymbol{\cdot}}(\bar{\boldsymbol{s}}', \bar{\boldsymbol{g}}')\bar{\boldsymbol{p}}, \\
\Longleftrightarrow\quad & (\boldsymbol{s}', \boldsymbol{g}', \boldsymbol{p}) \text{ associated with } Q' \text{ are identifiable.}
\end{aligned}$$

Therefore we have shown identifiability of DINA associated with Q in the form of (S.2) is equivalent to that of DINA associated with submatrix Q' in (S.2) and the proof of the proposition is complete.

A4: Proof of Lemma 1

To facilitate the proof of the lemma, we introduce the following proposition, which is from Proposition 3 in Xu (2017). We first generalize the definition of the T-matrix. For any $\boldsymbol{x} = (x_1, \ldots, x_J)^\top \in \mathbb{R}^J$ and $\boldsymbol{y} = (y_1, \ldots, y_J)^\top \in \mathbb{R}^J$, we still define the T-matrix $T(\boldsymbol{x}, \boldsymbol{y})$ to be a $2^J \times 2^K$ matrix, where the entries are indexed by row index $\boldsymbol{r} \in \{0,1\}^J$ and column index $\boldsymbol{\alpha}$. For any row indexed by $\boldsymbol{e}_j$ with $j = 1, \ldots, J$, we let $t_{\boldsymbol{e}_j,\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{y}) = (1 - x_j)^{\xi_{j,\boldsymbol{\alpha}}} y_j^{1-\xi_{j,\boldsymbol{\alpha}}}$; for any $\boldsymbol{r} \neq \boldsymbol{0}$, let the $\boldsymbol{r}$th row vector of $T(\boldsymbol{x}, \boldsymbol{y})$ be $T_{\boldsymbol{r},\boldsymbol{\cdot}}(\boldsymbol{x}, \boldsymbol{y}) = \bigodot_{j:r_j=1} T_{\boldsymbol{e}_j,\boldsymbol{\cdot}}(\boldsymbol{x}, \boldsymbol{y})$.

Proposition S.1 *If $T(\boldsymbol{s}, \boldsymbol{g})\boldsymbol{p} = T(\bar{\boldsymbol{s}}, \bar{\boldsymbol{g}})\bar{\boldsymbol{p}}$, then for any $\boldsymbol{\theta} \in \mathbb{R}^J$, $T(\boldsymbol{s} + \boldsymbol{\theta}, \boldsymbol{g} - \boldsymbol{\theta})\boldsymbol{p} = T(\bar{\boldsymbol{s}} + \boldsymbol{\theta}, \bar{\boldsymbol{g}} - \boldsymbol{\theta})\bar{\boldsymbol{p}}$.*

Let G be the set of items whose guessing parameters have been identified in the sense that $g_j = \bar{g}_j$, for any $j \in G$. Let $G^c := \{1, \ldots, J\} \backslash G$ be the complement of G. Note that $\{K+1, \ldots, J\} \cup S_k^- \cup S_k^+ \subseteq G$. Define

$$\boldsymbol{\theta} = \sum_{j \in G^c} (1 - s_j) \boldsymbol{e}_j + \sum_{j \in G} g_j \boldsymbol{e}_j. \tag{S.3}$$

Denote $T := T(\boldsymbol{s} = \boldsymbol{0}, \boldsymbol{g} = \boldsymbol{0})$ and denote the $(\boldsymbol{r}, \boldsymbol{\alpha})$-entry of T by $t_{\boldsymbol{r},\boldsymbol{\alpha}}$, then by definition,

$$t_{\boldsymbol{r},\boldsymbol{\alpha}} = \prod_{j:\, r_j = 1} 1^{I(\boldsymbol{\alpha} \succeq \boldsymbol{q}_j)} 0^{1 - I(\boldsymbol{\alpha} \succeq \boldsymbol{q}_j)} = I(\boldsymbol{\alpha} \succeq \boldsymbol{q}_j \ \forall j \text{ s.t. } r_j = 1), \tag{S.4}$$

where $I(\cdot)$ denotes the indicator function. Proposition S.1 implies that $T_{\boldsymbol{r},\boldsymbol{\cdot}}(\boldsymbol{s} + \boldsymbol{\theta}, \boldsymbol{g} - \boldsymbol{\theta}) = T_{\boldsymbol{r},\boldsymbol{\cdot}}(\boldsymbol{s} + \boldsymbol{\theta}, \bar{\boldsymbol{g}} - \boldsymbol{\theta}) \bar{\boldsymbol{p}}$ for $\boldsymbol{\theta}$ defined in (S.3). We use $\theta_{j,\boldsymbol{\alpha}}$ to denote the positive response probability of attribute profile $\boldsymbol{\alpha}$ to item j, i.e., $\theta_{j,\boldsymbol{\alpha}} = 1 - s_j$ for $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} \succeq \boldsymbol{q}_j$, and $\theta_{j,\boldsymbol{\alpha}} = g_j$ for $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} \not\succeq \boldsymbol{q}_j$. For any response pattern $\boldsymbol{r}$ such that $r_j = 0$ for all $j \in G^c$,

$$\begin{aligned} T_{\boldsymbol{r},\boldsymbol{\cdot}}(\boldsymbol{s} + \boldsymbol{\theta}, \boldsymbol{g} - \boldsymbol{\theta}) \boldsymbol{p} &= \sum_{\boldsymbol{\alpha} \in \{0,1\}^K} p_{\boldsymbol{\alpha}} \prod_{j \in G} [\theta_{j,\boldsymbol{\alpha}} - g_j]^{r_j} \prod_{j \in G^c} [\theta_{j,\boldsymbol{\alpha}} - (1 - s_j)]^{r_j} \\ &= \sum_{\boldsymbol{\alpha} \in \{0,1\}^K} p_{\boldsymbol{\alpha}} \prod_{j \in G} (\theta_{j,\boldsymbol{\alpha}} - g_j)^{r_j}, \end{aligned} \tag{S.5}$$

where in the above summation over $\boldsymbol{\alpha} \in \{0,1\}^K$, one can see that the product term $\prod_{j \in G} (\theta_{j,\boldsymbol{\alpha}} - g_j)^{r_j}$ is nonzero only for those $\boldsymbol{\alpha}$ such that $\theta_{j,\boldsymbol{\alpha}} = 1 - s_j > g_j$ for all j where $r_j = 1$; and when the product term is nonzero, it equals $\prod_{j \in G} (1 - s_j - g_j)^{r_j}$. Further examining those $\boldsymbol{\alpha}$ that make the product term nonzero in (S.5), one can find it is exactly those $\boldsymbol{\alpha}$ such that $t_{\boldsymbol{r},\boldsymbol{\alpha}} = 1$ according to (S.4). Noting that $t_{\boldsymbol{r},\boldsymbol{\alpha}}$ can either be 1 or 0, (S.5) can be further written as

$$
\begin{aligned}
T_{\boldsymbol{r},\bullet}(\boldsymbol{s}+\boldsymbol{\theta},\boldsymbol{g}-\boldsymbol{\theta})\boldsymbol{p} &= \sum_{\boldsymbol{\alpha}:\, t_{\boldsymbol{r},\boldsymbol{\alpha}}=1} p_{\boldsymbol{\alpha}} \prod_{j\in G}(1-s_j-g_j)^{r_j} \\
&= \sum_{\boldsymbol{\alpha}\in\{0,1\}^K} t_{\boldsymbol{r},\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} \prod_{j\in G}(1-s_j-g_j)^{r_j}.
\end{aligned}
\tag{S.6}
$$

Following the same argument, we also have

$$
T_{\boldsymbol{r},\bullet}(\boldsymbol{s}+\boldsymbol{\theta},\bar{\boldsymbol{g}}-\boldsymbol{\theta})\bar{\boldsymbol{p}} = \sum_{\boldsymbol{\alpha}\in\{0,1\}^K} t_{\boldsymbol{r},\boldsymbol{\alpha}} \bar{p}_{\boldsymbol{\alpha}} \prod_{j\in G}(1-s_j-g_j)^{r_j},
$$

then Proposition S.1 implies

$$
\sum_{\boldsymbol{\alpha}\in\{0,1\}^K} t_{\boldsymbol{r},\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha}\in\{0,1\}^K} t_{\boldsymbol{r},\boldsymbol{\alpha}} \bar{p}_{\boldsymbol{\alpha}}, \text{ for any } \boldsymbol{r} \text{ such that } r_j = 0 \text{ for all } j \in G^c. \tag{S.7}
$$

We then define a response vector $\boldsymbol{r}^* = (r_1^*, \ldots, r_J^*)^\top$ to be $\boldsymbol{r}^* = \sum_{j\in G}(1-q_{j,k})\boldsymbol{e}_j$, that is, $\boldsymbol{r}^*$ has correct responses to and only to those items among the set G that do not require the kth attribute. Let $S_{\boldsymbol{r}^*}$ denote the set of items that $\boldsymbol{r}^*$ has correct responses to, i.e., $S_{\boldsymbol{r}^*} = \{j : r_j^* = 1\}$. Since $S_k^- \subseteq G$ and $q_{j,k} = 0$ for any $j \in S_k^-$, we know $S_{\boldsymbol{r}^*}$ is nonempty. Now consider the row vector in the transformed T-matrix $T(\boldsymbol{s}+\boldsymbol{\theta},\boldsymbol{g}-\boldsymbol{\theta})$ corresponding to response vector $\boldsymbol{r}^* + \boldsymbol{e}_k$, then we have that $T_{\boldsymbol{r}^*+\boldsymbol{e}_k,\boldsymbol{\alpha}}(\boldsymbol{s}+\boldsymbol{\theta},\boldsymbol{g}-\boldsymbol{\theta}) \neq 0$ if and only if

$$
\boldsymbol{\alpha} \succeq \boldsymbol{q}_j \text{ for any item } j \in S_{\boldsymbol{r}^*}, \text{ and } \alpha_k = 0.
$$

In other words, $T_{\boldsymbol{r}^*+\boldsymbol{e}_k,\boldsymbol{\alpha}}(\boldsymbol{s}+\boldsymbol{\theta},\boldsymbol{g}-\boldsymbol{\theta}) \neq 0$ if and only if $\boldsymbol{\alpha}$ satisfies $t_{\boldsymbol{r}^*,\boldsymbol{\alpha}} = 1$ and $t_{\boldsymbol{r}^*+\boldsymbol{e}_k,\boldsymbol{\alpha}} = 0$.

This implies that

$$\begin{aligned}&T_{\boldsymbol{r}^*+\boldsymbol{e}_k,\bullet}(\boldsymbol{s}+\boldsymbol{\theta},\boldsymbol{g}-\boldsymbol{\theta})\boldsymbol{p}\\&=(g_k+s_k-1)\prod_{j\in S_{\boldsymbol{r}^*}}(1-s_j-g_j)\sum_{\boldsymbol{\alpha}\in\{0,1\}^K}(t_{\boldsymbol{r}^*,\boldsymbol{\alpha}}-t_{\boldsymbol{r}^*+\boldsymbol{e}_k,\boldsymbol{\alpha}})p_{\boldsymbol{\alpha}}\end{aligned}\tag{S.8}$$

and

$$\begin{aligned}&T_{\boldsymbol{r}^*+\boldsymbol{e}_k,\bullet}(Q,\boldsymbol{s}+\boldsymbol{\theta},\bar{\boldsymbol{g}}-\boldsymbol{\theta})\cdot\bar{\boldsymbol{p}}\\&=(\bar{g}_k+s_k-1)\prod_{j\in S_{\boldsymbol{r}^*}}(1-s_j-g_j)\sum_{\boldsymbol{\alpha}\in\{0,1\}^K}(t_{\boldsymbol{r}^*,\boldsymbol{\alpha}}-t_{\boldsymbol{r}^*+\boldsymbol{e}_k,\boldsymbol{\alpha}})\bar{p}_{\boldsymbol{\alpha}}.\end{aligned}\tag{S.9}$$

Note that (S.8) = (S.9) by Proposition 2.

We next show that the summation terms in (S.8) and (S.9) satisfy

$$\sum_{\boldsymbol{\alpha}\in\{0,1\}^K}(t_{\boldsymbol{r}^*,\boldsymbol{\alpha}}-t_{\boldsymbol{r}^*+\boldsymbol{e}_k,\boldsymbol{\alpha}})p_{\boldsymbol{\alpha}}=\sum_{\boldsymbol{\alpha}\in\{0,1\}^K}(t_{\boldsymbol{r}^*,\boldsymbol{\alpha}}-t_{\boldsymbol{r}^*+\boldsymbol{e}_k,\boldsymbol{\alpha}})\bar{p}_{\boldsymbol{\alpha}}\neq 0.\tag{S.10}$$

Note $\boldsymbol{r}^*$ satisfies the condition in (S.7) that $r_j^*=0$ for all $j\in G^c$. Therefore,

$$\sum_{\boldsymbol{\alpha}\in\{0,1\}^K}t_{\boldsymbol{r}^*,\boldsymbol{\alpha}}p_{\boldsymbol{\alpha}}=\sum_{\boldsymbol{\alpha}\in\{0,1\}^K}t_{\boldsymbol{r}^*,\boldsymbol{\alpha}}\bar{p}_{\boldsymbol{\alpha}}.\tag{S.11}$$

We further consider the response vector $\boldsymbol{r}^*+\boldsymbol{e}_k$. Under the conditions of Lemma 1, there exists some item $h\in G$ such that

$$q_{h,k}=1\text{ and }\{l:q_{h,l}=1,\ l\neq k\}\subseteq\bigcup_{j\in S_{\boldsymbol{r}^*}}\{l:q_{j,l}=1\}.$$

That is, the item h requires the kth attribute and h's any other required attribute is also required by some item in the set $S_{\boldsymbol{r}^*}$. Therefore we have $T_{\boldsymbol{r}^*+\boldsymbol{e}_k,\bullet}=T_{\boldsymbol{r}^*\vee\boldsymbol{r}^{\#},\bullet}$, where $\boldsymbol{r}^{\#}:=\sum_{h\in S_j^+\setminus S_j^-}\boldsymbol{e}_h$; in addition, since the response vector $\boldsymbol{r}^*\vee\boldsymbol{r}^{\#}$ satisfies the condition in (S.7) that its jth element $(\boldsymbol{r}^* \vee \boldsymbol{r}^{\#})_j = 0$ for any $j \in G^c$, we have

$$\begin{aligned}\sum_{\boldsymbol{\alpha}\in\{0,1\}^K} t_{\boldsymbol{r}^*+\boldsymbol{e}_k,\boldsymbol{\alpha}} \cdot p_{\boldsymbol{\alpha}} &= \sum_{\boldsymbol{\alpha}\in\{0,1\}^K} t_{\boldsymbol{r}^*\vee\boldsymbol{r}^{\#},\boldsymbol{\alpha}} \cdot p_{\boldsymbol{\alpha}} \\ &= \sum_{\boldsymbol{\alpha}\in\{0,1\}^K} t_{\boldsymbol{r}^*\vee\boldsymbol{r}^{\#},\boldsymbol{\alpha}} \cdot \bar{p}_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha}\in\{0,1\}^K} t_{\boldsymbol{r}^*+\boldsymbol{e}_k,\boldsymbol{\alpha}} \cdot \bar{p}_{\boldsymbol{\alpha}}.\end{aligned} \tag{S.12}$$

The first equation in (S.10) then follows from (S.11) and (S.12). The inequality in (S.10) also holds since $t_{\boldsymbol{r}^*,\boldsymbol{\alpha}} \geq t_{\boldsymbol{r}^*+\boldsymbol{e}_k,\boldsymbol{\alpha}}$ for any $\boldsymbol{\alpha}$ and $t_{\boldsymbol{r}^*,\boldsymbol{\alpha}} > t_{\boldsymbol{r}^*+\boldsymbol{e}_k,\boldsymbol{\alpha}}$ for those $\boldsymbol{\alpha}$ with $\alpha_k = 0$ and $\boldsymbol{\alpha} \succeq \boldsymbol{q}_j$ for any item $j \in S_{\boldsymbol{r}^*}$.

With the results in (S.10), we have $g_k = \bar{g}_k$ from the equality of (S.8) and (S.9). This completes the proof. □

References

Xu, G. (2017). Identifiability of restricted latent class models with binary responses. *The Annals of Statistics*, 45:675–707.